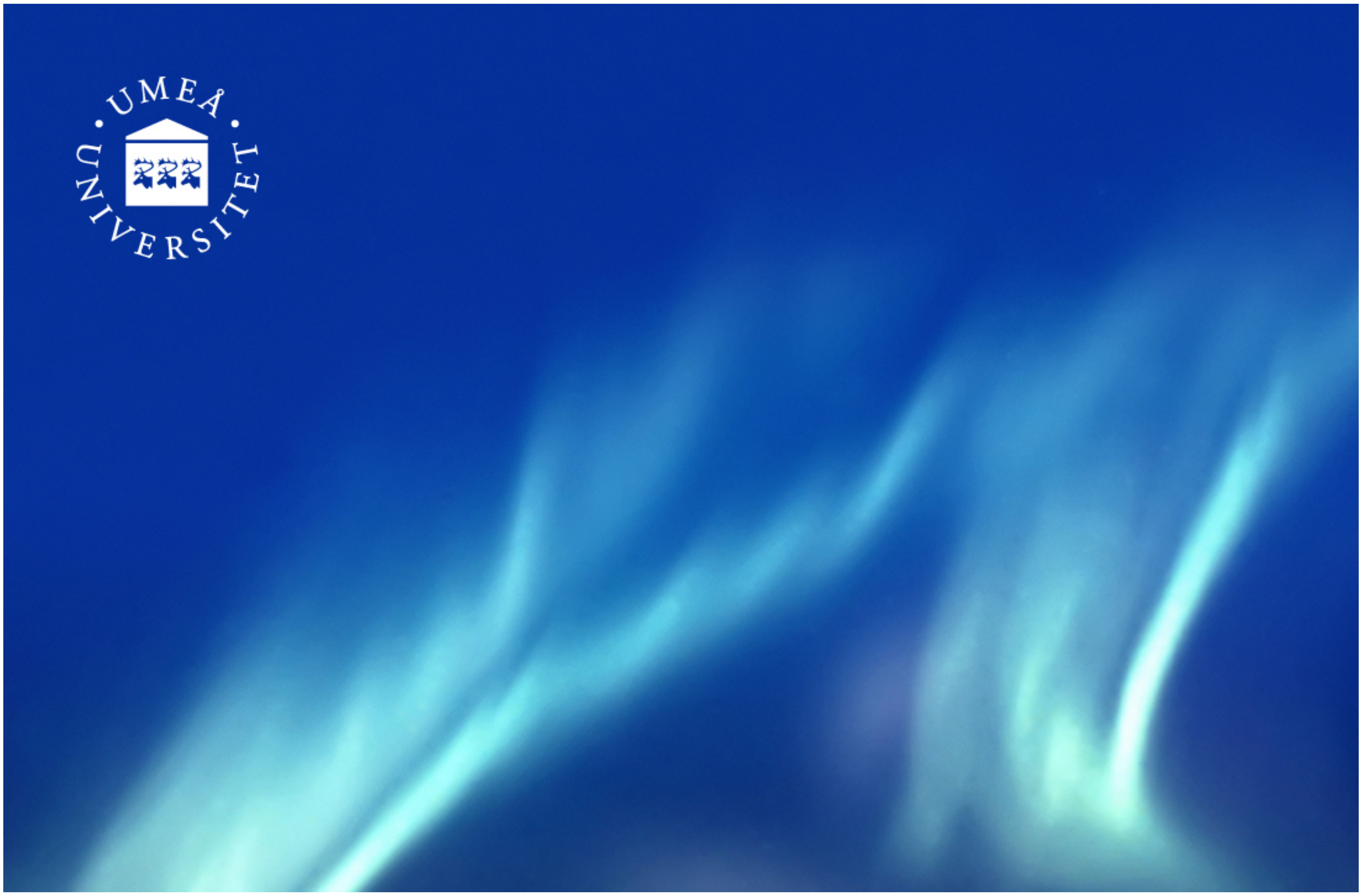

The Automorphism Groups on the Complex Plane

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Abstract

The automorphism groups in the complex plane are defined, and we prove that they satisfy the group axioms. The automorphism group is derived for some domains. By applying the Riemann mapping theorem, it is proved that every automorphism group on simply connected domains that are proper subsets of the complex plane, is isomorphic to the automorphism group on the unit disc.

Sammanfattning

Automorfigrupperna i det komplexa talplanet definieras och vi bevisar att de uppfyller gruppaxiomen. Automorfigruppen på några domän härleds. Genom att applicera Riemanns avbildningssats bevisas att varje automorfigrupp på enkelt sammanhängande, öppna och äkta delmängder av det komplexa talplanet är isomorf med automorfigruppen på enhetsdisken.

Contents

1. Introduction

To study the geometry of complex domains one central tool is the *automorphism group*. Let $D \subseteq \mathbb{C}$ be an open and connected set. The automorphism group of D is then defined as the collection of all biholomorphic mappings $D \to D$ together with the composition operator. Loosely, automorphic functions are functions that preserve the structure of a object.

The young Poincaré was one of the first to do work on automorphic functions, and ultimately it helped clarify a many different areas of mathematics. He introduced a language by combining different areas of mathematics, and his work on automorphic functions have most famously led to the proof of Fermat´s last theorem. [11]

It began as the Academy of Sciences of Paris proposed a challenge in 1880 to perfect some of the language in differential equations in one variable. As Poincaré entered his second memoir in the 1880s, he built the first groundwork on automorphic functions [2]. This memoir focused on *Fuchsian functions* which was inspired from an article by L. Fuchs. It was Klein who coined the modern term automorphic functions in 1890 [11]. Later, from all the work on automorphic functions, Poincaré and Koebe managed to prove the uniformization theorem, it lead further to the proof that the generalization of the uniformization theorem to arbitrary dimensions does not hold [4].

In Section 4, we shall explicitly deduce the following automorphism groups. Later in this introduction, we will see the importance of knowing explicitly what these groups are.

- $\mathrm{Aut}(\mathbb{C}) = \{az + b : a, b \in \mathbb{C},\ a \neq 0\}$
  (Theorem 4.9);
- $\mathrm{Aut}(\hat{\mathbb{C}}) = \left\{ \dfrac{az+b}{cz+d} : a, b, c, d \in \mathbb{C},\ \text{and } ad - bc \neq 0 \right\}$,
  where $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ (Theorem 4.11);
- $\mathrm{Aut}(B_1(0)) = \left\{ e^{i\phi} \dfrac{a - z}{1 - \bar{a}z} : a \in B_1(0),\ \text{and } \phi \in (0, 2\pi] \right\}$,
  where $B_1(0) \subset \mathbb{C}$ is the unit disc (Theorem 4.13);
- $\{f \in \mathrm{Aut}(D) : f(A) = A\}$ and $\mathrm{Aut}(D \backslash A)$ are group isomorphic
  where D is a bounded domain, and $A \subset D$ (Theorem 4.19).

The main result of this essay is the following theorem.

Theorem 5.10. *Let $D \subset \mathbb{C}$ be a simply connected and open set not equal $\mathbb{C}$. Then* $\mathrm{Aut}(D)$ *is isomorphic to* $\mathrm{Aut}(B_1(0))$ *in the sense of groups.*

Theorem 5.10 is a direct consequence of the celebrated Riemann mapping theorem (Theorem 5.8). This emphasises the importance of the above examples. Consider a simply connected domain $D \subseteq \mathbb{C}$, then from Theorem 5.10, we need only be interested in $\mathrm{Aut}(\mathbb{C})$, and $\mathrm{Aut}(B_1(0))$. The domain $B_1(0)\backslash\{0\}$ is not simply connected, and $\mathrm{Aut}(B_1(0)\backslash\{0\})$ is not isomorphic to $\mathrm{Aut}(B_1(0))$, which shows that Theorem 5.10 does not hold for connected and open sets that are not simply connected. Even so, it is possible to generalise Theorem 5.10 to Riemann surfaces, i.e. 1-dimensional complex manifolds. The uniformization theorem, as it is called, states: A simply connected Riemann surface, S, is biholomorphically equivalent to either $\mathbb{C}$, $\hat{\mathbb{C}}$ or $B_1(0)$ (see e.g. [1]). In the early 1900 Poincaré showed that Theorem 5.10 is not possible to generalise to higher dimension with the following famous example. He showed that the automorphism group of the bi-disc in $\mathbb{C}^2$,

$$\{(z_1, z_2) \in \mathbb{C}^2 : |z_1| < 1, |z_2| < 1\} ,$$

and automorphism group of the unit ball in $\mathbb{C}^2$,

$$\{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 < 1\} .$$

is not isomorphic. Even as they both are simply connected [6].

The overview of the essay is as follows. In Section 2, we introduce some background theory. The background related to abstract algebra is based on [3], and the background related to complex analysis is mainly inspired by [8, 10]. In the third Section, the automorphism group is defined. Ultimately, by some theorems taken from [8], the group axioms of the automorphism group are proved. In Section four, the automorphism group on the complex plane, extended complex plane and unit disc is derived. Also, biholomorphic equivalence is introduced, and automorphism groups on punctured domains are considered. Mainly, this section is inspired by [4–6, 8]. In Section 5, Theorem 5.10 is proved by applying the Riemann mapping theorem, this section is based on [5, 6].

2. Preliminaries

In this section, we shall introduce some concepts from group theory and complex analysis that will be needed in this essay.

2.1. Group and Set Theory.

For our purposes we will take the notion of a *set* to be obvious, it is well understood what a collection of *something* is. Thus we take the definition of a set intuitively. However, the concept of *supremum* is very useful, but since not all readers may have encountered it before, we shall state it here.

Definition 2.1. Let $X \subset \mathbb{R}$, then if there exists a

$$a = \min\{\alpha \in \mathbb{R} | x \leq \alpha, \text{ for all } x \in X\}$$

then we say that a is the *supremum* of X. We denote this as $\sup(X)$.

Notice that this is not the general definition of supremum on ordered sets, a see e.g. Definition 1.8 in [9]. For our purposes it is sufficient to assume that we take the supremum of some subset of $\mathbb{R}$. In many cases the supremum will be analogous to the maximum of a set, and it may help the reader to intuitively *read* maximum whenever we talk about supremum. Though, remember they are not always the same.

Throughout this essay, we will make frequent use of discs centered at a point, $c \in \mathbb{C}$, with radius $r > 0$, and the one point compactification of the complex plane (or extended complex plane) . We will denote them as $B_r(c)$ and $\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$, respectively. Furthermore, we denote the unit disc as $B_1(0)$.

Now, we will briefly go through some fundamental properties from group theory. Loosely speaking group theory is about generalising algebra. There is no need to limit oneself to only talk about numbers and addition, etc. What if we were to replace all the integers with different kinds of bananas (now imagine there are countably infinite amount of different bananas), then we could simply talk about putting together two different bananas and you would potentially get a new kind banana. For these bananas we can not apply directly our regular addition operator, rather in generality we talk about *binary operators*. Our goal is then to apply binary operators to sets, as in our example, the set of different kinds of bananas.

Definition 2.2. A *binary operator*, $*$, on a set A is a mapping from $A \times A$ into A. For $(a, b) \in A \times A$, we denote $*((a, b))$ as $a * b$.

Both ordinary addition and multiplication are examples of binary operators on $\mathbb{Z}$. In particular, it is important for our purposes to realise that the composition operator, $\circ$, is a binary operator on the set of all complex valued functions.

Definition 2.3. Let B be a subset of A, and $*$ be a binary operator on A. Then B is said to be *closed under* $*$, if for every $a, b \in B$, $a * b \in B$.

It is easy to check that $\circ$ is closed under the set of all complex valued functions. By restricting the binary operator to three properties, we can define a so called *group*.

Definition 2.4. Let A be a set closed under a binary operator $*$. We call $(A, *)$ a *group* if:

G1: the binary operator, $*$, satisfies

$$a * (b * c) = (a * b) * c,$$

for all $a, b, c \in A$, i.e. $*$ is associative in A.

G2: there exists an identity element $e \in A$ for $*$, such that

$$e * a = a * e = a,$$

for all $a \in A$;

G3: for each $a \in A$ there exists an inverse element $a' \in A$ such that

$$a * a' = a' * a = e.$$

The definition of a group may seem redundant for the reader with little experience with group theory, but it is really useful. One such use is when we can relate two different groups, we want a mapping between groups that preserves some structure. More formally, we call such mappings *group homomorphisms*, or simply *homomorphisms* if there is no risk for ambiguity.

Definition 2.5. A mapping $\phi : G \to G'$ between two groups $(G, *)$, $(G', *')$ is called a *homomorphism* if

$$\phi(g_1 * g_2) = \phi(g_1) *' \phi(g_2)$$

for all $g_1, g_2 \in G$.

One class of such homomorphisms are *isomorphisms*. Intuitively, when there is a isomorphism between two groups the groups are equivalent. For example, imagine if there is a isomorphic mapping between $(\mathbb{Z}, +)$ and our banana group, then we could hypothetically replace the number system with bananas, and there would be no algebraic difference between them. In spite of this, the number system may still be favourable for convenience reasons.

Definition 2.6. Two groups $(A, *)$, $(B, *')$ are said to be *isomorphic* if there exists a bijective homomorphism $\phi : A \to B$.

There are two properties of groups, in particular, that we will make use of. These are the *abelian* property and *normal* property. Observe that these properties are examples of properties that are transferred under isomorphisms.

Definition 2.7. A group, G, is said to be *abelian* if for all $a, b \in G$, $a * b = b * a$.

Note, for example that $(\mathbb{Z}, +)$ is abelian, but the group of all complex functions under composition is not abelian.

Definition 2.8. A subgroup, H, of a group, G, is said to be *normal* if for every $g \in G$, we have that $\{g * h \ : h \in H\} = \{h * g \ : h \in H\}$.

Observe that if G is abelian, then it follows directly that all subgroups H of G are normal, but the converse relation does not hold in generality.

2.2. **Topology and Complex Analysis.** Now we shall go through some concepts from topology and complex analysis. We will keep some of the concepts from topology relatively simple, and most readers should have encountered most of them in an introductory course in complex analysis. Topologically, *open sets* in $\mathbb{C}$, are by definition arbitrary unions and finite intersections of open discs. To explain this requires too much machinery, we will instead start by defining open sets in $\mathbb{C}$ less generally.

Definition 2.9. A point, $z_0 \in U \subseteq \mathbb{C}$, is said to be an *interior point of* U if there exists an open disc around z_0 that lies in U, i.e. $B_r(z_0) \subset U$ for some $r > 0$. A set $A \subseteq \mathbb{C}$ is said to be *open* if every element $a \in A$ is an interior point of A.

Further, the concept of connected sets is essential in complex analysis, so we will mention it.

Definition 2.10. Let $D \subseteq \mathbb{C}$ be a subset, then D is said to be *connected* if there do not exist two non-empty open subsets $U, V \subset D$ such that U and V are disjoint, and $U \cup V = D$. A subset $D \subseteq \mathbb{C}$ is said to be *simply connected*, if every closed curve on D is continuously rectifiable to a point.

For a more rigorous description of simply connected sets, see e.g. Definition 4.6 in [10]. This brings us to domains.

Definition 2.11. A *domain* $D \subseteq \mathbb{C}$ is an open connected set.

Note that domains are sometimes also referred to as regions in some literature. Also, we shall need the concept of *homeomorphisms*. Notice the difference of spelling between a group homomorphism and a homeomorphism, they are two very different functions.

Definition 2.12. A bijective function $\phi : \mathbb{C} \to \mathbb{C}$ is said to be an *homeomorphism* if it is continuous, and its inverse is also continuous.

Since we want to talk about holomorphic functions, it is essential to define complex differentiability.

Definition 2.13. Let $U \subseteq \mathbb{C}$ be an open set and let $f : U \to \mathbb{C}$ be a function. Then, f is said to be *(complex) differentiable at a point* $z_0 \in U$ if the limit,

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists.

This allows us to define *holomorphic functions*.

Definition 2.14. Let $D \subseteq \mathbb{C}$ be an open set and let $f : D \to \mathbb{C}$ be a function. Then, f is called *holomorphic* if it is differentiable at every point in D.

Observe that it is not uncommon to use the word analytic instead of holomorphic, but we will stick with holomorphic throughout this essay. To simplify reading, we shall use the following definition.

Definition 2.15. Let $U, V \subseteq \mathbb{C}$ be open sets, and let $f : U \to V$ be a bijective, holomorphic function. If the inverse $f^{-1} : V \to U$ is holomorphic, then we call f a *biholomorphic mapping*.

Remark 2.16. One often define biholomorphic functions as holomorphic and bijective functions in literature. By Theorem 3.8, we will see that such a definition of biholomorphy and Definition 2.15 are equivalent.

Harmonic functions are very essential in complex analysis as well. Notice that in $\mathbb{C}$, we only need to consider functions in $\mathbb{R}^2$, and we will restrict our definition as such.

Definition 2.17. Let $f : U \to \mathbb{R}$ be a twice differentiable function and let $U \subset \mathbb{R}^2$ be a open subset. Then, f is said to be *harmonic* if it holds that

$$\triangle f(x, y) = 0$$

for all $(x, y) \in U$, where $\triangle$ is the laplace-operator.

It is well known that for a holomorphic function, $f : \mathbb{C} \to \mathbb{C}$, we can write f in terms of functions $u, v : \mathbb{R}^2 \to \mathbb{R}$ as

$$f(x + iy) = u(x, y) + iv(x, y).$$

Furthermore, u and v are then harmonic. Also, the concept of *neighborhoods* are widely used, but take care observing that different literature might define neighborhood differently.

Definition 2.18. An open set $U \subseteq X$ is called a *neighborhood* of a point $p \in X$ if $p \in U$.

Another useful topological concept are *cluster points*.

Definition 2.19. A point $p \in \mathbb{C}$ is called a *cluster point* of a set $M \subset \mathbb{C}$ if

$$U \cap (M \backslash \{p\}) \neq \varnothing$$

for every neighborhood U of p. A set $A \subset \mathbb{C}$ is called *discrete*, if no point $a \in A$ is a cluster point of A.

Note for example that the set $\{1, 2, 3\}$ is discrete in $\mathbb{C}$. Furthermore, cluster points is a very useful concept, as we can now define the *closure* of sets.

Definition 2.20. Let $A \subset \mathbb{C}$ be a set, the *closure* of a set A is defined as

$$A \cup \{a \in \mathbb{C} : a \text{ is a cluster point of } A\}.$$

We denote the closure of A as $\bar{A}$.

One should also familiarize oneself with the concept of *uniform convergence* of a sequence of functions. Take note of the subtle difference between uniform convergence, and point-wise convergence.

Definition 2.21. Let $A \subseteq \mathbb{C}$ be an open set and let $\{f_n\}$, $f_n : A \to \mathbb{C}$, be a sequence of functions. Then we say that $\{f_n\}$ converges *uniformly* to $f : B \to \mathbb{C}$ if for every $\epsilon > 0$, there exists a $N \in \mathbb{N}$ such that

$$|f_n(z) - f(z)| < \epsilon, \qquad \text{for all} \quad n \geq N, \quad \text{and all} \quad z \in B.$$

Since we have not defined topological space, we will make use of a less general definition of *compact* sets. As always, we need only consider subsets of the complex plane.

Definition 2.22. A subset K of $\mathbb{C}$ is said to be *compact* if for every collection of open sets $\{U_\alpha\}$ whose union contains K, then there exists a finite subset of $\{U_\alpha\}$ whose union also contains K.

More commonly, we shall make use of that that compact sets in $\mathbb{C}$ are equivalent to bounded and closed sets. This property follows from the famous Heine Borel's Theorem.

Theorem 2.23. *A subset $K \subset \mathbb{C}$ is compact if, and only if, K is bounded and closed.*

Proof. See e.g. Theorem 27.3 in [7]. □

Since we shall treat *meromorphic functions*, we need to define *isolated singularities*.

Definition 2.24. Let D be a domain with a point $c \in D$ and let $f : D\backslash\{c\} \to \mathbb{C}$ be a holomorphic function. Then, c is said to be an *isolated singularity* of f, furthermore c is one of the following singularities;

i) if $\lim_{z\to c}(z-c)f(z) = 0$, then c is called a *removable singularity* of f;

ii) if there exists a $m \in \mathbb{N}$ such that $\lim_{z\to c}(z-c)^m f(z) = 0$, then c is called a *pole* of f;

iii) otherwise, c is said to be an *essential singularity* of f.

The last part of Definition 2.24 is actually a theorem, for details see e.g. Proposition 4.3.3. [5], but we shall take it as definition. Then, we can define meromorphy.

Definition 2.25. Let $A \subset D$ be a discrete, possibly empty, set in an open set, $D \subseteq \mathbb{C}$. Then, $f : D \to \hat{\mathbb{C}}$ is said to be a *meromorphic function* if, $f : D\backslash A \to \mathbb{C}$ is holomorphic, and if every point $a \in A$ is a pole of f.

Furthermore, we shall use the following theorems.

Theorem 2.26. *Let $D \subseteq \mathbb{C}$ be a domain, and let $f : D \to \mathbb{C}$ be a holomorphic function. Then, f is expandable around each point $c \in D$ into a Taylor series $\sum_j a_j(z-c)^j$, and the series converges uniformly around c on each compact subset of $B_d(c) \subset D$. The Taylor coefficients a_j are given by the integrals,*

$$a_j = \frac{f^{(j)}(c)}{j!} = \frac{1}{2\pi i}\int_{\partial B_r(c)} \frac{f(\zeta)d\zeta}{(\zeta-c)^{j+1}} \tag{2.1}$$

where $0 < r < d$.

In particular, f is infinitely complex differentiable at every point in D, and in every disc $B_r(c)$ the Cauchy integral formula hold:

$$f^{(j)}(z) = \frac{j!}{2\pi i}\int_{\partial B_r(c)} \frac{f(\zeta)d\zeta}{(\zeta-z)^{j+1}}, \quad z \in B, \quad \text{for all } j \in \mathbb{N}. \tag{2.2}$$

Proof. See e.g. Theorem 7.3.2 in [8]. □

Theorem 2.26, also called Cauchy-Taylor representation theorem, most importantly says that all holomorphic functions are infinitely differentiable, and can be expressed as a Taylor expansion. Notice that infinite differentiablility is special to holomorphic functions as this is not true for real differentiable functions.

From the Cauchy integral formula, we have the argument principle which rather nicely sums up everything we need to know about integration in the complex plane.

Theorem 2.27 (Argument principle)**.** *Let D be a domain and let $f : D \to \mathbb{C}$ be a holomorphic function. If f is non-zero at each point of a simple, closed, positively oriented and smooth curve C, and if f is meromorphic inside C then,*

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}dz = N_0(f) - N_p(f),$$

where $N_0(f)$, $N_p(f)$ are the number of zeros and poles of f inside C, respectively.

Proof. See e.g. Theorem 6.3 in [10]. □

Furthermore, the Cauchy integral formula also implies a theorem on estimates of holomorphic functions.

Theorem 2.28. *Let D be a domain, $f : D \to \mathbb{C}$ a holomorphic function and let $\bar{B}_r(c) \subset D$ be a closed disc. If*

$$M = \sup_{z\in\bar{B}_r(c)} |f(z)|$$

is well-defined and finite. Then, the Cauchy inequality

$$|f^{(j)}(c)| \le \frac{j!M}{r^j}$$

holds.

Proof. See e.g. Theorem 3.4.1 in [5]. □

Lastly, we will frequently use the maximum principle.

Theorem 2.29 (Maximum principle)**.** *Let D be a domain and let $f : D \to \mathbb{C}$ be a holomorphic function. If f has a local maximum at some point $c \in D$, then f is constant.*

Proof. See e.g. Theorem 4.23 in [10]. □

Remember that the maximum principle (Theorem 2.29), holds only for holomorphic functions, and is most certainly not true for real differentiable functions.

3. The Definition of the Automorphism Group

In this section, we will introduce the concept of *automorphism groups*. In short, an automorphism group is the set of all biholomorphic functions that maps a domain to itself, where the binary operator is the composition operator. The main goal of this section is to prove Theorem 3.10, which states that the automorphism group fulfil the group axioms in Definition 2.4. But in order to do this, some machinery is needed to be developed and understood. The overall structure of this section is summarized in figure 3.1.

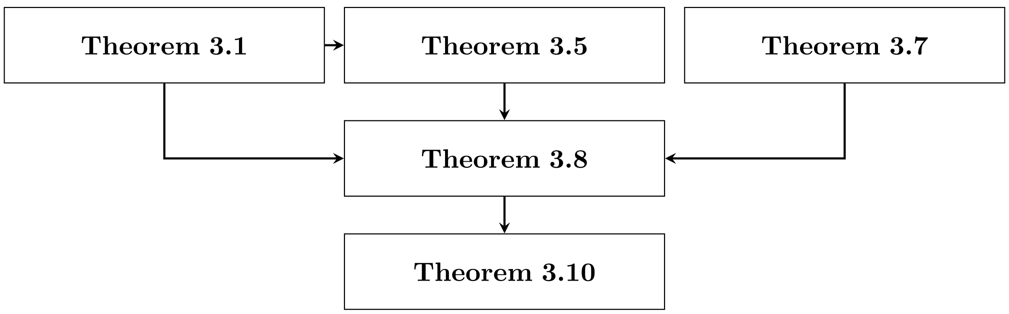

Figure 3.1 – A flowchart of the structure of this section.

We will start with the so called identity theorem. Observe that a *coincidence point* between two functions, f, g, is a point ω such that $f(\omega) = g(\omega)$. We shall refer to the set of all such points as the *coincidence set*.

Theorem 3.1. *Let D be a domain and let $f, g : D \to \mathbb{C}$ be holomorphic functions. Then, the following are equivalent:*

i) $f = g$;

ii) the coincidence set $\{w \in D : f(w) = g(w)\}$ *has a cluster point in* D;

iii) there is a point $c \in G$ *such that* $f^{(n)}(c) = g^{(n)}(c)$ *for all* $n \in \mathbb{N}$.

Proof. $i) \Rightarrow ii)$: Assume that $f = g$. Then, the coincidence set is all of D. It then follows directly by the construction of the open sets, that any neighborhood, U, of a point $p \in D$ has infinitely many elements. Hence, the intersection of D and U has infinitely many elements, and it is therefore non-empty. By the definition of coincidence set there is at least one cluster point p in D.

$ii) \Rightarrow iii)$ In this implication, we shall use proof by contradiction. Let c be a cluster point of the coincidence set of f and g. Let $h = f - g$, then $h(c) = 0$. Assume that there is a $m \in \mathbb{N}$ with

$$h^{(m)}(c) \neq 0,$$

we consider the smallest such m. Notice that h is holomorphic since f and g are holomorphic. Furthermore, we know that the zero set,

$$M = \{w \in D : h(w) = 0\},$$

of h has a cluster point c in D. Then, it follows from Theorem 2.26, that we can factorise h so that

$$h(z) = (z - c)^m h_m(z), \tag{3.1}$$

where

$$h_m(z) := \sum_{\mu \geq m} \frac{h^{(\mu)}(c)}{\mu!}(z - c)^{\mu - m},$$

and $h_m(c) \neq 0$. This holds for every open disc $B_r(c) \subset D$. Furthermore, h_m is zero-free in a small enough neighborhood $U \subset D$ of c, since h_m is continuous and $h_m(c) \neq 0$. It then follows from Equation (3.1) that

$$M \cap (U \backslash \{c\}) = \varnothing.$$

Thus, c is not a cluster point of M. But this is a contradiction to that c is a cluster point of M, and therefore, there does not exist any m such that

$$h^{(m)}(c) \neq 0.$$

That is,

$$h^{(n)}(c) = 0 \qquad \text{for all } n \in \mathbb{N}$$

and

$$f^{(n)}(c) = g^{(n)}(c) \qquad \text{for all } n \in \mathbb{N}.$$

iii) $\Rightarrow$ *i*): Set $h = f - g$. Each set

$$S_k := \{w \in D : h^{(k)}(w) = 0\}$$

is closed in D, which follows from the continuity of $h^{(k)}$. Hence, the intersections of all S_k,

$$S = \bigcap_{k=0}^{\infty} S_k$$

is also closed in D. Furthermore, S is also open in D. This comes from that if $z_1 \in S$, then we have that the Taylor series of h around z_1 is term-wise zero in a open disc $B_r(z_1) \subseteq D$. Then, $h^{(k)}|_{B_r(z_1)} = 0$ for every $k \in \mathbb{N}$. But then $B_r(z_1) \subseteq S$ from construction of S. It follows that all points in S are interior points and S is open in D. By hypothesis, we know $c \in S$, so S is not empty, but the only non-empty subset of D that is both open and closed is D itself. We can conclude that $S = D$, hence $f = g$ in all of D. □

Remark 3.2. Note also that it is indeed important to require that D is connected. Suppose for example that D can be split into two open non-intersecting sets B_1, B_2 so that $D = B_1 \cup B_2$. Then, we can set $f(D) := 1$, $g(B_1) := 1$ and $g(B_2) := 2$. Here we have that the properties (*ii*) and (*iii*) are fulfilled, but obviously it is not true that $f = g$ in all of D.

Furthermore, we need the open mapping theorem. It states that non-constant holomorphic functions maps open sets to open sets. In order to show this, we need two lemmas that are on their own important. Firstly, we will introduce a lemma, that is sometimes referred to as the existence theorem for zeros.

Lemma 3.3. *Let $B_r(c)$ be an open disc with $\bar{B}_r(c) \subset D$, where D is a domain, and let $f : D \to \mathbb{C}$ be a holomorphic function that satisfies $\min_{z \in \partial B_r(c)} |f(z)| > |f(c)|$. Then, f has a zero in $B_r(c)$.*

Proof. We shall use a proof by contradiction. Let f be a function satisfying the assumptions of the Lemma. Assume that f is non-zero in $B_r(c)$. Then, it is necessarily non-zero in some neighborhood $U \subset D$ that contains $\bar{B}_r(c)$, since f is continuous. We can now construct a function $g : U \to \mathbb{C}$, by

$$g(z) := \frac{1}{f(z)}.$$

By construction, g is well defined and holomorphic on U. Theorem 2.28 implies that,

$$\begin{aligned} |f(c)|^{-1} &= |g(c)| \\ &\leq \max_{z \in \partial B_r(c)} |g(z)| \\ &= \max_{z \in \partial B_r(c)} \frac{1}{|f(z)|} \\ &= \Big(\min_{z \in \partial B_r(c)} |f(z)| \Big)^{-1}. \end{aligned}$$

It follows that

$$|f(c)| \geq \min_{z \in \partial B_r(c)} |f(z)|,$$

which is a contradiction to the assumption that $\min_{z \in \partial B_r(c)} |f(z)| > |f(c)|$. □

Secondly, from Lemma 3.3, we can show a quantitative version of the open mapping theorem.

Lemma 3.4. *Let $B_r(c)$ be an open disc with $\bar{B}_r(c) \subset D$ and let $f : D \to \mathbb{C}$ be a holomorphic function satisfying $2\delta := \min_{z \in \partial B_r(c)} |f(z) - f(c)| > 0$. Then, $f(B_r(c)) \supset B_\delta(f(c))$.*

Proof. For every $b \in \mathbb{C}$ such that $|b - f(c)| < \delta$, we have by the triangle inequality that

$$|f(z) - b| \geq |f(z) - f(c)| - |b - f(z))| > \delta, \qquad \text{for all } z \in \partial B_r(c).$$

This means that

$$\min_{z \in \partial B_r(c)} |f(z) - b| > |f(c) - b|.$$

From Lemma 3.3, we have that $f(z) - b$ has a zero somewhere in $B_r(c)$. Hence, there exists a $z' \in V$ such that $f(z') = b$, and one can conclude that $B_\delta(f(c)) \subset f(B_r(c))$. □

Now, we can prove the open mapping theorem.

Theorem 3.5 (Open mapping theorem)**.** *Let $f : \mathbb{C} \to \mathbb{C}$ be a holomorphic function and let f be nowhere locally constant on a domain $D \subseteq \mathbb{C}$, then $f(D)$ is open in $\mathbb{C}$.*

Proof. We shall show that for every open subset $U \subseteq D$, and for every $c \in U$, $f(U)$ contains an open disc around $f(c)$. By assumption, f is not constant in a neighborhood of each point $c \in U$. Because f is continuous and not locally constant on a disc, $B_r(c)$, with $\bar{B}_r(c) \subset U$, it follows that the set $\{\omega \in B_r(c) : f(\omega) = f(c)\}$ does not have a cluster point. From Theorem 3.1, we have that $f(c) \notin f(\partial B_r(c))$ for a r sufficiently small. Therefore,

$$2\delta := \min_{z \in \partial B_r(c)} |f(z) - f(c)| > 0.$$

From Lemma 3.4, it follows that

$$B_\delta(f(c)) \subset f(B_r(c)) \subset f(U).$$

Hence, all points of $f(D)$ are interior points, and therefore, $f(D)$ is open by definition. $\square$

Remark 3.6. It is important to realise that Theorem 3.5 is not true for all functions. Take for example the function $f : (-1, 1) \to \mathbb{R}$ defined on the open interval $(-1, 1) \subset \mathbb{R}$, by $f(x) := x^2$. Then, $f((-1, 1)) = [0, 1)$, and this set is not open in $\mathbb{R}$.

Next, we shall introduce the Riemann continuation theorem. Note that we call a function, $f : D \to \mathbb{C}$, *extendable* over $B \supset D$ if $f : B \to \mathbb{C}$ is well-defined. An extendable function $f : D \to \mathbb{C}$ is said to be *holomorphically extendable* and *continuously extendable* over $B \supset D$, if $f : B \to \mathbb{C}$ is holomorphic and continuous, respectively.

Theorem 3.7. *Let $A \subset D$ be a discrete and closed set, then the following statements about a holomorphic mapping $f : D\backslash A \to \mathbb{C}$ are equivalent:*

i) f is holomorphically extendable over D;

ii) f is continuously extendable over D;

iii) f is bounded in a neighbourhood of each point of A;

iv) $\lim_{z \to c} (z - c)f(z) = 0$ for each point $c \in A$.

Proof. *i)* $\Rightarrow$ *ii)* $\Rightarrow$ *iii)* $\Rightarrow$ *iv)*; follows directly since if the extension of f is holomorphic, then it is obviously continuous. Furthermore, f is then bounded in a sufficiently small disc around every $c \in A$, and it holds that $\lim_{z \to c} (z - c)f(z) = 0$ for every point $c \in A$.

iv) $\Rightarrow$ *i)*: Assume, without loss of generality, that $A = \{a\} \subset D$. We construct $g, h : D \to \mathbb{C}$ by setting

$$g(z) := (z - a)f(z) \text{ for } z \in D\backslash\{a\}, \quad g(a) := 0 \quad \text{and} \quad h(z) := (z - a)g(z).$$

By assumption, we have that g is continuous at $z = a$. Then, the identity

$$h(z) = h(a) + (z - a)g(z),$$

implies that h is differentiable at $z = 0$ where

$$h'(a) = g(a) = 0.$$

We know that, $f : D\backslash\{a\} \to \mathbb{C}$ is holomorphic, which implies that $h : D\backslash\{a\} \to \mathbb{C}$ also is holomorphic by the product rule. But, from the previous arguments, h must then be differentiable in D. By Theorem 2.26, we have that the Taylor expansion of h around a converges on some disc $B_r(a) \subset D$ for some $r > 0$, and we have that

$$h(z) = \sum_{n=0}^{\infty} c_n (z - a)^n.$$

Since, $h(a) = h'(a) = 0$, it follows

$$h(z) = (z - a)^2 \sum_{n=2}^{\infty} c_n (z - a)^{n-2}.$$

Thus, $h(z) = (z - a)^2 f(z)$, and we can define an extension, F, of f around a as

$$F(z) = \sum_{n=2}^{\infty} a_n z^{n-2}.$$

Again, by Theorem 2.26, it follows that F is necessarily holomorphic.

The generalization of this holds for when A consists of more than one point, since A is closed and discrete. □

From Theorem 3.7, one can conclude that it is always possible to construct holomorphic functions on domains from functions defined on punctured domains. That is, if the function has removable singularities at the punctured points.

There is one last theorem that we will need to consider before we can start with our formulation of group automorphisms.

Theorem 3.8. *Let $D \subseteq \mathbb{C}$ be a domain and let $f : D \to \mathbb{C}$ be an injective and holomorphic function. Then $D' := f(D)$ is a domain in $\mathbb{C}$ and $f'(z) \neq 0$ for all $z \in D$. Furthermore, the mapping $f : D \to D'$ is biholomorphic and the inverse function f^{-1} satisfies*

$$(f^{-1})'(w) = \frac{1}{f'(f^{-1}(w))} \quad \text{for all } w \in D'.$$

Proof. We prove that $f(D)$ is a domain. Since f is injective, it has to be nowhere locally constant. The open mapping theorem (Theorem 3.5), implies that $f(D)$ is open, and $f(D)$ is connected since f is continuous. We can conclude that $D' = f(D)$ is a domain.

Now, we shall prove that f' is non-zero in D, it will follow from the proof that the identity

$$(f^{-1})'(w) = \frac{1}{f'(f^{-1}(w))}$$

holds for the inverse. Again, the open mapping theorem (Theorem 3.5) implies that the inverse map, $f^{-1} : D' \to D$, is continuous, since for every open subset $U \subset D$ we have that $f(U)$ is open in D'. By injectivity of f, the derivative cannot be identically zero in any open set of D. From Theorem 3.1, it follows that the set of all zeros of f^{-1}, $N(f^{-1})$, is discrete and closed in D. Since f is an open mapping, the image

$$M := f(N(f^{-1}))$$

is discrete and closed in D'. This follows from that a homeomorphism maps discrete and closed sets to discrete and closed sets, see e.g Theorem 18.1 in [7].

Consider $d \in D' \backslash M$, and set $c = f^{-1}(d)$. We have that

$$f(z) = f(c) + (z - c) f_1(z),$$

where $f_1(z) : D \to \mathbb{C}$ is continuous at c, and $f_1(c) = f'(c) \neq 0$. For $z = f^{-1}(w)$, $w \in D'$, it follows that

$$w = d + (f^{-1}(w) - c) f_1(f^{-1}(w)).$$

Furthermore, set $q := f_1 = f^{-1}$. Our construction implies that q is continuous at d and $q(d) = f'(c) \neq 0$. Therefore, we have that

$$f^{-1}(w) = f^{-1}(d) + (w - d)\frac{1}{q(w)},$$

for some sufficiently small neighborhood $U \subset D'$ of d. Since f is holomorphic on D, and its derivative is non-zero at $D \backslash N$, we have that f^{-1} is differentiable at every point d, and

$$(f^{-1})'(d) = \frac{1}{f'(c)} = \frac{1}{f'(f^{-1}(d))}, \quad \text{for all } d \in D' \backslash M. \tag{3.2}$$

From Equation (3.2), we have that f^{-1} is holomorphic on $D'\backslash M$, and we already know that f^{-1} is continuous on D'. Then Theorem 3.7 applies, and we can conclude that f^{-1} is holomorphic on D'. We have that

$$(f^{-1})'(w)f'(f^{-1}(w)) = 1$$

holds on $D'\backslash M$, and by continuity it holds on the whole D'. This implies that $M = \varnothing$, and hence $f'(z) \neq 0$ for all $z \in D$. □

In particular, take note that Theorem 3.8 implies that the inverse mapping is well-defined, and so is its derivative. This is specially important since the existence of inverses are essential for groups, hence it will be quite useful in our construction of the group automorphism.

Now, we are ready to define a group of biholomorphic functions that maps a domain to itself.

Definition 3.9. Let $U \subseteq \hat{\mathbb{C}}$ be a domain and let $f : U \to U$ be a biholomorphic mapping. Then f is called an *automorphism*. The set of all such automorphisms of U is denoted as $\mathrm{Aut}(U)$. The group $(\mathrm{Aut}(U), \circ)$ is called the *automorphism group* of U.

We have just stated that $(\mathrm{Aut}(U), \circ)$ is a group, we shall prove this in Theorem 3.10. We will use, other than Theorem 3.8, some fundamental facts from complex analysis. Henceforth, we will simplify the notation for groups, whenever we talk about a group $(G, *)$ we shall instead write G, as long as the operator, $*$, is obvious.

Theorem 3.10. *The automorphism group,* $\mathrm{Aut}(U)$*, is a group under the composition operator.*

Proof. Firstly, we need to show that $\mathrm{Aut}(U)$ is closed under $\circ$. Let $f, g \in \mathrm{Aut}(U)$ be automorphisms and let the composition of f, g be denoted as, $h := f \circ g$. By the chain rule we have that h is holomorphic on U. From the definition of automorphisms, f and g are onto U, so $f(U) = U$ and $g(U) = U$. Hence, $h(U) = (f \circ g)(U) = U$, and h is therefore onto U. Also, it follows from the definition of automorphisms, that f and g are injective on U. Thus, for $z_1, z_2 \in U$, assume that

$$f(g(z_1)) = f(g(z_2)).$$

This implies that $g(z_1) = g(z_2)$, and then $z_1 = z_2$. Hence h is injective. We can conclude that h is an automorphism on U, and $h \in \mathrm{Aut}(U)$, and therefore $\mathrm{Aut}(U)$ is closed under composition. Now, we shall prove all the group axioms as seen in Definition 2.4.

G1: We have to show that $\circ$ is associative. Let $f, g, h \in \mathrm{Aut}(U)$, then we have that,

$$f \circ (g \circ h)(z) = f(g \circ h(z)) = f(g(h(z))), \quad \text{and}$$
$$(f \circ g) \circ h(z) = (f \circ g)(h(z)) = f(g(h(z))).$$

Thus, $\circ$ is associative.

G2: We need to show that there exists an identity element. Define the function $e : U \to U$ by $e(z) := z$. It is clear that e is holomorphic on U, since all polynomials are entire. Furthermore, e is bijective. This is the case, since e maps every element of U to itself, so it is onto U. Moreover, for every $z_1, z_2 \in U$,

$$e(z_1) = e(z_2)$$

directly implies $z_1 = z_2$, and we can conclude e is injective. Thus, $e \in \text{Aut}(U)$. We have that for any $f \in \text{Aut}(U)$,

$$\begin{aligned}(e \circ f)(z) = (z \circ f)(z) &= f(z), \quad \text{and} \\ (f \circ e)(z) = (f \circ z)(z) &= f(z).\end{aligned}$$

Hence, $e(z) = z$ is the identity element of $\text{Aut}(U)$.

G3: Lastly, we need to show that for every $f \in \text{Aut}(U)$, there exists an inverse $f^{-1} \in \text{Aut}(U)$. It follows from Theorem 3.8 that the inverse, f^{-1}, is well-defined on U, since by assumption f is bijective and holomorphic and U is a domain. $\square$

4. Automorphism Groups on Sets in the Complex Plane

In this section, we will explicitly go through some examples of automorphisms groups. In particular, the automorphism group of the complex plane, extended complex plane and the unit disc are emphasized. These three sets are all simply connected, and are of special interest in Section 5 where we further discuss automorphism groups on simply connected domains. In Section 4.5 we shall briefly consider automorphisms on punctured domains and annuli.

4.1. The Automorphism Group on the Complex Plane.

Firstly, we will determine the automorphism group on $\mathbb{C}$, it consists of all complex polynomials of degree one. This might seem easy enough to prove, but we shall need to introduce Theorem 4.4 and Theorem 4.8 in order to show it. Before we formulate the first theorem, we need to define the concept of *dense* subsets.

Definition 4.1. Let $A, D \subseteq \mathbb{C}$, $A \subseteq D$, be two non-empty sets. The set A is called *dense* in D if all open subsets of D intersects A.

To visualise the concept of dense sets, we shall do so considering the unit disc $B_1(0)$. We can define the set $A = B_1(0)\backslash\{0\}$. It is easy to see that all neighborhoods of 0 in U, contains at least some small disc centred at zero. But then we can find a point inside such a disc that is not zero, which obviously also belongs to $B_1(0)$. Hence, A is a dense subset of $B_1(0)$. From this example, we can directly realize that for a discrete set, $C \subset U \subset \mathbb{C}$, then $D\backslash C$ is dense in D.

Furthermore, we need to generalise the concept of convergence in the extended complex plane, $\hat{\mathbb{C}}$. Informally, we say that either our regular definition of converging sequences applies, and the sequence converges to some $c \in \mathbb{C}$, or it converges to infinity.

Definition 4.2. A sequence $\{p_i\}$ in $\hat{\mathbb{C}}$ *converges* to a point $p_0 \in \hat{\mathbb{C}}$ if:

i) $\lim_{i \in \mathbb{N}} p_i = p_0 = \infty$ where $p_i \in \mathbb{C}$ for all $i \in \mathbb{N}$, or;

ii) $\lim_{i \to \infty} p_i = p_0 \in \mathbb{C}$ and all but finitely many $p_i \in \mathbb{C}$, i.e. it is identical to classical convergence in $\mathbb{C}$.

Remark 4.3. Remember that we say that a sequence $\{c_j\}$ converges to $c \in \mathbb{C}$, if for every neighborhood U of c all but finitely many $c_j \in U$. Note that this definition generalises quite nicely to $\hat{\mathbb{C}}$ by the Alexandorff one-point compactification, for details see e.g. page 185 in [7].

As we will see later, this new definition of convergence allows us to properly deal with functions with poles. The sequence of the image for a function, does not convergence to infinity on essential singularities. Then, the sequence diverges as one should expect by Definition 2.24. This fact is illustrated by the theorem of Casorati and Weierstrass.

Theorem 4.4. *Let $D \subseteq \hat{\mathbb{C}}$ be a domain containing a point $c \in D$ and let $f : D\backslash\{c\} \to \mathbb{C}$ be a holomorphic function. The following statements are equivalent:*

i) the point $c \in D$ is an essential singularity of f;

ii) for every neighborhood $U \subset D$ of c, the image set $f(U\backslash\{c\})$ is dense in $\mathbb{C}$;

iii) there exists a sequence, $\{z_n\}$ *in* $D\backslash\{c\}$*, with* $\lim_{n\to\infty} z_n = c$ *such that the sequence of the image,* $\{f(z_n)\}$*, has no limit in* $\hat{\mathbb{C}}$*.*

Proof. *i)* $\Rightarrow$ *ii)*: We show this by contradiction. Assume that there is a neighborhood, $U \subset D$, of c such that $f(U\backslash\{c\})$ is not dense in $\mathbb{C}$. This means that there are some disc, $B_r(a)$, with $r > 0$ small enough that does not intersect $f(U\backslash\{c\})$. This is equivalent to

$$|f(z) - a| \geq r, \qquad \text{for all } z \in U\backslash\{c\}.$$

It follows that we can construct a function,

$$g(z) := \frac{1}{f(z) - a},$$

which is holomorphic for all $z \in U\backslash\{c\}$. This is possible, since the image of f is never a by assumption, and it is also bounded by $1/r$. Thus, we have a removable singularity at $z = c$. It follows that

$$f(z) = a + \frac{1}{g(z)}$$

has a removable singularity at c if

$$\lim_{z\to c} g(z) \neq 0,$$

or a pole, if

$$\lim_{z\to c} g(z) = 0.$$

This contradicts that c is an essential singularity of $f(z)$, and we can conclude that $f(U\backslash\{c\})$ is dense in $\mathbb{C}$.

ii) $\Rightarrow$ *iii)*: For every neghborhood, U, of c, $f(U\backslash\{c\})$ is dense in $\mathbb{C}$ by assumption. Furthermore, assume there is a sequence, $\{z_n\}$, that converges to c. It follows that $\{f(z_n)\}$ is dense in $\mathbb{C}$, therefore it does not hold that for all but finitely many $f(z_n) \in V$, for every neighborhood V of $f(c)$. Hence, $\{f(z_n)\}$ diverges.

iii) $\Rightarrow$ *i)*: Follows from definition of essential singularity. □

Remark 4.5. Theorem 4.4 can be significantly improved by the famous Picard's great theorem. Remember that Picard's great theorem states that a holomorphic function, $f : D\backslash\{c\} \to \mathbb{C}$, with an essential singularity at c attains every complex number, except maybe one, in every neghborhood of c. Though, for our purposes, Theorem 4.4 will be sufficient.

Since we already know that we will deal with polynomials of degree one, it may not surprise the reader then that we will need use of *algebraic functions*, or its opposite *transcendental functions*.

Definition 4.6. Let f be a function. Then, f is called *algebraic* if it can be expressed as a polynomial

$$a_n(z)f^n(z) + \ldots + a_1(z)f(z) + a_0(z) = 0,$$

where a_i are polynomials. Otherwise, f is said to be *transcendental*.

For convenience, we shall denote the whole complex plane without the point zero as $\mathbb{C}^*$, i.e. $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$. Note that in Lemma 4.7, we restrict ourselves to entire functions. For our purposes this will be enough, since our goal is to define $\mathrm{Aut}(\mathbb{C})$ which obviously consists of entire functions. Further, we already know entire functions can be written as a sum

$$f(z) = \sum_{n=1}^{\infty} a_n z^n.$$

Then, we need only observe that entire functions that can be written as a finite sum

$$f(z) = \sum_{n=1}^{N} a_n z^n$$

are algebraic.

Lemma 4.7. *Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function and let $f^\times : \mathbb{C}^* \to \mathbb{C}$ be a function defined as $f^\times(z) = f(1/z)$. Then, f is transcendental if, and only if, $f^\times$ has an essential singularity at zero.*

Proof. $\Rightarrow$: Let

$$f(z) := \sum_{\alpha=0}^{\infty} a_\alpha z^\alpha$$

be an entire function and assume that 0 is not an essential singularity of $f^\times$, where $f^\times(z) = f(1/z)$. Then, by definition of essential singularity, there exists some $N \in \mathbb{N}$ such that

$$z^N f^\times = \sum_{\alpha=0}^{\infty} a_\alpha z^{N-\alpha},$$

and the extension of $f^\times$ to $\mathbb{C}$ is holomorphic. For all such N we have from the Cauchy integral theorem that

$$0 = \int_{\partial B_1(0)} \zeta^N f^\times(\zeta) d\zeta = \sum_{\alpha=0} a_\alpha \int_{\partial B_1(0)} \zeta^{N-\alpha} d\zeta = 2\pi i a_{N+1},$$

where we have uniformal convergence of $\sum_{\alpha}^{\infty} a_\alpha z^\alpha$ by construction. We can conclude that f is necessarily a polynomial of degree at most N, but this is a contradiction to that f is transcendental.

$\Leftarrow$: Conversely, assume f is not transcendental. Since f is entire, it follows that f is a polynomial of some finite order N, and set

$$f(z) := a_0 + a_1 z + \ldots + a_N z^N.$$

Then, we have that

$$f^\times(z) = f(z^{-1}) = \sum_{\nu=0}^{N} a_\nu z^{-\nu}.$$

This can be identified as the Laurent series expansion of $f^\times$ around 0, so we have directly that $f^\times$ does not have an essential singularity at 0. ☐

Lastly, we need to make use of the concept of isolated singularities for holomorphic and injective functions.

Theorem 4.8. *Let A be a closed and discrete subset of a domain $D \subset \mathbb{C}$ and let $f : D \backslash A \to \mathbb{C}$ be holomorphic and injective. Then it holds that:*

i) no point $c \in A$ is an essential singularity of f;

ii) if f has a pole at some $c \in A$, then it is of order 1;

iii) if every point of A is a removable singularity of f, then the extension of f, $\tilde{f} : D \to \mathbb{C}$, is injective and holomorphic.

Proof. *i*): Let B be an open disc containing c such that

$$B \cap A = \{c\},$$

and

$$D' = D \backslash (A \cup \bar{B}) \neq \varnothing.$$

From the the open mapping theorem (Theorem 3.5), $f(D')$ is non-empty and open, since D' is open and non-empty. Because f is injective, the sets $f(B\backslash\{c\})$ and $f(D')$ does not intersect. But then $f(B\backslash\{c\})$ is not dense, and by Theorem 4.4 the function f does not have an essential singularity at c.

ii): Assume f has a pole of order $m \geq 1$ at $c \in A$. Then we know that we can write

$$f(z) = \frac{1}{g(z)}$$

such that g is holomorphic in a sufficiently small neighborhood U of c, and g has a zero of order m at $z = c$. This implies that $1/f = g$ is holomorphic in U. Because f is injective, so is g, and by Theorem 3.8, we have that g' is never zero, hence $m = 1$.

iii): We shall prove this by contradiction. Assume there are two points $a, a' \in A$ such that $a \neq a'$ and

$$p := \tilde{f}(a) = \tilde{f}(a').$$

Choose disjoint open discs B, B' that contain a and a' respectively such that

$$B\backslash\{a\} \subset D\backslash A; \quad B'\backslash\{a'\} \subset D\backslash A.$$

Then $\tilde{f}(B) \cap \tilde{f}(B')$ is a neighborhood of p, and there exists points

$$b \in B\backslash\{a\}, \text{ and } \quad b' \in B'\backslash\{a'\}$$

such that $f(b) = f(b')$. By assumption, $b, b' \in D\backslash A$ and $b \neq b'$, this is a contradiction that f is injective. Hence, $\tilde{f}$ must also be injective. It then follows directly from Theorem 3.7 that $\tilde{f}$ is holomorphic. □

With the developed machinery it is fairly straightforward to prove that automorphisms on the complex plane are all the polynomials of degree one.

Theorem 4.9. *The automorphism group on $\mathbb{C}$ can be written as,*

$$\text{Aut}(\mathbb{C}) = \{az + b : a, b \in \mathbb{C}, \ a \neq 0\}.$$

Proof. For some holomorphic and injective function, $f : \mathbb{C} \to \mathbb{C}$, we can define $f^\times : \mathbb{C}^* \to \mathbb{C}$ as

$$f^\times(z) := f\Big(\frac{1}{z}\Big).$$

Since

$$f(z_1) = f(z_2)$$

implies that

$$z_1 = z_2$$

by injectivity. We have that from

$$f\Big(\frac{1}{z_1}\Big) = f\Big(\frac{1}{z_2}\Big),$$

we get

$$\frac{1}{z_1} = \frac{1}{z_2},$$

and it follows that

$$z_1 = z_2.$$

Hence, $f^\times$ is also injective. Then by setting $D := \mathbb{C}$ and $A := \{0\}$ we have from Theorem 4.8 that 0 is not an essential singularity for $f^\times$. By Lemma 4.7, f has to be a polynomial. Since f is injective, $f^\times$ must be a pole of order one according to Theorem 4.8. Hence, all injective holomorphic functions on $\mathbb{C}$ are of the form

$$f(z) = az + b.$$

Lastly, it is trivial to show that all such functions are surjective, since for every $y \in \mathbb{C}$, there is a $w \in \mathbb{C}$ such that $f(w) = y$, given by $w = \frac{y-b}{a}$. □

4.2. **The Automorphism Group on the Extended Complex Plane.** Here, we shall study the automorphism group on $\hat{\mathbb{C}}$. As we shall see, it consists of all *Möbius transfromations*. Note also that the way we choose to define the Möbius transformation is not universal, for example Krantz [5] have some further restrictions, and call our Möbius transfromations for *rational transformations*. Instead, we shall use the same definition as used by Saff and Snider [10]. From Definition 4.2, we can conjecture that Möbius transformations are bijective, continuous, and that they have well-defined inverses.

Lemma 4.10. *Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a Möbius transformation, i.e.*

$$f(z) = \frac{az+b}{cz+d}, \qquad \text{where } a, b, c, d \in \mathbb{C}, \quad ad - bc \neq 0,$$

then f is bijective and continuous. We also have that $f^{-1} : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a Möbius transformation. Furthermore, if $g : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a Möbius transformation, then $f \circ g$ is also a Möbius transformation.

Proof. From Definition 4.2, it follows that f is continuous. To show that f is injective, let $z_1, z_2 \in \hat{\mathbb{C}}$, and let $f(z_1) = f(z_2)$. Then, we have

$$(az_1 + b)(cz_2 + d) = (az_2 + b)(cz_1 + d),$$

which implies that

$$(z_1 - z_2)(ad - bc) = 0.$$

By assumption,

$$(ad - bc) \neq 0$$

so $z_1 = z_2$. Next, we shall show surjectivity. Let $\tilde{z} \in \hat{\mathbb{C}}$ be an arbitrary point, we want to show that $\tilde{z} = f(z')$ for some $z' \in \hat{\mathbb{C}}$. From straightforward algebra we have that

$$z' = \frac{b - d\tilde{z}}{c\tilde{z} - a}. \tag{4.1}$$

Equality (4.1) is at least well-defined for $\tilde{z} \in \mathbb{C} \backslash \{a/c\}$. For the special cases $z' = \infty$ we set $\tilde{z} = a/c$ and for $z' = -d/c$ we set $\tilde{z} = \infty$. Thus, f is bijective. From the proof of surjectivity we have directly that the inverse is also a Möbius transformation, and that it is well-defined on $\hat{\mathbb{C}}$.

Finally, let

$$g(z) := \frac{Az + B}{Cz + D}$$

where $A, B, C, D \in \mathbb{C}$ and $AD - BC \neq 0$. Now we have that

$$(g \circ f)(z) = \frac{(Aa + Bc)z + (Ab + Bd)}{(aC + cD)z + (bC + Dd)},$$

where

$$(Aa + Bc)(bC + Dd) - (Ab + Bd)(aC + cD) = (AD - BC)(ad - bc) \neq 0$$

by assumption of f and g. Hence, the group of möbius transformations is closed under composition. □

For our purposes, we could either re-define automorphisms to bijective meromorphisms, or extend the definition for holomorphic functions to $\hat{\mathbb{C}}$. We shall do the latter. Remember, by Definition 2.14 a function, $f : D \to \mathbb{C}$ on a domain is said to be holomorphic if the limit

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \tag{4.2}$$

is well-defined for all $z_0 \in D$. If we let $D \subseteq \hat{\mathbb{C}}$ then we shall take the convergence of the limit in Identity (4.2), by Definition 4.2. In principle, we shall just call meromorphisms in $\hat{\mathbb{C}}$ holomorphic functions, we do this out of convenience and to dodge some mathematical bullets (even as we do so we might cheat a bit).

Theorem 4.11. *The automorphism group on $\hat{\mathbb{C}}$ can be written as*

$$\operatorname{Aut}(\hat{\mathbb{C}}) = \left\{ \frac{az + b}{cz + d} : a, b, c, d \in \mathbb{C},\ ad - bc \neq 0 \right\},$$

i.e it can be uniquely determined as all the Möbius transformations.

Proof. By Lemma 4.10, and Theorem 4.8, we have that the only bijections on $\hat{\mathbb{C}}$ are the Möbius transformations. What is left to prove is that Möbius transformations are holomorphic on the extended complex plane. If we consider f as in Lemma 4.10, then we have that f is a meromorphic function with pole of order one at $z = -d/c$. This means that f is holomorphic on $\mathbb{C} \backslash \{-d/c\}$ in the classical sense. It follows that

$$f'(z) = \frac{ad - bc}{(cz + b)^2}$$

is well-defined and continuous on $\mathbb{C}\backslash\{-d/c\}$. Next, we want to construct an extension of f' to $\hat{\mathbb{C}}$. Set

$$f'(-d/c) := \infty,$$

and

$$f'(\infty) := 0,$$

then we have from Definition 4.2 that f' is continuous on $\hat{\mathbb{C}}$. Hence, the Möbius transformations are, in the sense of convergence as in Defininition 4.2, biholomorphic on $\hat{\mathbb{C}}$. □

4.3. **The Automorphism Group on the Unit Disc.** Now, we will get to automorphisms on the unit disc. As we shall see $\mathrm{Aut}(B_1(0))$ are Möbius transformations of the form

$$f(z) = e^{i\phi}\frac{a-z}{1-\bar{a}z},$$

where $\phi \in (0, 2\pi]$ and $a \in B_1(0)$, which is a subgroup of Möbius tranformations. Many authors define functions of this form to be Möbius transformations, but as we have discussed earlier, we will not follow this convention.

Before we go further, we introduce Schwarz's lemma.

Theorem 4.12 (Schwarz's Lemma)**.** *Let $f : B_1(0) \to B_1(0)$ be a biholomorphic function. Assume that $f(0) = 0$, then*

$$f(z) \leq |z|, \qquad \text{for all } z \in D,$$

and

$$|f'(0)| \leq 1.$$

If either $|f(z)| = |z|$ for some $z \neq 0$, or if $|f'(0)| = 1$, then $f(z) = \alpha z$, where $\alpha \in \mathbb{C}$ and $|\alpha| = 1$.

Proof. Construct the function

$$g(z) := \frac{f(z)}{z}.$$

This function is holomorphic on $B_1(0)\backslash\{0\}$. Furthermore,

$$\lim_{z\to 0} g(z) = f'(0)$$

by L'Hopital's rule. Then we define $g(0) := f'(0)$, so that g is continuous on $B_1(0)$. Thus, g is holomorphic on $B_1(0)$.

Now consider the restriction of the domain of definition to the closed disc $\bar{B}_{1-\epsilon}(0)$ for some $1 > \epsilon > 0$. By construction of g, we have that

$$|g(z)| \leq \left|\frac{f(z)}{z}\right| \leq \frac{1}{1-\epsilon}$$

on the boundary of $\bar{B}_{1-\epsilon}(0)$. So,

$$|g| \leq \frac{1}{1-\epsilon}, \qquad \text{for all } z \in \bar{B}_{1-\epsilon}(0).$$

As $\epsilon \to 0^+$, we have that $|g(z)| \leq 1$ for all $z \in B_1(0)$. Thus, we can conclude that

$$|f(z)| \leq |z|,$$

and since $g(0) = f'(0)$ we have that $f'(0) \leq 1$.

If $|f(z)| = |z|$ for some $z \neq 0$, then $|g(z)| = 1$ for all such z. Since $|g(z)| \leq 1$ on the entire disc, the Maximum Priniple (Theorem 2.29), gives that g is a constant with modulus 1. Hence, $f(z) = \alpha z$ where $|\alpha| = 1$.

If $|f'(0)| = 1$, then we know that $|g(0)| = 1$. Once again, by applying the Maximum Principle (Theorem 2.29), we have that $|g(z)| = 1$ on $B_1(0)$. So $f(z) = \alpha z$ where $|\alpha| = 1$. □

Geometrically, Schwarz's lemma is important, indeed since if we know that either $|f(z)| = |z|$ for some $z \neq 0$, or if $|f'(0)| = 1$, then we can interpret f as a rotation around zero. Note that all such rotations will form a subgroup of $\mathrm{Aut}(B_1(0))$. Furthermore, the group of rotations is a subgroup of $\mathrm{Aut}(\mathbb{C})$ and $\mathrm{Aut}(\hat{\mathbb{C}})$ as well. Hence, in the context of algebraic structures of automorphism groups, it is an important subgroup.

Now, we can go into the characterization of automorphisms on $B_1(0)$.

Theorem 4.13. *The automorphism group on the unit disc can be written as*

$$\mathrm{Aut}(B_1(0)) = \left\{ e^{i\phi} \frac{a - z}{1 - \bar{a}z} \;\middle|\; a \in B_1(0),\ \textit{and } \phi \in (0, 2\pi] \right\}.$$

Proof. Firstly, we shall show that

$$\varphi_a := \frac{a - z}{1 - \bar{a}z} \in \mathrm{Aut}(B_1(0)), \qquad \text{where } a \in B_1(0).$$

We have directly that φ_a is holomorphic on $B_1(0)$, and that the pole of φ_a is at $1/\bar{a}$, which is outside of $\bar{B}_1(0)$. At the boundary, $|z| = 1$, we have that

$$|\varphi_a| = \left|\frac{a - z}{1 - \bar{a}z}\right| = \left|\frac{1}{\bar{z}} \frac{a - z}{1 - \bar{a}z}\right| = \left|\frac{a - z}{\bar{z} - \bar{a}}\right| = 1.$$

Thus, φ_a maps the boundary of $B_1(0)$ to itself. From the Maximum Principle (Theorem 2.29), we have that φ_a maps the interior of $B_1(0)$ to the interior of $B_1(0)$.

Here, we show that φ_a is bijective on $B_1(0)$. Assume that $z_1, z_2 \in B_1(0)$ and

$$\varphi_a(z_1) = \varphi_a(z_2).$$

It follows that

$$(z_1 - a)(1 - \bar{a}z_2) = (z_2 - a)(1 - \bar{a}z_1),$$

and

$$(z_1 - z_2)(1 - |a|^2) = 0.$$

Since $|a| < 1$, this implies that $z_1 = z_2$. For surjectivity, assume that $\tilde{z} \in B_1(0)$, and we want to show that there exists at least one $z' \in B_1(0)$ such that $\tilde{z} = \varphi_a(z')$. From straightforward algebra, we have that

$$z' = \frac{a - \tilde{z}}{1 - \bar{a}\tilde{z}}.$$

Hence, φ_a is bijective. Furthermore, the existence of the inverse follows directly from our proof of the surjectivity, and the inverse is also of the form of φ_a. Thus, $\varphi_a \in \mathrm{Aut}(B_1(0))$.

Also, it is clear from Schwarz's Lemma (Theorem 4.12), that

$$e^{i\phi} z \in \mathrm{Aut}(B_1(0)).$$

Left to show is that all automorphisms on $B_1(0)$ is generated by rotations and functions of the form of φ_a. Suppose $f \in \text{Aut}(B_1(0))$ and $f(0) = b$ for some $b \in B_1(0)$, then we can construct

$$g = \varphi_b \circ f$$

which has to be an automorpism by definition. Therefore, g is biholomorphic on $B_1(0)$, and

$$g(0) = \varphi_b(b) = 0.$$

From Schwarz's Lemma (Theorem 4.12), we have that $|g'(0)| \leq 1$. By assumption, there has to exist an inverse g^{-1} that is also biholomorphic on $B_1(0)$ with $g^{-1}(0) = 0$. From Theorem 3.8, and once again Schwarz's Lemma (Theorem 4.12), we have that

$$\frac{1}{|g'(0)|} = |(g^{-1})'(0)| \leq 1.$$

Then, it follows that $g'(0) = 1$. Thus, g is a rotation, i.e. it can be written as

$$g(z) = e^{i\phi}z, \quad \text{where } \phi \in (0, 2\pi].$$

Hence,

$$(\varphi_b \circ f)(z) = e^{i\phi}z,$$

but since $\varphi_b^{-1} = \varphi_b$, we arrive at

$$f(z) = \varphi_b \circ (e^{i\phi}z).$$

We can conclude that all $f \in \text{Aut}(B_1(0))$ can be written of the form $f = e^{i\phi}\frac{a-z}{1-\bar{a}z}$, where $a \in B_1(0)$. □

4.4. **Automorphism Groups and Biholomorphic Mappings.** Observe that $\text{Aut}(B_1(0))$, $\text{Aut}(\mathbb{C})$ and $\text{Aut}(\hat{\mathbb{C}})$ all seems to be somewhat related to each other. This gives us incitement to wonder whether they are isomorphic. As we shall show, this is not the case.

Theorem 4.14. *The groups* $\text{Aut}(\mathbb{C})$, $\text{Aut}(\hat{\mathbb{C}})$ *and* $\text{Aut}(B_1(0))$ *are not isomorphic with each other.*

Proof. In the first part of the proof, we shall show that $\text{Aut}(\mathbb{C})$ is not isomorphic to either $\text{Aut}(\hat{\mathbb{C}})$ or $\text{Aut}(B_1(0))$. It is easy to check that the group of all translations,

$$H := \{z + b : b \in \mathbb{C}\},$$

is a normal and abelian subgroup of $\text{Aut}(\mathbb{C})$. But the only abelian subgroup of $\text{Aut}(B_1(0))$ is some subgroup of the group of rotations

$$G := \{e^{\phi i}z : \phi \in (0, 2\pi]\}.$$

Left to show is that no non-trivial subgroup of G is normal in $\text{Aut}(B_1(0))$. Consider the function $\phi \in \text{Aut}(B_1(0))$ defined as

$$\phi(z) = \frac{\frac{1}{2} - z}{1 - \frac{1}{2}z}.$$

The left coset, $\phi \circ G$, consists only of functions that maps 0 to $\frac{1}{2}$. The right coset, $G \circ \phi$, consists of functions that map 0 to the circle of radius $\frac{1}{2}$. Thus, there is no subgroup of $\text{Aut}(B_1(0))$ that is both normal and abelian. Hence, $\text{Aut}(\mathbb{C})$ and $\text{Aut}(B_1(0))$ are not isomorphic. Furthermore, we have that H is a subgroup of $\text{Aut}(\hat{\mathbb{C}})$, we shall now show that H is not a normal subgroup of $\text{Aut}(\hat{\mathbb{C}})$. Consider the function, $\psi \in \text{Aut}(\hat{\mathbb{C}})$ defined as

$$\psi(z) = \frac{z+1}{z-1}.$$

The left coset, $\psi \circ H$, consists of functions that maps 1 to ∞. The right coset, $H \circ \psi$ consists of functions that each maps a point in $\mathbb{C}$ to ∞. We can conclude that H is not normal, and thus there is no non-trivial subgroup of $\text{Aut}(\hat{\mathbb{C}})$ that is both normal and abelian. Hence, by the same argument as before $\text{Aut}(\hat{\mathbb{C}})$ and $\text{Aut}(\mathbb{C})$ are not isomorphic.

Finally, we need to show that $\text{Aut}(\hat{\mathbb{C}})$ and $\text{Aut}(B_1(0))$ are not isomorphic. We will show this through a contradiction, so assume there exists an isomorphism

$$\phi : \text{Aut}(B_1(0)) \to \text{Aut}(\hat{\mathbb{C}}).$$

It is easy to see that G is the biggest abelian subgroup of $\text{Aut}(B_1(0))$ and all other abelian subgroups of $\text{Aut}(B_1(0))$ are themselves subgroups of G. The group $\text{Aut}(\hat{\mathbb{C}})$ have two such abelian subgroups, the group of all magnifications

$$F := \{az : a \in \mathbb{C}\backslash\{0\}\}$$

and the group of all translations H. We notice here that $H \cap F = \{z\}$, i.e. the trivial subgroup of $\text{Aut}(\hat{\mathbb{C}})$. This means that G must have two proper abelian subgroups $G', G'' \subset G$ such that $G' \cap G'' = \{z\}$, i.e. the trivial subgroup of $\text{Aut}(B_1(0))$, and $G' \cup G'' \subseteq G$. All this such that $\phi(G') = F$ and $\phi(G'') = H$, but since ϕ is bijective

$$\phi(G' \cup G'') = \phi(G') \cup \phi(G'') \subseteq \phi(G).$$

But $\phi(G)$ is abelian so $\phi(G') \cup \phi(G'')$ is also abelian, but it is easy to show that the union of all translations and all magnifications is not abelian, which is a contradiction. We can conclude that $\text{Aut}(\hat{\mathbb{C}})$ and $\text{Aut}(B_1(0))$ are not isomorphic. □

This motivates us to look at when there actually *is* a meaningful connection between two distinct subsets of $\hat{\mathbb{C}}$. For our purposes, this property we want to construct shall be called *biholomorphic equivalency.*

Definition 4.15. Two domains A, B are said to be *biholomorphically equivalent* if there exists a biholomorphic function $\phi : A \to B$.

The main reason we are interested in biholomorphic equivalency, is that it gives us a tool to prove isomorphy between automorphism groups. Consider two domains, A and B, that are biholomorphically equivalent, then we can say that there is a isomorphism between $\text{Aut}(A)$ and $\text{Aut}(B)$.

Theorem 4.16. *If there exists a biholomorphic mapping, $f : A \to B$, between two domains, A, B, then $\phi : \text{Aut}(A) \to \text{Aut}(B)$ defined by*

$$\phi(h) = f \circ h \circ f^{-1},$$

is an isomorphism of the automorphism groups.

Proof. We shall first prove that ϕ is a homomorphism. We have for every $h, h' \in \text{Aut}(A)$, that

$$\begin{aligned}\phi(h \circ h') &= f \circ h \circ h' \circ f^{-1} \\ &= f \circ h \circ f^{-1} \circ f \circ h' \circ f^{-1} \\ &= \phi(h) \circ \phi(h').\end{aligned}$$

Since f, f^{-1} and h are all biholomorphic it is easy to see that $\phi(h)$ is biholomorphic on B and is thus an automorphism on B. Hence, ϕ is a homomorphism.

Furthermore, we want to show that ϕ is injective. Suppose there exists distinct $h, h' \in A$ such that $\phi(h) = \phi(h')$, but

$$\begin{aligned} f \circ h \circ f^{-1} &= f \circ h' \circ f^{-1} && \text{if, and only if,} \\ f^{-1} \circ (f \circ h \circ f^{-1}) \circ f &= f^{-1} \circ (f \circ h' \circ f^{-1}) \circ f && \text{if, and only if,} \\ h &= h' \end{aligned}$$

Finally, we need to show that ϕ is surjective. For every $g \in \text{Aut}(B)$, we have that

$$\phi^{-1}(g) = f^{-1} \circ g \circ f = \tilde{g}$$

is an automorphism on A. So, for every $g \in \text{Aut}(B)$ there is a $\tilde{g} \in \text{Aut}(A)$ such that

$$\phi(\tilde{g}) = g.$$

Hence, ϕ is an isomorphism. □

Remark 4.17. From here, there is a natural progression to the famous uniformization theorem. It requires tools from complex geometry, and is outside the scope of this essay. Therefore, we shall not consider it in this essay. In short, the uniformization theorem states that, any simply connected Riemann surface, or one dimensional complex manifolds, S, is biholomorphically equivalent to either $\mathbb{C}$, $\hat{\mathbb{C}}$ or $B_1(0)$. For details see [1]. From Theorem 4.14 and Theorem 4.16 together with the uniformization theorem, we can conclude that any automorphism group of a simply connected Riemann surface is isomorphic to either $\text{Aut}(\mathbb{C})$, $\text{Aut}(\hat{\mathbb{C}})$ or $\text{Aut}(B_1(0))$.

We shall consider one example of an application of Theorem 4.16. Notice how Theorem 4.16 makes the proof much more straightforward, compared to the pure algebraic proof of Theorem 4.14.

Proposition 4.18. *The unit disc is biholomorphically equivalent to the upper half plane,* $\mathbb{H} := \{z \in \mathbb{C} : \text{Im}(z) > 0\}$. *One such biholomorphic mapping is the inverse Cayley mapping,* $\phi_C^{-1} : \mathbb{H} \to B_1(0)$, *defined as*

$$\phi_C^{-1}(z) := \frac{z - i}{z + i}.$$

Furthermore,

$$\text{Aut}(\mathbb{H}) = \{\phi_C \circ h \circ \phi_C^{-1} : h \in \text{Aut}(B_1(0))\},$$

where the Cayley mapping, $\phi_C : B_1(0) \to \mathbb{H}$, *is defined by*

$$\phi_C(z) := i\frac{z + 1}{1 - z}.$$

Proof. It is obvious that $\phi_C^{-1} : \mathbb{H} \to B_1(0)$ is an injective and holomorphic mapping by our previous results concerning Möbius transformations. Then, we only need to show that ϕ_C^{-1} maps onto $B_1(0)$. Suppose that,

$$\left|\frac{z-i}{z+i}\right| < 1.$$

This holds, if, and only if,

$$|z-i|^2 < |z+i|^2,$$

which is equivalent to,

$$|z|^2 + zi - \bar{z}i + 1 < |z|^2 - zi + \bar{z}i + 1,$$

and we get,

$$-2\,\mathrm{Im}(z) < 2\,\mathrm{Im}(z)$$

if, and only if,

$$\mathrm{Im}(z) > 0.$$

Hence, ϕ_C^{-1} is biholomorphic.

By straightforward algebra we can show that ϕ_C^{-1} is the inverse of ϕ_C. Then, from Theorem 4.16, it follows that

$$\mathrm{Aut}(\mathbb{H}) = \{\phi_C \circ h \circ \phi_C^{-1} | h \in \mathrm{Aut}(B_1(0))\}.$$

□

4.5. **Automorphism Groups on Punctured Domains.** The observant reader might have noticed that all previous discussion deals only with simply connected domains. This gives us some incitement to ask about automorphism groups on domains where we do not require that the domains should be simply connected. It gets quite a lot messier in this section, as it gets harder to deal with domains that has different amount of holes in a good generalised way. Thus, we limit ourselves to some selected theorems, special cases and properties.

There is one thing we can say about automorphism groups on connected domains, $D\backslash A$, constructed by removing a closed set, A, from a simply connected domain, D. Actually, as we shall see in the next theorem, the automorphism group on $D\backslash A$ is directly related to the subgroup,

$$\{f \in \mathrm{Aut}(D) : f(A) = A\}$$

of $\mathrm{Aut}(D)$

Theorem 4.19. *Let D be a bounded domain with no isolated boundary point, then for every discrete and closed subset, $A \subset D$, we have that*

$$\mathrm{Aut}_A(D) := \{f \in \mathrm{Aut}(D) : f(A) = A\},$$

is group isomorphic to $\mathrm{Aut}(D\backslash A)$.

Proof. What we want to show is that for every $f \in \mathrm{Aut}(D\backslash A)$, we can relate a $\tilde{f} \in \mathrm{Aut}_A(D)$ such that

$$f = \tilde{f}|_{D\backslash A}.$$

By definition of automorphism and by our construction of f, we have that both $f|_{D\backslash A}$ and $f^{-1}|_{D\backslash A}$ are bijective mappings. Since D is bounded and A is closed and discrete, both f

and f^{-1} are bounded and holomorphic in a punctured neighborhood of each point of A. By Theorem 3.7, the holomorphic extensions

$$\hat{f}, \hat{f}^{-1} : D \to \mathbb{C}$$

of f and f^{-1} are injective.

Here, we want to show that $\hat{f}(D) \subset D$. By our construction we know that

$$\hat{f}(D \backslash A) = D \backslash A.$$

For a continuous function, h, it holds that

$$h(\bar{U}) \subseteq \overline{h(U)}$$

for every open set U. For further details see e.g. Theorem 18.1 in [7]. So

$$\hat{f}(D) \subseteq \hat{f}(\overline{D \backslash A}) \subseteq \overline{D \backslash A} = \bar{D},$$

since A is discrete and closed. Then, we can guarantee that

$$\hat{f}(D) \subseteq \bar{D}.$$

Now, we shall construct a contradiction. Assume there exists a $p \in D$ such that $\hat{f}(p) \in \partial D$. If $f \in D \backslash A$, then $p \in A$ since $\hat{f}(z) \in D \backslash A$. By Theorem 3.7, $\hat{f}$ is holomorphic and the open mapping theorem (Theorem 3.5) applies, so the image of an open set under $\hat{f}$ is also open. Then there has to exist some ball $B_r(p)$ with r small enough such that

$$B_r(p) \backslash \{p\} \subset D \backslash A,$$

since A is discrete. Then, $\hat{f}(B_r(p))$, is a neighborhood of $\hat{f}(p)$. Because $\hat{f}$ is injective, we have that

$$\bar{B}_r(p) \backslash \{\hat{f}(p)\} = \hat{f}(B_r(p) \backslash \{p\}) = f(B_r(p) \backslash \{p\}) \subseteq D,$$

which means that p is an isolated boundary point of D. This is a contradiction, and hence $\hat{f}(D) \subseteq D$.

By the same arguments, it holds that

$$\hat{f}^{-1}(D) \subseteq \bar{D}.$$

Thus,

$$(\hat{f} \circ \hat{f}^{-1}), (\hat{f}^{-1} \circ \hat{f}) : D \to \mathbb{C}$$

are well-defined. Since we have that

$$\hat{f} \circ \hat{f}^{-1}(z) = \hat{f}^{-1} \circ \hat{f}(z) = z$$

on $D \backslash A$, it follows from Theorem 3.7 that

$$(\hat{f} \circ \hat{f}^{-1})(z) = (\hat{f}^{-1} \circ \hat{f})(z) = z$$

on D. Hence, $\hat{f} \in \mathrm{Aut}(D)$. Furthermore, we have that

$$\hat{f}(D \backslash A) = D \backslash A,$$

which implies that $\hat{f}(A) = A$, because $\hat{f}$ is an automorphism on D and is thus surjective on D. Hence, we have that $\hat{f} \in \mathrm{Aut}_A(D)$. $\square$

Remark 4.20. Remember that Theorem 4.19 is only true for bounded domains, for if $D = \mathbb{C}$ and

$$A = \{z \in \mathbb{C} : 1 \leq z < \infty\},$$

then $D \backslash A$ is the unit disc. But we have already observed that if $f \in \operatorname{Aut}(B_1(0))$ then in general $\tilde{f} \notin \operatorname{Aut}(\mathbb{C})$, where $\tilde{f} : \mathbb{C} \to \hat{\mathbb{C}}$ is the extension of f.

In Theorem 4.26, we shall introduce how automorphism groups on the punctured unit disc are correlated to the permutation group. Before we do so, we shall go through some important concepts.

Definition 4.21. A domain D in $\mathbb{C}$ is called *homogeneous* with respect to a subgroup $L \subseteq \operatorname{Aut}(D)$ if, for every $z_1, z_2 \in D$, we have that there is an automorphism $h \in L$ such that $z_1 = h(z_2)$

One such domain is the unit disc, which is homogeneous with respect to $\operatorname{Aut}(B_1(0))$. In order to prove this, we shall need a lemma.

Lemma 4.22. *If there is a point $c \in D$ whose orbit*

$$\{g(c) : g \in L \subseteq \operatorname{Aut}(D)\}$$

is all of D, then D is homogeneous with respect to L.

Proof. Assume the orbit of c with respect to $L \subseteq \operatorname{Aut} D$ is all of D. Let $z_1, z_2 \in D$ be arbitrary points. Then, there exists $g, \hat{g} \in L$ such that $g(c) = z_1$ and $\hat{g}(c) = z_2$. We can construct

$$h := \hat{g} \circ g^{-1} \in L$$

for which $h(z_1) = z_2$. Hence, D is homogeneous with respect to L. □

Now, it is easy to prove that $B_1(0)$ is homogeneous with respect to $\operatorname{Aut}(B_1(0))$.

Theorem 4.23. *The unit disc $B_1(0)$ is homogeneous with respect to* $\operatorname{Aut}(B_1(0))$.

Proof. We can observe that the orbit of 0 is in all of $B_1(0)$. This follows from that for each $\omega \in B_1(0)$, there is a function

$$h(z) := \frac{z - \omega}{\bar{\omega} z - 1}$$

such that $h(0) = \omega$. We have seen in Theorem 4.13 that $h \in \operatorname{Aut}(B_1(0))$. Then, by Lemma 4.22, it follows that $B_1(0)$ is homogeneous with respect to $\operatorname{Aut}(B_1(0))$. □

From Theorem 4.23, we can prove another useful theorem. Remember that we say that a function, $f : D \to \mathbb{C}$, has a *fixed point*, $z_1 \in D$, if $f(z_1) = z_1$.

Theorem 4.24. *Every automorphism h of $B_1(0)$ with two distinct fixed points in $B_1(0)$ is the identity automorphism.*

Proof. We shall prove this with a contradiction. Assume there exists a non-identity automorphism, $f \in \operatorname{Aut}(B_1(0))$, such that it has two distinct fixed points. Since $B_1(0)$ is homogeneous with respect to $\operatorname{Aut}(B_1(0))$, by Theorem 4.23, it is sufficient to take one of the fixed points as zero. From Schwarz's lemma (Theorem 4.12), we have that for any $f \in \operatorname{Aut}(B_1(0))$ with $f(0) = 0$,

$$f(z) = \alpha z,$$

where $|\alpha| = 1$. In order for there to be another fixed point, $p \neq 0$, of f, we must have that $f(p) = \alpha p = p$. This implies that $\alpha = 1$. This is a contradiction to the assumption that f is not the identity automorphism. □

Remark 4.25. Observe, that Theorem 4.24 is for example not true for the extended complex plane. Rotations, $f = \alpha z \in \mathrm{Aut}(\hat{\mathbb{C}})$, where $|\alpha| = 1$, on $\hat{\mathbb{C}}$ maps $0 \mapsto 0$ and $\infty \mapsto \infty$.

From Theorem 4.24 we can actually say something about the subgroup of automorphisms on $B_1^{\times}(0) = B_1(0)\backslash\{0\}$. The automorphism group on $B_1^{\times}(0)\backslash A$, where $A \neq \varnothing$ is a discrete and closed set in $B_1^{\times}(0)$, is closely related to the permutation group of $A \cup \{0\}$, how these are rleated we will show in Theorem 4.26.

Theorem 4.26. *Let A be a finite non-empty subset of $B_1^{\times}(0) := \{z \in \mathbb{C} | 0 < |z| < 1\}$. Then there is a injective homomorphism, $\pi : \mathrm{Aut}(B_1^{\times}(0)\backslash A) \to Perm(A \cup \{0\})$.*

Proof. Since $B_1^{\times}(0)\backslash A = B_1(0)\backslash(A \cup \{0\})$, we have from Theorem 4.19 that $\mathrm{Aut}(B_1^{\times}(0)\backslash A)$ and $\mathrm{Aut}_{(A\cup\{0\})}(B_1(0))$ are isomrophic. Thus, every automorphism $f \in \mathrm{Aut}(B_1^{\times}(0)\backslash A)$ maps $A \cup \{0\}$ bijectively to $A \cup \{0\}$. In other words, f induces a permutation $\pi(f)$ on $A \cup \{0\}$. This means that $f \mapsto \pi(f)$ is a group homomorphism,

$$\pi : \mathrm{Aut}(B_1^{\times}(0)) \to \mathrm{Perm}(A \cup \{0\}).$$

This follows from that a non-identity automorphism on $B_1(0)$ fixes at most one point point by Theorem 4.24. Finally, $A \neq \varnothing$ implies that π is injective. □

Intuitively, one should be able to cut out enough points from $B_1(0)$, such that the only automorphism on thedomain is the identity automorphism. We shall call such a domain *rigid*.

Definition 4.27. A domain D is said to be biholomorphically *rigid* if the only automorphism on D is the identity automorphism.

Remark 4.28. From Theorem 4.26, we know that it is possible to homomorphically map the automorphism group

$$\mathrm{Aut}(B_1^{\times}(0)\backslash A)$$

injectively to the permutation group $\mathrm{Perm}(A \cup \{0\})$. Take care observing that we can not guarantee that this mapping is surjective. As a matter of fact when $B_1^{\times}(0)\backslash A$ is rigid the only element in $\mathrm{Perm}(A \cup \{0\})$ we can map to, is the identity permutation. Without going into further details one example of such a rigid domain is the set

$$B_1(0)\backslash\{0, 0.5, 0.75\}.$$

Lastly, we will look at the biholomorphic equivalency of annuli. One may initially guess that all annuli of the form $A := \{z \in \mathbb{C} : 0 < r < |z| < R < \infty\}$ are biholomorphically equivalent. Surprisingly, this is not true at all. As we shall see, the only biholomorphically equivalent annuli are geometrically similar annuli.

Theorem 4.29. *Two annuli $A_1 := \{z \in \mathbb{C} : r_1 < |z| < R_1\}$, $0 < r_1 < R_1 < \infty$ and $A_2 := \{z \in \mathbb{C} : r_2 < |z| < R_2\}$, $0 < r_2 < R_2 < \infty$ are biholomorphically equivalent if, and only if, $R_1/r_1 = R_2/r_2$.*

Proof. $\Leftarrow$: If

$$\frac{R_1}{r_1} = \frac{R_2}{r_2},$$

then we have the biholomorphic map $\chi : A_1 \to A_2$,

$$\chi(z) := \frac{R_2}{R_1} z,$$

which is obviously biholomorphic. Hence, A_1 and A_2 are biholomorphically equivalent.

This is analogous to that every annuli is biholomorphically equivalent to some annuli with inner radius 1. Hence, without loss of generality, we can set $r_1 = r_2 = 1$, for ease of computation in our second part of the proof.

$\Rightarrow$: Assume there exists a biholomorphic mapping $\chi : A_1 \to A_2$, now we want to show that this implies that

$$\frac{R_1}{r_1} = \frac{R_2}{r_2},$$

or since we can set $r_1 = r_2 = 1$, it is sufficient to show that $R_1 = R_2$. We show initially that χ maps circles in A_1 to circles in A_2. Let S be a circle in A_2 with radius $1 < r_2' < R_2$. Since χ is an homeomorphism, we know that $\chi^{-1}(S)$ is compact in A_1, thus it does not lie on the boundary of A_1. For details see e.g. Theorem 26.5 in [7]. Thus, there exists an $\epsilon > 0$ such that the set

$$\chi(\{z \in \mathbb{C} : 1 < |z| < 1 + \epsilon\}),$$

does not intersect S. This implies that either

$$\chi(\{z \in \mathbb{C} : 1 < |z| < 1 + \epsilon\})$$

is completely inside or outside of S. Consider a sequence, $\{z_n\}$, inside $\{z \in \mathbb{C} : 1 < |z| < 1 + \epsilon\}$ such that

$$\lim_{n\to\infty} |z_n| = 1.$$

Then, the sequence $\{\chi(z_n)\}$ does not have any accumulation points inside of A_2. If

$$\chi(\{z \in \mathbb{C} : 1 < |z| < 1 + \epsilon\})$$

is inside S then, we have that $|\chi(z_n)|$ converges to 1. Conversely, for

$$\lim_{n\to\infty} |z_n| = R_1,$$

we have that

$$\lim_{n\to\infty} |\chi(z_n)| = R_2$$

by a similar argument. We can conclude that $\chi(z_n)$ does not oscillate between 1 and R_2.

Furthermore, for the case that

$$\tilde{\chi}(\{z \in \mathbb{C} : 1 < |z| < 1 + \epsilon\})$$

is outside of S, the argument goes exactly the same as before but we have instead that

$$\tilde{\chi}(z) := \frac{R_2}{\chi(z)}.$$

Also, we have that

$$\lim_{n\to\infty} |\tilde{\chi}(z_n)| = R_2$$

whenever

$$\lim_{n\to\infty} |z_n| = 1,$$

and

$$\lim_{n\to\infty} |\tilde{\chi}(z_n)| = 1$$

whenever

$$\lim_{n\to\infty} |z_n| = R_1.$$

Now, consider the function

$$u(z) = \log|\chi(z)| = \mathrm{Re}(\log(\chi(z))),$$

which is the real part of a holomorphic function. Then, we know that $u(z) : A_1 \to A_2$ is harmonic, and from Theorem 3.7, we have, by setting

$$u(z) := \begin{cases} 0 & \text{when } |z| = 1, \\ \log(R_2) & \text{when } |z| = R_1, \end{cases}$$

that the extension of $u(z)$ is continuous on the closure of A_1. Another such function, with the same boundaries on A_1, is the function

$$\frac{\log(R_2)}{\log(R_1)} \log|z|.$$

From the maximum principle (Theorem 2.29), we have that both these functions are the same, and

$$u(z) = \frac{\log(R_2)}{\log(R_1)} \log|z|.$$

This implies that

$$|\chi| = |z|^\alpha,$$

where

$$\alpha := \frac{\log(R_2)}{\log(R_1)}.$$

We have that $\chi = cz^\alpha$ for some $|c| = 1$. By assumption, χ needs to be injective in order to be a biholomorphic mapping, it follows that α must be equal to one. Hence, we have that $R_1 = R_2$. □

Remark 4.30. Note, that this theorem is an counterexample for when the converse relation of Theorem 4.16 does not hold. It is easy to show that all automorphisms on some annuli, $A := \{z \in \mathbb{C} : 0 < r < |z| < R < \infty\}$, is generated by rotations $e^{i\phi}z$ and the inversion $\frac{rR}{z}$ under composition. There is an obvious group isomorphism, $\phi : \mathrm{Aut}(A_1) \to \mathrm{Aut}(A_2)$, where we can map every element directly by replacing r_1 and R_1 in $h_1 \in \mathrm{Aut}(A_1)$ with r_2 and R_2 in $h_2 \in \mathrm{Aut}(A_2)$, respectively.

5. Automorphism Groups and the Riemann Mapping Theorem

As we saw in Proposition 4.18 the upper half plane is biholomorphically equivalent to the unit disc. In this section, our goal is to generalize Proposition 4.18, and prove that $\mathrm{Aut}(D)$ is isomorphic to $\mathrm{Aut}(B_1(0))$, if D is a simply connected domain and a proper subset of $\mathbb{C}$. To show this, we shall use the famous Riemann mapping theorem. In short, it states that any simply connected domain, D, with at least two boundary points will be biholomorphically equivalent to the unit disc. Though, before we go into further details, we will need some serious machinery to prove the theorem.

The first theorem we shall introduce is Hurwitz' theorem. Before we state it, we shall need the following lemma.

Lemma 5.1. *Let $\{f_j\}$ be a sequence of holomorphic functions, $f_j : D \to \mathbb{C}$ where D is a domain. If, $\{f_j\}$ converges uniformly to a non-constant function, f, on each compact subset of D. Then for every $c \in D$, there is a number $\eta \in \mathbb{N}$ and a sequence of points, $\{c_j\}$, $c_j \in D$ for $j \geq \eta$ such that,*

$$\lim_{n\to\infty} c_n = c, \text{ and } f_j(c_j) = f(c) \text{ for all } j \geq \eta.$$

Proof. Let f, f_j be functions satisfying the assumptions of the lemma.Assume, without loss of generality, that $f(c) = 0$. Since f is non-constant Theorem 3.1 applies, and there exists an open ball, $B_r(c)$, with $\bar{B}_r(c) \subset D$, such that f does not have any zeros in $\bar{B}_r(c)\backslash\{c\}$. Since f_j converges uniformly on $\partial B_r(c) \cup \{c\}$, there is an $\eta \in \mathbb{N}$ such that

$$|f_j(c)| < \min\{|f_j(z)| : z \in \partial B_r(c)\}, \qquad \text{for all } j \geq \eta.$$

By Theorem 3.3, each f_j, with $j \geq \eta$, has a zero in $B_r(c)$. Then, it holds that $\lim_{j\to\infty} c_j = c$. Otherwise, there would exist a subsequence, c'_j, that converges to a $c' \in \bar{B}_r(c)\backslash\{c\}$. From the continuity of f_n, we have that

$$0 = \lim_{j\to\infty} f_j(c'_j) = f(c'),$$

which is a contradiction to that f has no zeros in $\bar{B_r}(c)\backslash\{c\}$. □

From the previous lemma, we can construct a proof by induction of Hurwitz' theorem.

Theorem 5.2 (Hurwitz' theorem)**.** *Let a sequence $\{f_j\}$ of holomorphic functions, $f_j : D \to \mathbb{C}$, converge uniformly to a non-constant function, f, on every compact subset of a domain, D. Let U be a bounded open subset of D, with $\bar{U} \subset D$, such that f has no zeros on ∂U. Then, there exists an index $\eta \in \mathbb{N}$, such that for all $j \geq \eta$, the functions f and f_j have the same amount of zeros in $\bar{U}$.*

Proof. Since $f \neq 0$, and $\bar{U}$ is compact, it follows from Theorem 3.1, that the number

$$m := \sum_{z\in\bar{U}} o_z(f)$$

is finite, where

$$o_z(f) := \begin{cases} 1 & \text{if } f(z) = 0, \\ 0 & \text{if } f(z) \neq 0. \end{cases}$$

We prove this by induction on m. For the case $m = 0$, the number

$$\epsilon := \min\{|f(z)| : z \in \bar{U}\}$$

is strictly positive. Since

$$|f_j - f|_{\bar{U}} < \epsilon$$

for all but finitely many j, then all but finitely many f_j do not have any zeros in $\bar{U}$.

Assume $m > 0$, and let $c \in U$ be a zero of f. According to Lemma 5.1, there exists an $\eta \in \mathbb{N}$ and a sequence $\{c_j\}$, where $c_j \in U$ for $j \geq \eta$, such that $f_j(c_j) = 0$ for all $j \geq \eta$, and $\lim c_j = c$. Then, for all such j, there exists holomorphic functions $h, h_j : D \to \mathbb{C}$ such that

$$f_j(z) = (z - c_j)h_j(z), \qquad f(z) = (z - c)h(z).$$

Since the limit,

$$\lim_{j\to\infty} (z - c_j) = z - c,$$

converges uniformly on the compact subset $\bar{U} \subset D$, we have that h_j must also converge uniformly to h on $\bar{U}$. Because f has m zeros, h must have $m - 1$ zeros in $\bar{U}$, and none in ∂U. Thus h_j must also have $m - 1$ zeros for all $j \geq \eta$. Then, by induction, it follows that for some $\eta \in \mathbb{N}$, f_j has exactly m zeros in U for all $j \geq \eta$. □

The next result we shall introduce is the Ascoli-Arzela theorem, but before we state it, we are going to define two properties that family of functions can attain. Note, that a *family of functions* is defined as a set of functions where every element of the set is exclusively associated with a natural number.

Definition 5.3. A family of functions, $\mathcal{F}$, on a domain $U \subseteq \mathbb{C}$ is called *equicontinuous*, if for any $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|f(z) - f(\omega)| < \epsilon$$

for every $f \in \mathcal{F}$, and for each pair of points $z, \omega \in U$ which satisfy $|z - \omega| < \delta$.

Definition 5.4. A family of functions, $\mathcal{F}$, on a domain $U \subseteq \mathbb{C}$ is called *equibounded*, if there exists a number $M > 0$, such that for every $z \in U$ and every $f \in \mathcal{F}$, it holds that

$$|f(z)| \leq M.$$

We shall restrict ourselves to a less general version of the Ascoli-Arzela theorem, as we only consider subsets of $\mathbb{C}$.

Theorem 5.5. *Let K be a compact set in $\mathbb{C}$. If a family of functions $\mathcal{F} = \{f_\eta\}$ is equicontinuous and equibounded, then there is a subsequence of $\mathcal{F}$ that is uniformly convergent on K.*

Proof. There exists a sequence, $\{\zeta_k\}$, that is dense in K, for further details why this is the case see e.g. Theorem 1.20 in [9]. Since $\mathcal{F}$ is equibounded on K, for ζ_1, there exists a convergent finite subsequence $\{f_{\nu 1k}(\zeta_1)\}$ in $\{f_\nu(\zeta_1)\}$. There also exists a convergent subsequence

$\{f_{\nu 2k}(\zeta_2)\}$ in $\{f_{\nu 1k}(\zeta_2)\}$, etc. By continuing this process we get,

$$\begin{gathered}\nu_{11} < \nu_{12} < \cdots < \nu_{1j} < \cdots, \\ \nu_{21} < \nu_{22} < \cdots < \nu_{2j} < \cdots, \\ \vdots \\ \nu_{k1} < \nu_{k2} < \cdots < \nu_{kj} < \cdots, \\ \vdots\end{gathered}$$

where every row is a subsequence of the preceeding row, and the limit,

$$\lim_{j\to\infty} f_{\nu_{kj}}(\zeta_k),$$

exists for every $k \in \mathbb{N}$. We have directly that the diagonal sequence, $\{\nu_{jj}\}$, is strictly increasing and is also a subsequence of each row, except for finitely many terms. Hence, $\{f_{\nu_{jj}}\}$ is a subsequence of $\{f_\nu\}$ that converges on every point, ζ_k. We denote ν_{jj} as ν_j for simplicity.

Since $\mathcal{F}$ is equicontinuous on K, we have that for every $\epsilon > 0$, there exists a $\delta > 0$ such that, for any $z, z_1 \in K$ and for any $f \in \mathcal{F}$, $|z - z_1| < \delta$ implies

$$|f(z) - f(z_1)| < \frac{\epsilon}{3}.$$

Since K is compact, it can be covered by finitely many discs with radius $\delta/2$. Choose a point, ζ_k, in each disc, then there exists an N such that

$$|f_{\nu_i}(\zeta_k) - f_{\nu_j}(\zeta_k)| < \frac{\epsilon}{3}$$

for all $i, j > N$. For every $z \in K$, there exists points ζ_k such that $|\zeta_k - z| < \delta$. Thus, we have that

$$|f_{\nu_i}(z) - f_{\nu_i}(\zeta_k)| < \frac{\epsilon}{3},$$

and

$$|f_{\nu_j}(z) - f_{\nu_j}(\zeta_k)| < \frac{\epsilon}{3}.$$

All of this, together with the triangle inequality, implies that

$$|f_{\nu_i}(z) - f_{\nu_j}(z)| < \epsilon,$$

for all $i, j \in \mathbb{N}$ and all $z \in K$. Hence, $\{f_{\nu_j}\}$ converges uniformly on K. □

Moreover, we shall need one more definition.

Definition 5.6. A family of functions $\mathcal{F}$ is said to be *normal* on a domain U, if for every sequence in $\mathcal{F}$, there exists a subsequence that converges uniformly on every compact subset of U.

With the use of the Theorem 5.5, we can now prove Montel's theorem.

Theorem 5.7 (Montel's theorem)**.** *Let $\mathcal{F}$ be a family of holomorphic functions on a domain $U \subseteq \mathbb{C}$. If there exists a number, $M > 0$, such that*

$$|f(z)| \leq M$$

for all $z \in U$ and all $f \in \mathcal{F}$, then $\mathcal{F}$ is normal.

Proof. For any $z_0 \in U$, there exists a $r > 0$ such that $\bar{B}_r(z_0) \subset U$. Since U is open, it holds that the complement U^c is closed in $\mathbb{C}$. From our construction we have that $\bar{B}_r(z_0)$ and U^c are disjoint. Furthermore, we have that the distance between these two sets is strictly positive. Then, there exists a $c > 0$ such that $|z - u| > c$ for all $z \in \bar{B}_r(z_0)$ and all $u \in U^c$. For any $z \in \bar{B}_r(z_0)$ and for any $f \in \mathcal{F}$, we can apply the Cauchy inequality (Theorem 2.28) on $B_c(z)$, so that

$$|f'(z)| \leq \frac{M}{c}.$$

Set

$$C := \frac{M}{c}.$$

Thus,

$$|f(z) - f(\omega)| \leq C|z - \omega|$$

for all $z, \omega \in \bar{B}_r(z_0)$. This implies that $\mathcal{F}$ is equicontinuous on $\bar{B}_r(z_0)$. Explicitly, we have that for every $\epsilon > 0$, we can choose

$$\delta = \frac{\epsilon}{C}.$$

If K is a compact subset of U, then there exists finitely many $B_r(z_0)$ that covers K. Then, it follows that $\mathcal{F}$ is also equicontinuous on K. According to Theorem 5.5, for any sequence, $\{f_\nu\}$, there exists a subsequence, $\{f_{\nu_k}\}$, that converges uniformly on K. Hence, $\mathcal{F}$ is normal. □

There, all of this is sufficient to show the Riemann Mapping Theorem. The proof of the theorem will make use of constructions of various functions.

Theorem 5.8 (Riemann Mapping Theorem)**.** *Any simply connected domain in $\mathbb{C}$ with at least two boundary points is biholomorphically equivalent to the unit disc.*

Proof. Let U be a simply connected domain. Firstly, we consider the case that U is bounded. Fix a point $z_0 \in U$, and let

$$\mathcal{F} := \{\sigma(z) : U \to B_1(0)\}$$

be a family of injective and holomorphic functions such that $\sigma(z_0) = 0$ for each $\sigma \in \mathcal{F}$. We shall show that $\mathcal{F}$ is well-defined and non-empty. This is the case, since U is bounded, then U lies completely within some disc $B_R(0)$, $R > 0$ large enough. The mapping, $\sigma_1 : D \to \mathbb{C}$, defined by $\sigma_1(\zeta) = \frac{1}{2R}(\zeta - z_0)$ is holomorphic, injective, maps $z_0 \mapsto 0$ and

$$|\sigma_1(\zeta)| < \frac{1}{2R}(R + R) < 1.$$

Hence, $\sigma_1 \in \mathcal{F}$ and we can conclude that $\mathcal{F}$ is non-empty. Since every function on $\mathcal{F}$ is holomorphic and bounded, we have, from Montel's theorem (Theorem 5.7), that $\mathcal{F}$ is a normal family.

Now, set

$$M := \sup\{|\sigma'(z_0)| : \sigma \in \mathcal{F}\}.$$

Suppose there is a closed disc, $\bar{B}_r(z_0)$, such that U is contained within it. Then, by the Cauchy inequality (Theorem 2.28), we have that

$$|\sigma'(z_0)| \leq \frac{1}{r}.$$

Hence, we have that

$$M \leq \frac{1}{r}.$$

Now, we want to show that there exists a function, $\sigma_0 \in \mathcal{F}$, such that $|\sigma_0(z_0)| = M$.

By construction of M, there exists a sequence, $\{\sigma_j\}$, in $\mathcal{F}$ such that

$$|\sigma_j'(z_0)| \to M.$$

Since $\mathcal{F}$ is a normal family, there exists a subsequence, $\{\sigma_{j_k}\}$, that converges uniformly to a function σ_0 on every compact subset of U. Because

$$|\sigma_j'(z_0)| \to M,$$

then

$$|\sigma_0'(z_0)| = M.$$

Next, we prove that σ_0 is injective on U. Let $z_1, z_2 \in U$ be two distinct points, such that there is a $s > 0$ with

$$s < |z_1 - z_2|.$$

Now, consider the function

$$\psi_k(z) := \sigma_{j_k}(z) - \sigma_{j_k}(z_1)$$

on $B_s(z_2)$. Since σ_j is injective, so is ψ_k, but that means that ψ_k has no zeros in $\bar{B}_s(z_2)$. By Hurwitz' theorem (Theorem 5.2), we have that the limit of ψ_k,

$$\sigma_0(z) - \sigma_0(z_1),$$

is either identically zero or never zero on $B_s(z_2)$. From the construction we have that σ_0 can not be identically zero, since $\sigma'(z_0) = M > 0$. Thus, for every $z \in \bar{B}_s(z_2)$, we have that

$$\sigma_0(z) \neq \sigma_0(z_1),$$

for which it follows that

$$\sigma_0(z_2) \neq \sigma_0(z_1).$$

Hence, σ_0 is injective.

Now, we shall show, with a contradiction, that $\sigma_0 : U \to B_1(0)$ is surjective. Suppose that σ_0 does not map U onto $B_1(0)$, then there exists a point $\beta \in B_1(0)$ such that

$$\beta \notin \sigma_0(U).$$

Define the function, $\phi_\beta : B_1(0) \to B_1(0)$, by

$$\phi_\beta(\zeta) = \frac{\zeta - \beta}{1 - \bar{\beta}\zeta}.$$

Then, we have that

$$\mu(\zeta) := (\phi_\beta \circ \sigma_0)^{1/2}(\zeta)$$

is a holomorphic function defined on U. Set

$$\tau := \mu(z_0),$$

and

$$\phi_\tau(\zeta) := \frac{\zeta - \tau}{1 - \bar{\tau}\zeta}.$$

Also, set

$$\eta(\zeta) := \phi_\tau \circ \mu(\zeta),$$

and

$$\nu(\zeta) := \frac{|\eta'(z_0)|}{\eta'(z_0)}\eta(\zeta).$$

Now, we have that $\nu \in \mathcal{F}$, but $\nu(z_0) = 0$ and

$$|\nu'(z_0)| = \frac{1 + |\beta|}{2|\beta|^{\frac{1}{2}}} M > M.$$

This is a contradiction of σ_0, since $\sigma_0(P) = \sup\{\sigma(P)|\sigma \in \mathcal{F}\}$ Thus, η can not belong to $\mathcal{F}$. Hence, σ_0 is surjective. We can conclude that $f := \sigma_0$ is precisely the biholomorphic mapping that is required.

Finally, we need to show that we can biholomorphically map an unbounded domain with at least two boundary points to a bounded region. Without loss of generality, choose these two points to be 0 and $a \notin \{0, \infty\}$. This is possible since we can map two arbitrary boundary points to 0 and a by a Möbius transformation.

Let $g(z)$ be a branch of $\sqrt{z - a}$ on U. Since U is simply connected, so is $g(U)$. By our choice of branch cut, g is injective. Let $z_1, z_2 \in U$, then

$$\sqrt{z_1 - a} = \sqrt{z_2 - a}$$

implies $z_1 - a = z_2 - a$. Also, we have that

$$g(U) \cap (-g(U)) = \varnothing.$$

If we suppose the contrary, i.e. there exists a $P \in g(U)$ such that $-P \in g(U)$, then we have for some $z_1, z_2 \in U$, that

$$\sqrt{z_1 - a} = P,$$

and

$$\sqrt{z_2 - a} = -P.$$

This implies that $z_1 = z_2$, so $P = -P$ and $P = 0$. But $P = 0 \notin U$, which is a contradiction.

Let $q \in g(U)$ be an arbitrary point. Since $g(U)$ is simply connected, it follows that there exists a neighborhood, U_q, of $g(q)$, and $U_q \subset g(U)$. Since $-U_q \not\subseteq g(U)$, we can choose a point $b \in -U_q$, and we can construct the function $\phi : g(U) \to D$,

$$\phi := \frac{1}{z - b}.$$

We have that ϕ maps $g(U)$ biholomorphically to a bounded simply connected region, D. Hence, the function $\phi \circ g$ maps U biholomorphically to a bounded region. □

Remark 5.9. More commonly the Riemann mapping theorem is stated for proper subsets of $\mathbb{C}$ instead of sets with at least two boundary points. As a matter of fact, they are equivalent for simply connected domains. It is obvious that if, D has at least two boundary points, then it is a proper subset of $\mathbb{C}$. Conversely, assume D is a proper subset of $\mathbb{C}$. Now, assume that D has exactly one boundary point, $c \in \mathbb{C}$. As we shall see, this will imply a contradiction. Since D is open, it holds that $c \notin D$. Then, for some neighborhood $B_r(c)$ of c, it holds that $B_r(c)\backslash\{c\} \subset D$. We can construct a curve on D around c, and we can conclude that D is not simply connected. This is a contradiction to the assumption of D, and we are done.

From the Riemann Mapping Theorem 5.8, we can conclude that automorphism groups on all simply connected domains, $D \subset \mathbb{C}$, where $D \neq \mathbb{C}$, have the same algebraic structure as $\mathrm{Aut}(B_1(0))$.

Theorem 5.10. *Let $D \subset \mathbb{C}$ be a simply connected domain not equal to $\mathbb{C}$, then $\mathrm{Aut}(D)$ and $\mathrm{Aut}(B_1(0))$ are isomorphic.*

Proof. From the Riemann mapping theorem (Theorem 5.8), we have that all simply connected domains $D \neq \mathbb{C}$ are biholomorphically equivalent to $B_1(0)$. Then, Theorem 4.16 applies, and $\mathrm{Aut}(D)$ is isomorphic to $\mathrm{Aut}(B_1(0))$ □

Remark 5.11. It is important to observe that the proof of Theorem 5.10 is only valid for $\mathbb{C}$, i.e. the complex plane in one dimension. The Riemann mapping theorem (Theorem 5.8), does not generalise to $\mathbb{C}^n$ [6].

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