Pricing of European and Asian options with Monte Carlo simulations

Variance reduction and low-discrepancy techniques

Alexander Ramström

Umeå University
Fall 2017
Bachelor Thesis, 15 ECTS
Department of Economics
Abstract

This thesis evaluates different models accuracy of option pricing by Monte Carlo simulations when changing parameter values and the number of simulations. By simulating the asset movements thousands of times and use well established theory one can approximate the price of one-year financial options and for the European options also compare them to the price from Black-Scholes exact pricing formula. The models in this thesis became two direct variance reducing models, a low-discrepancy model and also the Standard model. The results show that the models that controls the generating of random numbers has the best estimating of the price where Quasi-MC performed better than the others. A Hybrid model was constructed from two established models and it showed that it was accurate when it comes to estimating the option price and even beat the Quasi-MC most of the times.
Contents

1 Introduction 4
  1.1 Delimitations ................................................. 5
  1.2 Earlier studies ................................................. 5

2 Theory 8
  2.1 European options ............................................. 8
  2.2 Arithmetic Asian options .................................... 9
  2.3 Brownian motion ............................................... 10
  2.4 Black-Scholes .................................................. 12
  2.5 Monte Carlo ..................................................... 13
  2.6 Variance reducing models .................................... 14
      2.6.1 Stratified sampling ...................................... 14
      2.6.2 Antithetic variates ....................................... 18
      2.6.3 Quasi Monte Carlo ....................................... 19

3 Method 22

4 Results 24
  4.1 European Options ............................................. 24
  4.2 Asian Options ................................................ 29

5 Discussion and Conclusions 31

6 References 33
Appendix

A.1 Tables ......................................................... 35
A.2 Strong Law of Large Numbers .............................. 38
A.3 Variance calculation ......................................... 38
   A.3.1 Antithetic variates ........................................ 38
   A.3.2 Stratified sampling ....................................... 40
1 Introduction

The intensity and tempo in the financial market are probably higher than most businesses. Values of financial instruments can change all the time and traders try their best to make the most profit and have the best pricing model for their options in the market. Models that can calculate and see if options are over or under valued, from that take the opportunity to do a good deal by either buy or sell these instruments.

This thesis considers models to price one year financial options by Monte Carlo simulations, with focus on accuracy of price estimation when changing its parameters. By different approaches implement the theories and test them through scenarios to find the week spots. Since there exist a theoretical exact formula for pricing European options in Black-Scholes it’s interesting to see if extreme values of the parameters would affect the models and make the pricing of the options more uncertain. As a research question I then have under what conditions does these models price European options close to Black-Scholes?

With the technological development the last decades computers these days are a strong tool to use when it comes to calculations. If one wants to simulate a path thousands of times it can be fixed in no time\(^1\) and the biggest focus can be put in the implementing of the theories. By different techniques in how to simulate a stock, which one is pricing the option best? Or are the difference too small to actually care? Could it be possible to combine the characteristics of different models to one hybrid model and how would it perform compared to the already established?

\(^1\)Compared to doing it by hand instead.
For more exotic options as arithmetic Asian options there is no formula to calculate the exact value, so one has to simulate the paths and see what they converge to instead. Since they are path-dependent options the option price is calculated differently than European options. But if one wants to see how the relation is between Black-Scholes exact price and arithmetic Asian options, how far away would the price be?

1.1 Delimitations

The limitations for this thesis are that it will only simulate one underlying stock as a financial option, where it will only be tested on call options. The call option will always be a one year old option with a monthly path\(^2\), with no premium involved. The parameters as volatility and interest rate will be held constant during the simulation paths, this is specially to compare the models to Black-Scholes formulas. During all simulations will the strike price and initial stock price be 200. Default for volatility is 0.3, the interest rate is 0.02 and number of simulations is \(3 \cdot 10^4\) until other things are said. The whole implementation of the methods and testing will be worked out with the help of the programing language \(R\) and \(Matlab\).

1.2 Earlier studies

Monte Carlo techniques is a practical tool for many research areas as computer power develops all the time. It can be used on big topics today as for medical purpose on cancer studies (Zamora et al, 2016) and could hopefully be able to be a part of saving life in the long run. From simulations in medical studies to different areas as

\(^2\)The monthly path here means that the options will be simulated with discrete time steps \(\frac{1}{12}\) as a monthly representation.
nuclear physics (Nagaya et al, 2014) or studies in biology as the DNA (Delage et al, 2015) shows the broad set of users in Monte Carlo simulations. Also in finance these techniques are used widely and could be all from foreseeing risks in portfolios (Sak, Basoglu, 2017) to pricing options (Buchner, 2015).

This thesis is mostly about analyzing different models and compare them in pricing financial options. Since there is a lot of different kinds of models I narrowed it down to four already known models in the area and also one combined model with the characteristics from two of the already known. From Håkan Andersson (2013) a variance reducing model and a low-discrepancy model were used for comparison together with the Standard model and a benchmark of Black-Scholes formulas. He concluded that the Antithetic variates model\(^3\) was the one to prefer when it came to both pricing the options (convergence) and also the computational time of the simulations. By his results and analysis it got me interested to test these assumptions on pricing and compare Antithetic variates to the Standard-MC and a Quasi-MC model myself and also one other variance reducing model that have a different approach then Antithetic, i.e. Stratified sampling. The study also showed how these methods worked on more exotic options, i.e. Asian options, but since there is no exact price as for European options they were just compared to each other.

In Erik Wiklund (2012) he compared the Asian arithmetic approximations to the exact value for European options, i.e. Black-Scholes. He came to the conclusion that Black-Scholes overvalues the Asian options for most of the times. This gave me the intuition to test the models also on Asian options and how much they would deviate from Black-Scholes.

\(^3\)More about this and the other models characteristics in the Theory section.
The idea of combining two models into one and create some kind of hybrid model was tempting already in the beginning of this thesis. In Farshid Mehrdoust (2015) he combines the techniques of Antithetic variates and Multiple Control variates. By his results the approximation of arithmetic Asian options got improved in comparison to the models alone. Here in this thesis I will combine the Arithmetic variates with an low-discrepancy technique that will treat the random numbers that Control variates doesn’t.
2 Theory

2.1 European options

Financial options are much what it sounds as, it is an option for a holder to buy or sell an financial asset which can be a stock, bond, etc. The holder is not obligated to fulfill this contract but have the option to buy/sell the asset for a strike price, $K$, which is determined beforehand. The contract can only be used at time $T$ on the interval $[0, T]$.

For a European option described by Björk (2009, 92-97) the criteria is that the option only can be exercised at the end of the contracts time interval, at time $T$. Here we have two different kinds of European options, both call- and put options. Call options gives the holder the chance to _buy_ for example the stock $S$ for a price $K$ at time $T$. The payoff for this type of option is defined as

$$C_{EO} = \max(S(T) - K, 0) \equiv (S(T) - K)^+, \quad (1)$$

where $S(T)$ is the stock price at time $T$ and $(S(T) - K)^+$ is a shorthand notation for the maximum value of $S(T) - K$ or 0. The payoff for a European call option is unlimited in theory since it’s no restriction of how high the stock price will be at time $T$ as one can see in Figure 1.

The other type to be displayed is a put option. A European put option works similar to the call option but instead of hoping for the stock price to be above $K$ the holder of the option is hoping for the stock price to be below $K$ at time $T$. The holder has
the chance to sell the stock $S$ for a price $K$ at time $T$ and its payoff looks like

$$P_{EO} = \max(K - S(T), 0) \equiv (K - S(T))^+. \tag{2}$$

Here the limitation for the payoff is the strike price $K$. Since stock price can’t go under zero the maximum payoff for a European put option is $K$. To illustrate these two examples of European options the Figure 1 shows how the two payoffs looks like.

![Figure 1 – Illustrations of the possible payoffs for European call and put options.](image)

From Figure 1 we see the two different kinds of European options illustrated and its possible payoffs at time $T$.

### 2.2 Arithmetic Asian options

An Asian option in relation to a European option is path-dependent and does not only depend on the price at time $T$. For an arithmetic Asian option the payoff comes from the mean value of the stock price at specific discrete times, $0 < t_1, \ldots, t_m = T$. 

9
The payoff for an arithmetic Asian call option (average price) can be written as

\[ C_{AO} = \left( \frac{1}{m} \sum_{i=1}^{m} S(t_i) - K \right)^+ \]  

and the respectively put option

\[ P_{AO} = \left( K - \frac{1}{m} \sum_{i=1}^{m} S(t_i) \right)^+ \]  

This kind of mean value calculation is a arithmetic average and will be the type to calculate Asian options.

2.3 Brownian motion

Here the starting point for pricing assets will be the Geometric Brownian Motion (GBM) from Björk (2009, 67-69) and it can be seen as a linear ordinary differential equation (ODE)

\[ dX(t) = \alpha X(t) dt + \sigma X(t) dW(t), \]

where \( X(t) \) is a stochastic process and \( dW(t) \) is the time derivative of the Wiener process. A Wiener process has the properties that can be summed up from Definition 4.1 in Björk (2009, 40).

**Definition 2.1.** A stochastic process \( W \) is called a **Wiener process** if the following conditions hold

1. \( W(0) = 0 \).

2. The process \( W \) has independent increments, i.e. if \( r < s \leq t < u \) then \( W(u) - W(t) \)
and $W(s)-W(r)$ are independent stochastic variables.

3. For $s < t$ the stochastic variable $W(t)-W(s)$ has the Gaussian distribution $N[0, \sqrt{t-s}]$.

4. $W$ has continuous trajectories.

The solution for Eq.5, assuming that $Z(t) = ln(X(t))$, can be worked out with the help of Ito formula

$$dZ(t) = \frac{dX}{X} + \frac{1}{2} \left[ \frac{-1}{X^2} \right] (dX)^2,$$

then using Eq.5 in Eq.6 and know that $(dW)^2 = dt$ we get

$$dZ(t) = \frac{1}{X} \left[ \alpha X dt + \sigma X dW \right] - \frac{1}{2} \left[ \frac{1}{X^2} \right] \sigma^2 X^2 dt$$

$$= \left( \alpha - \frac{1}{2} \sigma^2 \right) dt + \sigma dW.$$

This leads up to

$$dZ(t) = \left( \alpha - \frac{1}{2} \sigma^2 \right) dt + \sigma dW,$$  
(8)

$$Z(0) = ln(x_0),$$

which can be integrated to

$$Z(t) = ln(x_0) + \left( \alpha - \frac{1}{2} \sigma^2 \right) dt + \sigma dW = ln(X(t))$$

$$= \Rightarrow X(t) = x_0 \cdot e^{\left( \alpha - \frac{1}{2} \sigma^2 \right) dt + \sigma dW}.$$

(9)

For a stock price $S$ at time $t$ with drift $\mu$ and volatility $\sigma$ the formulation of Eq.9
will look like

\[ S(t + 1) = S(t) \cdot e^{\left(\mu - \frac{1}{2} \sigma^2\right)dt + \sqrt{dt} \sigma Z(t + 1)}, \]  

(10)

where \( Z(t_i) \) are standard normal distributed variables.

### 2.4 Black-Scholes

For this thesis a benchmark for the models will be the well known *Black-Scholes* pricing formulas, which sees as the theoretical exact price for an European option. Though it gives an exact price there are some limitations one should know about it. For these formulas to work the volatility and risk-free interest rate are constant and known, which is not always the reality usually. It can only be exercised at time \( T \) and the payoff is not path-dependent, so it can not be used at Asian options since they are path-dependent. But it will be tested in this thesis to see how the formulas relates to the Asian options. Maybe a pattern can be seen. For a European call option the formulation (Björk 2009, 105) is like the following way

\[ C(t, S) = S \cdot N[d_1(t, S)] - e^{-r(T-t)}K \cdot N[d_2(t, S)], \]  

(11)

and its respectively put option as

\[ P(t, S) = K \cdot e^{-r(T-t)}N[-d_2(t, S)] - S \cdot N[-d_1(t, S)]. \]  

(12)

Where \( K \) is the strike price for the option, \( S \) is the initial stock price, \( T \) is the exercise time in years, \( r \) is the risk-free interest rate and \( t \) the starting time of the option. There are also some more difficult terms, the cumulative standard normal
distributions of \( d_1 \) and \( d_2 \). These two terms have the relations as

\[
d_1(t, S) = \frac{1}{\sigma \sqrt{T-t}} \left[ \ln \left( \frac{S}{K} \right) + \left( r + \frac{1}{2} \sigma^2 \right) (T-t) \right], \quad (13)
\]

\[
d_2(t, S) = d_1(t, S) - \sigma \sqrt{T-t} = \frac{1}{\sigma \sqrt{T-t}} \left[ \ln \left( \frac{S}{K} \right) + \left( r - \frac{1}{2} \sigma^2 \right) (T-t) \right]. \quad (14)
\]

where \( \sigma \) is the volatility. These call option equations will be the benchmark to see how the European option models approximate the option price for given and constant risk-free interest rates and volatility. Also the Asian option models will be checked against the formulas but not as a confirmation of its correctness but more as a test to see how they behave in comparison to Black-Scholes.

### 2.5 Monte Carlo

The base for all the upcoming methods in this thesis will be the Standard Monte Carlo model (Standard-MC). The model is approximating the option price as

\[
C = E \left[ e^{-rT} (S(T) - K)^+ \right],
\]

\[
P = E \left[ e^{-rT} (K - S(T))^+ \right],
\]

where \( C \) represents the European call option and \( P \) the put option. The factor \( e^{-rT} \) is for discounting the price for time 0 instead of \( T \). Since \( S \) evolves according to a GBM the methodology is (Glasserman 2004, 5)

**Definition 2.2.** Methodology for Monte Carlo Simulations, here a call option.

1. Generate \( m \) independent standard normal random variables, i.e. \( z_1, z_2, \ldots, z_m \sim N(0, 1) \).
2. Let $S$ be defined as $S(T) = S(0)e^{\left((r-\frac{1}{2}\sigma^2)T+\sigma\sqrt{T}Z_i\right)}$, for each $Z_i$.

3. Let $C_i = e^{-rT}(S(T) - K)^+$, for each $Z_i$.

4. Set $C \approx \hat{C}_m = \frac{1}{m} \sum_{i=1}^{m} C_i$.

By the *The strong law of large numbers*, in the fourth statement above, the call option approximates as $\hat{C}_m \to C$ and the same goes for the put option $\hat{P}_m \to P$ as one can read more about in Appendix. The Monte Carlo method is in many cases straightforward and easy\textsuperscript{4} to implement an given algorithm. The method works for both non path-dependent\textsuperscript{5} options, as European, but also for path-dependent options as Asian options that also will be handled in this thesis. All the different models that will be used later use this Standard-MC technique as a groundwork but then with some own specifications of handling the random numbers, reducing the variance of the payoffs and calculating the option price.

### 2.6 Variance reducing models

#### 2.6.1 Stratified sampling

One type of these variance reduction techniques is the one called *Stratified sampling* where the method uses an interesting approach. To start up this method an intuitive example could be a good start to see how Stratified sampling in general works. Let say we have 3 different populations according to the Figure 2.

If the populations have different sizes and we for example want to do a poll for some election. We could do a random selection from the whole population, with the

\textsuperscript{4}This is of course always relatively, but in this area of mathematics it can be seen as easy.

\textsuperscript{5}Non path-dependent here means that we only need the starting value $S(0)$ and last value $S(T)$ to calculate the option price.
risk of being biased since the opinions could shift drastically between the groups in the populations. Instead we can reduce this risk by dividing the population into subgroups and take an proportional number of persons from each subgroup and collect them to the poll. In this way we avoid estimation errors in the margin of our control.

Here we are working with pricing financial assets so for an European option the value at expiration is the only option value that the price depends on, i.e. at $T = 1$. By stratifying (Glasserman 2004, 210-221) along the options value at expiration one can get rid of majority of the variance from the payoff of the option. For this thesis GBM is used as a base for the option path and since it has a constant volatility we can use so called *Brownian bridges* construction to produce the Brownian path, here a discrete path, for all $W(t_i)$ where $i = 1,...,M$. The idea behind Brownian bridges (Glasserman 2004, 82-86) is that one constructs a so called *bridge* for the simulated path, given the values $W(t_0)$ and $W(t_m)$ for $0 = t_0 < t_1 < ... < t_m = T$. These values are the first and last in the path and it uses the distribution for the in between values $W(t_j)$, which depends on both the earlier values $W(t_{j-1})$ and the last $W(t_m)$.

This method is using an inverse transformation method to stratify the last value...
\( W(t_m) \) (Glasserman 2004, 221), which is going to used for the construction of the earlier values \( W(t_1), \ldots, W(t_{m-1}) \). The inverse functions \( \Phi^{-1}(V_i), i = 1, \ldots, K \), form a stratified sample as \( \Phi^{-1}(V_i) \sim N(0,1) \) where \( V_i \) is defined as

\[
V_i = \frac{i-1}{K} + \frac{U_i}{K},
\]

where \( U_i \) are independent and uniformed variables on the interval \([0,1]\). Given this it logically implies that the following stratified sample has the distribution \( \sqrt{t_m} \Phi^{-1}(V_i) \sim N(0,t_m) \). This could be seen as a start by stratifying the last value \( W(t_m) \), then to fill in the gap with the values before, \( W(t_1), \ldots, W(t_{m-1}) \), we use Algorithm 1, given that the conditional distribution of \( W(t_j) \) is

\[
W(t_j) \sim N \left( \frac{t_m-t_j}{t_m-t_{j-1}} W(t_{j-1}) + \frac{t_j-t_{j-1}}{t_m-t_{j-1}} W(t_{m}), \frac{(t_m-t_j)(t_j-t_{j-1})}{t_m-t_{j-1}} \right).
\]

**Algorithm 1:** An algorithm implementation of Brownian paths (Glasserman 2004, 221).

\(^6\)Also remember that \( W(0)=0 \) from Definition 2.1.
With the Stratified sampling it is possible to control the random numbers and how they are generated. This is an advantage with proportional stratification to match the underlying distribution. To write the option price as a expected value we uses the discounted payoff, $Y$, stratification variable, $X$, and $\{A\}_{i=1}^{m}$ which is disjointed $^7$ subsets of $\mathbb{R}^d$. These subsets are chosen $P(X \in \bigcup_i A_i)$ and the expected value of the discounted option price can be written as

$$E[Y] = \sum_{i=1}^{M} E[Y|X \in A_i] \cdot P(X \in A_i) = p_i \cdot E[Y|X \in A_i].$$

(16)

We will let $X=W(T)$ and the effect of this method could is illustrated in the Figure 3.

![Figure 3](image-url)

**Figure 3** – Left figure illustrates with stratification (control) and the right without.

In this way X is stratified and Y will depend on it by the price at time T. One can $^7$Disjointed subsets doesn’t have any common elements. Such as subset M=$\{1,2,3\}$ and N=$\{4,5,6\}$ means that M and N are disjointed subsets.

17
2.6.2 Antithetic variates

One other type of variance reduction techniques is called *Antithetic variates* and it is probably the most straightforward method of them all. From the Standard-MC model we have the simulating stock price movement as

\[ S_i = S_0 \cdot \exp \left( \left( r - \frac{1}{2} \sigma^2 \right) \cdot T + \sigma \sqrt{T} Z_i \right), \]

and since \( Z_i \) are standard normal variables, \( Z_i \sim N(0, 1) \), then also \( -Z_i \) should be standard normal variables, i.e. \( -Z_i \sim N(0, 1) \). By replicating the Standard-MC stock price movement using \( -Z_i \) instead (Glasserman 2004, 205-208) it would look like

\[ \tilde{S}_i = S_0 \cdot \exp \left( \left( r - \frac{1}{2} \sigma^2 \right) \cdot T + \sigma \sqrt{T} (-Z_i) \right). \]

The formulation for the price of a European call option would then be seen as

\[ Y_i = e^{-rT} \left( S_0 \cdot \exp \left( \left( r - \frac{1}{2} \sigma^2 \right) \cdot T + \sigma \sqrt{T} Z_i \right) - K \right)^+, \]

\[ \tilde{Y}_i = e^{-rT} \left( S_0 \cdot \exp \left( \left( r - \frac{1}{2} \sigma^2 \right) \cdot T + \sigma \sqrt{T} (-Z_i) \right) - K \right)^+. \]

From these two equations of \( Y_i \) and \( \tilde{Y}_i \) conclusions can be drawn that they are identically distributed but not independent. To make them also independent distributed
we use this kind of substitution

\[
\hat{Y}_i = \frac{Y_i + \bar{Y}_i}{2},
\]

and then all \( \hat{Y}_i \) are independent and identically distributed variables. To get the option price is just then to take the mean of all \( \hat{Y}_i \) as

\[
\hat{Y}_{AV} = \frac{1}{n} \sum_{i=1}^{n} \hat{Y}_i.
\]

2.6.3 Quasi Monte Carlo

For the Quasi-MC model we need a tool to handle the random numbers that are generated. Let’s assume a sequence \( X = \{x_1, x_2, ..., x_n\} \) that is a deterministic sequence in \([0, 1]^d\) designed to fill \([0, 1]^d\) uniformly. A sequence that fulfills this is called a low-discrepancy sequence. It can be hard to really understand what this actually means so let’s make a simple example to illustrate the meaning of discrepancy in general.

**Definition 2.3.** "Given a collection \( A \) of subsets to \([0, 1]^d\) the discrepancy in a sequence \( \{x_1, x_2, ..., x_n\} \) is relative to \( A \)" (Glasserman 2004, 283).

\[
D(x_1, x_2, ..., x_n) = \sup_{a \in A} \left| \frac{\# \{x_i \in a\}}{n} - vol(a) \right|,
\]

where \( a \) is subsets of the whole volume \( A \).

In Figure 4 we have eight points form a sequence, \( X = \{x_1, ..., x_4\} = \{1, ..., 8\} \), distributed in the given space. If we then would like to calculate the discrepancy of the sequences with respect of the definition it would be

\[
a_1: \frac{1}{8} - \frac{1}{4} = \frac{1}{8}, \quad a_2: \frac{3}{8} - \frac{1}{4} = \frac{1}{8}, \quad a_3: \frac{2}{8} - \frac{1}{4} = 0, \quad a_4: \frac{2}{8} - \frac{1}{4} = 0.
\]
Which gives the discrepancy for the sequence $sup\{\frac{1}{8}, \frac{1}{8}, 0, 0\} = \frac{1}{8}$. One can see this as some kind of manipulation of the random numbers but this is a way to fill up the gaps in the space where already random numbers have been generated. There are many different kinds of low-discrepancy algorithms and the one that will be used in this thesis are Sobol sequence. Sobol sequence is a so called Van der Corput sequence (VdC-sequence) but with the base of 2 ($b=2$). A VdC-sequence is a class of low-discrepancy sequences with $b \geq 2$, and every positive integer $k$ can be presented as

$$k = \sum_{i=1}^{\infty} a_j(k)b^i,$$ (23)

where $a_j(b) \in \{0, 1\}$ and gives a binary representation of $k$. It also has a inverse function, $\Psi_b(k)$, that can represent the values of $k$ on the line of $[0, 1)$. The definition of $\Psi_b(k)$ is

$$\Psi_b(k) = \sum_{i=1}^{\infty} \frac{a_j(k)}{b^j+1}.$$ (24)

This are two general representations for VdC-sequences (Glasserman 2004, 286) but since we will work with Sobol sequences we use $b = 2$. A more formal explanation

\begin{center}
<table>
<thead>
<tr>
<th>$a_2$</th>
<th>$a_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_2 = 3$</td>
<td>$x_3 = 2$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$a_1$</th>
<th>$a_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1 = 1$</td>
<td>$x_4 = 2$</td>
</tr>
</tbody>
</table>
\end{center}

Figure 4 – Example of discrepancy.
for these two equations can be seen in Table 1.

**Table 1** – VdC-sequences with base b=2 (Glasserman 2004, Table 5.1, 286.).

<table>
<thead>
<tr>
<th>k</th>
<th>k (binary)</th>
<th>$\Psi_2(k)$ (binary)</th>
<th>$\Psi_2(k)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0.1</td>
<td>1/2</td>
</tr>
<tr>
<td>2</td>
<td>10</td>
<td>0.01</td>
<td>1/4</td>
</tr>
<tr>
<td>3</td>
<td>11</td>
<td>0.11</td>
<td>3/4</td>
</tr>
<tr>
<td>4</td>
<td>100</td>
<td>0.001</td>
<td>1/8</td>
</tr>
<tr>
<td>5</td>
<td>101</td>
<td>0.101</td>
<td>5/8</td>
</tr>
<tr>
<td>6</td>
<td>110</td>
<td>0.011</td>
<td>3/8</td>
</tr>
<tr>
<td>7</td>
<td>111</td>
<td>0.111</td>
<td>7/8</td>
</tr>
</tbody>
</table>

As one can see from the radical inverse function $\Psi_2$ it represents a uniformly spread of points on a line, which is illustrated in Figure 5.

![Figure 5](image-url)  
**Figure 5** – Example of VdC sequence from Table 1.

From Figure 5 the $\Psi_2$ fills the space on the line uniformly by reflecting k’s base 2 coefficient around the decimal point. Similar to Stratified sampling I use the construction of paths in the same way for the option path. But here using Sobol sequences instead of the inverse function as in Stratified sampling.

---

*Similar to Algorithm 1.*
3 Method

To start of this thesis a literature study was done to choose suitable models for Monte Carlo simulations of financial options and also that they had clear distinctions from each other. Also collect information about financial options and how they are calculated. For this thesis stocks will represent the type of financial options that will be simulated. The choice of models became totally five of them, first a Standard Monte Carlo model to have as a ground for the others to be build on, the other models have the same look as the Standard model but with an own twist. In Monte Carlo simulations there is a whole pile of different methods to choose from, especially variance reducing models.

Here I chose a simple\(^9\) kind of variance reducing model that replicates itself by its random numbers and then take a mean of them both to get the option price. This is called Antithetic variates. The second variance reducing model fell on Stratified sampling where it controls the generated random numbers to get a better match of the underlying distribution, here I will be using 100 strata during the simulations. Quasi Monte Carlo is a method that goes under the category low-discrepancy method and it is also specified in the random numbers that are generated. Here the random numbers are generated with Sobol sequences which will be implemented with build in functions.

When testing these models there were also pretty obvious to use Black-Sholes formulas as a benchmark for the European options to test the accuracy of the models.

\(^9\)To say that a model is simple is always relative to ones competent in the field, but relative to the others one can see it as a simple technique.
Since Black-Scholes formulas are seen as the exact price of a European option the models can be tested and see the deviations in price when all the parameters are known and constant (during simulation). Also Arithmetic Asian options were tested with these methods, but the problem here is that it doesn’t exist an exact formula for pricing them. Instead one uses simulations to see the price and with increasing number of simulations the price will converge.

To test these models one can for different number of simulations see how the price of the option will be and for European if they converge to Black-Scholes price. I also wanted to see if the European options relation to Black-Scholes changes for different initial volatilities and rates. It’s interesting to see if the methods behave the same for example low volatilities as high ones.

I also combined two of these methods characteristics into a hybrid model. By using the handle of generating random numbers from Quasi-MC and the methodology of replicating the stock movement from Antithetic variates a fifth model will be tested to see if their attributes together will perform better than separately.
4 Results

4.1 European Options

To start of the results two figures will illustrate how a change in volatility and the interest rate will affect the option price. Since these two are the essential parts in the Monte Carlo formula, where the maturity time, strike price and initial price are hold constant, they will also affect the option price depending on the values. For these two figures the exact value of Black-Scholes is being used for representation of the call option price. Figure 6 shows how the option price changes with increased volatility.

![Volatility vs Price](image)

**Figure 6** – Volatility vs Price calculated with Black-Scholes for a call option.

From Figure 6 we see that with an increased volatility the option price increases
Almost linearly. The volatility span goes from \([0.01, 0.8]\) with step size of 0.01. For the interest rate the change in price for a call option will be presented in Figure 7. The span of the rate is \([0.01, 0.5]\) with step size of 0.01.

![Rate vs Price](image)

**Figure 7** – Interest rate vs Price calculated with Black-Scholes for a call option.

Almost the same as for the volatility is the option price increasing almost linear when the interest rate is increasing. These two figures were for illustrating how the price changes when the parameters changes for the upcoming results.

To find out how the models react under these conditions and manage to approximate Black-Scholes all the models get tested for parameter change. In Tables 2 and 3 the Black-Scholes exact price stands as the only actual price and for the other
models the relative errors\textsuperscript{10} are presented.

Table 2 – The relative error for the models combined with the exact value for $\sigma = [0.1, 0.8]$.

<table>
<thead>
<tr>
<th>Method</th>
<th>$\sigma$</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
<th>0.7</th>
<th>0.8</th>
</tr>
</thead>
<tbody>
<tr>
<td>BS</td>
<td>10.0340</td>
<td>17.8321</td>
<td>25.6432</td>
<td>33.4088</td>
<td>41.1024</td>
<td>48.7029</td>
<td>56.1917</td>
<td>63.5515</td>
<td></td>
</tr>
<tr>
<td>Standard</td>
<td>-0.0011</td>
<td>0.0069</td>
<td>0.0112</td>
<td>0.0107</td>
<td>0.0240</td>
<td>0.0110</td>
<td>0.0064</td>
<td>0.0177</td>
<td></td>
</tr>
<tr>
<td>Antithetic</td>
<td>-0.0045</td>
<td>0.0043</td>
<td>-0.0008</td>
<td>0.0076</td>
<td>0.0033</td>
<td>0.0065</td>
<td>0.0088</td>
<td>-0.0129</td>
<td></td>
</tr>
<tr>
<td>Stratified</td>
<td>0.0003</td>
<td>0.0003</td>
<td>-9.36E-05</td>
<td>0.0010</td>
<td>0.0001</td>
<td>0.0006</td>
<td>-0.0019</td>
<td>0.0012</td>
<td></td>
</tr>
<tr>
<td>Quasi</td>
<td>-4.98E-05</td>
<td>-7.85E-05</td>
<td>-9.36E-05</td>
<td>-0.0001</td>
<td>-0.0001</td>
<td>-0.0002</td>
<td>0.0177</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 3 – The relative error for the models combined with the exact value for $r = [0.01, 0.3]$.

<table>
<thead>
<tr>
<th>Method</th>
<th>$r$</th>
<th>0.01</th>
<th>0.05</th>
<th>0.10</th>
<th>0.15</th>
<th>0.20</th>
<th>0.25</th>
<th>0.30</th>
</tr>
</thead>
<tbody>
<tr>
<td>BS</td>
<td>24.7365</td>
<td>28.4625</td>
<td>33.4683</td>
<td>38.8057</td>
<td>44.4070</td>
<td>50.1991</td>
<td>56.1084</td>
<td></td>
</tr>
<tr>
<td>Standard</td>
<td>0.0141</td>
<td>0.0247</td>
<td>0.0051</td>
<td>-0.0003</td>
<td>0.0031</td>
<td>-0.0067</td>
<td>-0.0084</td>
<td></td>
</tr>
<tr>
<td>Antithetic</td>
<td>0.0113</td>
<td>0.0072</td>
<td>0.0046</td>
<td>0.0037</td>
<td>0.0045</td>
<td>0.0040</td>
<td>-0.0003</td>
<td></td>
</tr>
<tr>
<td>Stratified</td>
<td>0.0009</td>
<td>-0.0009</td>
<td>0.0001</td>
<td>-0.0007</td>
<td>0.0006</td>
<td>-0.0002</td>
<td>-0.0001</td>
<td></td>
</tr>
<tr>
<td>Quasi</td>
<td>-0.0001</td>
<td>-8.78E-05</td>
<td>-7.17E-05</td>
<td>-6.44E-05</td>
<td>-5.63E-05</td>
<td>-4.98E-05</td>
<td>-4.28E-05</td>
<td></td>
</tr>
</tbody>
</table>

In Table 2 and 3 one can see that for an increase in both the volatility and the rate the models have similar deviation from the real option price. Clearly the Hybrid model has the overall best results while the Standard-MC model deviates the most overall. Tables with all the values simulated for the models can be found in Appendix.

A 95\% confidence interval for the values in Table 2 and 3 can be seen in Figure 8 and and 9 where all the models except the Standard-MC keeps the values of Black-Scholes inside the interval. Just for two values of the Standard-MC intervals don’t contain Black-Scholes, one in the volatility test and one in the rate test.

The confidence interval is constructed in a regular way as $\hat{C} \pm 1.96 \cdot \frac{s_C}{\sqrt{n}}$, where $\hat{C}$ is

\textsuperscript{10}Relative error here is basically $error = 1 - \frac{approx}{true}$. 

26
the simulated option price, \( \sigma_C \) is the volatility of the options payoffs and \( n \) is the number of simulations.

![Figure 8](image)

**Figure 8** – 95% confidence interval for Table 2 for all the models, with Black-Scholes in blue, the models values in red and its 95% confidence interval in black dashed lines. The Standard-MC is in the top left, Antithetic in top right, Stratified in middle to left, Quasi-MC middle to right and Hybrid in the bottom.

To see the convergence of the option price for the models Figure 10 illustrates the approximated prices with the number of simulations \([10^4, 20^5]\) with step size \(10^4\). Here for the European options the Black-Scholes value is included to have as a benchmark for the other models.

It can be bit hard to see all the different models in Figure 10 and this is since a couple of them are so close to Black-Scholes already from just \(10^4\) simulated paths. From Figure 10 we see that there are two models that fluctuates relatively much from Black-Scholes, these two are the Standard model and Antithetic. The other three in Stratified, Quasi and Hybrid are the ones who follow the exact values close for all the different paths. All these values can be found in Appendix.
Figure 9 – 95% confidence interval for Table 3 for all the models, with Black-Scholes in blue, the models values in red and its 95% confidence interval in black dashed lines. The Standard-MC is in the top left, Antithetic in top right, Stratified in middle to left, Quasi-MC middle to right and Hybrid in the bottom.

Figure 10 – Converge test for European options.
4.2 Asian Options

To see how the Asian options relate to Black-Scholes for different volatility scenarios, Table 4 includes all the same volatilities as for the European test. The difference here is that the models actually prices will be seen instead of the relative error as was the case earlier.

Table 4 – The prices for the models with Black-Scholes European option prices for the same conditions as the models.

<table>
<thead>
<tr>
<th>Method</th>
<th>σ</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
<th>0.7</th>
<th>0.8</th>
</tr>
</thead>
<tbody>
<tr>
<td>BS</td>
<td>10.0340</td>
<td>17.8321</td>
<td>25.6432</td>
<td>33.4088</td>
<td>41.1024</td>
<td>48.7029</td>
<td>56.1917</td>
<td>63.5515</td>
<td></td>
</tr>
<tr>
<td>Antithetic</td>
<td>5.9886</td>
<td>10.7633</td>
<td>15.4883</td>
<td>20.5576</td>
<td>24.9167</td>
<td>30.0268</td>
<td>34.5611</td>
<td>39.1051</td>
<td></td>
</tr>
<tr>
<td>Stratified</td>
<td>5.9950</td>
<td>10.7522</td>
<td>15.5310</td>
<td>20.2699</td>
<td>25.0980</td>
<td>30.2340</td>
<td>34.5600</td>
<td>39.4865</td>
<td></td>
</tr>
<tr>
<td>Hybrid</td>
<td>6.0019</td>
<td>10.8016</td>
<td>15.5303</td>
<td>20.3207</td>
<td>25.0256</td>
<td>29.8310</td>
<td>34.3651</td>
<td>39.1894</td>
<td></td>
</tr>
</tbody>
</table>

The difference between Black-Scholes prices and the arithmetic Asian approximations doesn’t seem to change in relation for the volatilities as one can see in Table 4.

In Figure 11 the Asian approximations are tested in convergence for the same number of simulations and step size as the European options did. Here the values of Black-Scholes are excluded since they are not even close to the Asian option prices as we saw from Table 4.
In contrast to Figure 10 here we can see all the models prices for the different number of simulations. In Figure 11 all models are fluctuating but still in a relatively small interval. Still the Standard model seems to be the one fluctuating the most but not as clear difference from the others as it did for the European options earlier.
5 Discussion and Conclusions

This thesis purpose was to investigate how the picked Monte Carlo models would behave under conditions with changes in parameters and different number of simulations. From the results that are given it says that the models that control the generating of random numbers, Stratified, Quasi-MC and Hybrid, are performing overall best. From Table 2 and 3 the increments in both volatility and rate don’t show any signs of that the models would deviate much more with the increments, it also answers the research question about under which conditions does the models behave like Black-Scholes formulas? Since the models deviate with such relatively small error, and Figure 8 and 9 confirm it with its confidence intervals, they seem to work fine when the rate and volatility values change.

When it comes to the Asian options we can directly see from Table 4 and Figure 10 that they are not even close to the price of Black-Scholes, which was expected but still interesting to investigate. As Wiklund (2012) I also got the results that Black-Scholes overvalues the Asian options, here almost proportional through the changes in volatility. On the other hand Figure 11 shows small\textsuperscript{11} deviation between the models when the number of simulations increases. From that one can assume that the price for that specific Asian option would converge even better if the number of simulation axis would continue.

The Hybrid model seems to work fine when it comes to price estimation and seems to benefit of being combined by two different techniques. Tho it is close with the Quasi-MC model and probably gains most of its qualities there rather than Antithet-

\textsuperscript{11}Smaller than for the European options.
ics. As Mehrdoust (2015) results showed that a Hybrid model is doing well compared to its original models\textsuperscript{12} and contributed from its properties rather than them cancel each other out.

With all the combined results I think it is fair to say that the Hybrid and Quasi-MC model got the overall best results. For this thesis controlling the random numbers where surely the best way to go, and rather using a low-discrepancy technique over variance reducing. When it came to price estimating the models were close for all the tests and also got confirmation from their confidence intervals. Surely all the models\textsuperscript{13} did well since their confidence intervals collected the Black-Scholes prices but with different marginal. The thinking of combining different models characteristics sounds really interesting and as we could see here the Hybrid model did really good in the price estimation. It would be interesting to see how other types of hybrids would do and how far one can go in the mixing theories before it gets to much of it. Further studies that I would recommend is to create new models out of already existing ones\textsuperscript{14} and see how they would perform.

Worth mention is also that in this thesis the models are exposed to almost perfect conditions. All the parameter values are known which is not always the case in the real world. This thesis is more of test the models against a benchmark and see if they can be reliable. One positive thing by using Monte Carlo simulations that hasn’t been mentioned is that one can generate their own data instead of spend a lot of time collecting it.

\textsuperscript{12}Not the same combination of models, here I used Quasi-MC and he had Control variates.
\textsuperscript{13}Except for two values in Standard-MC.
\textsuperscript{14}More than just the hybrid model in this thesis.
6 References


Buchner, A., 2015, Equilibrium option pricing: A Monte Carlo approach, Article, University of Passau.


A Appendix

A.1 Tables

**Table 5 – Volatility European**

<table>
<thead>
<tr>
<th>Method</th>
<th>$\sigma$</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
<th>0.7</th>
<th>0.8</th>
</tr>
</thead>
<tbody>
<tr>
<td>BS</td>
<td>10.0340</td>
<td>17.8321</td>
<td>25.6432</td>
<td>33.4088</td>
<td>41.1024</td>
<td>48.7029</td>
<td>56.1917</td>
<td>63.5515</td>
<td></td>
</tr>
<tr>
<td>Standard</td>
<td>10.0450</td>
<td>17.7085</td>
<td>25.3570</td>
<td>33.0515</td>
<td>40.1175</td>
<td>48.1657</td>
<td>55.8303</td>
<td>62.4240</td>
<td></td>
</tr>
<tr>
<td>Antithetic</td>
<td>10.0794</td>
<td>17.7563</td>
<td>25.6640</td>
<td>33.1549</td>
<td>40.9648</td>
<td>48.3851</td>
<td>55.6967</td>
<td>64.3721</td>
<td></td>
</tr>
<tr>
<td>Stratified</td>
<td>10.0313</td>
<td>17.8274</td>
<td>25.6456</td>
<td>33.3767</td>
<td>41.0981</td>
<td>48.6714</td>
<td>56.2966</td>
<td>63.4770</td>
<td></td>
</tr>
<tr>
<td>Quasi</td>
<td>10.0345</td>
<td>17.8335</td>
<td>25.6456</td>
<td>33.4125</td>
<td>41.1549</td>
<td>48.7100</td>
<td>56.2005</td>
<td>63.5618</td>
<td></td>
</tr>
<tr>
<td>Hybrid</td>
<td>10.0344</td>
<td>17.8332</td>
<td>25.6451</td>
<td>33.4116</td>
<td>41.1063</td>
<td>48.7079</td>
<td>56.1975</td>
<td>63.5576</td>
<td></td>
</tr>
</tbody>
</table>

**Table 6 – Rate European**

<table>
<thead>
<tr>
<th>Method</th>
<th>$r$</th>
<th>0.01</th>
<th>0.05</th>
<th>0.10</th>
<th>0.15</th>
<th>0.20</th>
<th>0.25</th>
<th>0.30</th>
</tr>
</thead>
<tbody>
<tr>
<td>BS</td>
<td>24.7365</td>
<td>28.4625</td>
<td>33.4683</td>
<td>38.8057</td>
<td>44.4070</td>
<td>50.1991</td>
<td>56.1084</td>
<td></td>
</tr>
<tr>
<td>Standard</td>
<td>24.3875</td>
<td>27.7590</td>
<td>33.2969</td>
<td>38.8157</td>
<td>44.2695</td>
<td>50.5359</td>
<td>56.5780</td>
<td></td>
</tr>
<tr>
<td>Antithetic</td>
<td>24.4575</td>
<td>28.2577</td>
<td>33.3159</td>
<td>38.6609</td>
<td>44.2053</td>
<td>50.0008</td>
<td>56.1228</td>
<td></td>
</tr>
<tr>
<td>Stratified</td>
<td>24.7146</td>
<td>28.4878</td>
<td>33.4636</td>
<td>38.8310</td>
<td>44.3824</td>
<td>50.2100</td>
<td>56.1160</td>
<td></td>
</tr>
<tr>
<td>Quasi</td>
<td>24.7390</td>
<td>28.4650</td>
<td>33.4707</td>
<td>38.8082</td>
<td>44.4095</td>
<td>50.2016</td>
<td>56.1108</td>
<td></td>
</tr>
<tr>
<td>Hybrid</td>
<td>24.7384</td>
<td>28.4645</td>
<td>33.4701</td>
<td>38.8076</td>
<td>44.4089</td>
<td>50.2010</td>
<td>56.1102</td>
<td></td>
</tr>
</tbody>
</table>
Table 7 – European convergence

<table>
<thead>
<tr>
<th>Paths ($10^4$)</th>
<th>Method</th>
<th>BS</th>
<th>Standard</th>
<th>Antithetic</th>
<th>Stratified</th>
<th>Quasi</th>
<th>Hybrid</th>
</tr>
</thead>
<tbody>
<tr>
<td>60</td>
<td>25.6432</td>
<td>25.6883</td>
<td>25.6238</td>
<td>25.6503</td>
<td>25.6425</td>
<td>25.6432</td>
<td></td>
</tr>
<tr>
<td>70</td>
<td>25.6432</td>
<td>25.8903</td>
<td>25.7444</td>
<td>25.6263</td>
<td>25.6424</td>
<td>25.6428</td>
<td></td>
</tr>
<tr>
<td>100</td>
<td>25.6432</td>
<td>25.6026</td>
<td>25.6081</td>
<td>25.6445</td>
<td>25.6430</td>
<td>25.6430</td>
<td></td>
</tr>
<tr>
<td>170</td>
<td>25.6432</td>
<td>25.4961</td>
<td>25.6429</td>
<td>25.6395</td>
<td>25.6429</td>
<td>25.6434</td>
<td></td>
</tr>
<tr>
<td>Paths ($10^4$)</td>
<td>Method</td>
<td>Standard</td>
<td>Antithetic</td>
<td>Stratified</td>
<td>Quasi</td>
<td>Hybrid</td>
<td></td>
</tr>
<tr>
<td>---------------</td>
<td>--------</td>
<td>----------</td>
<td>------------</td>
<td>------------</td>
<td>-------</td>
<td>--------</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>15.34282</td>
<td>15.55592</td>
<td>15.38739</td>
<td>15.40056</td>
<td>15.56010</td>
<td></td>
<td></td>
</tr>
<tr>
<td>20</td>
<td>15.73676</td>
<td>15.64534</td>
<td>15.59578</td>
<td>15.64140</td>
<td>15.5953</td>
<td></td>
<td></td>
</tr>
<tr>
<td>30</td>
<td>15.56422</td>
<td>15.55630</td>
<td>15.59771</td>
<td>15.52383</td>
<td>15.52923</td>
<td></td>
<td></td>
</tr>
<tr>
<td>40</td>
<td>15.61560</td>
<td>15.48531</td>
<td>15.63885</td>
<td>15.68148</td>
<td>15.61563</td>
<td></td>
<td></td>
</tr>
<tr>
<td>50</td>
<td>15.52050</td>
<td>15.52724</td>
<td>15.56488</td>
<td>15.55540</td>
<td>15.57898</td>
<td></td>
<td></td>
</tr>
<tr>
<td>60</td>
<td>15.57224</td>
<td>15.55404</td>
<td>15.53738</td>
<td>15.62651</td>
<td>15.63206</td>
<td></td>
<td></td>
</tr>
<tr>
<td>70</td>
<td>15.52408</td>
<td>15.51929</td>
<td>15.52214</td>
<td>15.53966</td>
<td>15.53758</td>
<td></td>
<td></td>
</tr>
<tr>
<td>80</td>
<td>15.53774</td>
<td>15.53244</td>
<td>15.55033</td>
<td>15.52890</td>
<td>15.51832</td>
<td></td>
<td></td>
</tr>
<tr>
<td>90</td>
<td>15.49645</td>
<td>15.55521</td>
<td>15.53457</td>
<td>15.51471</td>
<td>15.56757</td>
<td></td>
<td></td>
</tr>
<tr>
<td>100</td>
<td>15.54840</td>
<td>15.56533</td>
<td>15.54940</td>
<td>15.58202</td>
<td>15.60313</td>
<td></td>
<td></td>
</tr>
<tr>
<td>110</td>
<td>15.40947</td>
<td>15.46389</td>
<td>15.54211</td>
<td>15.50632</td>
<td>15.54784</td>
<td></td>
<td></td>
</tr>
<tr>
<td>120</td>
<td>15.61553</td>
<td>15.55581</td>
<td>15.54760</td>
<td>15.55189</td>
<td>15.51032</td>
<td></td>
<td></td>
</tr>
<tr>
<td>130</td>
<td>15.57684</td>
<td>15.57412</td>
<td>15.54580</td>
<td>15.56462</td>
<td>15.57372</td>
<td></td>
<td></td>
</tr>
<tr>
<td>140</td>
<td>15.57911</td>
<td>15.56910</td>
<td>15.58077</td>
<td>15.54441</td>
<td>15.53457</td>
<td></td>
<td></td>
</tr>
<tr>
<td>150</td>
<td>15.57498</td>
<td>15.52522</td>
<td>15.56534</td>
<td>15.57747</td>
<td>15.55769</td>
<td></td>
<td></td>
</tr>
<tr>
<td>160</td>
<td>15.51167</td>
<td>15.57004</td>
<td>15.51301</td>
<td>15.51884</td>
<td>15.54893</td>
<td></td>
<td></td>
</tr>
<tr>
<td>170</td>
<td>15.55411</td>
<td>15.53514</td>
<td>15.54887</td>
<td>15.52572</td>
<td>15.52798</td>
<td></td>
<td></td>
</tr>
<tr>
<td>180</td>
<td>15.56601</td>
<td>15.56442</td>
<td>15.52595</td>
<td>15.56690</td>
<td>15.55982</td>
<td></td>
<td></td>
</tr>
<tr>
<td>190</td>
<td>15.55952</td>
<td>15.59579</td>
<td>15.51592</td>
<td>15.55406</td>
<td>15.57015</td>
<td></td>
<td></td>
</tr>
<tr>
<td>200</td>
<td>15.54055</td>
<td>15.53475</td>
<td>15.54246</td>
<td>15.55076</td>
<td>15.55350</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
A.2 Strong Law of Large Numbers

To describe the Strong Law of Large Numbers (SLLN) we assume a sequence of independent and identically distributed random variables \( X = \{x_1, x_2, \ldots\} \) with a mean \( \mu = E[X] < \infty \) and variance \( \sigma^2 = Var[X] < \infty \). Then we have

\[
\frac{1}{n} \sum_{i=1}^{n} x_i \to \mu, \text{ as } n \to \infty. \tag{25}
\]

This indicates that almost any realization of \( \{X_1, X_2, \ldots\} \), in this case our simulated sequence \( \{x_1, x_2, \ldots\} \) satisfies

\[
\frac{1}{n} \sum_{i=1}^{n} x_i \to \mu = E[x]. \tag{26}
\]

This means that for big values of \( n \) the approximation \( \frac{1}{n} \sum_{i=1}^{n} x_i \) should be a good estimate of \( \mu \).

A.3 Variance calculation

A.3.1 Antithetic variates

Since this thesis handles variance reduction techniques it would be interesting of how it actually reducing the variance. For the Antithetic (Glasserman 2004, 206-208) we have

\[
\hat{Y}_{AV} = \frac{1}{n} \sum_{i=1}^{n} \hat{Y}_i = \frac{1}{n} \sum_{i=1}^{n} \frac{Y_i + \bar{Y}_i}{2} = \frac{1}{2n} \left( \sum_{i=1}^{n} Y_i + \sum_{i=1}^{n} \bar{Y}_i \right). \tag{27}
\]

For the Standard-MC model we have

\[
\hat{Y}_{2n} = \frac{1}{2n} \sum_{i=1}^{2n} Y_i. \tag{28}
\]
Let's start by calculating the variance for the Standard-MC by some basic calculations

\[
\text{Var}\left[\hat{Y}_{2n}\right] = \frac{1}{(2n)^2} \cdot \text{Var}\left[\sum_{i=1}^{2n} Y_i\right],
\]  

(29)

since we have independent \( Y_i \) we can move out the summation and furthermore replace it with just \( 2n \) since it is the length of the sum.

\[
\text{Var}\left[\hat{Y}_{2n}\right] = \frac{1}{4n^2} \cdot 2n \cdot \text{Var}[Y_i] = \frac{1}{2n} \cdot \text{Var}[Y_i].
\]  

(30)

Now we will see what the variance for Antithetic variates becomes.

\[
\text{Var}\left[\hat{Y}_{AV}\right] = \text{Var}\left[\frac{1}{n} \sum_{i=1}^{n} \frac{(Y_i + \bar{Y}_i)}{2}\right] = \frac{1}{4n^2} \cdot \sum_{i=1}^{n} \text{Var}\left[Y_i + \bar{Y}_i\right] = \frac{1}{4n} \cdot \text{Var}\left[Y_i + \bar{Y}_i\right].
\]  

(31)

Here we can use the fact that \( \text{Var}[a+b] = \text{Var}[a] + \text{Var}[b] + 2 \text{cov}[a,b] \) and that \( \text{cov}[a,b] = \sigma_a \cdot \sigma_b \cdot \rho \). This gives the expression

\[
\text{Var}\left[\hat{Y}_{AV}\right] = \frac{1}{4n} \cdot \left( \text{Var}[Y_i] + \text{Var}[\bar{Y}_i] + 2\rho \sigma_Y \sigma_{\bar{Y}} \right).
\]  

(32)

Since we have identically distributed variables we can set \( \text{Var}[Y_i] = \text{Var}[\bar{Y}_i] \) and get

\[
\text{Var}\left[\hat{Y}_{AV}\right] = \frac{1}{4n} \cdot \left( 2\text{Var}[Y_i] + 2\rho \sigma_Y \sigma_{\bar{Y}} \right) = \frac{1}{4n} \cdot \left( 2\text{Var}[Y_i] + 2\rho \text{Var}[Y_i] \right)
\]  

(33)

\[
\text{Var}\left[\hat{Y}_{AV}\right] = \frac{1}{2n} \cdot (1 + \rho) \cdot \text{Var}[Y_i].
\]  

(34)

From this we can put the two variances against each other as

\[
\frac{\text{Var}\left[\hat{Y}_{AV}\right]}{\text{Var}\left[\hat{Y}_{2n}\right]} = \frac{\frac{1}{2n} \cdot (1 + \rho) \cdot \text{Var}[Y_i]}{\frac{1}{2n} \cdot \text{Var}[Y_i]} = 1 + \rho.
\]  

(35)
As one can see here the condition for $\hat{Y}_{AV}$ to have a lower variance than $Y_{2n}$ is when $\rho < 0$ and since it is a negative dependence in the input of $(Z, -Z)$ we have a negative value on $\rho$. This will give a negative dependence in output and a better approximation from $\hat{Y}_{AV}$. When constructing $\hat{Y}_{AV}$ as $\hat{Y}_i = \frac{\tilde{Y}_i + Y_i}{2}$, a potentially large value of $Y_i$ will be balanced by a small value of $\tilde{Y}_i$ which will reduce the variance.

A.3.2 Stratified sampling

We first have some notation for mean and variance as (Glasserman 2004, 215-220)

$$\mu_i = E[Y_{ij}] = E[Y|X \in A_i], \quad (36)$$

$$\sigma^2_i = Var[Y_{ij}] = Var[Y|X \in A_i]. \quad (37)$$

For variance without stratification, i.e. Standard-MC gets the variance

$$Var[\hat{Y}] = Var\left[ \frac{1}{m} \sum_{i=1}^{m} Y_i \right] = \frac{1}{n^2} \cdot nVar[Y_1] = \frac{1}{n} \left( E[Y^2_1] - E[Y_1]^2 \right) = \frac{1}{n} \left( E[Y^2_1] - \mu^2 \right). \quad (38)$$

We here have that the mean $\mu$ is equal to

$$\mu = E[Y_1] = \sum_{i=1}^{m} p_i E[Y_i|X \in A_i] = \sum_{i=1}^{m} p_i \cdot \mu_i, \quad (39)$$

and that the expectation of $Y^2_1$ can be written as

$$E[Y^2_1] = \sum_{i=1}^{m} p_i E[Y^2_i|X \in A_i] = \sum_{i=1}^{m} p_i (\sigma^2_i + \mu^2_i), \quad (40)$$
since

\[ \text{Var}[Y_1 | X \in A_i] = E[Y_1^2 | X \in A_i] - E[Y_1 | X \in A_i]^2 \iff \sigma^2 = E[Y_1^2 | X \in A_i] - \mu^2. \] (41)

So the variance without stratification is then

\[
\text{Var}[\hat{Y}] = \frac{1}{n} \left( \sum_{i=1}^{m} p_i \sigma_i^2 + \mu_i^2 \right) = \frac{1}{n} \left( \sum_{i=1}^{m} p_i \sigma_i^2 \right) + \frac{1}{n} \left( \sum_{i=1}^{m} p_i \mu_i^2 \right) - \frac{1}{n} \left( \sum_{i=1}^{m} p_i \mu_i \right)^2. \] (42)

With stratification the variance will be

\[
\text{Var}[\hat{Y}_S] = \text{Var} \left[ \sum_{i=1}^{m} p_i \cdot \frac{1}{n_i} \sum_{i=1}^{n_i} Y_{ij} \right] = \sum_{i=1}^{m} p_i^2 \cdot \text{Var} \left[ \frac{1}{n_i} \sum_{i=1}^{n_i} Y_{ij} \right] = \sum_{i=1}^{m} p_i^2 \frac{\sigma_i^2}{n_i} = \frac{\sigma^2(q)}{n}, \] (44)

where

\[
\sigma^2(q) = \sum_{i=1}^{m} \frac{p_i^2}{q_i} \cdot \sigma_i^2, \quad q_i = \frac{n_i}{n}. \] (45)

With proportional stratification \( p_i = q_i \) we get

\[
\sigma^2(q) = \sum_{i=1}^{m} \frac{p_i^2}{q_i} \cdot \sigma_i^2 = \sum_{i=1}^{m} p_i \sigma_i^2, \] (46)

and the variance with proportional stratification becomes

\[
\text{Var}[\hat{Y}_S] = \frac{1}{n} \sum_{i=1}^{m} p_i \sigma_i^2. \] (47)
Comparing these two variance calculations we get

$$Var[\hat{Y}] - Var[\hat{Y}_S] = \frac{1}{n} \left( \sum_{i=1}^{m} p_i \mu_i^2 \right) - \frac{1}{n} \left( \sum_{i=1}^{m} p_i \mu_i \right)^2,$$

(48)

and by Jensen’s inequality we get that

$$\sum_{i=1}^{m} p_i \mu_i^2 \geq \left( \sum_{i=1}^{m} p_i \mu_i \right)^2,$$

(49)

which clarifies that the method with stratified sampling (proportional) only can give lower variance compared to Standard-MC.