Subgraphs with large minimum ℓ-degree in hypergraphs where almost all ℓ-degrees are large

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Abstract

Let G be an r-uniform hypergraph on n vertices such that all but at most $\varepsilon\binom{n}{\ell}$ ℓ-subsets of vertices have degree at least $p\binom{n-\ell}{r-\ell}$. We show that G contains a large subgraph with high minimum ℓ-degree.

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1 Introduction

Given $r \in \mathbb{N}$ and a set A, we write $A^{(r)}$ for the collection of all r-subsets of A and $[n]$ for the set $\{1, 2, \dots n\}$. An *r-graph*, or *r-uniform hypergraph*, is a pair $G = (V, E)$, where $V = V(G)$ is a set of vertices and $E = E(G) \subseteq V^{(r)}$ is a collection of r-subsets, which constitute the edges of G. We say G is *nonempty* if it contains at least one edge and set $v(G) = |V(G)|$ and $e(G) = |E(G)|$. A *subgraph* of G is an r-graph H with $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. The subgraph of G *induced* by a set $X \subseteq V(G)$ is $G[X] = (X, E(G) \cap X^{(r)})$.

Let $\mathcal{F}$ be a family of nonempty r-graphs. If G does not contain a copy of a member of $\mathcal{F}$ as a subgraph, we say that G is *$\mathcal{F}$-free*. The *Turán number* $\mathrm{ex}(n, \mathcal{F})$ of a family $\mathcal{F}$ is the maximum number of edges in an $\mathcal{F}$-free r-graph on n vertices, and its *Turán density* is the limit $\pi(\mathcal{F}) = \lim_{n\to\infty} \mathrm{ex}(n, \mathcal{F})/\binom{n}{r}$ (this is easily shown to exist). Let $K_t^{(r)} = ([t], [t]^{(r)})$ denote the complete r-graph on t vertices. Determining $\pi(K_t^{(r)})$ for *any* $t > r \geqslant 3$ is a

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major problem in extremal combinatorics. Turán [19] famously conjectured in 1941 that $\pi(K_4^{(3)}) = 5/9$, and despite much research effort this remains open [8]. In this paper we shall be interested in some variants of Turán density.

The *neighbourhood* $N(S)$ of an ℓ-subset $S \in V(G)^{(\ell)}$ is the collection of $(r-\ell)$-subsets $T \in V(G)^{(r-\ell)}$ such that $S \cup T$ is an edge of G. The *degree* of S is the number $\deg(S)$ of edges of G containing S, that is, $\deg(S) = |N(S)|$. The minimum ℓ-degree of G, $\delta_\ell(G)$, is defined to be the minimum of $\deg(S)$ over all ℓ-subsets $S \in V(G)^{(\ell)}$. The *Turán ℓ-degree threshold* $\mathrm{ex}_\ell(n, \mathcal{F})$ of a family $\mathcal{F}$ of r-graphs is the maximum of $\delta_\ell(G)$ over all $\mathcal{F}$-free r-graphs G on n vertices. It can be shown [11, 9] that the limit $\pi_\ell(\mathcal{F}) = \lim_{n\to\infty} \mathrm{ex}_\ell(n, \mathcal{F})/\binom{n-\ell}{r-\ell}$ exists; this quantity is known as the *Turán ℓ-degree density* of $\mathcal{F}$. A simple averaging argument shows that

$$0 \leqslant \pi_{r-1}(\mathcal{F}) \leqslant \ldots \leqslant \pi_2(\mathcal{F}) \leqslant \pi_1(\mathcal{F}) = \pi(\mathcal{F}) \leqslant 1,$$

and it is known that $\pi_\ell(\mathcal{F}) \neq \pi(\mathcal{F})$ in general (for $\ell \notin \{0, 1\}$). In the special case where $(r, \ell) = (r, r-1)$, $\pi_{r-1}(\mathcal{F})$ is known as the *codegree density* of $\mathcal{F}$.

There has been much research on Turán ℓ-degree threshold for r-graphs when $(r, \ell) = (3, 2)$. In the late 1990s, Nagle [12] and Nagle and Czygrinow [2] conjectured that $\pi_2(K_4^{(3)-}) = 1/4$ and $\pi_2(K_4^{(3)}) = 1/2$, respectively. Here $K_4^{(3)-}$ denotes the 3-graph obtained by removing one edge from $K_4^{(3)}$. Falgas-Ravry, Pikhurko, Vaughan and Volec [6, 7] recently proved $\pi_2(K_4^{(3)-}) = 1/4$, settling the conjecture of Nagle, and showed all near-extremal constructions are close (in edit distance) to a set of quasirandom tournament constructions of Erdős and Hajnal [3]. The lower bound $\pi_2(K_4^{(3)}) \geqslant 1/2$ also comes from a quasirandom construction, which is due to Rödl [17]. For $t > r \geqslant 3$, the codegree density $\pi_{r-1}(K_t^{(r)})$ has been studied by Falgas-Ravry [4], Lo and Markström [9] and Sidorenko [18]. Recently, Lo and Zhao [10] showed that $1 - \pi_{r-1}(K_t^{(r)}) = \Theta(\ln t/t^{r-1})$ for $r \geqslant 3$.

One variant of ℓ-degree Turán density is to study r-graphs in which almost all ℓ-subsets have large degree. To be precise, given $\varepsilon > 0$, let $\delta_\ell^\varepsilon(G)$ be the largest integer d such that all but at most $\varepsilon\binom{v(G)}{\ell}$ of the ℓ-subsets $S \in V(G)^{(\ell)}$ satisfy $\deg(S) \geqslant d$. Note that r-graphs with large $\delta_\ell^\varepsilon(G)$ but with small $\delta_\ell(G)$ arise naturally. For instance, the reduced graphs R obtained from r-graphs with large minimum ℓ-degree after an application of hypergraph regularity lemma have large $\delta_\ell^\varepsilon(R)$.

Definition 1 (**(r, ℓ)-sequence**)**.** Let $1 \leqslant \ell < r$. We say that a sequence $\mathbf{G} = (G_n)_{n\in\mathbb{N}}$ of r-graphs is an *(r, ℓ)-sequence* if

(i) $v(G_n) \to \infty$ as $n \to \infty$ and

(ii) there is a constant $p \in [0, 1]$ and a sequence of nonnegative reals $\varepsilon_n \to 0$ as $n \to \infty$ such that $\delta_\ell^{\varepsilon_n}(G_n) \geqslant p\binom{v(G_n)-\ell}{r-\ell}$ for each n.

We refer to the supremum of all $p \geqslant 0$ for which (ii) is satisfied as the *density* of the sequence $\mathbf{G}$ and denote it by $\rho(\mathbf{G})$.

We can define the analogue of Turán density for (r, ℓ)-sequences.

Definition 2. Let $1 \leqslant \ell < r$. Let $\mathcal{F}$ be a family of nonempty r-graphs. Define

$$\pi^\star_\ell(\mathcal{F}) := \sup\Big\{\rho(\mathbf{G}) : \ \mathbf{G} \text{ is an } (r,\ell)\text{-sequence of } \mathcal{F}\text{-free } r\text{-graphs}\Big\}.$$

Our main result show that every large r-graph G contains a 'somewhat large' subgraph H with minimum ℓ-degree satisfying $\delta_\ell(H)/\binom{v(H)-\ell}{r-\ell} \approx \delta^\varepsilon_\ell(G)/\binom{v(G)-\ell}{r-\ell}$. Here 'somewhat large' means $v(H) = \Omega(\varepsilon^{1/\ell})$.

Theorem 3. *Let $1 \leqslant \ell < r$. For any fixed $\delta > 0$, there exists $m_0 > 0$ such that any r-graph G on $n \geqslant m \geqslant m_0$ vertices with $\delta^\varepsilon_\ell(G) \geqslant p\binom{n-\ell}{r-\ell}$ for some $\varepsilon \leqslant m^{-\ell}/2$ contains an induced subgraph H on m vertices with*

$$\delta_\ell(H) \geqslant (p-\delta)\binom{m-\ell}{r-\ell}.$$

This immediate implies the $\pi^\star_\ell(\mathcal{F}) = \pi_\ell(\mathcal{F})$ for all families $\mathcal{F}$ of r-graphs.

Corollary 4. *For any $1 \leqslant \ell < r$ and any family $\mathcal{F}$ of nonempty r-graphs, $\pi^\star_\ell(\mathcal{F}) = \pi_\ell(\mathcal{F})$.*

We note that the (tight) upper bounds for codegree densities $\pi_2(F)$ for 3-graphs F obtained by flag algebraic methods in [5, 6, 7] actually relied on giving upper bounds for $\pi^\star_\ell(F)$. Corollary 4 provides theoretical justification for why this strategy could give optimal bounds.

1.1 Quasirandomness in 3-graphs

One of the main motivations for this note comes from recent work of Reiher, Rödl and Schacht [13, 14, 15, 16] on extremal questions for quasirandom hypergraphs. These authors studied the following notion of quasirandomness for 3-graphs.

Definition 5 ((1,2)-quasirandomness)**.** A 3-graph G is $(p, \varepsilon, (1, 2))$-*quasirandom* if for every set of vertices $X \subseteq V$ and every set of pairs of vertices $P \subseteq V^{(2)}$, the number $e_{1,2}(X, P)$ of pairs $(x, uv) \in X \times P$ such that $\{x\} \cup \{uv\} \in E(G)$ satisfies:

$$\Big|e_{1,2}(X,P) - p|X| \cdot |P|\Big| \leqslant \varepsilon v(G)^3.$$

We define a $(1, 2)$-quasirandom sequence and the corresponding extremal density, denoted by $\pi_{(1,2)-qr}(\mathcal{F})$, analogously to the way we defined (r, ℓ)-sequences and $\pi^\star_\ell(\mathcal{F})$ in Definitions 1 and 2. It is not difficult to see that $\pi_{(1,2)-qr}(\mathcal{F}) \leqslant \pi(\mathcal{F})$ for all families $\mathcal{F}$ of 3-graphs. Moreover, a $(p, \varepsilon, (1, 2))$-quasirandom 3-graph G satisfies $\delta_2^{\sqrt{\varepsilon}}(G) \geqslant (p-4\sqrt{\varepsilon})v(G)$. Hence, Theorem 3 and Corollary 6 imply the following.

Corollary 6. *For any family of nonempty 3-graphs $\mathcal{F}$, $\pi_{(1,2)-qr}(\mathcal{F}) \leqslant \pi_2(\mathcal{F})$.*

Consider a $(p, \varepsilon, (1,2))$-quasirandom 3-graph G for some $p > 4\sqrt{\varepsilon} > 0$. As noted above, $\delta_2^{\sqrt{\varepsilon}}(G) \geqslant (p - 4\sqrt{\varepsilon})v(G)$. Thus provided $v(G)$ is sufficiently large, Theorem 3 tells us we can find a subgraph H of G on $m = \Omega(\varepsilon^{-1/4})$ vertices with strictly positive minimum codegree (at least $(p - 4\sqrt{\varepsilon})m$).

However, as we show below, we cannot guarantee the existence of any subgraph with strictly positive codegree on more than $2/\varepsilon + 1$ vertices: our lower bound on m above in terms of an inverse power of the error parameter ε is thus sharp up to the value of the exponent.

Proposition 7. *For every $p \in (0,1)$ and every $\varepsilon > 0$, there exists n_0 such that for all $n \geqslant n_0$ there exist $(p, 2\varepsilon, (1,2))$-quasirandom 3-graphs in which every subgraph on $m \geqslant \lfloor \varepsilon^{-1} \rfloor + 1$ vertices has minimum codegree equal to zero.*

Proof. Let $G = (V, E)$ be a $(p, \varepsilon, (1,2))$-quasirandom 3-graph on n vertices. Such a 3-graph can be obtained for example by taking a typical instance of an Erdős–Rényi random 3-graph with edge probability p. Consider a balanced partition of V into $N = \lfloor \varepsilon^{-1} \rfloor$ sets $V = \bigcup_{i=1}^{N} V_i$ with $\lfloor n/N \rfloor \leqslant |V_1| \leqslant |V_2| \leqslant \ldots \leqslant |V_N| \leqslant \lceil n/N \rceil$. Now let G' be the 3-graph obtained from G by deleting all triples that meet some V_i in at least two vertices for some i: $1 \leqslant i \leqslant N$.

By construction, every set of $N+1$ vertices in G' must contain at least two vertices from the same V_i, and thus must induce a subgraph of G' with minimum codegree zero. Note that $e(G) - e(G') \leqslant Nn\binom{\lceil n/N \rceil}{2} \leqslant n^3/N \leqslant \varepsilon n^3$. Since G is $(p, \varepsilon, (1,2))$-quasirandom, it follows that G' is $(p, 2\varepsilon, (1,2))$-quasirandom. ☐

2 Finding high minimum ℓ-degree subgraphs in r-graphs with large δ_ℓ^ε

In this section we show how we can extract arbitrarily large subgraphs with high minimum ℓ-degree from sufficiently large r-graphs with sufficiently small error ε. To do so, we will need Azuma's inequality (see e.g. [1]).

Lemma 8 (Azuma's inequality)**.** *Let $\{X_i : i = 0, 1, \ldots\}$ be a martingale with $|X_i - X_{i-1}| \leqslant c_i$ for all i. Then for all positive integers N and $\lambda > 0$,*

$$\mathbb{P}(X_N \leqslant X_0 - \lambda) \leqslant \exp\left(\frac{-\lambda^2}{2\sum_{i=1}^{N} c_i^2}\right).$$

Proof of Theorem 3. We may assume without loss of generality that $\delta > 0$ is small enough to ensure $\delta^{-1} \geqslant 26\ell(r-\ell)^2 \log(1/\delta)$ and $\ell \log(1/\delta) \geqslant \log 2$ as this only makes our task harder. Set $m_0 = \lceil 26\ell(r-\ell)^2 \delta^{-2} \log(1/\delta) \rceil$. Note that this implies that

$$2\ell \log m_0 \leqslant 4\ell \log\left(26\ell(r-\ell)^2\delta^{-2}\log(1/\delta)\right) \leqslant 12\ell \log(1/\delta). \tag{1}$$

Fix $m \geqslant m_0$. Let $n \geqslant m \geqslant m_0$ and $\varepsilon = m^{-\ell}/2$.

Suppose $G = (V, E)$ is an r-graph on n vertices with $\delta_\ell^\varepsilon(G) \geqslant p\binom{n-\ell}{r-\ell}$. We claim that it contains an induced subgraph on m vertices with minimum ℓ-degree at least $(p-\delta)\binom{m-\ell}{r-\ell}$. For $p \leqslant \delta$, we have nothing to prove, so we may assume that $1 \geqslant p > \delta$.

Call an ℓ-subset $S \in V^{(\ell)}$ *poor* if $\deg(S) < p\binom{n-\ell}{r-\ell}$, and *rich* otherwise. Let $\mathcal{P}$ be the collection of all poor ℓ-subsets. By our assumption on $\delta_\ell^\varepsilon(G)$, $|\mathcal{P}| \leqslant \varepsilon\binom{n}{\ell}$. As each poor ℓ-subset is contained in $\binom{n-\ell}{m-\ell}$ m-subsets, it follows that there are at least

$$\binom{n}{m} - |\mathcal{P}|\binom{n-\ell}{m-\ell} > \left(1 - \varepsilon m^\ell\right)\binom{n}{m} = \frac{1}{2}\binom{n}{m} \tag{2}$$

m-subsets of vertices which do not contain any poor ℓ-subsets.

Given an ℓ-subset $S \in V^{(\ell)} \setminus \mathcal{P}$, we call an m-subset T of V *bad for* S if $S \subseteq T$ and $\left|N(S) \cap T^{(r-\ell)}\right| \leqslant (p-\delta)\binom{m-\ell}{r-\ell}$. Let ϕ_S be the number of bad m-subsets for S. We claim that

$$\phi_S \leqslant \binom{n-\ell}{m-\ell} \exp\left(-\frac{\delta^2 m}{2(r-\ell)^2}\right). \tag{3}$$

Observe that

$$\phi_S = \left|\left\{T \in (V \setminus S)^{(m-\ell)} \colon \left|N(S) \cap T^{(r-\ell)}\right| \leqslant (p-\delta)\binom{m-\ell}{r-\ell}\right\}\right|.$$

Let X be the random variable $\left|N(S) \cap T^{(r-\ell)}\right|$, where T is an $(m-\ell)$-subset of $V \setminus S$ picked uniformly at random. We consider the vertex exposure martingale on T. Let Z_i be the ith exposed vertex in T. Define $X_i = \mathbb{E}(X|Z_1, \dots, Z_i)$. Note that $\{X_i : i = 0, 1, \dots, m-\ell\}$ is a martingale and $X_0 \geqslant p\binom{m-\ell}{r-\ell}$. Moreover, $|X_i - X_{i-1}| \leqslant \binom{m-\ell-1}{r-\ell-1} < \binom{m-1}{r-\ell-1}$. Thus, by Lemma 8 applied with $\lambda = \delta\binom{m}{r-\ell}$ and $c_i = \binom{m-1}{r-\ell-1}$, we have

$$\begin{aligned}\mathbb{P}\left(X_m \leqslant (p-\delta)\binom{m-\ell}{r-\ell}\right) &\leqslant \mathbb{P}(X_m \leqslant X_0 - \lambda) \leqslant \exp\left(\frac{-\delta^2\binom{m}{r-\ell}^2}{2m\binom{m-1}{r-\ell-1}^2}\right) = \left(\frac{-\delta^2\binom{m}{r-\ell}}{2(r-\ell)}\right)\\ &\leqslant \exp\left(-\frac{\delta^2 m}{2(r-\ell)^2}\right).\end{aligned}$$

Hence (3) holds.

An m-subset T of V is called *bad* if it is bad for some $S \in V^{(\ell)} \setminus \mathcal{P}$. The number of bad m-subsets is at most

$$\begin{aligned}\sum_{S \in V^{(\ell)} \setminus \mathcal{P}} \phi_S &\leqslant \binom{n}{\ell}\binom{n-\ell}{m-\ell} \exp\left(-\frac{\delta^2 m}{2(r-\ell)^2}\right) = \binom{n}{m}\binom{m}{\ell} \exp\left(-\frac{\delta^2 m}{2(r-\ell)^2}\right)\\ &\leqslant \binom{n}{m} m_0^\ell \exp\left(-\frac{\delta^2 m_0}{2(r-\ell)^2}\right) \leqslant \binom{n}{m} \exp\left(2\ell \log m_0 - 13\ell \log(1/\delta)\right)\\ &\leqslant \binom{n}{m} \exp\left(-\ell \log(1/\delta)\right) \leqslant \frac{1}{2}\binom{n}{m},\end{aligned}$$

where the last three inequalities hold by our choice of m_0, by inequality (1), and by our assumption on δ, respectively. Together with (2), this shows there exists an m-subset inside which there is no poor ℓ-subsets and in which every rich ℓ-subset has degree at least $(p-\delta)\binom{m-\ell}{r-\ell}$. Such a set clearly gives us an induced subgraph of G on m vertices with minimum ℓ-degree at least $(p-\delta)\binom{m-\ell}{r-\ell}$. □

3 Concluding remarks

A 3-graph G is $(p, \varepsilon, (1,1,1))$-*quasirandom* if for every triple of sets of vertices X, Y and $Z \subseteq V$, the number $e_{1,1,1}(X,Y,Z)$ of triples $(x,y,z) \in X \times Y \times Z$ such that $xyz \in E(G)$ satisfies $\Big| e_{1,1,1}(X,Y,Z) - p|X| \cdot |Y| \cdot |Z| \Big| \leqslant \varepsilon v(G)^3$. Define $\pi_{(1,1,1)-qr}(\mathcal{F})$ analogously to $\pi_{(1,2)-qr}(\mathcal{F})$. Note that $\pi_{(1,2)-qr}(\mathcal{F}) \leqslant \pi_{(1,1,1)-qr}(\mathcal{F}) \leqslant \pi(\mathcal{F})$ for all 3-graph families $\mathcal{F}$. An obvious open question is whether we have

$$\pi_{(1,1,1)-qr}(\mathcal{F}) \leqslant \pi_2(\mathcal{F}).$$

Even more: can one always extract subgraphs with large minimum codegree from $(1,1,1)$-quasirandom graphs? Even obtaining large subgraphs with non-zero minimum codegree remains an open problem for this weaker notion of quasirandomness.

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