On using a zero lower bound on the physical density in Material Distribution Topology Optimization

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Abstract

Material distribution topology optimization methods aim to place optimally material within a given domain or space. These methods use a so-called material indicator function ρ to determine for each point within the design domain whether it contains material ($\rho = 1$) or void ($\rho = 0$).

The most common topology optimization problem is the minimum compliance problem. This thesis studies the problem to minimize the compliance of a cantilever beam subject to a given load. The displacement of the beam is governed by a differential equation. Here, we use the finite element method to solve this continuous problem numerically. This method approximates the problem by partitioning the given domain into a finite number of elements. In material distribution topology optimization, each element E_n is assigned a design variable ρ_n that indicates whether this element is void ($\rho_n = 0$) or contains material ($\rho_n = 1$). There is a problem of allowing $\rho_n = 0$ to represent the voids: the linear system arising from the finite element approximation may be (will almost surely be) ill-conditioned. Generally, by using a weak material to approximate the voids (letting $\rho_n = \underline{\rho}$, where $0 < \underline{\rho} \ll 1$, represent void), the finite-dimensional problem is solvable.

The choice of parameter $\underline{\rho}$ in the weak material approximation is trade-off between accuracy (a smaller $\underline{\rho}$ gives a smaller error) and conditioning (the condition number grows as $\underline{\rho}$ decreases). Therefore, instead of using a weak material approximation, we use $\rho_n = 0$ and $\rho_n = 1$ to represent void and material, respectively. To alleviate the ill-conditioning problem, we introduce a preconditioner, which for each degree of freedom is based on the sum of design variables in elements neighbouring the corresponding node.

To study the effect of the preconditioning, we consider a one-dimensional bar and show that, under certain assumptions, the linear system becomes well-conditioned after preconditioning. Moreover, we use the proposed preconditioning method as part of a material distribution based algorithm to solve the problem to minimize the compliance of a cantilever beam subject to a given load. We present results obtained by solving large-scale optimization problems; these results illustrate that the proposed preconditioning approach ensures a well-conditioned linear system through the optimization process.

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1 Introduction

Over the past few decades, the design optimization field has been studied widely. The main purpose of design optimization is to find how to design a device with desirable properties. However, there may exist not only one but many designs that would fulfill the given requirements. In most cases, it seems to be impossible to enumerate all feasible designs for evaluating them in order to choose the best one. Rather than manually modifying the design in the search for a better one, an appealing alternative (which will be employed in this thesis) is to let a computer algorithm systematically search for an improved design.

Design optimization methods are classified in order of generality as sizing, boundary shape and topology optimization. In a sizing problem, one would like to find the optimal size of parts of a given structure. On the other hand, boundary shape optimization aims to determine not only the optimal size of the parts of structure but also the optimum shape of a given domain. However, the most general design optimization method, which also optimizes the connectivity of the design.

The field of topology optimization has been subject to intense research since the article regarding material distribution method for the design of elastic continuum structures by Bendsøe and Kikuchi [2] was published in 1988. Since then, such material distribution has been widely applied to a range of different physics research fields. The most popular research in this field is a well-known comprehensive account on topology optimization and a variety of its applications was published in the monograph by Bendsøe and Sigmund [3]. Furthermore, there is a number of researchs have been done in different fields such as electromagnetics (by Wadbro et al [4], by Elesin et al [5], by Hassan et al [6] and by Erentok et al [7]), fluid-structure interaction (by Andreasen et al [8] and by Yoon [9]), acoustics (by Kook et al [10], by Wadbro [11] and by Christiansen et al [12]) and linear and nonlinear elasticity (by Clausen et al [13], by Park et al [14] and by Klarbring et al [15]).

In material distribution topology optimization, a so-called material indicator function $\rho : \Omega_G \subset \mathbb{R}^d \rightarrow \{0, 1\}$ is introduced to illustrate presence ($\rho = 1$) or absence ($\rho = 0$) of material within a given design area Ω_G [3]. The topology optimization problem is typically solved numerically by partitioning the domain Ω into N elements and solving it by finite element approximation. Then each element has a value of $\rho_i \in \{0, 1\}, i = 1, 2, \dots, N$, that determines whether this element contains material or void. However, from a mathematical viewpoint, this material-void problem is ill-posed. One approach to resolve the ill-posedness is to relax the feasible set of the material indicator by allowing $\rho_i \in [0, 1], i \in \{1, \dots, N\}$. In the continuous cases, such relaxation is necessary to ensure that there exists the solution. The drawback here is the solution of the relaxation approach is not the solution of the original problem. Then a penalization technique is applied. However, this penalization may cause mesh-dependent solution of this problems. The mesh-dependency problem is solved by using filtering procedures such as sensitive filtering or density filtering.

However, there still exists a problem when using material indicator, that is if $\rho_i = 0$ for some

$i \in \{1, \dots, N\}$ then there is no the existence and uniqueness of solution for the problem. The most useful popular technique to apply a lower bound on the material indicator function, so that $\rho_i \in [\underline{\rho}, 1]$, $i \in \{1, \dots, N\}$ where $0 < \underline{\rho} \ll 1$. The approach have been used commonly in the most researches (see for example references [3], [16], [17]). The main drawback of this method is that the material indicator function is not exactly zero for indicating an absence of material as it should be in the original problem. In this thesis, we use another approach to attack this problem, that is preconditioning the stiffness matrix on the finite element approximation to ensure that there exists a unique solution to the problem.

In the following chapters, we discuss more about the material distribution topology optimization of the minimization of compliance for the cantilever beam problem and the solution to avoid the singular stiffness matrix by using preconditioning method. Chapter 2 will show the finite element method for the linear elastostatics. In Chapter 3, we will show the topology optimization by distribution isotropic material. After that, Chapter 4 will be discussed about using nonlinear fW -mean filters to avoid mesh-dependent problems of topology optimization using Solid Isotropic Material with Penalization (SIMP) approach. The most important chapter is Chapter 5, that describes an approach for an uniqueness and an existence of the solution of the linear elastic problem when using $\underline{\rho} = 0$. The last two chapters will show the result of the thesis as well as discussing about them, respectively.

2 The finite elements methods for the linear elastostatics

Consider an area Ω that contains an elastic material, let ω be an arbitrary subdomain of Ω , $\partial\omega$ is its boundary, and $\mathbf{n}$ is the exterior normal vector. There is two types of force apply on ω . The first one is so-called internal forces (or body forces) $\mathbf{b}$ acting on the entire domain. The second one is the forces acting on the boundary $\partial\omega$, these forces take the form $\boldsymbol{\sigma} \cdot \mathbf{n}$, where $\boldsymbol{\sigma}$ is stress tensor. By summing those types of force we attain the total force f on Ω

$$f = \int_{\omega} \mathbf{b} dx + \int_{\partial\omega} \boldsymbol{\sigma} \cdot \mathbf{n} ds. \quad (2.1)$$

From the divergence theorem on the boundary integral we obtain

$$f = \int_{\omega} (\mathbf{b} + \nabla \cdot \boldsymbol{\sigma}) dx. \quad (2.2)$$

In equilibrium $f = 0$ and because of arbitrary ω , we can conclude that

$$\mathbf{b} + \nabla \cdot \boldsymbol{\sigma} = 0. \quad (2.3)$$

The displacement of a material point is defined as the vector $\mathbf{u} = \mathbf{x} - \mathbf{x}_0$, where $\mathbf{x}$, $\mathbf{x}_0$ is the current and the initial position of the material point, respectively.

Therefore, the basic problem of linear elastostatics is to find the stress tensor $\boldsymbol{\sigma}$ and the displacement vector $\mathbf{u}$ such that [1]

$$\begin{cases} -\nabla \cdot (\mathbf{E}\boldsymbol{\varepsilon}(\mathbf{u})) = \mathbf{b}, & \text{in } \Omega \\ \mathbf{u} = 0, & \text{on } \Gamma_D \\ (\mathbf{E}\boldsymbol{\varepsilon}(\mathbf{u})) \cdot \mathbf{n} = \mathbf{t}, & \text{on } \Gamma_L \end{cases} \quad (2.4)$$

where Γ_D and Γ_L are two boundary portions associated with the Dirichlet and Neumann boundary conditions, respectively. Furthermore, $\mathbf{b} \in L^2(\Omega)^d$ denotes the internal force in Ω and $\mathbf{t} \in L^2(\Gamma_L)^d$ is surface force densities on boundary $\Gamma_L = \partial\Omega \setminus \Gamma_D$. The strain tensor $\boldsymbol{\varepsilon}(\mathbf{u})$ is defined by

$$\boldsymbol{\varepsilon}(\mathbf{u}) = \frac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^T). \quad (2.5)$$

The generalized Hooke's law of the theory of linear elasticity for isotropic materials implies

$$\boldsymbol{\sigma} = \mathbf{E}\boldsymbol{\varepsilon}(\mathbf{u}) = 2\mu\boldsymbol{\varepsilon}(\mathbf{u}) + \lambda(\nabla \cdot \mathbf{u})\mathbf{I}, \quad (2.6)$$

where $\mathbf{E}$ is the fourth-order elasticity tensor. The Lamé parameters λ and μ are defined as

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \quad (2.7)$$

Figure 1: Demonstration of linear elastostatics problem of body Ω deforming under a internal force $\mathbf{b}$ and boundary force $\mathbf{t}$. The solid line illustrates the initial state and the dashed line shows the deformed state of configuration [1].

and

$$\mu = \frac{E}{2(1+\nu)}, \quad (2.8)$$

where ν are constant throughout the material.

2.1 Variational form

In order to derive the variational formulation of (2.4), let

$$\mathcal{U} = \{\mathbf{u} \in H^1(\Omega)^d \mid \mathbf{u}|_{\Gamma_D} \equiv 0\}. \quad (2.9)$$

be the set of kinematically admissible displacement of structure as it was defined in (3.4). By multiplying $\mathbf{b} = \nabla \cdot \boldsymbol{\sigma}$ with a test function $\mathbf{v} \in \mathcal{U}$ and then integrating by parts, we get

$$\int_{\Omega} \mathbf{b} \cdot \mathbf{v} = \int_{\Omega} (-\nabla \cdot \boldsymbol{\sigma}) \mathbf{v} \quad (2.10)$$

$$\Leftrightarrow \int_{\partial\Omega} (\boldsymbol{\sigma} \cdot \mathbf{n}) \mathbf{v} + \int_{\Omega} \boldsymbol{\sigma} : \nabla \mathbf{v} = \int_{\Omega} \mathbf{b} \cdot \mathbf{v} \quad (2.11)$$

where the colon ":" denotes the contraction operator of the two matrices. By using boundary condition $\boldsymbol{\sigma} \cdot \mathbf{n} = \mathbf{t}$ on Γ_L and the Dirichlet boundary condition $\mathbf{u} = 0$ on Γ_D , we get

$$\int_{\Omega} \boldsymbol{\sigma} : \nabla \mathbf{v} = \int_{\Omega} \mathbf{b} \cdot \mathbf{v} + \int_{\Gamma_L} \mathbf{t} \cdot \mathbf{v} \quad \forall \mathbf{v} \in \mathcal{U}. \quad (2.12)$$

For any matrix, one can decompose it into symmetric and anti-symmetric part i.e. $A = (A + A^T)/2 + (A - A^T)/2$. Then we apply that into $\boldsymbol{\sigma} : \nabla \mathbf{v}$ to get

$$\boldsymbol{\sigma} : \nabla \mathbf{v} = \boldsymbol{\sigma} : \frac{1}{2}(\nabla \mathbf{v} + \nabla \mathbf{v}^T) + \boldsymbol{\sigma} : \frac{1}{2}(\nabla \mathbf{v} - \nabla \mathbf{v}^T) \quad (2.13)$$

$$= \boldsymbol{\sigma} : \boldsymbol{\varepsilon}(\mathbf{v}) + 0, \quad (2.14)$$

where $\nabla \mathbf{v} - \nabla \mathbf{v}^T$ is skew-symmetric matrix and because $\boldsymbol{\sigma}$ is symmetric matrix so the second term of the right hand side (2.13) is cancelled out

$$\boldsymbol{\sigma} : \frac{1}{2}(\nabla \mathbf{v} - \nabla \mathbf{v}^T) = 0. \quad (2.15)$$

Then (2.12) will become

$$\int_{\Omega} \boldsymbol{\sigma} : \boldsymbol{\varepsilon}(\mathbf{v}) = \int_{\Omega} \mathbf{b} \cdot \mathbf{v} + \int_{\Gamma_L} \mathbf{t} \cdot \mathbf{v} \quad \forall \mathbf{v} \in \mathcal{U}. \quad (2.16)$$

Now, we can insert Hooke's law from equation (2.4) which is $\boldsymbol{\sigma} = E_{ijkl}\boldsymbol{\varepsilon}(u)$ then we end up with

$$\int_{\Omega} (\mathbf{E}\boldsymbol{\varepsilon}(\mathbf{u})) : \boldsymbol{\varepsilon}(\mathbf{v}) = \int_{\Omega} \mathbf{b} \cdot \mathbf{v} + \int_{\Gamma_L} \mathbf{t} \cdot \mathbf{v} \quad \forall \mathbf{v} \in \mathcal{U}. \quad (2.17)$$

Hence, the variational form of (2.17) is: Find $\mathbf{u} \in \mathcal{U}$ such that

$$a(\mathbf{u}, \mathbf{v}) = l(\mathbf{v}) \quad \forall \mathbf{v} \in \mathcal{U}. \quad (2.18)$$

where the bilinear form $a(\cdot, \cdot)$ and the linear form $l(\cdot)$ are defined as

$$a(\mathbf{u}, \mathbf{v}) = \int_{\Omega} (\mathbf{E}\boldsymbol{\varepsilon}(\mathbf{u})) : \boldsymbol{\varepsilon}(\mathbf{v}) = \int_{\Omega} E_{ijkl}\boldsymbol{\varepsilon}_{ij}(\mathbf{u})\boldsymbol{\varepsilon}_{kl}(\mathbf{v}), \quad (2.19)$$

$$l(\mathbf{v}) = \int_{\Omega} \mathbf{b} \cdot \mathbf{v} + \int_{\Gamma_L} \mathbf{t} \cdot \mathbf{v}. \quad (2.20)$$

Remark 1: The bilinear form of $a(\mathbf{u}, \mathbf{v})$ can be written in the form

$$a(\mathbf{u}, \mathbf{v}) = \int_{\Omega} (2\mu\boldsymbol{\varepsilon}(\mathbf{u}) : \boldsymbol{\varepsilon}(\mathbf{v}) + \lambda(\nabla \cdot \mathbf{u})(\nabla \cdot \mathbf{v}))d\Omega. \quad (2.21)$$

2.2 Existence and Uniqueness of the Solution

We now need to show that there are existence and uniqueness of the solution of the weak formulation (2.18). In order to do that, firstly we should define the norm on $\mathcal{U}$ to measure the size of various matrices, tensors, and vectors involved as the following. For any $n \times n$ matrix or tensor A , and any $n \times 1$ vector b , the norm of them on $\mathcal{U}$ are defined as

$$\|A\|_{\mathcal{U}}^2 = \sum_{i,j=1}^n \|A_{ij}\|_{H^1(\Omega)}^2 \quad (2.22)$$

$$\|b\|_{\mathcal{U}}^2 = \sum_{i=1}^n \|b_i\|_{H^1(\Omega)}^2 \quad (2.23)$$

The semi-norms of $\mathbf{A}$ and $\mathbf{b}$ on $\mathcal{U}$ are obtained analogously. Using the Cauchy-Schwarz inequality, the continuity of $a(\cdot, \cdot)$ obeys

$$a(\mathbf{u}, \mathbf{v}) = \int_{\Omega} (2\mu \boldsymbol{\varepsilon}(\mathbf{u}) : \boldsymbol{\varepsilon}(\mathbf{v}) + \lambda (\nabla \cdot \mathbf{u})(\nabla \cdot \mathbf{v})) d\Omega \quad (2.24)$$

$$\leq 2\mu \|\boldsymbol{\varepsilon}(\mathbf{u})\| \|\boldsymbol{\varepsilon}(\mathbf{v})\| + \lambda \|\nabla \cdot \mathbf{u}\| \|\nabla \cdot \mathbf{v}\| \quad (2.25)$$

$$\leq C \|\nabla \mathbf{u}\| \|\nabla \mathbf{v}\| \quad (2.26)$$

$$\leq C \|\mathbf{u}\|_{\mathcal{U}} \|\mathbf{v}\|_{\mathcal{U}}. \quad (2.27)$$

From the trace inequality

$$\|\mathbf{v}\|_{\Gamma_N} \leq C(\|\nabla \mathbf{v}\| + \|\mathbf{v}\|) \leq C \|\mathbf{v}\|_{\mathcal{U}} \quad (2.28)$$

Then using Cauchy-Schwarz inequality, the continuity of $l(\cdot)$ follows

$$l(\mathbf{v}) = \int_{\Omega} \mathbf{b} \cdot \mathbf{v} + \int_{\Gamma_L} \mathbf{t} \cdot \mathbf{v} \quad (2.29)$$

$$\leq \|\mathbf{b}\| \|\mathbf{v}\| + \|\mathbf{t}\|_{\Gamma_N} \|\mathbf{v}\|_{\Gamma_N} \quad (2.30)$$

$$\leq C \|\mathbf{v}\|_{\mathcal{U}} \quad (2.31)$$

By using Korn's Inequality

$$C \|\nabla \mathbf{v}\|^2 \leq \|\boldsymbol{\varepsilon}(\mathbf{v})\|^2 = \int_{\Omega} \sum_{i,j=1}^n \varepsilon_{ij}(\mathbf{v}) \varepsilon_{ij}(\mathbf{v}) dx \quad (2.32)$$

we can prove coercivity of $a(\cdot, \cdot)$

$$a(\mathbf{u}, \mathbf{u}) = 2\mu \|\boldsymbol{\varepsilon}(\mathbf{u})\|^2 + \lambda \|\nabla \cdot \mathbf{u}\|^2 \leq 2\mu \|\boldsymbol{\varepsilon}(\mathbf{u})\|^2 \leq C \|\nabla \mathbf{u}\|^2 \leq m \|\mathbf{u}\|_{\mathcal{U}}^2 \quad (2.33)$$

Therefore, now we can conclude that there exist a unique solution $\mathbf{u} \in \mathcal{U}$ to the weak form (2.18).

2.3 Finite Element Approximation

From the previous section 2.2, we have shown that the variational form (2.18) obeys the Lax-Milgram theorem, which has a unique solution $\mathbf{u} \in \mathcal{U}$; thus it can be approximated with finite elements. We decide to approximate each component of $\mathbf{u}$ using continuous piecewise bilinears. Let $Q = \{(r, s) : -1 < r, s < 1\}$ be a reference square and $\mathcal{Q} = \{Q\}$ be a shape quadrilateral mesh on Ω and let $P(Q)$ be the space of bilinear functions. Then we let $\mathcal{U}_h$ be the polynomial space

$$\mathcal{U}_h = \{\mathbf{u} \in \mathcal{U} \mid \mathbf{u}|_Q \in [P(Q)]^2, \forall Q \in \mathcal{Q}\} \quad (2.34)$$

in which, all displacement vector with continuous piecewise bilinear components vanishing on Γ_D . Hence, the finite element method take the form: Find $\mathbf{u}_h \in \mathcal{U}_h$, such that

$$a(\mathbf{u}_h, \mathbf{v}) = l(\mathbf{v}), \quad \forall \mathbf{v} \in \mathcal{U}_h. \quad (2.35)$$

3 Topology optimization by isotropic material distribution

In the previous chapter, we defined variational formulation (2.18). In this chapter, from the domain Ω , we define the optimal compliance design into the problem of finding the optimal stiffness tensor $\mathbf{E}$ that is chosen to be a design variable over the domain Ω .

We consider the minimization of compliance for the cantilever beam in the domain that is shown in figure 2. The beam is held at its left hand side, that boundary portion is denoted as Γ_D . A downward force is applied vertically that is uniformly distributed over Γ_F , which is a portion of 10% at the middle of the right hand side boundary of the domain Ω_G .

As we discussed in Chapter 1, we define that the presence/absence of material by a material indicator function (design variables) ρ . The variational formulation (3.1) in the following is solved by using bilinear finite elements. Find $\mathbf{u} \in \mathcal{U}$ such that

$$a(\rho; \mathbf{u}, \mathbf{v}) = l(\mathbf{v}), \quad \forall \mathbf{v} \in \mathcal{U}. \quad (3.1)$$

The energy bilinear form a and the load linear form l are defined as

$$a(\rho; \mathbf{u}, \mathbf{v}) = \int_{\Omega} \tilde{\rho}(\rho) \mathbf{E} \boldsymbol{\varepsilon}(\mathbf{u}) : \boldsymbol{\varepsilon}(\mathbf{v}), \quad (3.2)$$

$$l(\mathbf{v}) = \int_{\Omega} \mathbf{b} \cdot \mathbf{v} + \int_{\Gamma_L} \mathbf{t} \cdot \mathbf{v}, \quad (3.3)$$

where $\mathbf{u} \in \mathcal{U}$ is equilibrium displacement field of the structure, $\tilde{\rho}(\rho)$ is penalized filtered design variables called by physical design, $\mathbf{b} \in L^2(\Omega)^d$ denotes the internal force in Ω and $\mathbf{t} \in L^2(\Gamma_L)^d$ is surface force densities on boundary $\Gamma_L = \partial\Omega \setminus \overline{\Gamma_D}$.

In which, $\mathcal{U}$ is introduced as the set of kinematically admissible displacements of the structure that is defined as

$$\mathcal{U} = \{\mathbf{u} \in H^1(\Omega)^d \mid \mathbf{u}|_{\Gamma_D} \equiv 0\}. \quad (3.4)$$

and Ω is generated domain that is defined as

$$\Omega = \{x \mid \rho(x) = 1\}. \quad (3.5)$$

Typically, to ensure that the bilinear form $a(\cdot; \cdot, \cdot)$ is coercive, the physical density is defined with $0 < \underline{\rho} \ll 1$ as

$$\tilde{\rho}(\rho) = \underline{\rho} + (1 - \underline{\rho})P(F(\rho)), \quad (3.6)$$

where $P : [0, 1] \rightarrow [0, 1]$ is a smooth and invertible penalty function and F is the filtering procedure that will be discussed in the next chapter. The above formulation includes the

Figure 2: Geometry for the cantilever beam problem.

case that uses SIMP [3] which is $P(x) = x^p$. However in this thesis, we would like to use physical density variables equation (3.6) as we mentioned in Chapter 1, which uses $\bar{\rho} = 1$ and $\underline{\rho} = 0$

$$\tilde{\rho}(\rho) = P(F(\rho)), \quad (3.7)$$

Because we used (3.7) instead of (3.6), the stiffness matrix for the finite element method will become singular and the solution for that will be discussed in Chapters 5. In the following, we will discuss about the topology optimization for the minimization of the compliance for the cantilever beam.

3.1 The minimization of the compliance for the cantilever beam

First of all, we introduce a material indicator function $\rho(x) \in \mathcal{U}$, where

$$\mathcal{U} = \{\rho(x) \in L^\infty(\Omega) \mid \rho(x) \in \{0, 1\} \text{ a.e. in } \Omega\}, \quad (3.8)$$

is the set of admissible design variables. The elastic properties varies spatially for any design $\rho \in \mathcal{U}$, that are assumed to be given by $\rho(x)\mathbf{E}$. Hence, the bilinear form in (2.19) is modified as below in order to take the effect of the spatially varying elasticity tensor

$$a(\rho; \mathbf{u}, \mathbf{v}) = \int_{\Omega} \rho(x) \mathbf{E} \boldsymbol{\varepsilon}(\mathbf{u}) : \boldsymbol{\varepsilon}(\mathbf{v}), \quad x \in \Omega \quad (3.9)$$

Hence, the minimum compliance problem is defined

$$\begin{aligned} \min_{\rho \in \mathcal{U}, \mathbf{u} \in \mathcal{U}} \quad & l(\mathbf{u}) \\ \text{s.t. : } \quad & a(\rho; \mathbf{u}, \mathbf{v}) = l(\mathbf{v}), \quad \forall \mathbf{v} \in \mathcal{U}, \\ & \int_{\Omega} \rho(x) dx \leq V, \end{aligned} \quad (3.10)$$

Figure 3: The physical density after relaxing as $\rho(x) \in [0, 1]$.

where V is the limit amount of isotropic material and $\rho \in \mathcal{U}$.

However, optimization problem (3.10) is ill-posed in that the problem lacks solutions within the feasible set of design variable $\mathcal{U}$, i.e. there exists non-convergent minimizing sequences of elements from $\mathcal{U}$. The most common approach to solve this problem is relaxing the range of the design variable into the continuum $[0, 1]$, i.e. we replace the set of admissible design variable $\mathcal{U}$ into

$$\widehat{\mathcal{U}} = \{\rho \in L^\infty(\Omega) \mid \rho(x) \in [0, 1] \text{ a.e. in } \Omega\}. \quad (3.11)$$

On one hand, by using $\rho \in \widehat{\mathcal{U}}$, problem (3.10) will become a convex optimization problem and guarantee that there exist a unique solution to the optimization problem. On another hand, that replacement will destroy the design variable with the intermediate values, that is the optimal design does not satisfy $\rho \in \{0, 1\}$ almost every where. In other words, the obtained design variable by replacing $\widehat{\mathcal{U}}$ into $\mathcal{U}$ is not the solution for the original problem. This problem is demonstrated in Figure 3. There is a lot of intermediate values on the physical design, that are the "gray" area on the physical design in Figure 3.

Fortunately, one possible approach which has proven very successful and exceedingly efficient is the so-called SIMP (Solid Isotropic Material with Penalization), in which the design variables ρ will be replaced with

$$\rho' = P(\rho), \quad (3.12)$$

where $P : [0, 1] \rightarrow [0, 1]$ is a smooth and invertible penalty function. The penalized design variable above includes the case that the design variable is penalized using SIMP [3], which is $P(x) = x^p$ for some penalty parameter $p > 1$. The purpose of this method is to promote the values 0 and 1 as well as suppress the intermediate values $0 < \rho < 1$. When $p \gg 1$, SIMP method suppresses intermediate values because the intermediate values contribute more to the volume constraint (cost more) than to the stiffness of the design. In other

Figure 4: The physical density with mesh-dependency problem. The left figure is solution for 96×64 elements discretization and the right figure is solution for 384×256 elements discretization.

words, the optimal designs contain intermediate values are uneconomical designs. The consequence of using penalization method is that the solutions of optimization problem is mesh-dependent. It is known that mesh-dependent may be because of lacking of existence of solutions to the corresponding continuous problem. The mesh-dependency problem is shown in Figure 4. The left sub-figure and right sub-figure show the optimal topology of a beam discretized by 96×64 and 384×256 finite elements, respectively. The beam in the right sub-figure includes much more fine-grained structural members compared to the beam in the left sub-figure. In other words, the mesh-dependency problem is that the granularity of the optimal topology structure is dependent on mesh-refinement. Meanwhile, by using the better mesh-refinement, we would like to obtain a better finite element approximation of the same optimal structure and a better description of boundaries [18]. Furthermore, there is some areas on the right side of the physical designs in Figure 4 that looks like a checkerboard, that is also a part of mesh-dependency problem.

The filtering procedure is the most popular technique to acquire the mesh-independent designs. Let $\tilde{\rho}$ denote the physical design

$$\tilde{\rho}(\rho) = P(F(\rho)) \quad (3.13)$$

where F is the filtering procedure that will be discussed in the next chapter. Figure 5 shows the physical design after using simple mean filter, the physical design is now mesh-independent. However, this filter seems to be not suitable to this problem because edges of the physical design is blurred out, that means there exist a lot of intermediate values of physical density on the edges of physical design. The better filters will be discussed in Chapter 4.

Eventually, the minimum compliance problem is therefore modified as

$$\begin{aligned} \min_{\substack{\rho \in \widehat{\mathcal{U}}, \mathbf{u} \in \mathcal{U}}} & l(\mathbf{u}) \\ \text{s.t. : } & a(\rho; \mathbf{u}, \mathbf{v}) = l(\mathbf{v}), \quad \forall \mathbf{v} \in \mathcal{U}, \\ & \tilde{\rho}(\rho) = P(F(\rho)), \\ & \int_{\Omega} F(\rho(x)) dx \leq V, \end{aligned} \quad (3.14)$$

Figure 5: The physical density after using mean filter.

where

$$a(\rho; \mathbf{u}, \mathbf{v}) = \int_{\Omega} \tilde{\rho}(\rho) \mathbf{E} \varepsilon(\mathbf{u}) : \varepsilon(\mathbf{v}), \quad (3.15)$$

$$l(\mathbf{v}) = \int_{\Omega} \mathbf{b} \cdot \mathbf{v} + \int_{\Gamma_L} \mathbf{t} \cdot \mathbf{v}. \quad (3.16)$$

3.2 Numerical solution and implementation

3.2.1 Discretization

In this thesis, the four nodes square mesh is used. That is we partition the domain Ω into $N = n_x \times n_y$ square elements, where n_x and n_y is the number of elements in the horizontal and vertical direction, respectively. Let $\mathbf{u} = (u_1, u_2, \dots, u_M)^T$, where $M = (n_x + 1)(n_y + 1)$ be the vector of degrees of freedom, that associates with the nodes of the elements for the displacement field. The displacements along coordinate axes x, y are defined by the displacement vector $\{u\} = \{u_x, u_y\}$. u_x, u_y are interpolate with the finite element shape functions N_i to obtain the continuous, piecewise bilinear approximate displacement field as

$$\begin{aligned} u_x &= \sum_{i=1}^M N_i u_{x,i} \\ u_y &= \sum_{i=1}^M N_i u_{y,i} \end{aligned} \quad (3.17)$$

at any point $x \in \Omega$. Let $\{q\} = \{u_{x,1}, u_{y,1}, u_{x,2}, u_{y,2}, \dots\}$ be the vector of nodal displacements, then the relation between $\{u\}$ and $\{q\}$ can be written in a matrix form as

$$\{u\} = [N]\{q\} \quad (3.18)$$

$$[N] = \begin{bmatrix} N_1 & 0 & N_2 & 0 & \dots & N_M & 0 \\ 0 & N_1 & 0 & N_2 & \dots & 0 & N_M \end{bmatrix} \quad (3.19)$$

Now, we approximate the design variable ρ with function ρ_h that are constant on each element. The degrees of freedom of ρ_h are denoted by vector $\boldsymbol{\rho} = (\rho_1, \rho_2, \rho_3, \dots, \rho_N)^T$, hence the discrete form of physical design variable can be written as

$$\tilde{\rho}(\boldsymbol{\rho}) = P(F(\boldsymbol{\rho})). \quad (3.20)$$

By applying a finite element method with N element and $P = 2M = P_1 + P_2$ nodes where P_1 is the number of free nodes and P_2 is the number of fixed nodes that are located on Γ_D . Hence, the approximation of the solution to weak formulation $a(\rho; u, v) = l(v)$ satisfies

$$\mathbf{K}(\boldsymbol{\rho})\mathbf{u} = \mathbf{f}, \quad (3.21)$$

where $\mathbf{K}(\boldsymbol{\rho}) \in \mathbb{R}^{P_1 \times P_1}$ is the symmetric and positive definite stiffness matrix, $\mathbf{u} \in \mathbb{R}^{P_1}$ and $\mathbf{f} \in \mathbb{R}^{P_1}$ are the free nodal displacement and load vector, respectively. The stiffness matrix $\mathbf{K}$ can be written as the summation of all element's stiffness matrices

$$\mathbf{K}(\boldsymbol{\rho}) = \sum_{i=1}^N P(F_i(\boldsymbol{\rho}))\mathbf{K}^{(i)}, \quad (3.22)$$

where $\mathbf{K}^{(i)}$ is the element stiffness matrix corresponding to an element filled with material.

By using $\mathbf{f}^T \mathbf{u}$ as the discrete analogue of compliance, the discrete counterpart optimization problem (3.14) is defined as

$$\begin{aligned} \min_{(\boldsymbol{\rho}, \mathbf{u}) \in [0, 1]^N \times \mathbb{R}^{P_1}} \quad & \mathbf{f}^T \mathbf{u} \\ \text{s.t. : } \quad & \mathbf{K}(\boldsymbol{\rho})\mathbf{u} = \mathbf{f}, \\ & \mathbf{1}_N^T F(\boldsymbol{\rho}) \leq NV, \end{aligned} \quad (3.23)$$

where $\mathbf{1}_N = (1, 1, \dots, 1)^T \in \mathbb{R}^N$. In the next section, we will discuss about the so-called optimality criterion (OC) method [3] for solving problem (3.23).

3.3 Condition of optimality

Because of using a gradient based optimization algorithm to solve (3.23), derivatives need to be calculated. Assume that the filter function F is differentiable, a design perturbation $\delta \boldsymbol{\rho}$ is introduced to derive an expression for the gradient of objective function. By comparing with the definition of $l(v)$ in (2.20) one knows that the load vector $\mathbf{f}$ is design independent. Hence, the first order of the objective function gradient in (3.23) is given by [19]

$$\delta(\mathbf{f}^T \mathbf{u}) = \mathbf{f}^T \delta \mathbf{u}, \quad (3.24)$$

where $\delta \mathbf{u}$ is the first order perturbation of the displacement vector. By substitute $\mathbf{f}$ as discrete equilibrium equation (3.21) and exploit the symmetry of $\mathbf{K}(\boldsymbol{\rho})$, one find that

$$\delta(\mathbf{f}^T \mathbf{u}) = \mathbf{u}^T \mathbf{K}(\boldsymbol{\rho}) \delta \mathbf{u}. \quad (3.25)$$

We now find for the relation between the first equilibrium of $\mathbf{u}$ and $\mathbf{K}(\boldsymbol{\rho})$. From the discrete equilibrium equation (3.21), one can find out that

$$0 = \delta(\mathbf{K}(\boldsymbol{\rho}) \mathbf{u} - \mathbf{f}) = \delta \mathbf{K}(\boldsymbol{\rho}) \mathbf{u} + \mathbf{K}(\boldsymbol{\rho}) \delta \mathbf{u} \quad (3.26)$$

Hence, we multiply both sides of (3.26) by $\mathbf{u}^T$ on the left side then we obtain

$$\mathbf{u}^T \delta \mathbf{K}(\boldsymbol{\rho}) \mathbf{u} + \mathbf{u}^T \mathbf{K}(\boldsymbol{\rho}) \delta \mathbf{u} = 0 \quad (3.27)$$

The equation (3.25) and the expression (3.27) yields that

$$\delta(\mathbf{f}^T \mathbf{u}) = -\mathbf{u}^T \delta \mathbf{K}(\boldsymbol{\rho}) \mathbf{u}. \quad (3.28)$$

By using the expression (3.22), the gradient of $\mathbf{K}(\boldsymbol{\rho})$ is evaluated by

$$\delta \mathbf{K}(\boldsymbol{\rho}) = \frac{\partial \mathbf{K}(\boldsymbol{\rho})}{\partial \rho_j} \delta \rho_j = \left(\sum_{i=1}^N \mathbf{K}^{(i)} P'(F_i(\boldsymbol{\rho})) \frac{\partial F_i}{\partial \rho_j} \right) \delta \rho_j. \quad (3.29)$$

Substitute (3.29) into (3.28), we finally get

$$\begin{aligned} \delta(\mathbf{f}^T \mathbf{u}) &= \left(\frac{\partial}{\partial \rho_j} \mathbf{f}^T \mathbf{u} \right) \delta \rho_j \\ &= \left(\sum_{i=1}^N (-\mathbf{u}^T \mathbf{K}^{(i)} \mathbf{u}) P'(F_i(\boldsymbol{\rho})) \frac{\partial F_i}{\partial \rho_j} \right) \delta \rho_j. \end{aligned} \quad (3.30)$$

Then we rewrite the expression (3.30) into

$$\frac{\delta(\mathbf{f}^T \mathbf{u})}{\delta \rho_j} = \left(\sum_{i=1}^N (-\mathbf{u}^T \mathbf{K}^{(i)} \mathbf{u}) P'(F_i(\boldsymbol{\rho})) \frac{\partial F_i}{\partial \rho_j} \right). \quad (3.31)$$

The expression (3.31) is the j -th component of $\nabla(\mathbf{f}^T \mathbf{u})$. Besides the gradient of the volume constraint function from (3.23) can be found by direct analyzation.

The computation of all derivatives of the objective function above facilitates an applying the OC method [3] to solve our the optimization problem. From the necessary condition of optimality for problem (3.23), it obeys that there is a $\Lambda \geq 0$ such that for those j where $0 < \rho_j < 1$,

$$\frac{\delta(\mathbf{f}^T \mathbf{u})}{\delta \rho_j} + \Lambda = 0. \quad (3.32)$$

where $\Lambda > 0$ is the Lagrange multiplier corresponding to the volume constrain

$$\mathbf{1}_n^T F(\boldsymbol{\rho}) \leq nV. \quad (3.33)$$

The optimality criterion algorithm update the design variable ρ during each step to reduce the objective function while satisfying the constraints. The update starts by guessing the value of Lagrange multiplier $\Lambda > 0$. Hence the quantity

$$B_j = -\frac{\delta(\mathbf{f}^T \mathbf{u})/\delta \rho_j}{\Lambda}, \quad (3.34)$$

is checked at each element. That means the algorithm attempts to increase ρ_j when $B_j > 1$ and decrease it otherwise $B_j < 1$; this only occurs if the update does not violate the bounds on ρ . In explicit terms, the optimality criterion update as the following scheme [17]

$$\rho_{j+1} = \begin{cases} \max\{\rho_j - \zeta, 0\} & \text{if } \rho_j B_j^\eta \leq \max\{\rho_j - \zeta, 0\} \\ \min\{\rho_j + \zeta, 1\} & \text{if } \rho_j B_j^\eta \geq \max\{\rho_j + \zeta, 1\} \\ \rho_j B_j^\eta & \text{otherwise,} \end{cases} \quad (3.35)$$

where η is tuning parameter (to soften the amount of update) and ζ is a move limit. Both η and ζ control the changes that can take place at each iteration step and they can be adjustable for efficiency of the method. The numerical implementation in this thesis use $\eta = 0.5$ and $\zeta = 0.2$ [20]. The first two cases in (3.35) are used for ensuring the bounds on ρ ($0 \leq \rho \leq 1$) and limiting the size of the update at each design element through ζ . After that, the volume constrain is checked and the value of Λ is adjusted using a bisection method. The bisection method is that if the volume is too large, Λ will be increased and vice versa if the volume is too small, Λ will be decreased.

However, as we mentioned above, there is a consequence of using penalization method, that is the solutions of optimization problem is mesh-dependent. In the next chapter, the nonlinear filters are used as the solution for that problem.

4 Nonlinear filters in topology optimization

The so-called mesh dependency problem is sometimes due to lack of existence of solutions to the corresponding continuous problem. There is several approaches to solve this problem; however, in this report we only consider a well-known techniques to achieve mesh-independent design that is filtering procedure. Filtering procedure are commonly classified as either sensitivity filtering or density filtering, that the gradient of the objective function are filtered or the design variables are filtered, respectively. As a part of this report, the density filtering procedure is used. The existence on solutions to a continuous version of the linearly filtered penalized minimum compliance problem was shown by Bourdin [21]. However, there is disadvantage of the linear filter, in which the linear filter counteracts the penalization by producing a relative large areas of intermediate design variables for the physical design.

Fortunately, there is a number of nonlinear filter that reduces the amount of intermediate densities has been researched recently ([22], [23], [16] and [24]). Furthermore, the class of generalized fW -mean filters that contains most useful filters used for topology optimization have been introduced to harmonize the use of filters [20]. In order to obtain the filters with desirable properties, the cascaded fW -mean filters is applied ([20] and [16]).

4.1 Conditions of filters and their implications

In the article [20], they show some typical requirements on a general filter function $\mathbf{F} : [0, 1]^N \rightarrow [0, 1]^N$ and the filtered design is $\mathbf{F}(\boldsymbol{\rho})$. The first requirement is the range must be conforming, that is

$$\mathbf{F}(\boldsymbol{\rho}) \in [0, 1]^N. \quad (4.1)$$

Furthermore, they also satisfy the requirement that the function $\mathbf{F}$ is coordinate-wise and component-wise non-decreasing; that is, for any i, j and any coefficient $\delta \geq 0$, we require that

$$F_i(\boldsymbol{\rho} + \delta \mathbf{e}_j) \geq F_i(\boldsymbol{\rho}), \quad (4.2)$$

where $\mathbf{e}_j$ is the j th basis vector of $\mathbb{R}^N$. The idea of condition (4.2) is that increasing a design variable should not decrease any value in the filtered design. The conditions (4.1) and (4.2) imply that

$$F_i(\mathbf{0}_N) \leq F_i(\boldsymbol{\rho}) \leq F_i(\mathbf{1}_N), \quad (4.3)$$

where $\mathbf{0}_N = (0, 0, \dots, 0)^T \in \mathbb{R}^N$ and $\mathbf{1}_N = (1, 1, \dots, 1)^T \in \mathbb{R}^N$. The condition (4.3) expresses that if binary filtered design is able to attain (the filtered design is able to attain the values 0

or 1), it forces to require the condition

$$\begin{aligned}\mathbf{F}(\mathbf{0}_N) &= \mathbf{0}_N, \\ \mathbf{F}(\mathbf{1}_N) &= \mathbf{1}_N.\end{aligned}\tag{4.4}$$

It is also natural to require a condition that the filtered design in element i is strictly increasing with the density in that element. A rather weaker assumption to guarantee sensitivity to design changes, that is to require that for each $\boldsymbol{\rho} \in [0, 1]^N$ there exists an i and a j such that $F_i(\boldsymbol{\rho})$ is strictly increasing in ρ_j in the vicinity of $\boldsymbol{\rho}$. In particular, for any sufficiently small positive δ there exists $\epsilon > 0$ such that

$$F_i(\boldsymbol{\rho} + \delta \mathbf{e}_j) > F_i(\boldsymbol{\rho}), \quad \forall 0 < \delta < \epsilon.\tag{4.5}$$

In the previous chapters, we have used the gradient based optimization algorithm; hence there is necessary condition that is filter functions $\mathbf{F}$ is differentiable. In this case, the condition (4.2) turns into

$$\frac{\partial F_i}{\partial \rho_j} \geq 0.\tag{4.6}$$

In addition, the requirement (4.5) may be replaced by the more restrictive requirement. That is: For each $\boldsymbol{\rho}$, there exists an i and a j such that

$$\frac{\partial F_i}{\partial \rho_j}(\boldsymbol{\rho}) > 0.\tag{4.7}$$

Another desirable property of a filter is volume-preservation, that is if

$$\mathbf{1}_N^T \mathbf{F}(\boldsymbol{\rho}) = \mathbf{1}_N^T \boldsymbol{\rho}, \quad \forall \boldsymbol{\rho} \in [0, 1]^N.\tag{4.8}$$

The explicit advantage of using a volume-preserving filter is that the volume constraint can be applied directly into the design variables $\boldsymbol{\rho}$.

4.2 The class of fW -mean filters

In this section, a short summary on fW -mean filters is discussed [20]. The fW -mean filters take the form

$$\mathbf{F}(\boldsymbol{\rho}) = \mathbf{f}^{-1}(\mathbf{W}\mathbf{f}(\boldsymbol{\rho})),\tag{4.9}$$

where function $f : [0, 1] \rightarrow \mathbb{R}$ is a smooth and invertible function with non-zero derivatives; in which $\mathbf{f}(\boldsymbol{\rho}) = (f(\rho_1), f(\rho_2), \dots, f(\rho_N))^T \in \mathbb{R}^N$ and $\mathbf{W} \in \mathbb{R}^{N \times N}$ is a weight matrix with non-negative entries such that

$$\mathbf{W}\mathbf{1}_N = \mathbf{1}_N.\tag{4.10}$$

We note that because

$$w_{ij} > 0 \iff j \in \mathcal{N}_i,\tag{4.11}$$

where the neighborhood $\mathcal{N}_i \subset \{1, 2, \dots, N\}$ of element i is defined by the weight matrix $\mathbf{W}$ implicitly. It is common to use a neighbourhood shape $\mathcal{N} \subset \mathbb{R}^d$ to define the neighbourhoods

$$\mathcal{N}_i = \{j : x_j - x_i \in \mathcal{N}\}, \quad i \in \{1, 2, \dots, N\}, \quad (4.12)$$

where $x_i \in \mathbb{R}^d$, $i \in \{1, 2, \dots, N\}$ are the element centroids. By using the equal weights within neighbourhood, then

$$\mathbf{W} = \mathbf{D}^{-1} \mathbf{G}, \quad (4.13)$$

where $\mathbf{D} = \text{diag}(|\mathcal{N}_1|, |\mathcal{N}_2|, \dots, |\mathcal{N}_N|)^T$ and $\mathbf{G}$ is the neighbourhood matrix with entries

$$g_{ij} = \begin{cases} 1 & \text{if and only if } j \in \mathcal{N}_i, \\ 0 & \text{otherwise.} \end{cases} \quad (4.14)$$

The ij entries of the Jacobian $\nabla \mathbf{F}$ of the fW -mean filters is given by

$$\frac{\partial F_i}{\partial \rho_j} = w_{ij} \frac{f'(\rho_j)}{f'(F_i(\rho))} \geq 0, \quad (4.15)$$

where the equality will occurs if and only if $w_{ij} = 0$. In other words, the inequality is strictly provided $w_{ij} > 0$, that is when j is in the neighbourhood of i .

Furthermore, all fW -mean filters have the property of mapping a vector with equal entries to itself, that is for any $c \in [0, 1]$

$$\mathbf{F}(c\mathbf{1}_N) = c\mathbf{1}_N. \quad (4.16)$$

However, in generally the fW -mean filters do not obey the volume-preservation, that is

$$\exists \rho \in [0, 1]^N \text{ such that } \mathbf{1}_N^T \mathbf{F}(\rho) \neq \mathbf{1}_N^T \rho. \quad (4.17)$$

In addition, there exists some fW -mean filter types that are not covered in the above fW -mean filter framework above. However, those filters can be fitted into the generalized fW -mean filter framework [20], that is

$$\mathbf{F}(\rho) = \mathbf{g}(\mathbf{W}\mathbf{f}(\rho)), \quad (4.18)$$

in which the function f^{-1} in (4.9) is replaced by a smooth function $g : f([0, 1]) \rightarrow [0, 1]$ that obeys $g(f(0)) = 0$ and $g(f(1)) = 1$.

4.3 Octagonal neighbourhood

In the previous sections, we have used a neighbourhood shape $\mathcal{N}$ for finding the neighbourhood elements in order to evaluate the weight matrix $\mathbf{W}$. The octagonal shaped neighbourhood in two-dimension is used [25]. As we mention above, the domain is partitioned into $N = n_x \times n_y$ element. With the positive filter radius r that satisfies $r < n_x/2$ and $r < n_y/2$, the two elements with center positions $\mathbf{p}_1 = (x_1, y_1)$ and $\mathbf{p}_2 = (x_2, y_2)$ are neighbours if

$$\begin{cases} \max\{|x_1 - x_2|, |y_1 - y_2|\} & \leq rh, \\ |x_1 - x_2| + |y_1 - y_2| & \leq (r + q)h \end{cases} \quad (4.19)$$

Figure 6: Octagonal shaped neighbourhood with filter radius $r = 10$ elements and parameter $q = 4$ elements.

where $0 < q < r$ is an integer that is used to define the neighbourhood shape. That means, $q = 0$ corresponds to a diamond shaped neighbourhood, $q = r$ corresponds to a square neighbourhood shape. In our case, the values of q is chosen as the nearest integer to $(2r + 1)/(2\sqrt{2} + 2)$ to make the sides of the octagonal shaped neighbourhood to be essentially the same length as $2q \approx \sqrt{2}(r + 1/2 - q)$. For example, the figure 6 below show the octagonal shaped neighbourhoods with $r = 10$ and then $q = 4$ elements.

4.4 Sensitivity evaluation

The effect of the filter on the sensitivities is found by determining $\mathbf{v}^T \nabla \mathbf{F}(\boldsymbol{\rho})$ for some vector $\mathbf{v} \in \mathbb{R}^N$. In the experiments, $\mathbf{v}$ is the gradient of the objective function or a constraint function with respect to the filtered design. By using the expression (4.15) we can find that

$$\sum_{i=1}^N v_i \frac{\partial F_i(\boldsymbol{\rho})}{\partial \rho_j} = \sum_{i=1}^N v_i w_{ij} \frac{f'(\rho_j)}{f'(F_i(\boldsymbol{\rho}))} = f'(\rho_j) \sum_{i=1}^N w_{ij} \frac{v_i}{f'(F_i(\boldsymbol{\rho}))}. \quad (4.20)$$

In other words, in order to modify sensitivities we need to implement matrix multiplication by $\mathbf{W}^T$. For the special case of equivalent weighting within the neighbourhoods, matrix multiplication by $\mathbf{W}^T$ can be changed into matrix multiplication by $\mathbf{G}^T$. If the neighbourhoods matrix are symmetric, then $\mathbf{G}^T = \mathbf{G}$ and we can exploit that symmetry to apply the same summation algorithm for both filtering and sensitivity calculation.

4.5 Cascaded fW -mean filters

However, in order to get filters with the desirable properties, the cascaded fW -mean filters are introduced [20]. Assume that we have a family of N fW -mean filters, hence

$$\mathbf{F}^{(K)}(\boldsymbol{\rho}) = \mathbf{f}_K^{-1} \left(\mathbf{W}^{(K)} \mathbf{f}_K(\boldsymbol{\rho}) \right), \quad K \in \{1, 2, \dots, n\}. \quad (4.21)$$

We define the cascaded filter function of order K that is denoted by $\mathbf{C}^{(K)} : [0, 1]^N \rightarrow [0, 1]^N$ to be the composition

$$\mathbf{C}^{(K)} = \mathbf{F}^{(K)} \circ \mathbf{F}^{(K-1)} \circ \dots \circ \mathbf{F}^{(1)}, \quad (4.22)$$

and we also denote

$$\boldsymbol{\rho}^{(K)} = \mathbf{C}^{(K)}(\boldsymbol{\rho}). \quad (4.23)$$

Therefore, the cascaded filter is applied sequentially

$$\boldsymbol{\rho}^{(K)} = \mathbf{F}^{(K)} \left(\mathbf{C}^{(K-1)}(\boldsymbol{\rho}) \right) = \mathbf{F}^{(K)} \left(\boldsymbol{\rho}^{(K-1)} \right), \quad (4.24)$$

where we use $\boldsymbol{\rho}^{(0)} = \boldsymbol{\rho}$.

Because of using gradient based optimization algorithm, we should modify sensitivities in the case of a cascade of n fW -mean filters by computing $\mathbf{v}^T \nabla \mathbf{C}^{(n)}(\boldsymbol{\rho})$. In order to implement that, assume that $n \geq 2$ and let $\mathbf{v}^{(n)} = \mathbf{v}$, by combining (4.23) and (4.24) as well as applying the chain rule [20]

$$\begin{aligned} \sum_{i=1}^N v_i^{(n)} \frac{\partial}{\partial \rho_j} C_i^{(n)}(\boldsymbol{\rho}) &= \sum_{i=1}^N v_i^{(n)} \frac{\partial}{\partial \rho_j} \left(F_i^{(n)} \left(\mathbf{C}^{(n-1)}(\boldsymbol{\rho}) \right) \right) \\ &= \sum_{i=1}^N v_i^{(n)} \sum_{k=1}^N \frac{\partial F_i^{(n)}}{\partial \rho_k} \bigg|_{\boldsymbol{\rho}^{(n-1)}} \frac{\partial}{\partial \rho_j} C_k^{(n-1)}(\boldsymbol{\rho}) \\ &= \sum_{k=1}^N \left(\sum_{i=1}^N v_i^{(n)} \frac{\partial F_i^{(n)}}{\partial \rho_k} \bigg|_{\boldsymbol{\rho}^{(n-1)}} \right) \frac{\partial}{\partial \rho_j} C_k^{(n-1)}(\boldsymbol{\rho}) \\ &= \sum_{k=1}^N v_k^{(n-1)} \frac{\partial}{\partial \rho_j} C_k^{(n-1)}(\boldsymbol{\rho}), \end{aligned} \quad (4.25)$$

where $v_k^{(n-1)}$ in the last step denotes

$$v_k^{(n-1)} = \sum_{i=1}^N v_i^{(n)} \frac{\partial F_i^{(n)}}{\partial \rho_k} \bigg|_{\boldsymbol{\rho}^{(n-1)}}. \quad (4.26)$$

The first and last lines of expression (4.25) have similar form. The expression (4.25) is evaluated by repeating the steps until the order of cascade in the last line is 1. Therefore, one can conclude that computing $\mathbf{v}^T \nabla \mathbf{C}^{(n)}(\boldsymbol{\rho})$ corresponds to computing the vectors $\mathbf{v}^{K-1}$ sequentially as

$$v_k^{(K-1)} = \sum_{i=1}^N v_i^{(K)} \frac{\partial F_i^{(K)}}{\partial \rho_k} \bigg|_{\boldsymbol{\rho}^{(K-1)}}, \quad K = N, N-1, \dots, 1, \quad (4.27)$$

where as above $\mathbf{v}^{(n)} = \mathbf{v}$. The right hand side of (4.25) is the same form with (4.20).

However, because of sing the physical design as (3.20), the linear elastic problem may not be unique solvable. The solution for that problem will be discussed in the next chapter.

5 Preconditioning of the linear system

Kasolis [26] proved that, if we use the physical design as (3.6) with $\underline{\rho} \in (0, 1/2]$ and $h \in (0, 1]$ the stiffness matrix $\mathbf{K}(\underline{\rho})$ associated with finite element of linear elastic problems (2.35) satisfies

$$\text{cond}(\mathbf{K}(\underline{\rho})) \leq Ch^{-2}\underline{\rho}^{-1}. \quad (5.1)$$

However, we would like to use $\underline{\rho} = 0$ and it will cause that

$$\text{cond}(\mathbf{K}(\underline{\rho})) \rightarrow \infty. \quad (5.2)$$

We propose the solution to this problem is precondition the stiffness matrix. In this chapter, we introduce the preconditioner D as

$$D = \text{diag}\left(\frac{1}{B_1}, \frac{1}{B_2}, \dots, \frac{1}{B_N}\right), \quad (5.3)$$

where

$$B_i = \sum_{\{n | \text{supp}(\varphi_i) \cup E_n \neq \emptyset\}} \rho_n. \quad (5.4)$$

Hence, the linear system (3.21) is modified as

$$D^{-1/2}\mathbf{K}(\underline{\rho})D^{-1/2}\tilde{\mathbf{u}} = D^{-1/2}\mathbf{f}, \quad (5.5)$$

or the reduced expression

$$\tilde{\mathbf{K}}(\underline{\rho})\tilde{\mathbf{u}} = \tilde{\mathbf{f}}, \quad (5.6)$$

where

$$\tilde{\mathbf{K}}(\underline{\rho}) = D^{-1/2}\mathbf{K}(\underline{\rho})D^{-1/2}, \quad (5.7)$$

$$\tilde{\mathbf{f}} = D^{-1/2}\mathbf{f}, \quad (5.8)$$

$$\mathbf{u} = D^{-1/2}\tilde{\mathbf{u}}. \quad (5.9)$$

For example: Consider a bar occupying the interval $I = [0, L]$ and subjected to a line load f . We would like to find the vertical displacement u of the bar. The equilibrium displacement of the bar is the solution to the following variational problem.

Find $\mathbf{u} \in \mathcal{V} = \{\mathbf{v} : \|\mathbf{v}\|_{L^2(I)} < \infty, \|\mathbf{v}'\|_{L^2(I)} < \infty\}$ such that

$$a(\underline{\rho}; \mathbf{u}, \mathbf{v}) = l(\mathbf{v}), \quad \forall \mathbf{v} \in \mathcal{V}. \quad (5.10)$$

Figure 7: One-dimension geometry of the minimization of compliance for a bar.

Where the energy bilinear form $a(\cdot, \cdot)$ and the load linear form $l(\cdot)$ are

$$a(\rho; \mathbf{u}, \mathbf{v}) = \int_0^L \tilde{\rho}(\rho) A E \mathbf{u}' \mathbf{v}' dx, \quad (5.11)$$

$$l(\mathbf{v}) = \int_0^L \mathbf{b} \mathbf{v} dx. \quad (5.12)$$

The problem (5.10) is discretized in order to be solve by finite element method. We partition the domain I in to $N = 3$ intervals with $M = 4$ nodes as demonstrated geometry in Figure 7. The physical design of the first element should be one because it is attached on the wall. Let $\mathbf{u} = (u_1, u_2, u_3, u_4)^T$ and by using interpolation with the finite element shape function ϕ_i , the displacement field $u_h(x)$ is defined as the continuous, piecewise bilinear approximation

$$\mathbf{u}_h(x) = \sum_{j=1}^4 u_j \phi_j(x), \quad i = 1, \dots, 4, \quad (5.13)$$

at any point $x \in I$. Let $\rho = (\rho_1, \rho_2, \rho_3)^T$ denotes the degrees of freedom for design variables ρ , then the physical design can be defined as (3.20). That is $\tilde{\rho}(\rho) = P(F(\rho))$. Hence, the problem (5.10) is approximated by solving linear system

$$\mathbf{K}(\rho) \mathbf{u} = \mathbf{f}, \quad (5.14)$$

where the $\mathbf{K}(\rho)$ is stiffness matrix that can be computed by using test function $\mathbf{v} = \phi_i$, $i = 1, \dots, 4$

$$K_{ij} = \int_0^L P(F(\rho)) A E \phi_j' \phi_i' dx, \quad (5.15)$$

and $\mathbf{f}$ is load vector that is defined as

$$f_i = \int_0^L \mathbf{b} \phi_i dx. \quad (5.16)$$

The stiffness matrix (5.15) can be formed by summing all the element stiffness matrices $\mathbf{K}^{(i)}$

$$\mathbf{K}(\rho) = \sum_{i=1}^3 P(F_i(\rho)) \mathbf{K}^{(i)}. \quad (5.17)$$

The element stiffness matrix $\mathbf{K}^{(i)}$ is defined as in [1]

$$\mathbf{K}^{(i)} = \frac{AE}{h_i} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}. \quad (5.18)$$

Substitute expression (5.18) into (5.17) then we can determine the stiffness matrix

$$\mathbf{K}(\boldsymbol{\rho}) = \frac{AE}{h} \begin{bmatrix} \tilde{\rho}_1 & -\tilde{\rho}_1 & 0 & 0 \\ -\tilde{\rho}_1 & \tilde{\rho}_1 + \tilde{\rho}_2 & -\tilde{\rho}_2 & 0 \\ 0 & -\tilde{\rho}_2 & \tilde{\rho}_2 + \tilde{\rho}_3 & -\tilde{\rho}_3 \\ 0 & 0 & -\tilde{\rho}_3 & \tilde{\rho}_3 \end{bmatrix} \quad (5.19)$$

$$= \frac{AE}{h} \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 + \tilde{\rho}_2 & -\tilde{\rho}_2 & 0 \\ 0 & -\tilde{\rho}_2 & \tilde{\rho}_2 + \tilde{\rho}_3 & -\tilde{\rho}_3 \\ 0 & 0 & -\tilde{\rho}_3 & \tilde{\rho}_3 \end{bmatrix} \quad (5.20)$$

where $\tilde{\boldsymbol{\rho}} = P(F(\boldsymbol{\rho})) = (\tilde{\rho}_1, \tilde{\rho}_2, \tilde{\rho}_3)^T$ is the vector of physical design and $h_1 = h_2 = h_3 = h$ is the mesh size. Here, we assume that $\tilde{\rho}_1 = 1$ because it contains the first node that is fixed node.

We then apply the essential boundary condition $\mathbf{u} = 0$ at $\mathbf{x} = 0$, that mean $u_1 = 0$. The stiffness matrix is reduced into $\mathbf{K}_r(\boldsymbol{\rho})$

$$\mathbf{K}_r(\boldsymbol{\rho}) = \frac{AE}{h} \begin{bmatrix} 1 + \tilde{\rho}_2 & -\tilde{\rho}_2 & 0 \\ -\tilde{\rho}_2 & \tilde{\rho}_2 + \tilde{\rho}_3 & -\tilde{\rho}_3 \\ 0 & -\tilde{\rho}_3 & \tilde{\rho}_3 \end{bmatrix} \quad (5.21)$$

In order to find the condition number of $\mathbf{K}_r(\boldsymbol{\rho})$, we need to compute its inverse

$$\mathbf{K}_r^{-1}(\boldsymbol{\rho}) = \frac{h}{AE} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 + \frac{1}{\tilde{\rho}_2} & 1 + \frac{1}{\tilde{\rho}_2} \\ 1 & 1 + \frac{1}{\tilde{\rho}_2} & 1 + \frac{1}{\tilde{\rho}_2} + \frac{1}{\tilde{\rho}_3} \end{bmatrix}. \quad (5.22)$$

Therefore, we can find the condition number of stiffness matrix with out preconditioning is

$$\text{cond}(\mathbf{K}_r) = \|\mathbf{K}_r\|_{\infty} \|\mathbf{K}_r^{-1}\|_{\infty} \quad (5.23)$$

$$= \left(\max_{1 \leq i \leq 3} \sum_{j=1}^3 |K_{ij}| \right) \left(\max_{1 \leq i \leq 4} \sum_{j=1}^3 |K_{ij}^{(-1)}| \right), \quad (5.24)$$

$$= \max(1 + 2\tilde{\rho}_2, 2\tilde{\rho}_2 + 2\tilde{\rho}_3) \left(3 + \frac{2}{\tilde{\rho}_2} + \frac{1}{\tilde{\rho}_3} \right), \quad (5.25)$$

$$= 2 \left(\max(0.5 + \tilde{\rho}_2, \tilde{\rho}_2 + \tilde{\rho}_3) \right) \left(3 + \frac{2}{\tilde{\rho}_2} + \frac{1}{\tilde{\rho}_3} \right), \quad (5.26)$$

where K_{ij} and $K_{ij}^{(-1)}$ is the entries of matrix $\mathbf{K}(\boldsymbol{\rho})$ and $\mathbf{K}^{-1}(\boldsymbol{\rho})$, respectively; and $\text{cond}(\mathbf{K}_r)$ denotes condition number of stiffness matrix $\mathbf{K}_r$. The expression (5.26) shows that the condition number of $\mathbf{K}_r$ depends on $\tilde{\rho}_3$ and it will go to infinity if either $\tilde{\rho}_2$ or $\tilde{\rho}_3$ is zero. In other words, there exists no solution to this problem.

Now we consider the stiffness matrix with preconditioning as

$$\tilde{\mathbf{K}}(\rho)\tilde{\mathbf{u}} = \tilde{\mathbf{f}}, \quad (5.27)$$

where

$$\tilde{\mathbf{K}}(\rho) = D^{-1/2}\mathbf{K}(\rho)D^{-1/2} \quad (5.28)$$

$$\tilde{\mathbf{f}} = D^{-1/2}\mathbf{f} \quad (5.29)$$

$$\mathbf{u} = D^{-1/2}\tilde{\mathbf{u}} \quad (5.30)$$

First of all, we need to find preconditioner matrix D . That is computed as

$$D_{ii} = \begin{cases} \tilde{\rho}_{i-1} + \tilde{\rho}_i & \text{if } 1 < i < N, \\ \tilde{\rho}_i & \text{if } i = 1, N. \end{cases} \quad (5.31)$$

Hence, matrix D is

$$D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 + \tilde{\rho}_2 & 0 & 0 \\ 0 & 0 & \tilde{\rho}_2 + \tilde{\rho}_3 & 0 \\ 0 & 0 & 0 & \tilde{\rho}_3 \end{bmatrix} \quad (5.32)$$

By substituting (5.32) into (5.28) we get

$$\tilde{\mathbf{K}}(\rho) = \begin{bmatrix} 1 & -\frac{1}{\sqrt{\tilde{\rho}_2+1}} & 0 & 0 \\ -\frac{1}{\sqrt{\tilde{\rho}_2+1}} & 1 & -\frac{\tilde{\rho}_2}{\sqrt{(1+\tilde{\rho}_2)(\tilde{\rho}_2+\tilde{\rho}_3)}} & 0 \\ 0 & -\frac{\tilde{\rho}_2}{\sqrt{(1+\tilde{\rho}_2)(\tilde{\rho}_2+\tilde{\rho}_3)}} & 1 & -\frac{\sqrt{\tilde{\rho}_3}}{\sqrt{\tilde{\rho}_2+\tilde{\rho}_3}} \\ 0 & 0 & -\frac{\sqrt{\tilde{\rho}_3}}{\sqrt{\tilde{\rho}_2+\tilde{\rho}_3}} & 1 \end{bmatrix} \quad (5.33)$$

As above, we also apply the essential boundary condition to reduce the stiffness matrix into

$$\tilde{\mathbf{K}}_r(\rho) = \begin{bmatrix} 1 & -\frac{\tilde{\rho}_2}{\sqrt{(1+\tilde{\rho}_2)(\tilde{\rho}_2+\tilde{\rho}_3)}} & 0 \\ -\frac{\tilde{\rho}_2}{\sqrt{(1+\tilde{\rho}_2)(\tilde{\rho}_2+\tilde{\rho}_3)}} & 1 & -\frac{\sqrt{\tilde{\rho}_3}}{\sqrt{\tilde{\rho}_2+\tilde{\rho}_3}} \\ 0 & -\frac{\sqrt{\tilde{\rho}_3}}{\sqrt{\tilde{\rho}_2+\tilde{\rho}_3}} & 1 \end{bmatrix}. \quad (5.34)$$

Then its inverse matrix is computed as

$$\tilde{\mathbf{K}}_r^{-1}(\rho) = \begin{bmatrix} \tilde{\rho}_2 + 1 & \sqrt{(\tilde{\rho}_2 + 1)(\tilde{\rho}_2 + \tilde{\rho}_3)} & \sqrt{(\tilde{\rho}_2 + 1)\tilde{\rho}_3} \\ \sqrt{(\tilde{\rho}_2 + 1)(\tilde{\rho}_2 + \tilde{\rho}_3)} & 1 + \tilde{\rho}_2 + \tilde{\rho}_3 + \frac{\tilde{\rho}_3}{\tilde{\rho}_2} & \frac{(\tilde{\rho}_2 + 1)\sqrt{(\tilde{\rho}_2 + \tilde{\rho}_3)\tilde{\rho}_3}}{\tilde{\rho}_2} \\ \sqrt{(\tilde{\rho}_2 + 1)\tilde{\rho}_3} & \frac{(\tilde{\rho}_2 + 1)\sqrt{(\tilde{\rho}_2 + \tilde{\rho}_3)\tilde{\rho}_3}}{\tilde{\rho}_2} & 1 + \tilde{\rho}_3 + \frac{\tilde{\rho}_3}{\tilde{\rho}_2} \end{bmatrix}. \quad (5.35)$$

Because of the complexity of computing condition number of stiffness matrix after preconditioning $\tilde{\mathbf{K}}_r(\rho)$, we therefore analyze the condition number of $\tilde{\mathbf{K}}_r(\rho)$ further by demonstrating it in the surface plot as figure 8 of a function of $\tilde{\rho}_2$ and $\tilde{\rho}_3$ where $(\tilde{\rho}_2, \tilde{\rho}_3) \in [0, 1]^2$.

Figure 8: The surface plot for the variation of condition number of stiffness matrix with preconditioning as the function of $\tilde{\rho}_2$ and $\tilde{\rho}_3$.

Figure 9: Geometry of the minimization of compliance for a 1D bar when $\tilde{\rho}_2 = \epsilon$, $\tilde{\rho}_3 = 1$ and $\epsilon \rightarrow 0$.

Figure 8 shows that, the condition number of $\tilde{\mathbf{K}}_r(\boldsymbol{\rho})$ heavily depends on $\tilde{\rho}_3$ when $\tilde{\rho}_2 = 0$. Hence, we consider in two cases. The first one is $\tilde{\rho}_2 = 0$ and $\tilde{\rho}_3 = 1$ and the second one is $\tilde{\rho}_2 = \tilde{\rho}_3 = 0$.

On one hand, the condition number will go to infinity if $\tilde{\rho}_2 = 0$ and $\tilde{\rho}_3 = 1$. However, it is reasonable because there is no connection between the first element and the third element (see Figure 9). Therefore, we cannot find the equilibrium displacement field for this problem. On another hand, when $\tilde{\rho}_2 = \tilde{\rho}_3 = 0$, the linear system will be well-conditioned if $f_2 = f_3 = 0$ because there is no reason to apply a force in to the void. This case is illustrated in Figure 10.

Furthermore, when $\tilde{\rho}_2 > 0$ the condition number drops significantly as well as it reduces with the descending order of $\tilde{\rho}_3$. Therefore, we can conclude that there exists a solution to the problems after using preconditioning approach.

Figure 10: Geometry of the minimization of compliance for a 1D bar when $\tilde{\rho}_2 = \tilde{\rho}_3 = \epsilon$, $\epsilon \rightarrow 0$.

6 Numerical experiments and results

The experiment is performed as considering the minimization of the compliance for the cantilever beam problems in the domain in 2. The beam is held along the left hand side that is the boundary portion denoted Γ_D in the figure. A downward vertical force is distributed uniformly over Γ_F that is the middle 10% of the beam's right side boundary. We used SIMP that is $P(x) = x^p$ with $p = 3$ as our penalization method. The optimality criteria method is used to minimize the compliance with damping parameter $\nu = 1/2$.

In all experiments, the open-close (open followed by close) filtering procedure over octagonal shaped neighbourhoods. More particularly, the harmonic open-close filter as cascade of four fW -mean filters that is defined by [20].

$$f_1(x) = f_4(x) = \frac{1}{x + \alpha}, \quad (6.1)$$

$$f_2(x) = f_3(x) = \frac{1}{1 - x + \alpha}, \quad (6.2)$$

and $g_K = f_K^{-1}$, for $K = 1, 2, \dots, 4$.

In which, the fixed parameter $\alpha = 10^{-4}$ is used. The experiments are performed on the different size domains which are 384×256 and 768×512 elements and the volume fraction is $V = 0.5$. The figures 11 and 12 show the results of the optimized cantilever beams using different neighbourhood sizes. In order to evaluate results of optimized physical design by using our approach. The physical design is evaluated by "measuring of discreteness". Hence, a so-called "measure of non-discreteness" quantity M_{nd} was introduced by Sigmund in [16] as

$$M_{nd} = \frac{4\mathbf{F}(\boldsymbol{\rho})^T(\mathbf{1}_N - \mathbf{F}(\boldsymbol{\rho}))}{N} \times 100 \in [0, 1], \quad (6.3)$$

to quantify the discreteness of a physical design. In other words, if there are no elements with intermediate density values, M_{nd} is 0 % and if there are all elements with density values is 0.5, that is the physical design is totally grey, M_{nd} is 1 %.

Figure 11 shows the optimized cantilever beams with the relative ratio is 1 between the filter radius in the open and close step on the different size domains. The top figure is the result of topology optimization for 384×256 elements domain after 100 iterations of optimizing and the measure of non-discreteness of the physical design is 0.13%. On the bottom figure, the experiment was performed on 768×512 elements domain and it took 98 iterations to optimize and the final measure of non-discreteness of the physical design is 0.21 %. Comparing these two figures, we can see the smaller size domain had jagged edges while the bigger one had smooth edges. Therefore the larger domain size is, the better physical design will be obtained.

In Figure 12, the upper-right and upper-left cantilever beams took 399 and 378 iterations to be optimized, respectively. The lower-left took 66 iterations and the lower-right took 392

Figure 11: Cantilever beams optimized using 384×256 (top) and 768×512 (bottom) elements and different harmonic open-close filter. The octagonal neighbourhoods are indicated in the upper-right corner of each sub-figure. The relative ratio between the filter radius used in the open and close step is 1.

iterations. The measure of non-discreteness of a physical design of the upper-left one is 0.17 %, the upper-right is 0.39 %, the lower-left and the lower-right are 0.09 %, 0.35 % respectively. These figures show the effect of filter radius on the physical designs.

The figure 13 illustrates the changes of condition number during each iteration. It is easy to see that without preconditioning, the condition number of stiffness matrix rockets to infinity rapidly; meanwhile with preconditioning, the condition number stays stably at very low values.

Figure 12: Cantilever beam optimized using 768×512 elements and different harmonic open-close filter. The octagonal neighbourhoods are indicated in the upper-right corner of each sub-figure. The relative ratio between the filter radius used in the open and close step of two top and bottom sub-figures and is 4 and $1/4$ respectively.

Figure 13: The variation of condition number of stiffness matrix with and without preconditioning during topology optimization iterations.

7 Conclusions

In this thesis, we have reviewed, analyzed and implement the topology optimization material distribution on the minimizing the compliance problem with zero lower bound on physical design by using preconditioning technique. The most theory and analyzing have been discussed along with the numerical solution except the general analytical theory on the use of the preconditioning method.

However, we have shown that there exists the solution of using the physical design with $\underline{\rho} = 0$ for the minimization of compliance for the cantilever beam problem through the example of the one-dimension linear elastic problem. The non-existence and non-uniqueness of solution when using lower bound $\underline{\rho} = 0$ have shown in according to theorems that are proving in [26]. After using the preconditioning, the linear system became well-conditioned. The minimization compliance of cantilever beam problem in two-dimension was also implemented from a many thousands elements scale topology optimization. The filtering procedure that is a cascaded of four fW -mean filter that were mentioned in [20] is used to gain a mesh-independent solution. That is shown in the results that the physical design is almost black and white. Because of limitation of my computer, I have not experimented on the large-scale problems. There is only 768×512 elements of problems have done with different filters radius.

In the future work, we will attempts to prove analytically the preconditioning method for the use of the zero lower bound of physical design as well as try to use different morphology-based filters.

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