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On avoiding and completing edge colorings

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Abstract

This thesis consists of the following papers.

I C.J Casselgren, K. Markström, L.A. Pham, *Restricted extension of sparse partial edge colorings of hypercubes*, submitted

II C.J Casselgren, K. Markström, L.A. Pham, *Edge precoloring extension of hypercubes*, submitted

III C.J Casselgren, K. Markström, L.A. Pham, *Latin cubes with forbidden entries*, manuscript

These papers are all related to the problem of avoiding and completing an edge precoloring of a graph. In more detail, given a graph G and a partial proper edge precoloring φ of G and a list assignment L for every non-colored edge of G, can we extend the precoloring to a proper edge coloring avoiding any list assignment? In the first paper, G is a d-dimensional hypercube graph Q_d, a partial proper edge precoloring φ and every list assignment L must satisfy certain sparsity conditions. The second paper still deals with d-dimensional hypercube graph Q_d, but the list assignment L for every edge of Q_d is an empty set and φ must be a partial proper edge precoloring of at most $d-1$ edges. For the third paper, G can be seen as a complete 3-uniform 3-partite hypergraph, every list assignment L must satisfy certain sparsity conditions but we do not have a partial proper edge precoloring φ on edges of G.

Contents

1 Introduction

A graph is a collection of points (called vertices) and the lines between some pairs of vertices (called edges). Graph theory is the branch of mathematics that studies of graphs, which was initiated by Leonhard Euler in 1763 when he tried to solve the "Königsberg bridges" problem. One of the most famous problems in the field of graph theory is graph coloring, which can be used to formulate a large number of practical problems, e.g., scheduling, timetable and register allocation. The study of graph coloring started in 1852 when Francis Guthrie observed that only four colors were needed to color a map of the counties of England in order to ensure that no neighboring counties had the same color. If we form a graph by representing each county with a vertex and drawing an edge between any pair of vertices if the corresponding counties share a border on the map, then the problem of coloring a map translates into a graph coloring problem. To be more specific, a coloring of a graph is an assignment of colors to the vertices (or edges) in such a way that any two adjacent vertices (or edges) receive different colors.

In this thesis, we consider a problem of edge coloring, given a list of acceptable colors to every edge, is it possible to find a coloring so that each edge gets a color from its list of allowed colors and no two adjacent edges share the same color? A complete bipartite graph $K_{p,q}$ is a graph in which the set of vertices can be decomposed into two disjoint sets with p and q vertices such that no two vertices within the same set are adjacent and every pair of vertices from the different sets are adjacent. Dinitz conjectured, and Galvin proved [1], that if each edge of a complete bipartite graph $K_{n,n}$ is given a list of n colors, then there is a proper edge coloring of $K_{n,n}$ with support in the lists. Motivated by the Dinitz's problem, Häggkvist [2] introduced the notion of *βn-arrays*, which correspond to list assignments L of forbidden colors for $E(K_{n,n})$, such that each edge e of $K_{n,n}$ is assigned a list $L(e)$ of at most βn forbidden colors from $\{1, \dots, n\}$, and at every vertex v each color is forbidden on at most βn edges adjacent to v; we call such a list assignment for $K_{n,n}$ *β-sparse*. Häggkvist conjectured that there exists a fixed $\beta > 0$, in fact also that $\beta = \frac{1}{3}$, such that for every positive integer n, every β-sparse list assignment for $K_{n,n}$, there is a proper n-edge coloring φ of $K_{n,n}$ *avoids* the list assignment L, i.e., $\varphi(e) \notin L(e)$ for every edge e of $K_{n,n}$. That such a $\beta > 0$ exists was proved by Andrén in her PhD thesis [3]. Our third paper demonstrates that the similar result holds for the family of complete 3-uniform 3-partite hypergraphs.

In the second paper we first prove that every proper partial edge coloring of at most $d-1$ edges of the d-dimensional hypercube Q_d can be extended to a proper d-edge coloring of Q_d. A similar edge precoloring extension problem appeared already in 1960, when Evans [4] stated his now classic conjecture that for every positive integer n, if $n-1$ edges in $K_{n,n}$ have been (properly) colored, then this partial coloring can be extended to a proper n-edge-coloring of $K_{n,n}$. This conjecture was solved for large n by Häggkvist [5] and later for all n by Smetaniuk [6], and independently by Andersen and Hilton [7]. Moreover, Andersen and Hilton [7] characterized which $n \times n$ partial Latin squares with exactly n non-empty cells are extendable, we establish an analogue result for hypercubes by proving that a proper precoloring φ of at most d edges in Q_d is always extendable unless the precoloring φ is unsatisfied some specific conditions.

Generalizing Evans' problem, Daykin and Häggkvist [8] proved several results on extending partial edge colorings of $K_{n,n}$, and they also conjectured that much denser partial colorings can be extended, as long as the colored edges are spread out in a specific sense: a partial n-edge coloring of $K_{n,n}$ is ϵ-*dense* if there are at most ϵn colored edges from $\{1, \ldots, n\}$ at any vertex and each color in $\{1, \ldots, n\}$ is used at most ϵn times in the partial coloring. Daykin and Häggkvist [8] conjectured that for every positive integer n, every $\frac{1}{4}$-dense partial proper n-edge coloring can be extended to a proper n-edge coloring of $K_{n,n}$ and proved a version of the conjecture for $\epsilon = o(1)$ (as $n \to \infty$) and n divisible by 16. Andrén et al. [9] proved that there are constants $\alpha > 0$ and $\beta > 0$, such that for every positive integer n, every α-dense partial edge coloring of $K_{n,n}$ can be extended to a proper n-edge-coloring avoiding any given β-sparse list assignment L, provided that no edge e is precolored by a color that appears in $L(e)$. The aim of the first paper is to study this type of problem for the family of hypercubes. We show that for a given partial proper d-edge coloring of the d-dimensional hypercube Q_d, and lists of forbidden colors for the non-colored edges of Q_d, if both the partial coloring and the color lists satisfy certain sparsity conditions, then it is possible to extend the partial coloring to a proper d-edge coloring avoiding colors from the lists. Our basic idea is similar to the one in [9], but the proof contains more technical details since compared to the complete bipartite graph $K_{n,n}$, the hypercube graph Q_d is much more sparse.

2 Summary of papers

2.1 Paper I: Restricted extension of sparse partial edge colorings of hypercubes

The *d-hypercube graph*, also called the d-cube graph and commonly denoted Q_d, is the graph whose vertices are the 2^d strings $a_1...a_d$ where $a_i = 0$ or 1 and two vertices are adjacent iff the strings differ in exactly one coordinate. Every vertex of the d-hypercube graph is incident to d edges and the total number of edges in Q_d is $2^{d-1}d$. A *dimensional matching* M of Q_d is a perfect matching of Q_d such that $Q_d - M$ is isomorphic to two copies of Q_{d-1}, evidently there are precisely d dimensional matchings in Q_d. The *distance* between two edges e and e' is the number of edges in a shortest path between an endpoint of e and an endpoint of e', the *t-neighborhood* of an edge e is the graph induced by all edges of distance at most t from e. An edge precoloring of Q_d with colors $1, \dots, d$ is called *α-dense* if

(i) there are at most αd precolored edges at each vertex;

(ii) for every 27-neighborhood W of an edge e of Q_d, there are at most αd precolored edges with color i in W, $i = 1, \dots, d$;

(iii) for every 27-neighborhood W, and every dimensional matching M, at most αd edges of M are precolored in W.

A list assignment L for $E(Q_d)$ is *β-sparse* if the list of each edge is a (possibly empty) subset of $\{1, \dots, d\}$, and

(i) $|L(e)| \leq \beta d$ for each edge $e \in E(Q_d)$;

(ii) for every vertex $v \in V(Q_d)$, each color in $\{1, \dots, d\}$ occurs in at most βd lists of edges incident to v;

(iii) for every 27-neighborhood W, and every dimensional matching M, any color appears at most βd times in lists of edges of M contained in W.

In this paper, we prove the following theorem: *There are constants $\alpha > 0$ and $\beta > 0$ such that for every positive integer d, if φ is an α-dense d-edge precoloring of Q_d, L is a β-sparse list assignment for Q_d, and $\varphi(e) \notin L(e)$ for every edge $e \in E(Q_d)$, then there is a proper d-edge coloring of Q_d which agrees with φ on any precolored edge and which avoids L.*

For the proof we shall start with the *standard* d-edge coloring h of Q_d where all edges of the ith dimensional matching in Q_d are colored i, $i = 1, \dots, d$. If h is a proper d-edge coloring of Q_d which agrees with φ on every precolored edge and which avoids L, then we are done. Otherwise, there will be some *unsatisfied edges* (e.g. an edge e such that $h(e) \neq \varphi(e)$ or $h(e) \in L(e)$). We will find a permutation ρ of the elements of the set $\{1, \dots, d\}$ such that in the proper d-edge-coloring h' obtained by applying ρ to the colors used in h, each vertex of Q_d and each dimensional matching in Q_d contain "sufficiently few" unsatisfied edges. Next, we define a suitable new edge precoloring φ' such that for every edge e of Q_d, $\varphi'(e) = \varphi(e)$ and $\varphi'(e) \notin L(e)$ and the number of unsatisfied edges (e.g. $\varphi'(e) \neq h'(e)$) is sufficiently few. It is obvious that if a proper d-edge coloring of Q_d agrees with φ', then it agrees with φ and avoids L. Finally, for each edge e of Q_d such that $h'(e) \neq \varphi'(e)$, we construct a subset of edges of Q_d such that performing a series of swaps on the coloring of these subsets yields a proper d-edge coloring of Q_d which is an extension of φ' (and thus φ), and which avoids L.

2.2 Paper II: Edge precoloring extension of hypercubes

We begin this paper by giving a short proof for the theorem: *Let $d \geq 2$ be a positive integer. If φ is a proper precoloring of at most $d-1$ edges of the hypercube Q_d, then φ can be extended to a proper d-edge coloring of Q_d.* The remaining part of this paper characterizes which partial edge colorings of Q_d with precisely d precolored edges are extendable to proper d-edge colorings of Q_d. We denote by $\mathcal{C}$ the set of all the colorings of Q_d, $d \geq 1$, satisfying any of the following conditions:

- there is an uncolored edge uv in Q_d such that u is incident with edges of $k \leq d$ distinct colors and v is incident to $d-k$ edges colored with $d-k$ other distinct colors (so uv is adjacent to edges of d distinct colors);

- there is a vertex u that is incident with edges of $d-k$ distinct colors $c_1, \dots, c_{d-k}$, and k vertices $v_1, \dots, v_k$ such that for $i = 1, \dots, k$, uv_i is uncolored but v_i is incident with an edge colored $c \notin \{c_1, \dots, c_{d-k}\}$;

- there is a vertex u such that every edge incident with u is uncolored but there is a color c satisfying that every edge incident with u is adjacent to another edge colored c;

- $d = 3$ and the precolored three edges use three different colors and is a subset of a dimensional matching.

Clearly, if φ is a precoloring of Q_d with exactly d precolored edges and $\varphi \in \mathcal{C}$, then φ is not extendable. We show that if φ is a proper d-edge precoloring of Q_d with exactly d precolored edges and $\varphi \notin \mathcal{C}$, then φ is extendable to a proper d-edge coloring of Q_d. The proof is quite long but the idea is not complicated. Using induction method, we consider the 3-dimensional hypercube graph Q_3 for the base case, the induction step is solved based on the observation: If M is a dimensional matching in Q_d and that H_1 and H_2 are the components of $Q_d - M$, then H_1 and H_2 are both isomorphic to Q_{d-1}.

In addition to the main results, we also consider the problem of completing an edge precoloring of a hypercube graph in different cases. We prove that the edge precoloring of Q_d is extendable if the precolored edges of Q_d form an induced matching all edges of which lie in two dimensional matchings or if the precolored edges of Q_d lie in two hypercube graphs contained in Q_d and satisfy some extra conditions.

2.3 Paper III: Latin cubes with forbidden entries

A *Latin cube* L of order n on the symbols $\{1, \ldots, n\}$ is an $n \times n \times n$ cube (n rows, n columns, n files) such that each symbol in $\{1, \ldots, n\}$ appears exactly once in each row, column and file. An $n \times n \times n$ cube where each cell contains a subset of the symbols in the set $\{1, \ldots, n\}$ is called an (m, m, m, m)*-cube (of order* n*)* if the following conditions are satisfied:

(a) No cell contains a set with more than m symbols.

(b) Each symbol occurs at most m times in each row.

(c) Each symbol occurs at most m times in each column.

(d) Each symbol occurs at most m times in each file.

Let $A(i,j,k)$ denote the set of symbols in the cell (i,j,k) of A, let $L(i,j,k)$ denote the symbol in position (i,j,k) of L. We say that those cells (i,j,k) of L where $L(i,j,k) \in A(i,j,k)$ are *conflict cells of L with A* (or simply *conflicts* of L). A Latin cube L of order n *avoids* A if L does not contain any conflict cells; if there is such a Latin cube, then A is *avoidable.* A 3-*cube* in L is a set of eight cells $\{(i_1,j_1,k_1), (i_1,j_2,k_1), (i_2,j_1,k_1), (i_2,j_2,k_1), (i_1,j_1,k_2), (i_1,j_2,k_2), (i_2,j_1,k_2), (i_2,j_2,k_2)\}$ in L such that $L(i_1,j_1,k_1) = L(i_2,j_2,k_1) = L(i_1,j_2,k_2) = L(i_2,j_1,k_2) = x_1$ and $L(i_1,j_2,k_1) = L(i_2,j_1,k_1) = L(i_1,j_1,k_2) = L(i_2,j_2,k_2) = x_2$. An *allowed* 3-*cube* of L is a 3-cube $\mathcal{C}$ in L such that swapping the two symbols x_1, x_2 of $\mathcal{C}$ yields a Latin cube where non of the cells of $\mathcal{C}$ is a conflict.

This paper answers the question: If every cells of a cube is assigned a list of symbols satisfying certain sparsity conditions, is there any Latin cube avoiding these lists? Our main result is the following: *There is a positive constant γ such that if $t \geq 30$ and $m \leq \gamma 2^t$, then any (m,m,m,m)-cube A of order 2^t is avoidable.*

Let us consider a 3-uniform 3-partite hypergraph in which its vertices can be partitioned into three parts such that every edge consists of exactly one vertex from each class. If we view the 3-uniform 3-partite hypergraph G as a cube C and an edge of G as a cell in C, then from another perspective, the problem of avoiding (m,m,m,m)-cube is actually the problem of avoiding list assignment of forbidden colors for every edge of a complete 3-uniform 3-partite hypergraph.

The basic proof strategy is similar to the one in [11]. Our starting point in the proof is the special Latin cube (called Boolean Latin cube) which has a very strong property that each cell in Boolean Latin cube belongs to $n-1$ 3-cubes; we permute its row layers, column layers, file layers and symbols so that the resulting Latin cube does not have too many conflicts with a given (m,m,m,m)-cube A. After that, we find a set of allowed 3-cubes such that each conflict belongs to one of them, with no two of the 3-cubes intersecting, and swap on those 3-cubes.

3 Concluding remarks and further problems

The proof in paper I holds when $\alpha = 10^{-622}$ and $\beta = 2 \times 10^{-622}$. For $K_{n,n}$ Daykin and Häggkvist [8] conjectured that $\alpha = 1/4$ is the optimal value, and Häggkvist conjectured that $\beta = 1/3$ is optimal. For the hypercube Q_d and the values of α and β in the main theorem of the paper I, we prove an upper bound that $\alpha + \beta < \frac{1}{2}$, and an open question is what are the optimal values for α and β?

In the second paper we have obtained analogues for hypercubes of some classic results on completing partial Latin squares; in general we believe that we can obtain similar results for $(K_{n,n})^d$. Here, G^d denotes the dth power of the cartesian product of G with itself.

The third paper shows that there exists a Latin cube avoiding an (m,m,m,m)-cube. Note that finding a Latin square avoiding an (m,m,m)-array was solved before. Thus, one future direction is to consider the similar problem on the Latin cubes with higher dimension.

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