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Subgraphs with large minimum $\ell$-degree
in hypergraphs where almost all $\ell$-degrees are large

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Abstract

Let $G$ be an $r$-uniform hypergraph on $n$ vertices such that all but at most $\varepsilon \binom{n}{r}$ $\ell$-subsets of vertices have degree at least $p \binom{n-\ell}{r-\ell}$. We show that $G$ contains a large subgraph with high minimum $\ell$-degree.

Keywords: $r$-uniform hypergraphs, $\ell$-degree, extremal hypergraph theory

Mathematics Subject Classifications: 05C65, 05D99

1 Introduction

Given $r \in \mathbb{N}$ and a set $A$, we write $A^{(r)}$ for the collection of all $r$-subsets of $A$ and $[n]$ for the set $\{1, 2, \ldots, n\}$. An $r$-graph, or $r$-uniform hypergraph, is a pair $G = (V, E)$, where $V = V(G)$ is a set of vertices and $E = E(G) \subseteq V^{(r)}$ is a collection of $r$-subsets, which constitute the edges of $G$. We say $G$ is nonempty if it contains at least one edge and set $v(G) = |V(G)|$ and $e(G) = |E(G)|$. A subgraph of $G$ is an $r$-graph $H$ with $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. The subgraph of $G$ induced by a set $X \subseteq V(G)$ is $G[X] = (X, E(G) \cap X^{(r)})$.

Let $\mathcal{F}$ be a family of nonempty $r$-graphs. If $G$ does not contain a copy of a member of $\mathcal{F}$ as a subgraph, we say that $G$ is $\mathcal{F}$-free. The Turán number $\text{ex}(n, \mathcal{F})$ of a family $\mathcal{F}$ is the maximum number of edges in an $\mathcal{F}$-free $r$-graph on $n$ vertices, and its Turán density is the limit $\pi(\mathcal{F}) = \lim_{n \rightarrow \infty} \text{ex}(n, \mathcal{F})/\binom{n}{r}$ (this is easily shown to exist). Let $K_t^{(r)}$ denote the complete $r$-graph on $t$ vertices. Determining $\pi(K_t^{(r)})$ for any $t > r \geq 3$ is a
major problem in extremal combinatorics. Turán [19] famously conjectured in 1941 that $\pi(K_4^{(3)}) = 5/9$, and despite much research effort this remains open [8]. In this paper we shall be interested in some variants of Turán density.

The *neighbourhood* $N(S)$ of an $\ell$-subset $S \in V(G)^{(\ell)}$ is the collection of $(r - \ell)$-subsets $T \in V(G)^{((r-\ell)})$ such that $S \cup T$ is an edge of $G$. The *degree* of $S$ is the number $\deg(S)$ of edges of $G$ containing $S$, that is, $\deg(S) = |N(S)|$. The minimum $\ell$-degree of $G$, $\delta_\ell(G)$, is defined to be the minimum of $\deg(S)$ over all $\ell$-subsets $S \in V(G)^{(\ell)}$. The *Turán $\ell$-degree threshold* $\operatorname{ex}_\ell(n, \mathcal{F})$ of a family $\mathcal{F}$ of $r$-graphs is the maximum of $\delta_\ell(G)$ over all $\mathcal{F}$-free $r$-graphs $G$ on $n$ vertices. It can be shown [11, 9] that the limit $\pi_\ell(\mathcal{F}) = \lim_{n \to \infty} \operatorname{ex}_\ell(n, \mathcal{F})/\binom{n \ell}{\ell}$ exists; this quantity is known as the *Turán $\ell$-degree density* of $\mathcal{F}$.

A simple averaging argument shows that

$$0 \leq \pi_{r-1}(\mathcal{F}) \leq \ldots \leq \pi_2(\mathcal{F}) \leq \pi_1(\mathcal{F}) = \pi(\mathcal{F}) \leq 1,$$

and it is known that $\pi_\ell(\mathcal{F}) \neq \pi(\mathcal{F})$ in general (for $\ell \notin \{0, 1\}$). In the special case where $(r, \ell) = (r, r - 1)$, $\pi_{r-1}(\mathcal{F})$ is known as the *codegree density* of $\mathcal{F}$.

There has been much research on Turán $\ell$-degree threshold for $r$-graphs when $(r, \ell) = (3, 2)$. In the late 1990s, Nagle [12] and Nagle and Czygrinow [2] conjectured that $\pi_2(K_4^{(3)}) = 1/4$ and $\pi_2(K_4^{(3)}) = 1/2$, respectively. Here $K_4^{(3)}$ denotes the 3-graph obtained by removing one edge from $K_4^{(3)}$. Falgas-Ravry, Pikhurko, Vaughan and Volec [6, 7] recently proved $\pi_2(K_4^{(3)}) = 1/4$, settling the conjecture of Nagle, and showed all near-extremal constructions are close (in edit distance) to a set of quasirandom tournament constructions of Erdős and Hajnal [3]. The lower bound $\pi_2(K_4^{(3)}) \geq 1/2$ also comes from a quasirandom construction, which is due to Rödl [17]. For $t > r \geq 3$, the codegree density $\pi_{r-1}(K_t^{(r)})$ has been studied by Falgas-Ravry [4], Lo and Markström [9] and Sidorenko [18]. Recently, Lo and Zhao [10] showed that $1 - \pi_{r-1}(K_t^{(r)}) = \Theta((t/r)^{r-1})$ for $r \geq 3$.

One variant of $\ell$-degree Turán density is to study $r$-graphs in which almost all $\ell$-subsets have large degree. To be precise, given $\varepsilon > 0$, let $\delta_\ell^\varepsilon(G)$ be the largest integer $d$ such that all but at most $\varepsilon\binom{v(G)}{\ell}$ of the $\ell$-subsets $S \in V(G)^{(\ell)}$ satisfy $\deg(S) \geq d$. Note that $r$-graphs with large $\delta^\varepsilon_\ell(G)$ but with small $\delta_\ell(G)$ arise naturally. For instance, the reduced graphs $R$ obtained from $r$-graphs with large minimum $\ell$-degree after an application of hypergraph regularity lemma have large $\delta^\varepsilon_\ell(R)$.

**Definition 1** $(r, \ell)$-sequence). Let $1 \leq \ell < r$. We say that a sequence $\mathbf{G} = (G_n)_{n \in \mathbb{N}}$ of $r$-graphs is an $(r, \ell)$-sequence if

(i) $v(G_n) \to \infty$ as $n \to \infty$ and

(ii) there is a constant $p \in [0, 1]$ and a sequence of nonnegative reals $\varepsilon_n \to 0$ as $n \to \infty$ such that $\delta^\varepsilon_n(G_n) \geq p\binom{v(G_n)^{\ell}}{r-\ell}$ for each $n$.

We refer to the supremum of all $p \geq 0$ for which (ii) is satisfied as the *density* of the sequence $\mathbf{G}$ and denote it by $\rho(\mathbf{G})$.

We can define the analogue of Turán density for $(r, \ell)$-sequences.
Definition 2. Let $1 \leq \ell < r$. Let $\mathcal{F}$ be a family of nonempty $r$-graphs. Define

$$\pi^*_\ell(\mathcal{F}) := \sup\{\rho(G) : G \text{ is an } (r, \ell)-\text{sequence of } \mathcal{F}\text{-free } r\text{-graphs}\}.$$

Our main result show that every large $r$-graph $G$ contains a ‘somewhat large’ subgraph $H$ with minimum $\ell$-degree satisfying $\delta_\ell(H)/(v(H)-\ell) \approx \delta^*_\ell(G)/(v(G)-\ell)$. Here ‘somewhat large’ means $v(H) = \Omega(\varepsilon^{1/\ell})$.

Theorem 3. Let $1 \leq \ell < r$. For any fixed $\delta > 0$, there exists $m_0 > 0$ such that any $r$-graph $G$ on $n \geq m \geq m_0$ vertices with $\delta^*_\ell(G) \geq p(n-\ell)$ for some $\varepsilon \leq m^{-\ell}/2$ contains an induced subgraph $H$ on $m$ vertices with

$$\delta_\ell(H) \geq (p - \delta)\left(\frac{m - \ell}{r - \ell}\right).$$

This immediate implies the $\pi^*_\ell(\mathcal{F}) = \pi_\ell(\mathcal{F})$ for all families $\mathcal{F}$ of $r$-graphs.

Corollary 4. For any $1 \leq \ell < r$ and any family $\mathcal{F}$ of nonempty $r$-graphs, $\pi^*_\ell(\mathcal{F}) = \pi_\ell(\mathcal{F})$.

We note that the (tight) upper bounds for codegree densities $\pi_2(\mathcal{F})$ for 3-graphs $F$ obtained by flag algebraic methods in [5, 6, 7] actually relied on giving upper bounds for $\pi^*_\ell(F)$. Corollary 4 provides theoretical justification for why this strategy could give optimal bounds.

1.1 Quasirandomness in 3-graphs

One of the main motivations for this note comes from recent work of Reiher, Rödl and Schacht [13, 14, 15, 16] on extremal questions for quasirandom hypergraphs. These authors studied the following notion of quasirandomness for 3-graphs.

Definition 5 ((1,2)-quasirandomness). A 3-graph $G$ is $(p, \varepsilon, (1,2))$-quasirandom if for every set of vertices $X \subseteq V$ and every set of pairs of vertices $P \subseteq V(2)$, the number $e_{1,2}(X, P)$ of pairs $(x, uv) \in X \times P$ such that $\{x\} \cup \{uv\} \in E(G)$ satisfies:

$$|e_{1,2}(X, P) - p|X| \cdot |P| \leq \varepsilon v(G)^3.$$

We define a $(1,2)$-quasirandom sequence and the corresponding extremal density, denoted by $\pi_{(1,2)-qr}(\mathcal{F})$, analogously to the way we defined $(r, \ell)$-sequences and $\pi^*_\ell(\mathcal{F})$ in Definitions 1 and 2. It is not difficult to see that $\pi_{(1,2)-qr}(\mathcal{F}) \leq \pi(\mathcal{F})$ for all families $\mathcal{F}$ of 3-graphs. Moreover, a $(p, \varepsilon, (1, 2))$-quasirandom 3-graph $G$ satisfies $\delta^2_{(1,2)}(G) \geq (p-4\sqrt{\varepsilon})v(G)$. Hence, Theorem 3 and Corollary 6 imply the following.

Corollary 6. For any family of nonempty 3-graphs $\mathcal{F}$, $\pi_{(1,2)-qr}(\mathcal{F}) \leq \pi_2(\mathcal{F})$. 

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Consider a \((p, \varepsilon, (1, 2))\)-quasirandom 3-graph \(G\) for some \(p > 4\sqrt{\varepsilon} > 0\). As noted above, \(\delta_2^c(G) \geq (p - 4\sqrt{\varepsilon})v(G)\). Thus provided \(v(G)\) is sufficiently large, Theorem 3 tells us we can find a subgraph \(H\) of \(G\) on \(m = \Omega(\varepsilon^{-1/4})\) vertices with strictly positive minimum codegree (at least \((p - 4\sqrt{\varepsilon})m\)).

However, as we show below, we cannot guarantee the existence of any subgraph with strictly positive codegree on more than \(2/\varepsilon + 1\) vertices: our lower bound on \(m\) above in terms of an inverse power of the error parameter \(\varepsilon\) is thus sharp up to the value of the exponent.

**Proposition 7.** For every \(p \in (0, 1)\) and every \(\varepsilon > 0\), there exists \(n_0\) such that for all \(n \geq n_0\) there exist \((p, 2\varepsilon, (1, 2))\)-quasirandom 3-graphs in which every subgraph on \(m \geq [\varepsilon^{-1}] + 1\) vertices has minimum codegree equal to zero.

**Proof.** Let \(G = (V, E)\) be a \((p, \varepsilon, (1, 2))\)-quasirandom 3-graph on \(n\) vertices. Such a 3-graph can be obtained for example by taking a typical instance of an Erdős–Rényi random 3-graph with edge probability \(p\). Consider a balanced partition of \(V\) into \(N = \lceil \varepsilon^{-1} \rceil\) sets \(V = \bigcup_{i=1}^{N} V_i\) with \(|n/N| \leq |V_1| \leq |V_2| \leq \ldots \leq |V_N| \leq |n/N|\). Now let \(G'\) be the 3-graph obtained from \(G\) by deleting all triples that meet some \(V_i\) in at least two vertices for some \(i: 1 \leq i \leq N\).

By construction, every set of \(N + 1\) vertices in \(G'\) must contain at least two vertices from the same \(V_i\), and thus must induce a subgraph of \(G'\) with minimum codegree zero. Note that \(e(G) - e(G') \leq N n \binom{n/N}{2} \leq n^3/N \leq \varepsilon n^3\). Since \(G\) is \((p, \varepsilon, (1, 2))\)-quasirandom, it follows that \(G'\) is \((p, 2\varepsilon, (1, 2))\)-quasirandom. \(\Box\)

## 2 Finding high minimum \(\ell\)-degree subgraphs in \(r\)-graphs with large \(\delta_\ell^c\)

In this section we show how we can extract arbitrarily large subgraphs with high minimum \(\ell\)-degree from sufficiently large \(r\)-graphs with sufficiently small error \(\varepsilon\). To do so, we will need Azuma’s inequality (see e.g. [1]).

**Lemma 8** (Azuma’s inequality). Let \(\{X_i : i = 0, 1, \ldots\}\) be a martingale with \(|X_i - X_{i-1}| \leq c_i\) for all \(i\). Then for all positive integers \(N\) and \(\lambda > 0\),

\[
P(X_N \leq X_0 - \lambda) \leq \exp \left( \frac{-\lambda^2}{2 \sum_{i=1}^{N} c_i^2} \right).
\]

**Proof of Theorem 3.** We may assume without loss of generality that \(\delta > 0\) is small enough to ensure \(\delta^{-1} \geq 26\ell (r - \ell)^2 \log(1/\delta)\) and \(\ell \log(1/\delta) \geq \log 2\) as this only makes our task harder. Set \(m_0 = \lceil 26\ell (r - \ell)^2 \delta^{-2} \log(1/\delta) \rceil\). Note that this implies that

\[
2\ell \log m_0 = 4\ell \log \left( 26\ell (r - \ell)^2 \delta^{-2} \log(1/\delta) \right) \leq 12\ell \log(1/\delta).
\]

Fix \(m \geq m_0\). Let \(n \geq m \geq m_0\) and \(\varepsilon = m^{-\ell}/2\).
Suppose \( G = (V, E) \) is an \( r \)-graph on \( n \) vertices with \( \delta^r(G) \geq p \binom{n-\ell}{r-\ell} \). We claim that it contains an induced subgraph on \( m \) vertices with minimum \( \ell \)-degree at least \( (p-\delta) \binom{m-\ell}{r-\ell} \). For \( p \leq \delta \), we have nothing to prove, so we may assume that \( 1 \geq p > \delta \).

Call an \( \ell \)-subset \( S \in V^{(\ell)} \) poor if \( \deg(S) < p \binom{n-\ell}{r-\ell} \), and rich otherwise. Let \( \mathcal{P} \) be the collection of all poor \( \ell \)-subsets. By our assumption on \( \delta^r(G) \), \( |\mathcal{P}| \leq \varepsilon \binom{n}{\ell} \). As each poor \( \ell \)-subset is contained in \( \binom{n-\ell}{m-\ell} \) \( m \)-subsets, it follows that there are at least

\[
\left( \binom{n}{m} - |\mathcal{P}| \binom{n-\ell}{m-\ell} \right) > (1 - \varepsilon m^\ell) \binom{n}{m} = \frac{1}{2} \binom{n}{m}
\]

\( m \)-subsets of vertices which do not contain any poor \( \ell \)-subsets.

Given an \( \ell \)-subset \( S \in V^{(\ell)} \setminus \mathcal{P} \), we call an \( m \)-subset \( T \) of \( V \) bad for \( S \) if \( S \subseteq T \) and \( |N(S) \cap T^{(r-\ell)}| \leq (p-\delta) \binom{m-\ell}{r-\ell} \). Let \( \phi_S \) be the number of bad \( m \)-subsets for \( S \). We claim that

\[
\phi_S \leq \binom{n-\ell}{m-\ell} \exp \left( -\frac{\delta^2 m}{2(r-\ell)^2} \right).
\]

Observe that

\[
\phi_S = \left\{ T \in (V \setminus S)^{(m-\ell)} : |N(S) \cap T^{(r-\ell)}| \leq (p-\delta) \binom{m-\ell}{r-\ell} \right\}.
\]

Let \( X \) be the random variable \( |N(S) \cap T^{(r-\ell)}| \), where \( T \) is an \( (m-\ell) \)-subset of \( V \setminus S \) picked uniformly at random. We consider the vertex exposure martingale on \( T \). Let \( Z_i \) be the \( i \)th exposed vertex in \( T \). Define \( X_i = \mathbb{E}(X|Z_1, \ldots, Z_i) \). Note that \( \{X_i : i = 0, 1, \ldots, m-\ell\} \) is a martingale and \( X_0 \geq p \binom{m-\ell}{r-\ell} \). Moreover, \( |X_i - X_{i-1}| \leq \binom{m-\ell-1}{r-\ell-1} < \binom{m-1}{r-\ell-1} \). Thus, by Lemma 8 applied with \( \lambda = \delta \binom{m}{r-\ell} \) and \( c_i = \binom{m-1}{r-\ell-1} \), we have

\[
\mathbb{P} \left( X_m \leq (p-\delta) \binom{m-\ell}{r-\ell} \right) \leq \mathbb{P}(X_m \leq X_0 - \lambda) \leq \exp \left( -\frac{\delta^2 \binom{m}{r-\ell}}{2 \binom{m-\ell-1}{r-\ell-1}} \right) = \left( \frac{-\delta^2 \binom{m}{r-\ell}}{2(r-\ell)} \right)
\]

Hence (3) holds.

An \( m \)-subset \( T \) of \( V \) is called bad if it is bad for some \( S \in V^{(\ell)} \setminus \mathcal{P} \). The number of bad \( m \)-subsets is at most

\[
\sum_{S \in V^{(\ell)} \setminus \mathcal{P}} \phi_S \leq \binom{n}{\ell} \binom{n-\ell}{m-\ell} \exp \left( -\frac{\delta^2 m}{2(r-\ell)^2} \right) = \binom{n}{m} \binom{m}{\ell} \exp \left( -\frac{\delta^2 m}{2(r-\ell)^2} \right)
\]

\[
\leq \binom{n}{m} \exp \left( -\ell \log(1/\delta) \right) \leq \frac{1}{2} \binom{n}{m},
\]
where the last three inequalities hold by our choice of $m_0$, by inequality (1), and by our assumption on $\delta$, respectively. Together with (2), this shows there exists an $m$-subset inside which there is no poor $\ell$-subsets and in which every rich $\ell$-subset has degree at least $(p - \delta)(m-\ell)$. Such a set clearly gives us an induced subgraph of $G$ on $m$ vertices with minimum $\ell$-degree at least $(p - \delta)(m-\ell)$.

\section{Concluding remarks}

A 3-graph $G$ is $(p, \varepsilon, (1, 1, 1))$-quasirandom if for every triple of sets of vertices $X$, $Y$ and $Z \subseteq V$, the number $e_{1,1,1}(X,Y,Z)$ of triples $(x,y,z) \in X \times Y \times Z$ such that $xyz \in E(G)$ satisfies $|e_{1,1,1}(X,Y,Z) - p|X| \cdot |Y| \cdot |Z| | \leq \varepsilon v(G)^3$. Define $\pi_{(1,1,1)-qr}(\mathcal{F})$ analogously to $\pi_{(1,2)-qr}(\mathcal{F})$. Note that $\pi_{(1,2)-qr}(\mathcal{F}) \leq \pi_{(1,1,1)-qr}(\mathcal{F}) \leq \pi(\mathcal{F})$ for all 3-graph families $\mathcal{F}$. An obvious open question is whether we have

$$\pi_{(1,1,1)-qr}(\mathcal{F}) \leq \pi_2(\mathcal{F}).$$

Even more: can one always extract subgraphs with large minimum codegree from $(1, 1, 1)$-quasirandom graphs? Even obtaining large subgraphs with non-zero minimum codegree remains an open problem for this weaker notion of quasirandomness.

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