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Edge precoloring extension of hypercubes

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Dedicated to the memory of Robin Thomas

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Abstract

We consider the problem of extending partial edge colorings of hypercubes. In particular, we obtain an analogue of the positive solution to the famous Evans' conjecture on completing partial Latin squares by proving that every proper partial edge coloring of at most $d-1$ edges of the d-dimensional hypercube Q_d can be extended to a proper d-edge coloring of Q_d. Additionally, we characterize which partial edge colorings of Q_d with precisely d precolored edges are extendable to proper d-edge colorings of Q_d.

KEYWORDS

edge coloring, hypercube, precoloring extension

1 | INTRODUCTION

An *edge precoloring* (or *partial edge coloring*) of a graph G is a proper edge coloring of some subset $E' \subseteq E(G)$; *a t-edge precoloring* is such a coloring with t colors. An edge t-precoloring φ is *extendable* if there is a proper t-edge coloring f such that $f(e) = \varphi(e)$ for any edge e that is colored under φ; f is called an *extension* of φ.

In general, the problem of extending a given edge precoloring is an $\mathcal{NP}$-complete problem, already for three-regular bipartite graphs [7]. One of the earlier references explicitly discussing the problem of extending a partial edge coloring is [15]; there a simple necessary condition for the existence of an extension is given and the authors find a class of graphs where this condition is also sufficient. More recently the question of extending a precoloring where the precolored edges form a matching has gathered interest. In [5] a number of positive results and conjectures are given. In particular it is conjectured that for every graph G, if φ is an edge precoloring of a matching M in G using $\Delta(G)+1$ colors, and any two edges in M are at distance at least 2 from each other, then φ can be extended to a proper $(\Delta(G)+1)$-edge coloring of G; this was first conjectured in [1], but then with distance 3 instead. Here, as usual, $\Delta(G)$ denotes the maximum degree of a graph G, and by the *distance* between two edges e and e' we mean the number of edges in a shortest path between an endpoint of e and an endpoint of e'; a *distance-t matching* is a matching where any two edges are at distance at least t from each other. A distance-2 matching is also called *an induced matching*.

Note that the conjecture on distance-2 matchings in [5] is sharp both with respect to the distance between precolored edges, and in the sense that $\Delta(G)+1$ can in general not be replaced by $\Delta(G)$ (for Class 1 graphs), even if any two precolored edges are at arbitrarily large distance from each other [5]. In [5], it is proved that this conjecture holds for, for example, bipartite multigraphs and subcubic multigraphs, and in [10] it is proved that a version of the conjecture with the distance condition increased to 9 holds for general graphs.

However, for one specific family of graphs, the balanced complete bipartite graphs $K_{n,n}$, the edge precoloring extension problem was studied far earlier than in the above-mentioned references. Here the extension problem corresponds to asking whether a partial Latin square can be completed to a Latin square. In this form the problem appeared already in 1960, when Evans [6] stated his now classic conjecture that for every positive integer n, if $n-1$ edges in $K_{n,n}$ have been (properly) colored, then the partial coloring can be extended to a proper n-edge coloring of $K_{n,n}$. This conjecture was solved for large n by Häggkvist [14] and later for all n by Smetaniuk [17], and independently by Andersen and Hilton [2]. Moreover, Andersen and Hilton [2] characterized which $n \times n$ partial Latin squares with exactly n nonempty cells are completable.

In this paper we consider the edge precoloring extension problem for the family of hypercubes. Although matching extendability and subgraph containment problems have been studied extensively for hypercubes (see, eg, [8,11,18,19] and references therein), the edge precoloring extension problem for hypercubes seems to be a hitherto quite unexplored line of research. As in the setting of completing partial Latin squares (and unlike the papers [5,10]) we consider only proper edge colorings of hypercubes Q_d using exactly $\Delta(Q_d)$ colors.

We prove that every edge precoloring of the d-dimensional hypercube Q_d with at most $d-1$ precolored edges is extendable to a d-edge coloring of Q_d, thereby establishing an analogue of the positive resolution of Evans' conjecture. Moreover, similarly to [2] we also characterize which proper precolorings with exactly d precolored edges are not extendable to proper d-edge colorings of Q_d. We also consider the cases when the precolored edges form an induced matching, or one or two hypercubes of smaller dimension. The paper is concluded by a conjecture and some examples and remarks on edge precoloring extension of general d-regular bipartite graphs.

2 | PRELIMINARIES

Unless otherwise stated all (partial) edge colorings (or just *colorings*) in this paper are proper. Moreover, all proper d-edge colorings use colors $1, \ldots, d$ unless otherwise stated. If φ is an edge precoloring of G, and an edge e is colored under φ, then we say that e is *φ-precolored*.

If φ is a (partial) proper t-edge coloring of G and $1 \leq a, b \leq t$, then a path or cycle in G is called *(a, b)-colored under φ* if its edges are colored by colors a and b alternately.

In the above definitions, we often leave out the explicit reference to a coloring φ, if the coloring is clear from the context.

Havel and Moravek [13] (see also [12]) proved a criterion for a graph G to be a subgraph of a hypercube:

Proposition 2.1. *A graph G is a subgraph of Q_d if and only if there is a proper d-edge coloring of G with integers $\{1, \ldots, d\}$ such that*

(i) in every path of G there is some color that appears an odd number of times;
(ii) in every cycle of G no color appears an odd number of times.

A *dimensional matching* M of Q_d is a perfect matching of Q_d such that $Q_d - M$ is isomorphic to two copies of Q_{d-1}; evidently there are precisely d dimensional matchings in Q_d. We shall need the following easy lemma.

Lemma 2.2. *Let $d \geq 2$ be an integer. There are d different dimensional matchings in Q_d; indeed Q_d decomposes into d such perfect matchings.*

The proof is left to the reader.

Intuitively, the colors in the proper edge coloring in Proposition 2.1 correspond to dimensional matchings in Q_d (as pointed out in [12]). In particular, Proposition 2.1 holds if we take the dimensional matchings as the colors. Furthermore we have the following.

Lemma 2.3. *The subgraph induced by r dimensional matchings in Q_d is isomorphic to a disjoint union of r-dimensional hypercubes.*

This simple observation shall be used quite frequently below.

We shall also need some standard definitions on list edge coloring. Given a graph G, assign to each edge e of G a set $\mathcal{L}(e)$ of colors. Such an assignment $\mathcal{L}$ is called a *list assignment* for G and the sets $\mathcal{L}(e)$ are referred to as *lists* or *color lists*. If all lists have equal size k, then $\mathcal{L}$ is called a *k-list assignment*. Usually, we seek a proper edge coloring φ of G, such that $\varphi(e) \in \mathcal{L}(e)$ for all $e \in E(G)$. If such a coloring φ exists, then G is *$\mathcal{L}$-colorable* and φ is called an *$\mathcal{L}$-coloring*. Denote by $\chi'_L(G)$ the minimum integer t such that G is $\mathcal{L}$-colorable whenever $\mathcal{L}$ is a t-list assignment. A fundamental result in list edge coloring theory is the following theorem by Galvin [9]. As usual, $\chi'(G)$ denotes the chromatic index of a multigraph G.

Theorem 2.4. *For any bipartite multigraph $G, \chi'_L(G) = \chi'(G)$.*

3 | EXTENDING EDGE PRECOLORINGS OF HYPERCUBES

We begin this section by giving a short proof of the following theorem, thereby establishing an analogue for hypercubes to the positive solution of the Evans' conjecture.

Theorem 3.1. *Let $d \geq 2$ be a positive integer. If φ is an edge precoloring of at most $d - 1$ edges of the hypercube Q_d, then φ can be extended to a proper d-edge coloring of Q_d.*

Proof. The proof is by induction on d. For $d = 2$, the statement is straightforward.

Suppose that $d > 2$ and that the theorem holds for Q_{d-1}. Let φ be an edge precoloring of at most $d - 1$ edges of Q_d. By Lemma 2.2, Q_d has d perfect matchings M such that $Q_d - M$ is the disjoint union of two copies of Q_{d-1}. Since at most $d - 1$ edges of Q_d are precolored, there is such a perfect matching $\hat{M}$ satisfying that no edge of $\hat{M}$ is precolored. Let H_1 and H_2 be the components of $Q_d - \hat{M}$. We distinguish between two different cases.

Case 1. H_1 has at least 1 and at most $d - 2$ precolored edges.

Without loss of generality we assume that the precoloring of Q_d uses colors $1, \ldots, d - 1$. Since H_1 contains at most $d - 2$ precolored edges, there is, by the induction hypothesis, a proper $(d - 1)$-edge coloring φ_1 of H_1 which is an extension of the restriction of φ to H_1. Similarly, there is a proper $(d - 1)$-edge coloring φ_2 of H_2 which is an extension of the restriction of φ to H_2. By coloring the edges of $\hat{M}$ with color d, we obtain a proper d-edge coloring of Q_d.

Case 2. H_1 has exactly $d - 1$ precolored edges.

Without loss of generality we assume that at least one edge in Q_d is precolored with color 1. Define a new edge precoloring φ' of Q_d by removing color 1 from any precolored edge of Q_d that is colored 1. By the induction hypothesis, there is a proper $(d - 1)$-edge coloring φ_1' of H_1 using colors $2, 3, \ldots, d$ which is an extension of φ'. From φ_1' we define a new proper edge coloring φ_1 of H_1 by setting $\varphi_1(e) = 1$ for every edge e with $\varphi(e) = 1$, and retaining the color of every other edge of H_1. Then φ_1 is an extension of φ on the graph H_1.

Let φ_2 be an edge coloring of H_2 obtained by coloring every edge of H_2 with the color of the corresponding edge of H_1 under φ_1.[1] Now, for any vertex v of H_1, if color t does not appear on an edge incident to v, $1 \le t \le d$, then color t does not appear on any edge incident to the corresponding vertex of H_2. Thus we may extend φ_1 and φ_2 to a proper edge coloring ψ of Q_d by, for any edge e of $\hat{M}$, coloring e with the color in $\{1, 2, \ldots, d\}$ not appearing on any edge incident to one of its endpoints. Clearly, ψ is an extension of φ.

By symmetry, it suffices to consider the two different cases above. Hence, the theorem follows. □

Ryser [16] proved a necessary and sufficient condition for an $n \times n$ partial Latin square where all nonempty cells lie in a completely filled $r \times s$ subrectangle to be completable. In particular, his result implies that any $n \times n$ partial Latin square, where all nonempty cells lie within an $\lfloor n/2 \rfloor \times \lfloor n/2 \rfloor$ subrectangle, is completable. We note the following analogue for hypercubes:

Proposition 3.2. *If φ is a proper d-edge coloring of $Q_r \subseteq Q_d$, then φ can be extended to a proper edge coloring of Q_d.*

We provide a brief sketch of the proof.

[1]Here, and in the following, two edges of H_1 and H_2 are *corresponding* if their endpoints are joined by two edges of M. Similarly, two vertices are *corresponding* if they are joined by an edge of M.

Proof. (Sketch). Evidently, Q_r is a component of the subgraph of Q_d induced by exactly r dimensional matchings in Q_d. It suffices to prove that if Q_{r+1} is a hypercube of dimension $r + 1$ which is contained in Q_d, and which contains Q_r, then there is a proper d-edge coloring of Q_d that agrees with φ. However, such a graph Q_{r+1} consists of two copies of Q_r and a dimensional matching joining corresponding vertices of the two copies of Q_r. We may thus obtain a proper d-edge coloring of Q_{r+1} as in the proof of the preceding theorem. □

If we do not insist that all edges in a subgraph of Q_d isomorphic to Q_r have to be precolored, then we have the following.

Corollary 3.3. *If $r \leq d/2$, then any partial proper edge coloring of $Q_r \subseteq Q_d$ with colors $1, \ldots, d$ can be extended to a proper d-edge coloring of Q_d.*

Proof. It suffices to prove that there is a proper d-edge coloring of Q_r that agrees with the given partial edge coloring φ of Q_r; invoking Proposition 3.2 then yields the desired result. Since $r \leq d/2$, such a proper d-edge coloring can be obtained by greedily coloring the uncolored edges of Q_r. □

Note that the bound on r is sharp, since there is a partial proper edge coloring of $Q_{d/2+1}$ with colors $1, \ldots, d$ that cannot be extended to a proper d-edge coloring of Q_d: Let uv be an edge of $Q_{d/2+1}$ and color the edges incident with u and distinct from uv by colors $1, \ldots, d/2$, respectively; color the edges incident with v and distinct from uv by colors $d/2 + 1, \ldots, d$, respectively. The resulting partial edge coloring can clearly not be extended to a proper d-edge coloring of Q_d.

Our next result establishes an analogue for hypercubes of the characterization of Browning et al [3] of when a partial Latin square, the nonempty cells of which constitute two Latin subsquares, is completable.

Theorem 3.4. *Let Q_{k_1} and O_{k_2} be two hypercubes of dimensions k_1 and k_2, respectively, contained in a d-dimensional hypercube Q_d, and let f be a proper edge coloring of $Q_{k_1} \cup O_{k_2}$ such that the restriction of f to $Q_{k_1}(O_{k_2})$ is a proper edge coloring using $k_1(k_2)$ colors $A_1(A_2)$ from $\{1, \ldots, d\}$. Then the coloring f is extendable to a proper d-edge coloring of Q_d unless Q_{k_1} and O_{k_2} are disjoint, a vertex of Q_{k_1} is adjacent to a vertex of O_{k_2}, and $d \leq |A_1 \cup A_2|$.*

We shall need the following easy lemma; the proof is left to the reader.

Lemma 3.5. *Let Q_{k_1} and O_{k_2} be hypercubes contained in a hypercube Q_d of larger dimension. If $Q_{k_1} \cap O_{k_2} \neq \emptyset$, then the intersection $Q_{k_1} \cap O_{k_2}$ is a hypercube of a smaller dimension.*

Proof of Theorem 3.4. Let f_1 (f_2) denote the restriction of the coloring f to Q_{k_1} (O_{k_2}). Let $\mathcal{M}$ be the set of dimensional matchings in Q_d, and denote by $\mathcal{M}_1$ and $\mathcal{M}_2$ the set of dimensional matchings that Q_{k_1} and O_{k_2} occupies, respectively. Assume that Q_{k_1} and O_{k_2} together contain edges from k-dimensional matchings, put $\mathcal{M}_k = \mathcal{M}_1 \cup \mathcal{M}_2$, and let $\mathcal{Q}_k$ be the set of subhypercubes of Q_d induced by all the dimensional matchings in $\mathcal{M}_k$.

Let H_1 and H_2 be the components of $\mathcal{Q}_k$ that contains Q_{k_1} and O_{k_2}, respectively. Suppose first that Q_{k_1} and O_{k_2} are disjoint subgraphs of Q_d. This implies that H_1 and H_2 are disjoint.

By Proposition 3.2, there is a proper edge coloring g_1 of H_1 which agrees with f_1 and uses exactly k colors from $\{1, \ldots, d\}$, and a proper edge coloring g_2 of H_2 which agrees with

f_2 and uses exactly k colors from $\{1, \ldots, d\}$ (possibly distinct from the ones used in the coloring of H_1). Additionally, we choose these edge colorings so that g_i uses as many colors from A_{3-i} as possible.

Note that if the coloring g_1 or g_2 uses some color not in $A_1 \cup A_2$, then $|A_1 \cup A_2| < k$, and both g_1 and g_2 use all colors in $A_1 \cup A_2$ and $k - |A_1 \cup A_2|$ additional colors from $\{1, \ldots, d\}$. Clearly, we may then assume that g_1 and g_2 use the same additional colors from $\{1, \ldots, d\} \setminus (A_1 \cup A_2)$.

Case 1. There is an edge e between a vertex of H_1 and a vertex of H_2.

We prove that the coloring f can be extended to a d-edge coloring of Q_d if $d - |A_1 \cup A_2| > 0$.

Let M be the dimensional matching that contains e. Consider the set of subhypercubes $\mathcal{Q}_{k+1}$ induced by the set of dimensional matchings $\mathcal{M}_k \cup \{M\}$. Since e is adjacent to both vertices of H_1 and H_2 we have that H_1 and H_2 are subgraphs of the same component H in $\mathcal{Q}_{k+1}$.

Now, if $|A_1 \cup A_2| < k$, then g_1 and g_2 use the same k colors from $\{1, \ldots, d\}$. Moreover, $d \geq k + 1$, because $M \notin \mathcal{M}_k$. This implies that there is a color $c \in \{1, \ldots, d\}$ which is not used in the coloring g_1 or g_2. By coloring all the edges of the dimensional matching M with one endpoint in H_1 and one endpoint in H_2 by color c, we obtain a proper edge coloring of H; by Proposition 3.2 this edge coloring can be extended to a proper d-edge coloring of Q_d. Clearly, this coloring is an extension of f.

If, on the other hand, $|A_1 \cup A_2| \geq k$, then g_1 and g_2 use only colors from $A_1 \cup A_2$, and since $d > |A_1 \cup A_2|$, there is a color $c \in \{1, \ldots, d\}$ which is not used in the coloring g_1 or g_2; as in the preceding paragraph, we conclude that f is extendable.

Case 2. There is no pair of adjacent vertices where one is in H_1 and the other in H_2.

Consider the graph $\mathcal{Q}_k$; by Lemma 2.3, $\mathcal{Q}_k$ consists of disjoint k-dimensional hypercubes. We define a new graph G where every component H_i in $\mathcal{Q}_k$ is represented by a vertex u_{H_i}, and where u_{H_i} and u_{H_j}, $i \neq j$, are adjacent if there is an edge joining a vertex of H_i with a vertex of H_j. It is easy to see that G is a regular bipartite graph with degree $d - k$.

We define a list assignment $\mathcal{L}$ for G by for every edge $e = u_{H_i} u_{H_j}$ of G and every color $c \in \{1, \ldots, d\}$ including c in $\mathcal{L}(e)$ if

- c does not appear in the coloring of H_1 if $i = 1$ or $j = 1$.
- c does not appear in the coloring of H_2 if $i = 2$ or $j = 2$.

Since H_1 and H_2 do not contain pairs of adjacent vertices, $|\mathcal{L}(e)| \geq d - k$ for all edges $e \in E(G)$. Thus, by Theorem 2.4, there is a proper edge coloring of G with support in the lists. By coloring all edges going between H_i and H_j by the color of the edge $e = u_{H_i} u_{H_j}$, and coloring every uncolored subhypercube H_i in $\mathcal{Q}_k$ by k colors which does not appear on the edges incident with u_{H_i} in G, we obtain a proper d-edge coloring of Q_d that is an extension of f.

Let us now consider the case when Q_{k_1} and O_{k_2} are not disjoint. If Q_{k_1} and O_{k_2} intersect in only one vertex, then Q_{k_1} and O_{k_2} occupy different dimensional matchings and

$A_1 \cap A_2 = \varnothing$. Hence, for $i = 1, 2$, by Lemma 2.3 and König's edge coloring theorem, there is a proper edge coloring g_i with colors only from A_i of the subgraph of Q_d induced by the matchings in $\mathcal{M}_i$ which agrees with f_i. Similarly, the subgraph of Q_d induced by $\mathcal{M} \setminus \mathcal{M}_k$ is $(d-k)$-regular; so if $d > k$, then there is, by König's edge coloring theorem, a proper $(d-k)$-edge coloring of this graph using colors only from the set $\{1, \ldots, d\} \setminus (A_1 \cup A_2)$. This coloring, along with g_1 and g_2, yields a proper d-edge coloring of Q_d that is an extension of f.

Suppose now that $Q_{k_1} \cap O_{k_2}$ contains at least one edge; by Lemma 3.5, this intersection is an r-dimensional hypercube D_r ($r \geq 1$). Also, $H_1 = H_2$.

We shall prove that there is a proper edge coloring of H_1 that agrees with f and uses at most d colors; the result then follows by invoking Proposition 3.2. If $D_r = O_{k_2}$ (or $D_r = Q_{k_1}$), then obviously f is extendable, so we assume that this is not the case. Thus $k_2 - r \geq 1$.

Let us consider the restriction f_r of the coloring f to D_r. Since Q_{k_1} and O_{k_2} are both regular bipartite graphs, and the restrictions of f to Q_{k_1} and O_{k_2} are both proper edge colorings using a minimum number of colors, the coloring f_r is a proper edge coloring using exactly r colors; that is, $|A_1 \cap A_2| = r$.

Consider the subgraph $\mathcal{Q}_{k_1}$ of Q_d induced by all dimensional matchings in $\mathcal{M}_1$. Consider a subhypercube Q'_{k_1} of dimension k_1 in $\mathcal{Q}_{k_1}$ that lies in H_1, and such that the vertices of Q_{k_1} and Q'_{k_1} are adjacent via a subset M_1 of edges lying in a dimensional matching. Note that some edges of M_1 and Q'_{k_1} are in O_{k_2}. Let $S_1 = E(Q'_{k_1}) \cap E(O_{k_2})$, $T_1 = M_1 \cap E(O_{k_2})$. By coloring the edges of $E(Q'_{k_1}) \setminus S_1$ by the colors of the corresponding edges in Q_{k_1} and coloring all the edges of $M_1 \setminus T_1$ by a fixed color $c \in A_2 \setminus A_1$ (such a color exists since $k_2 - r \geq 1$), we obtain an edge coloring of the subhypercube Q_{k_1+1} containing Q_{k_1} and Q'_{k_1}. This edge coloring is proper, since all common colors of A_1 and A_2 appear in the coloring of D_r and are therefore not used in the coloring of $E(Q'_{k_1}) \setminus S_1$. Moreover, $O_{k_2} \cap Q_{k_1+1}$ is an $(r+1)$-dimensional hypercube D_{r+1} containing D_r, and if u is an arbitrary vertex of D_{r+1}, then the set of colors incident with u in $Q_{k_1+1} - E(D_{r+1})$ is disjoint from A_2.

If $k_2 - r = 1$, then we are done; the constructed edge coloring of H_1 can by Proposition 3.2 be extended to a proper d-edge coloring of Q_d.

Suppose now that $k_2 - r \geq 2$. Let A_{k_1+1} be the set of colors in A_2 that has not been used in the coloring of $Q_{k_1+1} - E(D_{r+1})$; since the coloring of $Q_{k_1+1} - E(D_{r+1})$ is a proper (k_1+1)-edge coloring in which k_1 colors are in A_1, we have $|A_{k_1+1}| = k_2 - r - 1 \geq 1$. Consider a subhypercube Q'_{k_1+1} of H_1 that occupy the same dimensional matchings as the subhypercube Q_{k_1+1}, and such that the vertices of Q_{k_1+1} and Q'_{k_1+1} are adjacent via a subset M_2 of edges lying in a dimensional matching. Note that some edges of M_2 and Q'_{k_1+1} are in O_{k_2}. Let $S_2 = E(Q'_{k_1+1}) \cap E(O_{k_2})$, $T_2 = M_2 \cap E(O_{k_2})$. By coloring the edges of $E(Q'_{k_1+1}) \setminus S_2$ by the colors of corresponding edges in Q_{k_1+1} and coloring all the edges of $M_2 \setminus T_2$ by a fixed color $c \in A_{k_1+1}$, we obtain a proper edge coloring of the subhypercube Q_{k_1+2} containing Q_{k_1+1} and Q'_{k_1+1}, and where $O_{k_2} \cap Q_{k_1+2}$ is an $(r+2)$-dimensional hypercube D_{r+2} containing D_{r+1}. Moreover, if u is an arbitrary vertex of D_{r+2}, then the set of colors incident with u in $Q_{k_1+2} - E(D_{r+2})$ is disjoint from A_2.

Now, if $k_2 - r = 2$, then we are done; otherwise, we continue the above process until we get a proper edge coloring of H_1, which can then be extended to a proper edge coloring of Q_d by Proposition 3.2. $\square$

Next, we consider the case when all precolored edges lie in a matching. We would like to propose the following:

Conjecture 3.6. *If φ is an edge precoloring of Q_d where all precolored edges lie in an induced matching, then φ is extendable to a proper d-edge coloring.*

In [4], we proved that this conjecture is true under the stronger assumption that every precolored edge is of distance at least 3 from any other precolored edge. Moreover, by results in [18], Conjecture 3.6 is true in the case when all precolored edges have the same color.

Here we prove that the conjecture is true when all precolored edges lie in at most two distinct dimensional matchings.

Proposition 3.7. *If the precolored edges of Q_d form an induced matching all edges of which lie in two dimensional matchings, then the precoloring is extendable.*

Proof. Let M_1 and M_2 be the two dimensional matchings of Q_d containing all precolored edges. Denote this precoloring by φ. By Lemma 2.3, $Q_d - M_1 \cup M_2$ is isomorphic to four copies $H_1, \ldots, H_4$ of the $(d-2)$-dimensional hypercube. Moreover, the graph $Q_d[M_1 \cup M_2]$ induced by $M_1 \cup M_2$ is a disjoint union of two-dimensional hypercubes, and every vertex of H_i is adjacent to precisely two edges from $Q_d[M_1 \cup M_2]$.

Since the precolored edges form an induced matching, at most one edge of each component of $Q_d[M_1 \cup M_2]$ is precolored. From the precoloring φ of $Q_d[M_1 \cup M_2]$ we define an edge precoloring φ' of $Q_d[M_1 \cup M_2]$ that satisfies the following:

- φ' agrees with φ on any edge that is colored under φ;
- for each component of $Q_d[M_1 \cup M_2]$, exactly two edges in this component are colored under φ'; moreover, these two edges are nonadjacent and have the same color under φ'.

Trivially, there is such a precoloring φ'; so to prove the theorem, it suffices to prove that there is a proper d-edge coloring f of H_1 such that for every edge e of H_1, there is no adjacent edge e' in $Q_d[M_1 \cup M_2]$ such that $f(e) = \varphi'(e')$. This follows from the observation that given such a coloring f of H_1, we may color the edges of H_2, H_3, and H_4 correspondingly, and thereafter color the uncolored edges of $Q_d[M_1 \cup M_2]$ by for each edge using the unique color not appearing at any of its endpoints.

To construct such a coloring of the edges of H_1 we define a list assignment L for H_1 by for every edge $e \in E(H_1)$ setting

$$L(e) = \{1, \ldots, d\} \setminus \{\varphi'(e') : e' \quad \text{is adjacent to } e\}.$$

Since every edge of H_1 is adjacent to two φ'-precolored edges, $|L(e)| \geq d - 2$ for every edge $e \in E(H_1)$. Hence, by Theorem 2.4, there is an L-coloring of H_1. □

Note that the condition on the matching being induced is the best possible in terms of size of a precolored subset of a dimensional matching that is extendable to a proper d-edge coloring of Q_d. To see this, color all 2^{d-2} edges of a maximal induced matching M_1 contained in a dimensional matching M with color 1. Note that any extension of this precoloring uses color 1 on all edges of M, because M_1 is a maximal induced matching of M. So by coloring one edge of $M \backslash M_1$ by color 2, we obtain a nonextendable edge precoloring.

Next, we shall establish an analogue for hypercubes of the characterization by Andersen and Hilton [2] of which $n \times n$ partial Latin squares with exactly n nonempty cells are completable. We shall prove that a proper precoloring of at most d edges in Q_d is always extendable unless the precoloring φ satisfies any of the following conditions:

(C1) There is an uncolored edge uv in Q_d such that u is incident with edges of $k \leq d$ distinct colors and v is incident to $d - k$ edges colored with $d - k$ other distinct colors (so uv is adjacent to edges of d distinct colors).
(C2) There is a vertex u and a color c such that u is incident with at least one colored edge, u is not incident with any edge of color c, and every uncolored edge incident with u is adjacent to another edge colored c.
(C3) There is a vertex u and a color c such that every edge incident with u is uncolored and every edge incident with u is adjacent to another edge colored c.
(C4) $d = 3$ and the three precolored edges use three different colors and form a subset of a dimensional matching.

For $i = 1, 2, 3, 4$, we denote by $\mathcal{C}_i$ the set of all colorings of Q_d, $d \geq 1$, satisfying the corresponding condition above, and we set $\mathcal{C} = \cup \mathcal{C}_i$. Let us briefly verify that if φ is a precoloring of Q_d with exactly d precolored edges and $\varphi \in \mathcal{C}$, then φ is not extendable.

Suppose first that the precoloring φ satisfies condition (C1). Since the edge uv is adjacent to edges of d distinct colors, there is no proper d-edge coloring of Q_d that agrees with φ. If φ, on the other hand, satisfies condition (C2), then since u has degree d, any extension of φ satisfies that the color c must appear on one of the uncolored edges incident with u. However, such a d-edge coloring cannot be proper since this implies that there is a vertex that is incident with two edges colored c.

Suppose now that φ satisfies condition (C3). If f is an extension of φ, then since u has degree d, at least one edge incident with u is colored c. However, such a d-edge coloring is not proper, so φ is not extendable. That φ is not extendable if it satisfies condition (C4) is a straightforward verification and is left to the reader.

Theorem 3.8. *If φ is a proper d-edge precoloring of Q_d with exactly d precolored edges and $\varphi \notin \mathcal{C}$, then φ is extendable to a proper d-edge coloring of Q_d.*

The proof of this theorem is rather lengthy so we devote Section 4 to this proof.

4 | PROOF OF THEOREM 3.8

The proof of Theorem 3.8 proceeds by induction. It is easily seen that the theorem holds when $d \in \{1, 2\}$; let us consider the case when $d = 3$.

Let φ be a precoloring of Q_3 and let us first assume that all precolored edges have the same color. If all three precolored edges lie in distinct dimensional matchings, then $\varphi \in \mathcal{C}_3$, and if all three edges lie in the same dimensional matching, then we may color all the edges in this dimensional matching by the same color, and then obtain an extension of φ by König's edge coloring theorem. Moreover, in the case when exactly two of the precolored edges are in the same dimensional matching, then these two edges must be at distance 1 from each other, and so there is a perfect matching containing all precolored edges; hence, φ is extendable.

Suppose now that two colors appear on the precolored edges. Let e_1, e_2, e_3 be the precolored edges of Q_3 and assume that two edges from $\{e_1, e_2, e_3\}$, say e_1 and e_2, have the same color and e_3 has another color under φ. If e_1 and e_2 lie in the same dimensional matching, then φ is extendable provided that there is a perfect matching of Q_3 containing e_1 and e_2, but not e_3. If e_1 and e_2 lie on a common 4-cycle, then there is certainly such a matching; if e_1 and e_2 do not lie on a common 4-cycle, then this holds unless $\varphi \in \mathcal{C}_2$.

Let us now assume that e_1 and e_2 lie in different dimensional matchings. By symmetry, we may assume that e_1 is any fixed edge of Q_3, which then yields four different choices for the edge e_2, because every edge of Q_3 is adjacent to exactly four other edges. In fact, again by symmetry, it suffices to consider the two different cases when e_2 is in different dimensional matchings (distinct from the one containing e_1). It is straightforward to verify that in both cases, the edges e_1 and e_2 are contained in a perfect matching not containing e_3 unless $\varphi \in \mathcal{C}_2$. Hence, if $\varphi \notin \mathcal{C}$, then φ is extendable.

Finally, let us consider the case when three distinct colors appear on edges under φ. If all three precolored edges e_1, e_2, e_3 lie in distinct dimensional matchings, then φ trivially is extendable. Moreover, since $\varphi \notin \mathcal{C}$, all three precolored edges do not lie in the same dimensional matching. Hence, it suffices to consider the case when exactly two of the precolored edges lie in the same dimensional matching. We assume $\varphi(e_i) = i$.

Suppose, without loss of generality, that e_1 and e_2 lie in the same dimensional matching. We first consider the case when e_1 and e_2 lie on a common 4-cycle. Since $\varphi \notin \mathcal{C}$, either e_3 is adjacent to both e_1 and e_2, or not adjacent to any of these edges. In both cases, φ is extendable by coloring all edges in the dimensional matching containing e_3 by color 3. If, on the other hand, e_1 and e_2 do not lie on a common 4-cycle, then we may extend φ by coloring all edges of the dimensional matching containing e_3 by color 3. This completes the base step of our inductive proof of Theorem 3.8.

Let us now assume that the theorem holds for any hypercube of dimension less than d, and consider a precoloring φ of Q_d. The induction step of the proof of Theorem 3.8 is done by proving a series of lemmas. We shall also need two preparatory lemmas.

Lemma 4.1. *Let Q_{d-1} be the $(d-1)$-dimensional hypercube, where $d - 1 \geq 3$. Suppose that $d-1$ edges are precolored with color 1 in Q_{d-1}, and that there is a vertex u not incident with any precolored edge, but every neighbor of u is incident with an edge colored 1. Let e_1 be an uncolored edge which is not incident with u, but adjacent to at least one precolored edge. Unless $d-1=3$ and one end x of e_1 is incident with three uncolored edges all of which are adjacent to precolored edges, then there is a cycle $C = v_1v_2 \ldots v_{2k}v_1$ in Q_{d-1} of even length with the following properties:*

(i) $v_1v_2 = e_1$ *and* $u \notin V(C)$,
(ii) *none of the edges in* $\{v_1v_2, v_3v_4, \ldots, v_{2k-1}v_{2k}\}$ *is precolored,*
(iii) *if any vertex in* $\{v_1, \ldots, v_{2k}\}$ *is incident with a precolored edge, then this edge lies on* C.

Proof. Let $M_1, \ldots, M_{d-1}$ be the $d-1$ dimensional matchings in Q_{d-1} and let $e_1 = wx \in M_1$. Let $e_2 = vw \in M_2$ be a precolored edge adjacent to e_1.

We first consider the case when e_1 is adjacent to two precolored edges. If the other precolored edge e_3 adjacent to e_1 is in M_2, then v is adjacent to an endpoint of e_3 via an edge from M_1, so there is trivially a 4-cycle satisfying (i)–(iii). So we assume that $e_3 \in M_3$. Moreover, since Q_{d-1} has no odd cycles, we may without loss of generality assume that v and x are both adjacent to u. Since any 4-cycle has edges from exactly two-dimensional matchings (which, eg, follows from Proposition 2.1(ii)), this implies that $uv \in M_1$ and $ux \in M_2$.

Consider the subgraph of Q_{d-1} induced by the edges in $M_1 \cup M_2 \cup M_3$; by Lemma 2.3, this is a disjoint union of three-dimensional hypercubes. Let F be the component of this subgraph containing e_1, e_2, and e_3. Since any precolored edge is adjacent to an edge incident with u, it follows that the edge of M_3 incident with u is adjacent to some precolored edge e' that lies in M_1 or in M_j for some $j \geq 4$. Moreover, e_3, e', and e_2 are the only precolored edges incident with vertices of F. If $e' \in M_1$, then there is a 6-cycle in F containing e_1, e_2, e_3, e' that satisfies (i) to (iii); if $e' \notin M_1$, then there is a 6-cycle in F containing e_1, e_2, e_3, but no vertex incident with e', which satisfies (i)–(iii).

Suppose now that $e_1 = wx$ is adjacent to precisely one precolored edge $e_2 = vw$. Since every precolored edge is adjacent to an edge incident with u, either v or w is adjacent to u. Let us first assume that w is adjacent to u. Since x is not incident to any precolored edge, and all precolored edges are adjacent to edges incident with u, the unique vertex $a \notin \{w, x, v\}$ in the component of the subgraph $Q_d[M_1 \cup M_2]$ containing e_1 is not incident with a precolored edge. Thus, there is a 4-cycle $vwxav$ whose edges lie in $M_1 \cup M_2$ and which satisfies (i)–(iii).

Let us now consider the case when v is adjacent to u. Then we may assume that $e_3 = uv$ is in some dimensional matching distinct from M_1 and M_2, since $uv \in M_1$ implies that x is adjacent to u and thus x is incident with some precolored edge, contradicting our assumption. We assume $e_3 \in M_3$. As above we consider the subgraph of Q_{d-1} induced by the edges in $M_1 \cup M_2 \cup M_3$. Let F be the component of this induced subgraph containing e_1, e_2, and e_3. Straightforward case analysis shows that there is a 4- or 6-cycle satisfying (i)–(iii) unless every edge incident with x in F is adjacent to a precolored edge of F. It remains to prove that if $d - 1 \geq 4$, and every edge incident with x in F is adjacent to a precolored edge of F, then there is a cycle C satisfying (i)–(iii). Consider the subgraph of Q_{d-1} induced by $M_1 \cup M_2 \cup M_3 \cup M_4$. Let K be the component of this induced subgraph containing F. Since all precolored edges are adjacent to edges incident with u, K contains at most one precolored edge not in F. Using these facts, it is straightforward that K has a cycle containing all three precolored edges of F and satisfying (i)–(iii). □

Lemma 4.2. *Let φ_1 be an edge precoloring of $d-1$ edges of Q_{d-1} such that there is a vertex u incident with an edge e' precolored 2, and where every other edge incident with u is not precolored but adjacent to an edge precolored 1. Let e_1 be some edge precolored 1 in Q_{d-1}. There is a partial proper edge coloring f_1 of Q_{d-1} with colors 1 and 2 satisfying the following:*

(i) *Any vertex of Q_{d-1} is incident with at least one edge that is colored under f_1.*
(ii) *The coloring f_1 agrees with φ_1 on any edge that is colored under φ_1.*
(iii) *e_1 is contained in a cycle that is $(1, 2)$-colored under f_1, and which does not contain e'.*

Proof. Note that the condition of the lemma implies that e_1 is no incident with u, but an end of e_1 is adjacent to u. Let M_1, M_2, M_3 be three dimensional matchings in Q_{d-1} that contain e_1, e' and an edge adjacent to both e' and e_1.

The spanning subgraph of Q_{d-1} induced by $M_1 \cup M_2 \cup M_3$ is a disjoint union of copies of Q_3; let F be the component containing e_1 and e'.

If e_1 and e' lie in distinct dimensional matchings, then it is easy to see that there is a 4-cycle C_1 in F containing e_1 and no other precolored edge, and that satisfies that no vertex of C_1 is incident to a precolored edge that is not in C_1. We color the edges of C_1 by colors 1 and 2 alternately such that the coloring agrees with φ_1. Additionally we retain the color of any precolored edges of F, and we possibly color one additional edge in F by color 2 so that every vertex of F is incident with a colored edge. Denote the obtained coloring of F by h_1.

Now, since every precolored edge has one endpoint adjacent to u, any component T of $Q_{d-1}[M_1 \cup M_2 \cup M_3]$ distinct from F contains at most one precolored edge. Hence, there is a perfect matching M_T of T that does not contain any precolored edge. We extend h_1 to a coloring of Q_{d-1} satisfying (i) to (iii) by retaining the color of any φ_1-precolored edge not in F, and for every component T of $Q_d[M_1 \cup M_2 \cup M_3]$ distinct from F we color every edge in M_T by color 2.

Suppose now that e_1 and e' lie in the same dimensional matching, M_1 say. Then e_1 and e' are contained in a 4-cycle of F. Suppose that the edges of this cycle are in $M_1 \cup M_3$. If $M_3 \cap E(F)$ contains no φ_1-precolored edge, then e_1 is contained in a 4-cycle such that no vertex of this cycle is incident with another φ-precolored edge. On the other hand, if $M_3 \cap E(F)$ contains some precolored edge, then e_1 is contained in a 6-cycle C_2 not containing e', but two other precolored edges colored 1. Moreover, no vertex of C_2 is incident to a precolored edge that is not in C_2. Thus there is a proper edge coloring h_2 of C_2 with colors 1 and 2 that agrees with φ_1.

The coloring h_2 can be extended to a partial proper edge coloring of Q_{d-1} satisfying (i)–(iii) by proceeding as above. □

We now turn to the induction step of the proof of Theorem 3.8. Henceforth, we shall always assume that φ is a proper d-edge precoloring of precisely d edges in Q_d. Moreover, we assume that M is a dimensional matching in Q_d and that H_1 and H_2 are the components of $Q_d - M$; so H_1 and H_2 are both isomorphic to Q_{d-1}. As in the proof of Theorem 3.1, two edges of H_1 and H_2 are corresponding if their endpoints are joined by two edges of M. Similarly, two vertices are corresponding if they are joined by an edge of M.

In the proofs of the lemmas we shall generally distinguish between the cases when there is a dimensional matching that contains no precolored edge, and when there is no such dimensional matching.

Lemma 4.3. *If all d precolored edges in Q_d have the same color and $\varphi \notin \mathcal{C}_3$, then φ is extendable.*

Proof. Suppose that the color used by φ is 1. It follows from König's edge coloring theorem that for proving the lemma, it suffices to show that there is a perfect matching in Q_d containing all edges precolored 1.

Case 1. Every dimensional matching contains a precolored edge.

The assumption implies that M contains precisely one edge u_1u_2 colored 1, where $u_i \in V(H_i)$.

Case 1.1. No precolored edges are in H_2.

The conditions imply that $d-1$ precolored edges are in H_1. By coloring the edges of H_2 corresponding to the precolored edges of H_1 by color 1, coloring all edges of M that are not adjacent to any colored edges by color 1, we obtain a partial coloring where the precolored edges form a perfect matching of Q_d; thus φ is extendable.

Case 1.2. Both H_1 and H_2 contain at most $d-3$ precolored edges.

Suppose that there is a vertex x_1 of H_1 adjacent to u_1 such that neither x_1 nor the vertex x_2 of H_2 corresponding to x_1 is incident with a precolored edge. Consider the precoloring of H_1 obtained from the restriction of φ to H_1 by in addition coloring x_1u_1 with 1. By Theorem 3.1, this precoloring is extendable to a proper $(d-1)$-edge coloring f_1 of H_1; and similarly there is an extension f_2 of the precoloring of H_2 obtained from the restriction of φ to H_2 by in addition coloring u_2x_2 by color 1; this is evident since the obtained precolorings of H_1 and H_2, respectively, both contain at most $d-2$ precolored edges. We now define a perfect matching containing all φ-precolored edges of Q_d by removing u_1x_1 and u_2x_2 from the union of all edges colored 1 under f_1 or f_2, and adding the edges u_1u_2 and x_1x_2. We conclude that φ is extendable.

Now suppose that for each neighbor x_1 of u_1 either x_1 or the corresponding vertex x_2 of H_2 is incident with a precolored edge. Since Q_d is d-regular and contains altogether d precolored edges, this implies that all precolored edges have one end which is adjacent to either u_1 or u_2. Now, since Q_d contains d precolored edges, M contains one precolored edge, and both H_1 and H_2 contain at most $d-3$ precolored edges, $(d-3)+(d-3)+1 \geq d$, and so $d \geq 5$. Thus u_1 is adjacent to at least two vertices incident with precolored edges in H_1, and u_2 is adjacent to two vertices of H_2 incident with precolored edges.

We shall need the following claim.

Claim 4.4. There is a dimensional matching M_j and a precolored edge $vv' \in M_j$ such that not every other precolored edge has one end adjacent to either v or v'.

Proof. Recall that Proposition 2.1 holds if we take the dimensional matchings of Q_d as the colors in the proposition. Let $M_1, \ldots, M_d$ be the dimensional matchings in Q_d, where $M_1 = M$. Without loss of generality, we assume that there are precolored edges $e_j = a_jb_j \in M_j$ and $e_k = a_kb_k \in M_k$, such that b_j and u_1 are adjacent and $u_1b_j \in M_2$, and b_k and u_1 are adjacent and $u_1b_k \in M_3$. If no endpoint of e_j is adjacent to an endpoint of e_k, then we are done, so suppose, without loss of generality, that a_j and b_k are adjacent. By Proposition 2.1(ii), this means that $a_jb_j \in M_3$ and $a_jb_k \in M_2$. Now, H_2 contains at least one precolored edge ab, where either a or b is adjacent to u_2 via an edge from a dimensional matching that is distinct from M_2 and M_3, because otherwise, as for H_1, it would follow that at least one precolored edge of H_2 would be in M_2 or M_3; a contradiction to the assumption that all precolored edges are in distinct dimensional matchings. Thus, without loss of generality, we assume that $u_2a \in M_4$. Moreover, since all precolored edges lie in distinct dimensional matchings $ab \notin M_1 \cup M_3$. Hence, all edges on the path $a_jb_ju_1u_2a$ are in different dimensional matchings. Again using Proposition 2.1(ii), it thus follows that no endpoint of ab is adjacent to an endpoint of a_jb_j. We conclude that there is a dimensional matching M_j and a precolored edge $vv' \in M_j$ such that not every other precolored edge has one end adjacent to either v or v'. □

Let M_j be a dimensional matching as in the preceding claim. Then the graph $Q_d - M_j$, consists of two copies J_1 and J_2 of Q_{d-1}. Moreover, if both J_1 and J_2 contain at most $d - 3$ precolored edges, then we may proceed as above for obtaining an extension of φ. Moreover, if $d - 1$ precolored edges lie in J_1, then we proceed as in Case 1.1. We conclude that it suffices to consider the case when $d - 2$ edges of H_1 (or H_2) are precolored.

Case 1.3. H_1 contains $d - 2$ precolored edges and H_2 contains one precolored edge.

Denote by v_2w_2 the precolored edge of H_2 and let v_1 and w_1 be the vertices of H_1 corresponding to v_2 and w_2, respectively. If no precolored edge is incident with v_1 or w_1, then we may color v_1w_1 with color 1, and then color all edges of H_2 corresponding to precolored edges of H_1 by color 1. The resulting coloring is extendable, since by coloring any edge of M (including u_1u_2), which is not adjacent to a colored edge, by color 1, the precolored edges form a perfect matching of Q_d, as required.

Thus, we may assume that some φ-precolored edge in H_1 is incident with v_1 or w_1, say w_1. Since there are $d - 2$ precolored edges in H_1, the restriction of φ to H_1 is extendable; in particular, there is a perfect matching M^* in H_1 containing all precolored edges of H_1. Note that the edge of M^* incident with u_1 is not incident with w_1. If $u_1v_1 \notin M^*$, then let e' be the edge of H_2 corresponding to the edge of M^* incident with u_1. Then the precoloring of H_2 where e' and v_2w_2 are colored 1 is extendable, in particular there is perfect matching M_2^* in H_2 containing both these edges. By removing the edge e' from M_2^*, removing the corresponding edge from M^* and including two edges from M, we obtain a perfect matching in Q_d containing all precolored edges of φ; hence, the coloring φ is extendable. Thus, we may assume that $u_1v_1 \in M^*$, and, consequently, v_1 is not incident to any φ-precolored edge. Moreover, if u_1 is the only neighbor of v_1 that is not incident with a precolored edge of H_1, then $\varphi \in \mathcal{C}_3$, because all neighbors of v_1 are incident with a precolored edge in Q_d. Thus, there is a neighbor $y \neq u_1$ of v_1 in H_1 that is not incident with any precolored edge.

Consider the precoloring ψ of H_1 obtained from the restriction of φ to H_1 by also coloring v_1y by color 1. If ψ is extendable to a proper $(d - 1)$-edge coloring ψ' of H_1, then in the matching of H_1 containing all edges with color 1 under ψ', u_1 is matched to some vertex distinct from v_1, and, as before, this implies that φ is extendable. Thus it suffices to consider the case when ψ is not extendable to a proper edge coloring of H_1. Since there are exactly $d - 1$ precolored edges under ψ, all of which have the same color, by the induction hypothesis, there is some vertex a of H_1 that is not incident with any ψ-precolored edge, but all neighbors of a are incident with ψ-precolored edges. We shall prove that this property also holds for the vertex u_1 unless φ is extendable.

Claim 4.5. Every neighbor of u_1 in H_1 is incident with a ψ-precolored edge unless φ is extendable.

Proof. Assume to the contrary that u_1 does not have this property. Then there is a neighbor $z \neq v_1$ of u_1 that is not incident to any φ-precolored edge. Let α be the precoloring of H_1 obtained from the restriction of φ to H_1 by coloring the edge u_1z by color 1. As we have seen above, if any of the precolorings ψ or α of H_1 is extendable (in H_1) to a proper $(d-1)$-edge coloring, then φ is extendable. (Because in both these extensions u_1 is matched to some other vertex than v_1 in the matching induced by color 1.)

We conclude that since neither of α and ψ is extendable, there are vertices b_1 and b_2 such that under α every neighbor of b_1 in H_1 is incident with a precolored edge, and under ψ every neighbor of b_2 in H_1 is incident with a precolored edge. Note that $b_1 \neq b_2$ because the vertices u_1, v_1, y, z are all distinct and all vertices in H_1 have degree $d-1$ in H_1. Since $d-1 \geq 3$, b_1 and b_2 are both adjacent to endpoints of at least two distinct φ-precolored edges. Hence, the distance $d(b_1, b_2)$ between b_1 and b_2 is at least 1 and at most 3. We consider some different subcases.

Subcase A. $d(b_1, b_2) = 1$.

Since $d(b_1, b_2) = 1$ and b_1 and b_2 are both adjacent to endpoints of at least two distinct φ-precolored edges e_1 and e_2 in H_1, there are two 4-cycles containing e_1 and b_1b_2, and e_2 and b_1b_2, respectively. However, this implies that e_1 and e_2 are in the same dimensional matching; a contradiction to the assumption of Case 1. We conclude that the case $d(b_1, b_2) = 1$ is not possible.

Subcase B. $d(b_1, b_2) = 2$.

In this case, it follows that b_1 and b_2 have a common neighbor which is incident to an edge which is precolored under φ. Then, since H_1 is bipartite, b_1 and b_2 are adjacent to the same end of every edge which is precolored under φ. If $d-1=3$, then H_1 contains two φ-precolored edges that lie in the same dimensional matching, because b_1 and b_2 lie on a common 4-cycle with edges from exactly two-dimensional matchings; a contradiction to the assumption of Case 1. If $d-1 \geq 4$, then H_1 has at least 3 φ-precolored edges, and thus two adjacent edges of H_1 lie on at least two distinct 4-cycles; a contradiction because H_1 is isomorphic to Q_{d-1}. We conclude that the case $d(b_1, b_2) = 2$ is not possible.

Subcase C. $d(b_1, b_2) = 3$.

If $d(b_1, b_2) = 3$, then b_1 and b_2 are adjacent to distinct ends of an edge which is precolored under φ. Since H_1 is bipartite, this implies that b_1 and b_2 are adjacent to distinct endpoints of every edge that is precolored under φ. If $d-1=3$, then H_1 contains two φ-precolored edges, and there is exactly one edge of H_1 that we can color 1 so that b_1 or b_2 is adjacent to three vertices all of which are incident with an edge colored 1. This contradicts that the vertices u_1, v_1, y, z are all distinct.

Assume now that $d-1 \geq 4$. Then b_1 and b_2 are adjacent to distinct endpoints of at least three φ-precolored edges that lie in distinct dimensional matchings. In fact, we must have $d-1=4$. Indeed, recall that Proposition 2.1 holds if we take the colors to be the dimensional matchings of Q_d. It then follows from Proposition 2.1(ii) that two vertices in a hypercube are endpoints of at most three distinct paths of length 3, where any two central edges of the paths are in distinct dimensional matchings. Furthermore, since all edges of these three distinct paths with endpoints b_1 and b_2 must lie in three distinct dimensional matchings (which again follows from Proposition 2.1(ii)), these paths induce a hypercube F of dimension 3. Now, since in H_1, u_1 is adjacent to at least two vertices that are not incident with any φ-precolored edges, $u_1 \notin V(F)$. Moreover, $v_1 \notin \{b_1, b_2\}$, because v_1 has at least two neighbors that are not incident with any φ-precolored edges of H_1. Now, since $d - 1 = 4$, and all φ-precolored edges of H_1 are in F, this implies that there is a perfect matching of H_1 containing all φ-precolored edges of H_1, and where u_1 is matched to some other vertex than v_1; as before, this implies that φ is extendable. □

From the preceding claim, we conclude that we may assume that u_1 is not incident to any ψ-precolored edge, but every neighbor of u_1 is incident with a ψ-precolored edge.

Now, since all φ-precolored edges of H_1 are also ψ-precolored, both ends of v_1w_1 are incident with ψ-precolored edges. Hence, by Lemma 4.1, there is a cycle $C = a_1a_2...a_{2k}a_1$ of even length such that

(i) $a_1 = v_1$, $a_2 = w_1$, and $u_1 \notin V(C)$,
(ii) none of the edges in $\{a_1a_2, a_3a_4, ..., a_{2k-1}a_{2k}\}$ is ψ-precolored in H_1,
(iii) if any vertex in $\{a_1, ..., a_{2k}\}$ is incident with a precolored edge, then this edge lies on C.

From the precoloring ψ of H_1 we define another precoloring ψ_1 of H_1 by coloring all uncolored edges in $\{a_2a_3, a_4a_5, ..., a_{2k}a_1\}$ by color 1 and retaining the color of every other edge. Next, we define a precoloring ψ_2 of H_2 by coloring all edges of H_2 corresponding to the edges in $\{a_1a_2, a_3a_4, ..., a_{2k-1}a_{2k}\}$ by color 1; furthermore, for any edge of H_1 which is ψ_1-precolored and does not lie on C, we color the corresponding edge of H_2 by 1.

Note that a vertex of H_2 is incident with a ψ_2-precolored edge if and only if the corresponding vertex of H_1 is incident with a ψ_1-precolored edge. Moreover, any edge in Q_d which is precolored under φ is also precolored under ψ_1 or ψ_2. Hence, we obtain an extension of φ from ψ_1 and ψ_2 by coloring any edge of M which is not incident with a ψ_1-precolored edge by color 1.

Case 2. There is a dimensional matching containing no precolored edge.

Without loss of generality, we assume that no edge of M is precolored.

Case 2.1. No precolored edges are in H_2.

If all precolored edges lie in H_1, then the precoloring is extendable, since by coloring the edges of H_2 corresponding to the precolored edges of H_1 by color 1, and then coloring the edges of M not adjacent to precolored edges by color 1, we obtain a monochromatic perfect matching of Q_d which contains all φ-precolored edges of Q_d.

Case 2.2. Both H_1 and H_2 contain at most $d - 2$ precolored edges.

If both H_1 and H_2 contain at most $d - 2$ precolored edges, then by Theorem 3.1, the restriction of φ to H_i is extendable to $(d - 1)$-edge coloring of H_i, $i = 1, 2$; thus φ is extendable.

Case 2.3. H_1 contains $d - 1$ precolored edges and H_2 contains one precolored edge.

As in Case 1.3, we may assume that the edge v_1w_1 of H_1, corresponding to the precolored edge v_2w_2 of H_2, is adjacent to at least one precolored edge of H_1, since otherwise φ is extendable.

Now, by the induction hypothesis, the restriction of φ to H_1 is extendable (and thus there is an extension of φ) unless there is a vertex $u \in V(H_1)$ not incident to any precolored edge, and satisfying that all neighbors of u in H_1 are incident with precolored edges. Furthermore, if $v_1 = u$ or $w_1 = u$, then clearly $\varphi \in \mathcal{C}_3$, so we assume that $u \notin \{v_1, w_1\}$.

If $d - 1 = 3$, and one end of v_1w_1 is not incident to any precolored edge, but all neighbors of v_1 or w_1 are incident with precolored edges, then $\varphi \in \mathcal{C}_3$. Thus, since $\varphi \notin \mathcal{C}_3$, and v_1w_1 is adjacent to at least one precolored edge, it follows from Lemma 4.1 that there is a cycle $C = v_1v_2...v_{2k}v_1$ of even length such that

(i) $v_2 = w_1$, $u \notin V(C)$,
(ii) none of the edges in $\{v_1v_2, v_3v_4, ..., v_{2k-1}v_{2k}\}$ is φ-precolored in H_1,
(iii) if any vertex in $\{v_1, ..., v_{2k}\}$ is incident with a precolored edge, then this edge lies on C.

We may now finish the proof in this case by proceeding exactly as in Case 1.3 above, using the cycle C to construct a precoloring of H_2. □

Lemma 4.6. *If only two distinct colors appear in the precoloring φ of Q_d and $\varphi \notin \mathcal{C}$, then φ is extendable.*

Proof. Without loss of generality we shall assume that colors 1 and 2 appear on edges under φ.

Case 1. Every dimensional matching contains a precolored edge.

Without loss of generality, we assume that M contains an edge $e_M = u_1u_2$ precolored 1 under φ, where $u_i \in V(H_i)$.

Case 1.1. No precolored edges are in H_2.

Suppose that color 1 does not appear in the restriction φ_1 of φ to H_1. If φ_1 is extendable to a proper edge coloring of H_1 using colors 2, ..., d, then we obtain an extension of φ by coloring H_2 correspondingly, and then coloring all edges of M by color 1. So assume that there is no such extension of φ_1. By the induction hypothesis, there is a vertex u in H_1 that is not incident with any precolored edge, but all vertices in H_1 adjacent to u are incident with an edge precolored 2. If u is an endpoint of e_M, then $\varphi \in \mathcal{C}_2$; so we assume that this is not the case. Thus, either there is an edge e' incident with u_1 colored 2, or we can select e' to be an arbitrary edge of H_1 that is incident with u_1 but not adjacent to any edge precolored 2. In both cases, we define a precoloring φ_1' of H_1 by coloring e' by color 1. Then trivially there is a proper edge coloring f_1 of H_1 using colors 1, 3, ..., d that agrees with φ_1'. From f_1, we define a proper edge coloring f_1' by recoloring all edges that are precolored 2 under φ by color 2 and also recoloring e' with color 2. This yields a coloring of H_1 that agrees with the restriction of φ to H_1 and where color 1 does not appear at an end of e_M. Hence, we may color H_2 correspondingly, and then color every edge of M by the color in $\{1, ..., d\}$ missing at its endpoints to obtain an extension of φ.

Suppose now that color 1 does appear on some edge of H_1. By removing the color from any edge of H_1 that is precolored 1, we obtain a precoloring φ_1 of H_1. By Theorem 3.1, there is a proper edge coloring of H_1 using colors 2, ..., d that agrees with φ_1. Now, by recoloring any edge of H_1 that is φ-precolored 1 by color 1, thereafter coloring H_2 correspondingly, and finally coloring all edges of M by the unique color missing at its endpoints, we obtain an extension of φ.

Case 1.2. Both H_1 and H_2 contain at most $d - 3$ precolored edges.

The conditions imply that $d \geq 5$. If there is an edge e_1 in H_1 adjacent to e_M, and such that neither e_1 nor the corresponding edge e_2 of H_2 is colored under φ, and neither of e_1 and e_2 is adjacent to an edge precolored 1 under φ distinct from e_M, then we color e_1 and e_2 by color 1, and consider the precolorings of H_1 and H_2 obtained from the restriction of φ to H_1 and H_2, respectively, along with coloring e_1 and e_2 by color 1. By Theorem 3.1, these colorings are extendable to proper $(d - 1)$-edge colorings f_1 and f_2 of H_1 and H_2, respectively. Now, by recoloring e_1 and e_2 by color d and then coloring all edges of M by the color missing at its endpoints we obtain the required extension of φ.

Now suppose that there are no edges e_1 and e_2 as described in the preceding paragraph. Since $Q_d - M$ contains exactly $d - 1$ precolored edges, and H_1 and H_2 are $(d - 1)$-regular bipartite graphs, this implies that any edge colored 2 under φ is adjacent to e_M, and any edge colored 1 under φ is adjacent to an edge e' that is adjacent to e_M. Thus either one or two edges in Q_d are colored 2 under φ.

Suppose first that there are (at least) two edges precolored 1 in H_1 or H_2, say H_1. Let e_1' and e_2' be two such edges. Consider the subgraph J_1 of Q_d induced by all dimensional matchings containing an edge precolored 1. Since there are at most two edges colored 2 under φ, the maximum degree of J_1 is $d - 1$ or $d - 2$. Moreover, there is a proper edge coloring of $J_2 = Q_d - E(J_1)$ using $\Delta(J_2)$ colors, and which agrees with the restriction of φ to J_2, because J_2 is a collection of disjoint one- or two-dimensional hypercubes, where every component contains at most one precolored edge. Thus, φ is extendable if there is an extension with $\Delta(J_1)$ colors of the restriction φ_1 of φ to J_1 (using distinct colors from the extension of the restriction of φ to J_2). Now, by the induction hypothesis, there is an extension of φ_1 if for no component T of J_1 the restriction of φ_1 to T satisfies the condition (C3) (with $d - 1$ or $d - 2$ in place of d). If there is such a component T of J_1, then clearly all precolored edges of J_1 are in T and there is a vertex u of T that is not incident with any precolored edge, but any vertex adjacent to u in T is incident with a precolored edge. Thus we may assume that e_1', e_2', and e_M are in the same component of J_1, and one endpoint of all these three edges is adjacent to u. Now, if u is adjacent to u_2, then since T is bipartite, this implies that e_M and u_2u lie on 2 common 4-cycles, which is not possible since T is isomorphic to a hypercube. On the other hand, if u is adjacent to u_1, then since T is bipartite, by Proposition 2.1, this implies that e_1' and e_2' lie in the same dimensional matching; a contradiction in both cases, so φ is extendable.

It remains to consider the case when only one edge in H_1 and one edge in H_2 is precolored 1 under φ. Since at most two edges are precolored 2 under φ, this implies that $d = 5$ and, consequently, there are exactly two edges colored 2 in Q_d. Suppose that u_1v_1 and u_2v_2 are the edges colored 2 under φ, where $u_iv_i \in E(H_i)$. Let M_2 be the dimensional matching containing u_1v_1, and let H_1' and H_2' be the components of $Q_d - M_2$. Note that u_1u_2 and u_2v_2 lie in the same component of $Q_d - M_2$, H_1' say. Let $u_2'v_2'$ be the edge of H_2'

corresponding to u_2v_2; then $u_2'v_2'$ is not precolored under φ, because every dimensional matching contains a single precolored edge. Consider the precoloring φ_1 of Q_d obtained from the restriction of φ to H'_1 by recoloring u_2v_2 by color 3, and the precoloring φ_2 obtained from the restriction of φ to H_2' by also coloring $u_2'v_2'$ by color 3. Let us verify that neither of φ_1 and φ_2 satisfies any of the conditions (C1) to (C3) (with 4 in place of d). Indeed, H_1' contains at most four precolored edges colored by exactly two distinct colors, and, moreover, two precolored edges are adjacent; H_2' contains at most three precolored edges. Thus, it follows from Theorem 3.1 and the induction hypothesis that there are proper edge colorings f_1 of H_1' and f_2 of H_2' using colors 1, 3, 4, 5 that agree with φ_1 and φ_2, respectively. Now, by recoloring u_2v_2 and $u_2'v_2'$ by color 2 and coloring all edges of M_2 by the unique color missing at its endpoints, we obtain an extension of φ.

By symmetry, it remains to consider the case when H_1 contains $d-2$ precolored edges, and H_2 contains one precolored edge.

Case 1.3. H_1 contains $d-2$ precolored edges and H_2 contains one precolored edge.

Suppose first that for every edge e_1 in H_1 that is adjacent to e_M, either e_1 or the corresponding edge e_2 of H_2 is colored 2 under φ, or one of e_1 and e_2 is adjacent to an edge colored 1 distinct from e_M. If there are at least two edges precolored 1 in H_1, then we proceed as in the preceding case and consider the subgraphs J_1 and J_2 defined as above. So suppose instead that there is only one edge precolored 1 in H_1; then $d=4$ and H_1 contains one edge precolored 1 and one edge precolored 2. If H_2 contains an edge precolored 2, then since all precolored edges lie in distinct dimensional matchings and all edges precolored 2 are adjacent to e_M, there is a perfect matching M^* in Q_d containing all edges precolored 1 and no edge precolored 2. Since H_1 and H_2 both contain only one edge precolored 2, this implies that φ is extendable. If H_2 contains an edge precolored 1, then one may proceed similarly; the details are omitted.

Let us now consider the case when there is an edge $e_1 \in E(H_1)$ adjacent to e_M and satisfying that neither e_1 nor its corresponding edge e_2 in H_2 is precolored or adjacent to an edge colored 1 in H_1 and H_2, respectively. If the precoloring φ_1 obtained from the restriction of φ to H_1 by in addition coloring e_1 by color 1 is extendable to a $(d-1)$-edge coloring of H_1, then there is a similar extension of H_2 of the restriction of φ to H_2 along with coloring e_2 by 1. By recoloring e_1 and e_2 by color d, it is easy to see that there is an extension of φ. Thus we assume that φ_1 is not extendable.

Suppose first that e_1 is the only edge colored 1 in H_1 under φ_1. If the φ-precolored edge of H_2 is colored 2, then H_1 and H_2 only contain φ-precolored edges with color 2, and by Theorem 3.1, for $i=1,2$, the restriction of φ to H_i is extendable to a proper edge coloring of H_i using colors $2, \ldots, d$; thus φ is extendable by coloring all edges of M by color 1. Hence, we may assume that H_2 contains a φ-precolored edge of color 1. Note that this implies that the precolored edge e_2' of H_2 is not adjacent to e_M. Moreover, the corresponding edge e_1' of H_1 is not φ-precolored, since all precolored edges lie in different dimensional matchings. Now, since the restriction of φ to H_1 consists of $d-2$ precolored edges with colors distinct from 1, Theorem 3.1 yields that there is an extension of H_1 using colors $2, \ldots, d$. We color H_2 correspondingly. Since e_M and e_2' are not adjacent, we now obtain an extension of φ by recoloring e_1' and e_2' by color 1, and thereafter coloring all edges of M by the color in $\{1, \ldots, d\}$ missing at its endpoints.

Now assume that there are several edges φ_1-precolored 1 in H_1. Since φ_1 is not extendable, only two colors are used in φ_1, and there are at least two edges in H_1 precolored 1 under φ_1, there is some vertex $v \in V(H_1)$ such that either

(a) v is not incident with any φ_1-precolored edge, but any edge incident to v is adjacent to some edge φ_1-precolored 1, or
(b) v is incident with an edge φ_1-precolored 2 and all other edges incident with v are not φ_1-precolored but adjacent to edges precolored 1.

Subcase A. (a) holds.

If (a) holds, then every φ-precolored edge of H_1 is colored 1 and thus the single φ-precolored edge in H_2 is colored 2. Moreover, the restriction of φ to H_1 is by Theorem 3.1 extendable to a proper $(d-1)$-edge coloring; in particular there is a perfect matching M_1^* in H_1 containing all edges precolored 1. Let e_1'' be the edge of M_1^* that is incident with u_1, and let e_2'' be the corresponding edge of H_2. Then there is a perfect matching M_2^* in H_2 which does not contain the φ-precolored edge of H_2 if it is distinct from e_2''. We now define a perfect matching M^* of Q_d by removing e_1'' and e_2'' from $M_1^* \cup M_2^*$ and adding two edges from M with the same endpoints as e_1'' and e_2''. Since M^* is a perfect matching containing all edges colored 1 under φ and no edges with color 2 under φ, and there is only one edge φ-precolored 2 in Q_d, φ is extendable.

Subcase B. (b) holds.

Suppose now that (b) holds. Then $u_1 \neq v$, because u_1 is incident with an edge colored 1 under φ_1. Suppose first that u_1 is not adjacent to v. Then u_1 and v have a common neighbor x, because H_1 is $(d-1)$-regular and contains exactly φ_1-precolored edges. Moreover, since u_1 and v are at distance 2 and H_1 is a $(d-1)$-dimensional hypercube, u_1 and v have precisely two common neighbors. Now, since H_1 is a $(d-1)$-regular bipartite graph and (b) holds, this means that there are $d-3$ edges of H_1 incident with u_1 that are neither φ_1-precolored nor adjacent to a φ-precolored edge of H_1. Thus if $d \geq 5$, then there is an edge e' incident with u_1 that is not precolored under φ_1, and not adjacent to an edge of H_1 precolored 1 under φ, and, moreover, the analogous statement holds for the corresponding edge of H_2. Now, since

- e_1 and e' are adjacent,
- there is exactly one edge φ-precolored 2 in H_1, and
- H_1 contains at least two φ-precolored edges of color 1 which lie in different dimensional matchings,

it follows that the precoloring obtained from the restriction of φ to H_1 by in addition coloring e' by color 1 is extendable to a $(d-1)$-edge coloring of H_1, and, as above, we obtain an extension of φ by constructing a coloring of H_2 as in the preceding subcase. Suppose now that $d = 4$. Then, since (b) holds, and all φ-precolored edges are in different dimensional matchings, there is a perfect matching M^* in Q_d containing all edges φ-precolored 1, but no edges precolored 2 under φ; thus φ is extendable, because H_1 and H_2 both contain at most one edge precolored 2 under φ.

Now assume that u_1 is adjacent to v. Note that u_1v is not colored 2, because e_1 is incident with u_1 and colored 1 under φ_1, and H_1 is bipartite and $(d-1)$-regular, and contains exactly $d-1$ precolored edges under φ_1. Let φ_1' be the precoloring of H_1 obtained from the restriction of φ to H_1 by coloring u_1v by color 1. Let v_2 be the vertex of H_2 corresponding to v. Note that no edge of H_2 incident with u_2 or v_2 is precolored 1, because in the former case this contradicts u_1u_2 being φ-precolored 1, and in the latter case $\varphi \in \mathcal{C}_2$. Let φ_2' be the precoloring of H_2 obtained from the restriction of φ to H_2 by in addition coloring (possibly recoloring) u_2v_2 by color 1. Then φ_1' and φ_2' are extendable to proper $(d-1)$-edge colorings; in particular for $i = 1, 2$, there is a perfect matching M_i^* in H_i containing all φ_i'-precolored edges with color 1. By removing u_1v and u_2v_2 from $M_1^* \cup M_2^*$ and adding two edges from M instead we get a perfect matching M^* of Q_d that contains all φ-precolored edges with color 1, but no such edges with color 2. Now, since H_1 and H_2 each contains only one edge φ-precolored 2, there is an extension of φ.

Case 2. There is a dimensional matching containing no precolored edge.

Without loss of generality, we assume that no edge of M is precolored.

Case 2.1. No precolored edges are in H_2.

Without loss of generality we assume that there are more colors precolored 1 than 2. Then by Theorem 3.1, the precoloring of H_1 obtained from the restriction of φ to H_1 by removing color 1 from all edges e with $\varphi(e) = 1$, is extendable to a proper edge coloring f of H_1 using colors $2, \ldots, d$. By recoloring all the edges e with $\varphi(e) = 1$ by color 1 we obtain, from f, a d-edge coloring f' of H_1. Moreover, by coloring every edge of H_2 by the color of its corresponding edge in H_1 under f', and then coloring every edge of M with the color in $\{1, \ldots, d\}$ missing at its endpoints, we obtain an extension of φ.

Case 2.2. Both H_1 and H_2 contain at most $d-2$ precolored edges.

By Theorem 3.1, for $i = 1, 2$, there is a $(d-1)$-edge coloring f_i of H_i that is an extension of the restriction of φ to H_i. By taking f_1 and f_2 together and coloring every edge of M by color d, we obtain an extension of φ.

Case 2.3. H_1 contains $d-1$ precolored edges and H_2 contains one precolored edge.

Let e_2 be the precolored edge of H_2, and let e_1 be the edge of H_1 corresponding to e_2. If the restriction of φ to H_1 is extendable to a $(d-1)$-edge coloring of H_1, then it follows, as above, that φ is extendable. So suppose that the restriction of φ to H_1 is not extendable. Then, since only two colors appear in the precoloring φ and $d \geq 4$, we may without loss of generality assume that either

(a) there is a vertex u incident with an edge e' precolored 2, and every edge in H_1 incident with u and distinct from e' is not precolored but adjacent to an edge precolored 1, or
(b) there is a vertex u of H_1 such that no edge incident with u is precolored, but every vertex adjacent to u in H_1 is incident with an edge precolored 1.

Subcase A. (a) holds.

Suppose that (a) holds, and let e' be the edge in H_1 that is precolored 2. We shall consider two different subcases.

Subcase A.1. $\varphi(e_2) = 1$.

If e_1 is incident with u, then the conditions imply that $\varphi \in \mathcal{C}_2$, so we assume that e_1 is not incident with u. If e' is not adjacent to e_1, then we define φ_1 to be the precoloring obtained from the restriction of φ to H_1 by removing color 2 from e'. By Theorem 3.1, φ_1 is extendable to a proper edge coloring f_1 of H_1 using colors $1, 3, \ldots, d$. Let φ_2 be the precoloring of H_2 obtained from the restriction of φ to H_2 by additionally coloring the edge of H_2 corresponding to e' by color $f_1(e')$; by Theorem 3.1, this precoloring is extendable to a proper edge coloring using colors $1, 3, \ldots, d$. Now, by recoloring e' and the corresponding edge of H_2 by color 2 and thereafter coloring every edge of M by the color missing at its endpoints, we obtain an extension of φ.

Let us now consider the case when e_1 is adjacent to e', but not incident to u. Then e_1 is not precolored under φ. If e_1 is not adjacent to any edge precolored 1 in H_1, then we proceed as follows: Let φ_1 be the precoloring of H_1 obtained from the restriction of φ to H_1 by removing color 1 from all edges φ-precolored 1. Then φ_1 is extendable to a proper edge coloring using colors $2, \ldots, d$. By coloring H_2 correspondingly, and thereafter recoloring all edges φ-precolored 1 in H_1 with color 1, recoloring e_1 by color 1, and recoloring H_2 correspondingly, we obtain an extension of φ by coloring every edge of M by the unique color in $\{1, \ldots, d\}$ missing at its endpoints.

Finally, assume that e_1 is adjacent to e', not incident to u, but adjacent to some edge precolored 1 in H_1. From φ we define a new precoloring φ' of Q_d with d precolored edges by removing the color 2 from e' and coloring the edge of M incident with u by color 1. Now, unless $\varphi' \in \mathcal{C}_3$, then by Lemma 4.3, φ' is extendable; in particular there is a perfect matching M^* containing all edges φ-precolored 1 but not the edge φ-precolored 2. Since Q_d contains only one edge φ-precolored 2, this implies that φ is extendable; hence, it suffices to prove that $\varphi' \notin \mathcal{C}_3$. Now, if $\varphi' \in \mathcal{C}_3$, then there is a vertex v that is not incident with any edge φ'-precolored 1, but all neighbors of v are incident with φ'-precolored edges of color 1. Since H_1 contains $d - 2 \geq 2$ edges with color 1 under φ', $v \in V(H_1)$. Moreover, since $\varphi'(e_2) = 1$ and the end x of e_1 that is not an end of e' is incident with an edge that is φ'-precolored, it follows that v must be the common end of e_1 and e'. However, since e' is colored 2 under φ, this implies that $\varphi \in \mathcal{C}_2$, a contradiction.

Subcase A.2. $\varphi(e_2) = 2$.

If $e' = e_1$, then we consider the precoloring φ_1 of H_1 obtained from φ by removing the color from e'. This coloring is, by Theorem 3.1, extendable to a proper edge coloring f_1 of H_1 using colors $1, 3, \ldots, d$. Let f_2 be the corresponding coloring of H_1. An extension of φ can now be obtained by recoloring e_1 and e_2 by color 2, and then coloring every edge of M by the color not appearing at its endpoints.

If e_1 is not adjacent to e' and not precolored 1, then we proceed as in the preceding paragraph, except that we color both e' and e_1, and their corresponding edges in H_2, by color 2 in the final step.

Suppose now that e_1 is adjacent to e'. Then e_1 is not precolored under φ, because H_1 contains exactly $d-1$ precolored edges and (a) holds. Moreover, since $d \geq 4$, there is an edge $e_3 \in E(H_1)$ precolored 1 that is not adjacent to e_1. Define a precoloring φ_1 of H_1 from φ by removing color 1 from e_3 and recoloring all other edges of H_1 precolored 1 under φ by color 3. By Theorem 3.1, the precoloring φ_1 is extendable to a proper edge coloring f_1 of H_1 using colors 2, 3, ..., d. Now, define a precoloring φ_2 of H_2 from the restriction of φ to H_2 by for every edge e in H_1 precolored 1 under φ, coloring the corresponding edge of H_2 by $f_1(e)$. The precoloring φ_2 does not satisfy any of the conditions (C1) to (C4) (with $d-1$ in place of d), because it is not monochromatic, and all precolored edges have one end which is at distance 1 from the vertex of H_2 corresponding to u. Hence, by the induction hypothesis, the coloring φ_2 is extendable to a proper edge coloring f_2 of H_2 using colors 2, 3, ..., d. From f_1 and f_2 we define an extension of φ by recoloring any edge of H_1 that is φ-precolored 1 by color 1, recoloring every edge of H_2 corresponding to such an edge by color 1, and thereafter coloring every edge of M by the unique color not appearing at its endpoints.

Finally, let us consider the case when e_1 is precolored 1 under φ. Let φ_1 be the restriction of φ to H_1. By Lemma 4.2, there is a partial proper edge coloring f_1 of H_1 satisfying the conditions (i)–(iii) of Lemma 4.2. Let E' be the set of edges colored under f_1. The graph $H_1 - E'$ has maximum degree $d-2$ so the coloring f_1 can be extended to a proper d-edge coloring f_1' of H_1 by using König's edge coloring theorem. Let f_2' be the corresponding coloring of H_2, except that we interchange colors on the $(1, 2)$-colored cycle containing e_2. Note that for every vertex x of H_1, the same colors appear at x under f_1' and at the corresponding vertex of H_2 under f_2'. Moreover, f_1' and f_2' agrees with φ. Hence, φ is extendable.

Subcase B. (b) holds.

Recall that if (b) holds, then there is a vertex u of H_1 such that no edge incident with u is φ-precolored, but every vertex adjacent to u in H_1 is incident with an edge precolored 1 under φ. Recall that e_2 is the unique edge of H_2 that is precolored, and e_1 is the corresponding edge of H_1. Since two colors appear in φ, $\varphi(e_2) = 2$. If e_1 is not precolored, then let f_2 be an extension of the restriction of φ to H_2 using colors 2, ..., d; such an extension exists by Theorem 3.1. Let f_1 be the corresponding edge coloring of H_1. From f_1 and f_2 we obtain an extension of φ by recoloring all edges precolored 1 under φ by color 1, recoloring all corresponding edges of H_2 by color 1, and thereafter coloring every edge of M by the unique color in $\{1, ..., d\}$ not appearing at its endpoints.

Suppose now that e_1 is precolored under φ; then $\varphi(e_1) = 1$. Since H_1 contains at least three φ-precolored edges, there are at most two vertices v_1 and v_2 of H_1 which are at distance 1 from $d-1$ vertices all of which are incident with edges precolored 1 (because otherwise two vertices of distance 2 lie in at least two distinct 4-cycles, which is not possible since H_1 is a $(d-1)$-dimensional hypercube). Now, since $d - 1 \geq 3$, there is an edge e' in H_2 that is adjacent to e_2, and satisfies that the corresponding edge of H_1 is not incident with v_1 or v_2. This implies that the precoloring φ' obtained from φ by coloring e' by color 1 and removing color 2 from e_2 is not in $\mathcal{C}_3$; so by Lemma 4.3, φ' is extendable to a proper d-edge coloring f. Now, $f(e') = 1$; so $f(e_2) \neq 1$, and since e_2 is the only edge colored 2 under φ, we obtain an extension of φ by permuting colors in f. $\square$

Lemma 4.7. *If at least three and at most $d-1$ colors appear on edges under φ, and $\varphi \notin \mathcal{C}$, then φ is extendable.*

Proof. Without loss of generality we shall assume that colors 1, 2, and 3 appear on edges under φ, and that color d does not appear under φ.

Case 1. Every dimensional matching contains a precolored edge.

Without loss of generality, we assume that M contains an edge e_M precolored 1 under φ, and first consider the case when all other precolored edges are in H_1.

Case 1.1. No precolored edges are in H_2.

Suppose first that color 1 does not appear in H_1. If the restriction of φ to H_1 is extendable to a $(d-1)$-edge coloring of H_1, then we may choose such an extension with colors $2, \ldots, d$, and thus φ is extendable. If, on the other hand, the restriction of φ to H_1 is not extendable, then, since at most $d-2$ different colors appear in H_1, φ satisfies (C2) or (C3) (with $d-1$ in place of d). Hence, there is a vertex u such that all edges in H_1 incident with u are either precolored, or non-precolored and adjacent to an edge of a fixed color, say 2. Note that this implies that at least two edges in H_2 are precolored 2. If u is an endpoint of e_M, then $\varphi \in \mathcal{C}$; otherwise, assuming $d > 4$, there is either some edge e' adjacent to e_M that is not colored under φ and not adjacent to any edge precolored 2 under φ, or an edge e' adjacent to e_M and colored 2. By removing the colors from all edges precolored 2 under φ and coloring e' by color 1, we obtain, from the restriction of φ to H_1, a precoloring that is extendable to a $(d-1)$-edge coloring of H_1, because at least two edges in H_1 are colored 2 under φ. Let f_1 be an extension of this precoloring using colors $1, 3, \ldots, d$. Now, by recoloring e' by color 2, and also recoloring all (other) edges precolored 2 under φ with color 2, we obtain a proper d-edge coloring of H_1. By coloring H_2 correspondingly and then coloring every edge of M with the color missing at its endpoints, we obtain an extension of φ.

It remains to consider the case when $d = 4$. However, it is easy to see that if $d = 4$ (and thus H_1 is isomorphic to Q_3) there cannot be a vertex u as described above and such that all precolored edges lie in different dimensional matchings.

Suppose now that color 1 appears in H_1 under φ. By removing the color from all edges precolored 1 under φ from the restriction of φ to H_1, we obtain an edge precoloring of H_1 that is extendable to a proper $(d-1)$-edge coloring of H_1. Let f_1 be an extension of this precoloring using colors $2, \ldots, d$. By recoloring all edges precolored 1 under φ by color 1, we obtain an extension of φ as above.

Case 1.2. Both H_1 and H_2 contain at most $d-3$ precolored edges.

If there is an edge e_1 in H_1 adjacent to e_M, and such that neither e_1 nor the corresponding edge e_2 of H_2 is colored under φ, and neither of e_1 and e_2 is adjacent to an edge precolored 1 under φ, then we consider the precolorings of H_1 and H_2 obtained from the restriction of φ to H_i along with coloring e_1 and e_2 by color 1. By the induction hypothesis, these colorings are extendable to $(d-1)$-edge colorings f_1 and f_2, respectively. Now, by recoloring e_1 and e_2 by color d and then coloring every edge of M by the color missing at its endpoints we obtain the required extension of φ.

Now suppose that there are no edges e_1 and e_2 as described in the preceding paragraph. Then any edge precolored by a color distinct from 1 under φ is adjacent to e_M, and any edge colored 1 under φ is adjacent to an edge e' that is adjacent to e_M.

Let J_1 be the subgraph of Q_d induced by all dimensional matchings containing edges precolored 1 or 2, and let $J_2 = Q_d - E(J_1)$. Suppose that J_1 has maximum degree q. Note that no component T of J_1 has the property that the restriction of φ to T satisfies condition (C2) (with q in place of d), because an edge precolored 2 is adjacent to an edge precolored 1. Moreover, no component T of J_2, with the restriction of φ to T, satisfies any of the conditions (C1) to (C4) (with $d - q$ in place of d), because if all precolored edges of J_2 are in T, then they are all incident to the same endpoint of e_M. Thus, by the induction hypothesis, the restriction φ_1 of φ to J_1 is extendable to a proper q-edge coloring, and the restriction φ_2 of φ to J_2 is extendable to a proper edge coloring with $d - q$ colors. Moreover, since φ_1 and φ_2 use distinct sets of colors, we may use distinct colors for the extensions of J_1 and J_2, respectively; thus we conclude that φ is extendable.

Case 1.3. H_1 contains $d - 2$ precolored edges and H_2 contains one precolored edge.

If for every edge e_1 in H_1 adjacent to e_M, either e_1 or the corresponding edge e_2 of H_2 is precolored under φ, or one of e_1 and e_2 is adjacent to an edge colored 1 distinct from e_M, then we proceed exactly as in the preceding case and construct an extension of φ by defining subgraphs J_1 and J_2 as above.

Thus we may assume that there is an edge $e_1 \in E(H_1)$ such that neither e_1 nor its corresponding edge e_2 in H_2 is precolored or adjacent to an edge colored 1 in H_1 and H_2, respectively. If the precoloring φ_1 obtained from the restriction of φ to H_1 by in addition coloring e_1 by color 1 is extendable to a $(d - 1)$-edge coloring of H_1, then we can obtain an extension of φ as follows: By Theorem 3.1, there is a similar extension of H_2 of the restriction of φ to H_2 along with coloring e_2 by 1. By recoloring e_1 and e_2 by color d, it is easy to see that there is an extension of φ. Thus we assume that φ_1 is not extendable.

Let $e_M = u_1u_2$, and suppose first that there is only one edge precolored 1 under φ_1. If the φ-precolored edge of H_2 is not colored 1, then by the induction hypothesis, the restriction of φ to H_i is extendable ($i = 1, 2$), to proper edge colorings using colors 2, ..., d; thus, φ is extendable. Hence, we may assume that color 1 appears in H_2 under φ. Note that this implies that the precolored edge e_2' of H_2 is not adjacent to e_M. Moreover, the corresponding edge e_1' of H_1 is not φ-precolored, since all precolored edges lie in different dimensional matchings. Now, since the restriction of φ to H_1 consists of $d - 2$ precolored edges with colors distinct from 1, Theorem 3.1 yields that there is an extension of H_1 using colors 2, ..., d. We color H_2 correspondingly. Since e_M and e_2' are not adjacent, we now obtain an extension of φ by recoloring e_1' and e_2' by color 1, and thereafter coloring all edges of M by the color in $\{1, ..., d\}$ missing at its endpoints.

Suppose now that color 1 appears on several edges in H_1 under φ_1. Note that since at least three colors, and at most $d - 1$ colors, are used by φ, this implies that $d \geq 5$. Since φ_1 is not extendable and thus satisfies one of the conditions (C1) to (C4), and color 1 appears on several edges under φ_1, there is some vertex $v \in V(H_1)$ such that every edge incident with v is φ_1-precolored or adjacent to an edge precolored with 1.

Since u_1 is incident with an edge precolored 1 under φ_1, $u_1 \neq v$. If u_1 is not adjacent to v, then since $d \geq 5$, and any two adjacent edges in H_1 are contained in exactly one 4-cycle, there is some edge e_1' in H_1 adjacent to e_M such that neither e_1' nor its corresponding edge e_2' in H_2 is precolored or adjacent to any edge precolored 1 under φ. Let us prove that the precoloring φ_1'' obtained from the restriction of φ to H_1 by in addition coloring e_1' by color 1 is extendable to a $(d-1)$-edge coloring of H_1. Indeed, if H_1 contains only one edge that is φ-precolored 1, then v is incident with edges of at least two distinct colors, so φ_1'' does not satisfy (C2) (with d in place of $d-1$); if H_1 contains at least two edges that are φ-precolored 1, then since u_1 and v have exactly two common neighbors and any two φ-precolored edges lie in distinct dimensional matchings, φ_1'' does not satisfy (C2). Furthermore, the precoloring of H_2 obtained from the restriction of φ to H_2 by also coloring e_2' by color 1 is extendable. By recoloring e_1' and e_2' by color d we obtain an extension of φ as before.

If, on the other hand, u_1 is adjacent to v, then we may color u_1v and proceed as above unless the edge e_2' of H_2 corresponding to u_1v is precolored or adjacent to an edge precolored 1. If the latter holds, then $\varphi \in \mathcal{C}$. On the other hand, if e_2' is φ-precolored, then let M' be a dimensional matching in Q_d containing a φ-precolored edge incident with v and colored by a color c that only occurs once under φ; such an edge exist since at least three colors are used in φ. Then both components of $Q_d - M'$ satisfy that the restriction of φ to this component is not in $\mathcal{C}$; thus, by the induction hypothesis, the restriction of φ to $Q_d - M'$ is extendable to a proper edge coloring of $Q_d - M'$ using colors in $\{1, \ldots, d\} \setminus \{c\}$. We conclude that φ is extendable.

Case 2. There is a dimensional matching containing no precolored edge.

Without loss of generality we assume that no edge of M is precolored.

The case when all precolored edges are in H_1, and the case when H_1 and H_2 both contain at most $d-2$ precolored edges can be dealt with exactly as in Case 2 of the proof of Lemma 4.6. Hence, we assume that H_1 contains exactly $d-1$ precolored edges. We shall assume that e_2 is the precolored edge of H_2, e_1 is the edge of H_1 corresponding to e_2, and that there is no edge colored d under φ.

If the restriction of φ to H_1 is extendable to a $(d-1)$-edge coloring of H_1, then since the same holds for the restriction of φ to H_2, φ is extendable to a d-edge coloring of Q_d; so assume that the restriction of φ to H_1 is not extendable. Since at least three distinct colors appear under φ, this implies that

(a) $d = 4$, and there is a dimensional matching in H_1 with three edges with three different colors; or
(b) there is an edge uv of H_1 that is not precolored, but uv is adjacent to an edge colored i, for $i = 1, \ldots, d-1$; or
(c) there is a vertex u incident to k precolored edges and every edge incident with u in H_1, which is not precolored, is adjacent to an edge precolored by some fixed color c_1.

Subcase A. (a) holds.

Without loss of generality we assume that $\varphi(e_2) = 1$. If e_1 is adjacent both to the edge precolored 2 and to the edge precolored 3, then it is straightforward that φ is extendable (because all precolored edges of H_1 lie in the same dimensional matching). Otherwise, either the edge colored 2 or the edge colored 3 is not adjacent to e_1, suppose, for example, that this holds for the edge e_1' colored 2. The precoloring obtained from the restriction of φ to H_1 by removing color 2 from e_1' is extendable to a proper edge edge coloring f_1 using colors 1, 3, 4, and the precoloring obtained from the restriction of φ to H_2 by in addition coloring the edge e_2', corresponding to e_1', by the color $f_1(e_1')$ is extendable to a proper edge coloring f_2 using colors 1, 3, 4. Now, by recoloring e_1' and e_2' by color 2, and thereafter coloring all edges of M by the color missing at its endpoints, we obtain an extension of φ.

Subcase B. (b) holds.

Without loss of generality, we assume that $\varphi(e_2) = 1$. If e_1 is not precolored and not adjacent to the edge e_1' in H_1 precolored 1, then we construct an extension of φ in the following way: remove color 1 from all edges colored 1 under φ. The resulting precoloring of H_1 is, by Theorem 3.1, extendable to a proper edge coloring using colors 2, ..., d. By coloring H_2 correspondingly, then recoloring e_2, e_1' and their corresponding edges in H_1 and H_2, respectively, by color 1, and thereafter coloring every edge of M by the color missing at its endpoints, we obtain an extension of φ.

Suppose now that e_1 is precolored or adjacent to e_1'. Let us first assume that there is some precolored edge e_1'' of H_1 that is not adjacent to e_1 and not colored 1. Suppose, for instance, that $\varphi(e_1'') = 2$. By removing the color 2 from e_1'', we obtain a precoloring of H_1 that is extendable to a proper edge coloring f_1 using colors 1, 3, 4, ..., d. Moreover, the precoloring of H_2 obtained from the restriction of φ to H_2 by additionally coloring the edge e_2'', corresponding to e_1'', by the color $f_1(e_1'')$ is extendable to a proper edge coloring f_2 using colors 1, 3, 4, ..., d. By recoloring e_1'' and e_2'' by color 2, and thereafter coloring every edge of M by the color missing at its endpoints, we obtain an extension of φ.

Suppose now that all φ-precolored edges with colors distinct from 1 are adjacent to e_1. If e_1 is precolored, then φ satisfies (C2) (with $d - 1$ in place of d), so we may assume that e_1 is not precolored; then $e_1 = uv$. Moreover if u or v is incident with only one precolored edge that is colored 1, then φ satisfies (C2), so we assume that either u or v is incident with two edges precolored 1 and 2, respectively, under φ. Now, by removing color 2 from the edge e' φ-colored 2, we obtain a precoloring that is extendable to a proper edge coloring f_1 using colors 1, 3, ..., d. Moreover, the precoloring obtained from the restriction of φ to H_2 by in addition coloring the edge corresponding to e' by the color $f_1(e')$ is extendable to a proper edge coloring f_2 using colors 1, 3, ..., d. From f_1 and f_2 we obtain an extension of φ by recoloring e' and its corresponding copy in H_2 by color 2, and thereafter coloring every edge of M by the color missing at its endpoints.

Subcase C. (c) holds.

Let us first assume that at least three colors appear in the restriction of φ to H_1. If e_1 is not incident with u, then there is an edge $e' \neq e_1$ in H_1, such that $\varphi(e') = c$, e' is not adjacent to e_1 and e' is the only edge in H_1 with color c under φ. Suppose that $\varphi(e_2) \neq c$. Then by removing the color c from the restriction of φ to H_1, we obtain a precoloring φ_1 that is extendable to a proper edge coloring f_1 of H_1 using colors $\{1, ..., d\} \setminus \{c\}$. (Note that $f_1(e') = c_1$.) Moreover, there is a similar extension f_2 of the restriction of φ to H_2, where the edge of H_2 corresponding to e' is colored $f_1(e')$. Now, by recoloring e' and its corresponding copy in H_2 by color c, we obtain an extension of φ as before.

If $\varphi(e_2) = c$ and e_1 is not precolored under φ, then we proceed similarly as in the preceding paragraph, except that after constructing the coloring f_1 of H_1 as in the preceding paragraph, we define f_2 as the coloring of H_2 corresponding to f_1 and then color e' and e_1, and the corresponding edges of H_2, by color c. On the other hand, if $\varphi(e_2) = c$ and e_1 is precolored under φ, then $\varphi(e_1) = c_1$ because e_1 is not incident with u; now, since at least three distinct colors are used by φ on edges in H_1, we may clearly choose another φ-precolored edge incident with u as our edge e', and then proceed as in the preceding paragraph.

Now assume that e_1 is incident with u. If $\varphi(e_2) = c_1$, then $\varphi \in \mathcal{C}$, so we assume that $\varphi(e_2) \neq c_1$. If there is a color $c \neq \varphi(e_2)$ appearing on precisely one edge $e' \neq e_1$ of H_1, then we consider the restriction of φ to H_1 where color c is removed, and proceed as before; otherwise, since at least three colors appear in H_1 under φ, it follows that e_1 is not adjacent to any edge precolored c_1 under φ. Thus by removing color c_1 from any edge in H_1 precolored by color c_1 under φ, we obtain a precoloring that is extendable to a proper edge coloring of H_1 using colors $\{1, ..., d\} \setminus \{c_1\}$. Moreover, there is a similar extension f_2 of the restriction of φ to H_2, where for any edge e'_2 corresponding to an edge e_1' of H_1 with $\varphi(e_1') = c_1$, we have $f_2(e_2') = f_1(e_1')$. From f_1 and f_2 we may construct an extension of φ by recoloring any such pair of edges by color c_1. Let us now consider the case when only two colors appear in the restriction of φ to H_1. Since at least three colors appear on edges under φ, it follows that $\varphi(e_2)$ does not appear in H_1 under φ. Without loss of generality we assume that $\varphi(e_2) = 2$, color 3 appears on exactly one edge e' in H_1, and color 1 is the third color used by φ. If $e' \neq e_1$, then we consider the precoloring of H_1 obtained from the restriction of φ to H_1 by removing color 3. There is an extension of this precoloring of H_1 using colors $\{1, ..., d\} \setminus \{3\}$ such that $\varphi(e') = 1$. Let e_2' be the edge of H_2 corresponding to e'. Then the precoloring obtained from the restriction of φ to H_2 by additionally coloring e_2' by color 1 is extendable to a coloring using colors $\{1, ..., d\} \setminus \{3\}$. Now, by recoloring e' and e_2' by color 3, we can construct an extension of φ.

If, on the other hand, $e' = e_1$, then e_1 is not adjacent to any edge colored 1. Let E' be the set of edges colored 1 under φ. If $d > 4$, then $|E'| \geq 3$ and we recolor all edges in E' by colors 2, 3, 4 so that at least one edge is colored i, $i = 2, 3, 4$. This yields a precoloring φ_1 that, by the induction hypothesis, is extendable to a proper edge coloring f_1 of H_1 using colors 2, ..., d, because the precolored edges form a matching which is colored by at least three distinct colors. Next, consider the precoloring φ_2 of H_2 obtained from the restriction of φ to H_2 by setting $\varphi_2(e_2') = f_1(e_1')$ for any edge $e_2' \in E(H_2)$ corresponding to an edge $e_1' \in E'$. The φ_2-precolored edges form a matching consisting of $d - 1$ edges, where edges corresponding to E' are colored by at least three distinct colors, so by the induction hypothesis, there is an extension f_2 of φ_2, where $f_2(e_2') = f_1(e_1')$ for any edge $e_2' \in E(H_2)$ corresponding to an edge $e_1' \in E'$. We may now obtain an extension of φ as before.

It remains to consider the case $d = 4$. By symmetry of the hypercube, it suffices to consider the two cases when all edges in H_1 are in the same dimensional matching, and the case when the two edges precolored 1 are in different dimensional matchings, one of which is necessarily the same as the dimensional matching containing e'. In both cases it is a straightforward exercise to check that there is an extension of φ where two edges in H_1, and their corresponding edges in H_2, are the only edges colored 4; and, moreover, these two edges of H_1 (H_2) lie in a dimensional matching with no precolored edges. □

Lemma 4.8. *If all colors* 1, ..., d *appear under* φ, *and* $\varphi \notin \mathcal{C}$, *then* φ *is extendable.*

Proof. Since all colors are present under φ, every color appears on precisely one edge. Let us first note that if every dimensional matching contains at most one precolored edge, then trivially φ is extendable. Thus, for the rest of the proof we assume that there is a dimensional matching M that does not contain any precolored edge. Let H_1 and H_2 be the components of $Q_d - M$.

Case 1. No precolored edges are in H_2.

If there is some edge e that is not precolored, and adjacent to all precolored edges in H_1, then $\varphi \in \mathcal{C}$. On the other hand, if there is a precolored edge e such that removing the color from e yields a precoloring φ_1 of H_1 that is not in $\mathcal{C}_1$ (with $d-1$ in place of d) or $\mathcal{C}_4$, then the induction hypothesis yields that there is an extension f of φ_1 using all colors except the removed one. Suppose, for example, that the color from e under φ was removed in φ_1; then by recoloring e with $\varphi(e)$ and retaining the color of every other edge in H_1 under f, we obtain a proper d-edge coloring of H_1 that is an extension of φ; by coloring H_2 correspondingly and then coloring every edge of M by the color missing at its endpoints, we obtain an extension of φ.

Now, suppose that e is a precolored edge of H_1, and removing the color of e yields a coloring φ_1 that satisfies (C1). Let e' be another φ-precolored edge of H_1 that is adjacent to a minimum number of other φ-precolored edges of H_1. Then the precoloring obtained from φ by removing the color from e' does not satisfy (C4); suppose that it satisfies (C1). Then either $\varphi \in \mathcal{C}_1$, or there are non-precolored edges $uv, ux \in E(H_1)$ satisfying that e' is incident with v, e is incident with x, and all other precolored edges are incident with u. Now, since u is incident with at least two precolored edges (from different dimensional matchings), by instead removing the color on a precolored edge incident with u, we obtain a precoloring that does not satisfy (C1) or (C4). We conclude that if $\varphi_1 \in \mathcal{C}_1$, then either φ is extendable or φ satisfies (C1).

It remains to consider the case when φ_1 satisfies (C4). Suppose, consequently, that $d = 4$ and that removing the color from any precolored edge of H_1 yields a precoloring that satisfies (C4); then the precolored edges of H_1 lie in a dimensional matching M'. It is easily seen that since all precolored edges lie in M', there is a proper 4-edge coloring of H_1 which agrees with φ. By coloring H_2 correspondingly and thereafter coloring all edges of M by the color in $\{1, 2, 3, 4\}$ missing at its endpoints, we obtain an extension of φ.

Case 2. Both H_1 and H_2 contain at most $d-3$ precolored edges.

Note that neither the restriction of φ to H_1 nor to H_2 satisfies any of the conditions (C1) to (C4) (with $d-1$ in place of d). Moreover, since H_1 and H_2 contain altogether d precolored edges, $d \geq 6$, and thus both H_1 and H_2 contain at least three precolored edges. We consider two different cases.

Case 2.1. There is some edge e in H_1 (or H_2) that is precolored and the corresponding edge of H_2 (H_1) is not precolored.

Without loss of generality we assume that e_1 is such an edge in H_1, $\varphi(e_1) = 1$, and that e_2 is the edge of H_2 corresponding to e_1. Since both H_1 and H_2 contain precolored edges, there is some color which appears in H_2 but not in H_1. Suppose first that some precolored edge of H_2 is not adjacent to e_2. Assume without loss of generality that such an edge is precolored d in H_2. Then we construct a new precoloring φ' from φ by coloring e_2 by color d, and recoloring e_1 by color d. The restrictions of φ' to both H_1 and H_2 are, by the induction hypothesis, extendable to proper edge colorings using colors $2, \ldots, d$, respectively. Now by recoloring e_1 and e_2 by color 1 we obtain proper edge colorings f_1 and f_2 of H_1 and H_2, respectively, satisfying that the color in $\{1, \ldots, d\}$ not appearing at a vertex v of H_1 is also missing at the corresponding vertex of H_2. Since for $i = 1, 2, f_i$ is an extension of the restriction of φ to H_i, φ is extendable.

Suppose now instead that every precolored edge of H_2 is adjacent to e_2. In fact, we may assume that if $e \in E(H_1)$, e is precolored under φ and the corresponding edge e' of H_2 is not precolored under φ, then e' is adjacent to all precolored edges of H_2; otherwise we proceed as in the preceding paragraph. If all precolored edges of H_2 are incident with a common vertex, then since there are at least three precolored edges in H_i, $i = 1, 2$, this means that all precolored edges of H_1 are incident with the corresponding vertex of H_1; and so, $\varphi \in \mathcal{C}_1$. Assume now that $e_i = u_i v_i$, $i = 1, 2$, and that both u_2 and v_2 are adjacent to precolored edges; since H_2 contains at least three precolored edges, e_2 is the unique edge with this property. Moreover, since any precolored edge of H_1 satisfies that if the corresponding edge of H_2 is not precolored, then it is adjacent to all precolored edges of H_2, it follows that any precolored edge in H_1 is incident with u_1 or v_1. This means that the dimensional matching M_1 in Q_d containing e_1, contains no other precolored edge. Hence, since both u_2 and v_2 are incident with precolored edges, both components of $Q_d - M_1$ contain at most $d - 2$ precolored edges using colors $2, \ldots, d$. Thus, by Theorem 3.1, the restriction of φ to $Q_d - M_1$ is extendable to a proper edge coloring of $Q_d - M_1$ using colors $2, \ldots, d$. By coloring all edges of M_1 by color 1, we obtain an extension of φ.

Case 2.2. Each precolored edge of H_1 corresponds to a precolored edge of H_2 and vice versa.

The conditions imply that H_i contains exactly $d/2$ precolored edges.

Suppose first that $d = 6$, and let u_1u_2 be a precolored edge of H_1, and v_1v_2 be the corresponding edge of H_2. Now, since H_1 contains two additional precolored edges which both correspond to precolored edges of H_2, and u_1u_2 is in four 4-cycles in H_1, there is a 4-cycle $u_1u_2u_3u_4u_1$ in H_1 such that u_3u_4 is not precolored and the dimensional matching M_2, containing u_2u_3 and u_4u_1, does not contain any precolored edge. Let H'_1 and H'_2 be the components of $Q_d - M_2$. Now, since all precolored edges lie on 4-cycles whose non-precolored edges are in M, either both or none of the precolored edges of such a cycle is in H_i'. Hence, H_i' contains an even number of precolored edges, and so, we may proceed as in Case 1 or Case 3 of the proof of the lemma.

Now assume that $d \geq 8$. If all precolored edges in H_1 are incident with one common vertex, then $\varphi \in \mathcal{C}$, so we assume that this is not the case; thus, there are two precolored edges in H_1 (and thus H_2) that are not adjacent. In H_2 we assume that these edges are colored $d/2 + 1$ and d, respectively. Let v_1v_2 be the edge precolored d in H_2 and let u_1u_2 be the corresponding edge of H_1. Without loss of generality, we assume that $\varphi(u_1u_2) = d/2$. Now, since there are exactly $d/2$ precolored edges in both H_1 and H_2, $d \geq 8$, and each edge in H_i is in $d - 2$ 4-cycles in H_i, there are 4-cycles $u_1u_2u_3u_4u_1$ and $v_1v_2v_3v_4v_1$ in H_1 and H_2, respectively, such that

- u_1u_2 and v_1v_2 are the only precolored edges of these 4-cycles,
- v_3v_4 is not adjacent to an edge precolored $d/2 + 1$.

We construct a precoloring φ_1 of H_1 from the restriction of φ to H_1 by in addition coloring u_1u_4 and u_2u_3 by color $d/2 + 1$ and by coloring u_3u_4 by color $d/2$. Similarly, we define a precoloring φ_2 of H_2 from the restriction of φ to H_2 by recoloring v_1v_2 by $d/2 + 1$, and in addition coloring v_3v_4 by $d/2 + 1$, and v_2v_3 and v_1v_4 by color $d/2$. Note that the obtained precolorings are proper. Now, since $d/2 + 3 \leq d - 1$ (because $d \geq 8$) and none of φ_1 and φ_2 satisfies any of the conditions (C1) to (C4) (with $d - 1$ in place of d), it follows from Theorem 3.1 and the induction hypothesis that for $i = 1, 2$, there is a proper edge coloring f_i of H_i using colors $1, \ldots, d - 1$ that is an extension of φ_i. Now by recoloring all the edges u_2u_3, u_1u_4, v_1v_2, v_3v_4 by color d we obtain two proper edge colorings such that by coloring every edge of M by the color in $\{1, \ldots, d\}$ missing at its endpoints, we obtain an extension of φ.

Case 3. H_1 contains $d - 2$ precolored edges and H_2 contains 2 precolored edges.

We consider two different subcases.

Case 3.1. No precolored edge of H_1 satisfies that the corresponding edge of H_2 is non-precolored.

The conditions imply that $d = 4$. Without loss of generality we assume that H_1 contains two edges e_1 and e_1' precolored 1 and 2, respectively. Let e_2 and e_2' be the corresponding edges of H_2. By symmetry, it suffices to consider the following different cases:

(a) e_1 and e_1' are adjacent;
(b) e_1 and e_1' are not adjacent but lie on a common 4-cycle;
(c) e_1 and e_1' are not adjacent and do not lie on a common 4-cycle.

If (a) holds, then $\varphi \in \mathcal{C}_4$. Suppose now that (b) holds. It suffices to prove that there are perfect matchings M_1 and M_2 in Q_d, where M_i contains all edges precolored i and no other precolored edges, and where M_1 and M_2 satisfy that the precolored edges of $Q_d - M_1 \cup M_2$ lie in different components. We construct M_1 in the following way: include e_1 and the unique non-precolored edge e_3 of H_1 that is in the same dimensional matching as e_1 and contained in a 4-cycle with e_1; from H_2 we select the two edges corresponding to the two opposite non-precolored edges of the 4-cycle containing e_1 and e_3; for the remaining edges of M_1 we choose four edges from M that are adjacent to none of the edges e_1 and e_3. We now define M_2 to consist of the edges from the unique perfect matching in $H_1 - M_1$ containing e_1' and of the edges from a perfect matching of H_2 with no precolored edges.

Suppose now that (c) holds. By symmetry, it suffices to consider the two cases when e_1 and e_1' are in the same dimensional matching and when they are not. If the former holds, then we define the matchings M_1 and M_2 exactly as in the preceding paragraph, and it follows that φ is extendable. If e_1 and e_1' are in different dimensional matchings, then we select M_1 as the union of the dimensional matching of H_1 containing e_1 and the unique dimensional matching of H_2 with no precolored edge. As before, we can then choose a perfect matching M_2 containing e_1' and no other precolored edges; the details are omitted.

Case 3.2. There is a precolored edge $e_1 = u_1v_1$ in H_1 such that the corresponding edge of H_2 is not precolored.

Let $e_2 = u_2v_2$ be the edge in H_2 corresponding to e_1. If some precolored edge of H_2 is not adjacent to e_2, then we may proceed as above: Assume without loss of generality that such an edge is precolored d in H_2, and that $\varphi(e_1) = 1$. Then we construct a new precoloring φ' from φ by coloring e_2 by color d, and recoloring e_1 by color d. H_1 contains $d - 2$ φ'-precolored edges, so the restriction of φ' to H_1 is extendable by Theorem 3.1. H_2 contains three φ'-precolored edges, so it is extendable unless $d = 4$ and the restriction φ_2 of φ' satisfies (C2) (with $d - 1$ in place of d). Assuming $d > 4$, we can choose these extensions so that they use colors 2, ..., d, respectively, and we obtain an extension of φ by recoloring e_1 and e_2 by color 1, and thereafter coloring the edges of M. If $d = 4$, and φ_2 satisfies (C2), then e_2 and the two φ-precolored edges of H_2 form a matching, and none of the φ-precolored edges in H_2 is adjacent to e_2. It follows that for at least one of these two precolored edges, the corresponding edge in H_1 is not precolored; denote this edge by e_2' and assume $\varphi(e_2') = 4$. Now, by Theorem 3.1, the restriction of φ to H_1 is extendable to a proper edge coloring f_1 using colors $1, 2, 3$. Moreover, the precoloring of H_2 obtained from the restriction of φ by recoloring e_2' by the color of the corresponding edge e_1' of H_1 under f_1 is, by Theorem 3.1, extendable to a proper edge coloring f_2 using colors $1, 2, 3$. By recoloring e_1' and e_2' by color 4, and thereafter coloring the edges of M, we obtain an extension of φ.

Let us now assume that both precolored edges of H_2 are adjacent to e_2. In fact, we may assume that every precolored edge in H_1 either corresponds to a precolored edge of H_2 or is adjacent to both precolored edges of H_2. Now, if both precolored edges of H_2 are incident with a common vertex v, then this implies that $\varphi \in \mathcal{C}$; so assume that u_2 is incident with one precolored edge and that v_2 is incident with one precolored edge. Clearly, this implies that at most four edges are precolored in H_1, and thus $d \leq 6$.

So let us assume that $d \leq 6$ and that $e_1 = u_1v_1$ is precolored 1. If the dimensional matching M_1 containing e_1 contains no other precolored edges, then the restriction φ' of φ to $Q_d - M_1$ is a precoloring of $d - 1$ edges using $d - 1$ colors. Furthermore, both components of $Q_d - M_1$ contain at most $d - 2$ precolored edges, so by Theorem 3.1, φ' is extendable to to a proper edge coloring of $Q_d - M_1$ using colors 2, ..., d. By coloring all edges of M_1 by color 1, we obtain an extension of φ.

If M_1 contains more than one precolored edge, then it contains exactly two precolored edges, u_1v_1 and z_1t_1, where $u_1v_1z_1t_1u_1$ is a 4-cycle in H_1. Let $u_2v_2z_2t_2u_2$ be the corresponding 4-cycle of H_2, where u_2t_2 and v_2z_2 are precolored. We define H_3 to be the three-dimensional hypercube containing vertices $u_1, v_1, z_1, t_1, u_2, v_2, z_2, t_2$; then all precolored edges of Q_d lie in H_3. Since $d \geq 4$, there is a dimensional matching M_3 in Q_d which does not contain any edge from H_3. It follows that if H_1' and H_2' are the components of $Q_d - M_3$, then either H_1' or H_2' contains all precolored edges of Q_d; thus we may proceed as in Case 1 above when H_1 contains exactly d precolored edges.

Case 4. H_1 contains $d - 1$ precolored edges and H_2 contains 1 precolored edge.

Without loss of generality we assume that the edge in H_2 is precolored d. We first consider the case when the restriction of φ to H_1 is extendable (as a precoloring of Q_{d-1}). Suppose first that there is some precolored edge e_1 in H_1 such that the corresponding edge of H_2 is not precolored or adjacent to the precolored edge of H_2. Without loss of generality we assume that $\varphi(e_1) = 1$. We define a new precoloring φ' from φ by recoloring e_1 by color d and by coloring e_2 by color d; this precoloring is proper, and, moreover, for $i = 1, 2$, the restriction of φ' to H_i is extendable to a proper edge coloring f_i using colors $2, \ldots, d$. By recoloring e_1 and e_2 by color 1 and coloring every edge of M by the color in $\{1, \ldots, d\}$ that is missing at its endpoints, we obtain an extension of φ.

Suppose now that every precolored edge of H_1 either corresponds to a precolored edge of H_2, or that the corresponding edge of H_2 is adjacent to a precolored edge of H_2. Since the restriction of φ to H_1 is extendable, it follows that if $e_1 = u_1v_1$ is the edge of H_1 corresponding to the precolored edge $e_2 = u_2v_2$ of H_2, then e_1 is precolored under H_1. Moreover, since $\varphi \notin \mathcal{C}$, there are at least two precolored edges of H_1 incident with u_1 and similarly for v_1. Suppose, for example, that $\varphi(e_1) = 1$ and that color 2 does appear at v_1 under φ, but not at u_1, and that color 3 appears at u_1. We define a new precoloring φ' of Q_d by recoloring the edge with color 3 under φ by color 2, and by coloring the corresponding edge of H_2 by color 2. Then, by Theorem 3.1, the restriction of φ' to H_2 is extendable to a proper edge coloring using colors $1, 2, 4, \ldots, d$, and the restriction of φ' to H_1 does not satisfy (C1), (C3), or (C4) (with $d - 1$ in place of d). Furthermore, since 2 is the only color that appears on two edges under φ', and these two edges are both adjacent to e_1, φ' does not satisfy (C2). Hence, by the induction hypothesis, the restriction of φ' to H_1 is extendable to a proper edge coloring f_1 using colors $1, 2, 4, \ldots, d$. By recoloring the edges incident with u_1 and u_2 with color 2 by color 3, we obtain proper edges colorings of H_1 and H_2, such that we may color any edge of M by the color missing at its endpoints to obtain an extension of φ.

Let us now consider the case when the restriction of φ to H_1 is not extendable. Then there is some edge u_1v_1 in H_1 such that all precolored edges of H_1 are incident with u_1 or v_1 and u_1v_1 is not precolored. Without loss of generality, we assume that the edge in H_2 is precolored d, there is some edge e_1 precolored 3 incident with u_1 such that the corresponding edge e_2 of H_2 is not precolored, and there is an edge precolored 2 incident with v_1. We define a new precoloring φ' from φ by recoloring e_1 by color 2 and coloring e_2 by color 2. We may now finish the proof by proceeding exactly as in the preceding paragraph. □

This completes the proof of Theorem 3.8.

5 | CONCLUDING REMARKS

In this paper we have obtained analogues for hypercubes of some classic results on completing partial Latin squares; in general we believe that the following might be true. Here, G^d denotes the dth power of the Cartesian product of G with itself.

Conjecture 5.1. *If n and d are positive integers, and φ is a proper edge precoloring of $(K_{n,n})^d$ with at most $nd - 1$ precolored edges, then φ extends to a proper nd-edge coloring of $(K_{n,n})^d$.*

Note that this is a generalization of both Evans' conjecture and the results obtained in this paper; Evans' conjecture is the case $d = 1$, and the results obtained in this paper resolve the cases when $n = 1$ and 2; thus this conjecture is open whenever $d \geq 2$ and $n \geq 3$.

Given that a precoloring of at most $d - 1$ precolored edges of Q_d or $K_{d,d}$ is always extendable, we might ask how many precolored edges of a general d-regular bipartite graph allow for an extension. Trivially, any precoloring of at most one edge of a graph G can be extended to a $\chi'(G)$-edge coloring of G. For larger sets of precolored edges, we have the following:

Proposition 5.2. *For any $d \geq 2$, there is a d-regular bipartite graph with a precoloring f of only two edges, such that f cannot be extended to a proper d-edge coloring.*

Proof. Let $r > 1$ be a positive integer, and let $G_1, \ldots, G_r$ be r copies of $K_{d,d} - e$, that is, the complete bipartite graph with $d + d$ vertices with exactly one edge removed. From $G_1, \ldots, G_r$ we form a d-regular graph H by for $i = 1, \ldots, r$ joining a vertex in G_i of degree $d - 1$ with a vertex in G_{i+1} of degree $d - 1$ by an edge so that all added edges have distinct endpoints (indices taken modulo r). Let e_1 and e_2 be two distinct edges in H joining vertices in distinct copies of $K_{d,d} - e$. We color e_1 with color 1, and e_2 with color 2. Since any perfect matching in H_1 that contains e_1 also contains e_2, this precoloring cannot be extended to a proper d-edge coloring of H. □

Note that in the proof of Proposition 5.2, there is a similar precoloring with two edges colored 1, which is not extendable to a proper d-edge coloring of the full graph. Also, the distance between the two precolored edges can be made arbitrarily large.

Furthermore, the examples given in the proof of Proposition 5.2 are 2-connected. One may construct examples of arbitrarily large connectivity by taking two copies G_1 and G_2 of $K_{n,n-1}$ and for each vertex v in G_1 of degree $n - 1$ adding an edge between v and its copy in G_2. The resulting graph is n-regular, $(n - 1)$-connected, and the edge precoloring obtained by coloring any two edges with one endpoint in G_1 and one endpoint in G_2 by color 1 is not extendable to a proper d-edge coloring of the full graph.

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