The Chichilnisky–Heal Approach to Arrow’s Impossibility Theorem

Anton Vernersson
Abstract
This essay explores a surprising intersection between economics and algebraic topology. On the economical side social choice is studied, a field which includes topics such as voting theory. Within standard economics this is often approached by discrete methods, such as Arrow’s impossibility theorem. This essay will instead follow a continuous proof by Chichilnisky and Heal, which is based on a considerable amount of algebraic topology. Therefore, this essay will cover a great deal of algebraic topology. In particular, the subject of obstruction theory, which provides clear conditions for the extension of a function to a larger space, is studied.

Sammanfattning
## Contents

1. Introduction 1
2. Economic Framework 7
   2.1. Introductory Economics 7
   2.2. The Chichilnisky-Heal setting 12
3. Algebraic Topology 15
   3.1. Deformation of Paths 15
   3.2. Category Theory 18
   3.3. H-groups 21
   3.4. Exact Sequences of Sets of Homotopy Classes 24
   3.5. Higher Homotopy Groups 32
   3.6. CW-Complexes 35
   3.7. Homotopy Functors 41
4. Obstruction Theory 45
   4.1. Eilenberg-Maclane Spaces and Cohomology 45
   4.2. The Tower of Postnikov 50
   4.3. The Final Obstruction 58
5. Proof of Theorem 2.16 63
6. Intriguing Equivalences 65
7. References 69
1. Introduction

All voting systems are fundamentally flawed! Already in 1785, Marie Jean Antoine Nicolas de Caritat, Marquis of Condorcet, a French political scientist pointed out this in his “Essai sur l’application de l’analyse à la probabilité des décisions rendues à la pluralité des voix”. Despite its age, the insights still have impact on modern day elections. Consider the following short example of the work done by Nicolas de Condorcet. Assume that there is a remarkably small country consisting only of three persons. They are aptly named A, B, and C. Despite the size of their country they decide that it needs a president. Since there are only three individuals they decide to elect the most agreeable candidate. The following are the preferences held by A, B, and C, respectively.

A:  A ≻ B ≻ C.
B:  B ≻ C ≻ A.
C:  C ≻ A ≻ B.

From this it follows that C is preferred to A by both B and C. Therefore it would seem reasonable begin by considering C as president. However, both A and B prefers B instead of C. It might then seem like B is the most suitable president. Then again, both A and C would rather prefer to have A as president. Thus, no matter who is elected, an unambiguously more popular candidate always exists. This is what is known as the Condorcet paradox.

Centuries later Kenneth Arrow a legendary American economist, and Nobel laureate, proved in 1950 what is known today as Arrow’s impossibility theorem. Informally, it says that no rank-order electoral system can be designed that always satisfies the following fairness-type properties:

(1) if every voter prefers alternative A over alternative B, then the group prefers A over B;
(2) if every voter’s preference between A and B remains unchanged, then the group’s preference between A and B will remain unchanged;
(3) there is no single voter that possess the power to always determine the group’s preference, i.e. there is no dictator

(see [4] for the formal statement and proof). This theorem has been argued among, generalized by, and inspired many (see e.g. [23] or [6]). From the work done by Arrow an entire field known as social choice theory has sprung. The statement of Arrow’s impossibility theorem is discrete in its nature, and the proof is not surprisingly solely combinatorial.

Contrary to the discrete origin of social choice theory, Chichilnisky introduced in 1980 a continuous, topological approach [10]. It is this approach we are interested in this essay, especially a topological version of Arrow’s impossibility theorem. We shall present here a fairly self-contained proof of the following theorem proved by Chichilnisky and Heal [11], see also [9].
Theorem 2.16. Let $P$ be the space of preference profiles. A continuous function $\phi : \prod_{k=1}^{n} P \to P$, which satisfies anonymity and unanimity, exists if and only if $P$ is contractible.

There are benefits using this continuous model. First, the framework used by Chichilnisky and Heal agrees with many common assumptions from modern microeconomics. Secondly, we can determine whether or not a social choice rule exists based solely on if the preference space $P$ is contractible. Finally, the topological construction also allows for an application to a wider section of economics, specifically to the issue of aggregating preferences held by individuals.

Given the benefits of continuous social choice it is somewhat surprising that it not widely used. One could argue that this perhaps is because of its position somewhere between mathematics and economics. On one hand the methods and tools used are not commonly known among economists, and on the other hand there are no new mathematical results for a topologist. This combined with the great success of the discrete model by Arrow does at least provide a partial answer for why the continuous model is not more well-known. Here it should also be mentioned that there have been attempts to unify and justify the interplay between the topological and traditional combinatorial approaches of the social choice theory, we restate one such result in Section 6.

The method used by Chichilnisky and Heal [11] in proving the sufficiency makes use of obstruction theory from algebraic topology. This is quite a general method of solving problems involving the extension of maps to larger domains. While one could go for a more modern approach such as [25] we will instead follow the classical textbook [24]. A benefit of this is that our treatment of algebraic topology, and obstruction theory, will closely follow what was available to Chichilnisky and Heal. The specific theorem we will use is the following.

Theorem 4.35. Let $\iota \in H^n(Y, y_0; \pi)$ be $n$-characteristic for a simple $(n-1)$-connected pointed space $Y$ where $n \geq 1$, and let $(X, A)$ be a relative CW complex such that $H^{q+1}(X, A; \pi_q(Y, y_0)) = 0$ for $q > n$. A map $f : A \to Y$ can be extended over $X$ if and only if $\delta f^*(\iota) = 0$ in $H^{n+1}(X, A; \pi)$.

As one can see this is quite a technical theorem and we will spend the entirety of Section 3 and Section 4 working towards a proof. The division between the sections about algebraic topology is made so that Section 3 introduce and addresses the fundamental parts of the subject and Section 4 deals with the topics specific to the obstruction theory. Within Sections 3 and Section 4 the theory, unless otherwise noted, is from the monograph [24] written by Edward Spanier.

Section 2 gives an introduction to economics. It starts with a broad presentation of modern microeconomics, with a focus on the mathematics. Following that is a presentation of the construction specific to the ideas used in the proof of Theorem 2.16, which is presented in Section 5. Following the proof of our main result in Section 5 we will continue with Section 6 and an exploration of some interesting interconnections between Brouwer’s Fixed Point Theorem, Arrow’s Theorem and the continuous social choice rule, as given by Chichilnisky and Heal. Specifically,
we will rely on [9] and [26] to show that both the economical theorems are equivalent to Brouwer’s Fixed Point Theorem, Theorem 6.1. This is quite remarkable, as the theorem by Arrow is entirely combinatorial while the work by Chichilnisky and Heal is continuous.

Even though this essay is, up to a point, self-contained the reader is assumed to have mathematical maturity and basic knowledge of general topology and abstract algebra.
Acknowledgement

There are a number of people without whose support this essay would never have matured into what it is today. I would like to thank, in alphabetical order, Jasper Gräflich, Christoffer Olsson, and Aron Persson for reading and commenting on my work. I only hope that my last days of writing have not undone what you so kindly helped me build. I would also like to thank Per Åhag, my supervisor, for your time and advise but especially for never doubting that this essay would be completed. Finally I would like to thank my fiancée, Clara Nygren, for supporting me in what has undoubtedly been some of my most difficult times.
2. Economic Framework

To understand social choice theory it is not necessary to have a solid foundation in economics. However, in order to see why the approach by Chichilnisky and Heal is appealing it is important to know the basics. This is what we shall begin to do in Section 2.1, where we introduce ordered spaces and the order isomorphisms that characterize modern economical theory. We continue by looking at differentiability of these morphisms. This will then allow us to formulate the specific setting of Chichilnisky and Heal and state the main theorem, Theorem 2.16, in Section 2.2.

2.1. Introductory Economics. The first point of interest in any treatment of modern economics is the concept of preference relations and their associated order isomorphisms. Within economics these are used to describe and model the behaviour of economical agents. The agents are of course not some kind of cheap versions of James Bond but an individual or firm acting within an economy. These tools are the tools of the trade for modern economists. To a mathematician a preference relation is just a binary relation and the order isomorphism is a monotone transformation.

In this section we will look at the mathematical details of how preference relations are used in economics. This treatment will be somewhat informal and a reader well versed in mathematics or economy will find that more than a few details are omitted. Ideally there would be no trade off between quality and quantity but this is not the case. Instead of a complete treatment this section aims at a brief introduction trying to get the reader up to speed. Perhaps breaking with what was just stated we begin by properly defining binary relations in general and the preference relation in particular.

**Definition 2.1.** A binary relation $R$ of a set $X$ is a subset $R \subset X \times X$. If $(x, y) \in R$, we write $x R y$.

**Definition 2.2.** Let $Y$ be a set and let $y_1, y_2$ and $y_3$ be elements in $Y$. If $\succsim$ is a binary relation on $Y$ such that it is:

1. reflexive, $y_1 \succsim y_1$ for all $y \in Y$;
2. transitive, if $y_1 \succsim y_2$ and $y_2 \succsim y_3$ then $y_1 \succsim y_3$;
3. complete, for all $y_1$ and $y_2 \in Y$ either $y_1 \succsim y_2$ or $y_2 \succsim y_1$.

Then we say that $\succsim$ is a preference relation on $Y$ and that $(Y, \succsim)$ is an ordered set.

A preference relation can be subdivided into a symmetric relation, i.e. one where $y_1 \succsim y_2$ is equivalent to $y_2 \succsim y_1$. And an asymmetric, i.e. one where $y_1 \succsim y_2$ implies that $y_2 \succsim y_1$ does not hold. On the real line, with its usual ordering, we have the symmetric relation $=$ and the asymmetric relation $>$.

Dealing with orderings in an economical setting the symmetric relation is usually denoted by $\sim$ and is said to be an indifference relation. The asymmetric relation is denoted $>$ (or sometimes $<$ when appropriate). We say that $>$ is a strict preference relation. In economics we use (order) preferences to represent preferences held by individuals. Consider the following example on how preferences held by an individual is captured by preference relations.

**Example 2.3.** Let Alice be an individual who consumes bread and wine and enjoys spending time away from work. Let $b, w, l \in \mathbb{R}_+$, then the triple $(b, w, l) \in \mathbb{R}_3^+$.
of bread, wine and leisure represents a possible consumption choice that Alice could make. Let's say that Alice currently consumes ten breads, drinks two bottles of wine and has eighty hours of free time a week, i.e. her consumption choice is: \((10, 2, 80)\). In this case she would certainly prefer that bundle to the bundle: \((10, 2, 70)\), because it is identical except that less work is necessary. But more than either of those she would prefer the bundle: \((12, 2, 80)\). Therefore \((10, 2, 70) \succeq (10, 2, 80) \succeq (12, 2, 80)\). □

There is an issue in how this framework is used in practice and we will address that before proceeding. Commonly it is assumed that an increase in one good, while the other remains the same, is always a good thing. Within the context of our previous example this means that there is no amount of bread Alice could eat such that a small increase would not make her happier. This is of course absurd: no one would prefer to eat a hundred loaves of bread for lunch over having just one. This is, however, not due to a fault of the theory but of the practitioner. All such cases, perhaps with the exception to some “off the wall” cases, are due to a model misspecification. In our example we have neglected to include effects to Alice health (and her presumed utility of it).

Whether this short exposition has convinced the reader that more is always better, or not, it is a fundamental assumption of economics. In the literature this property is commonly known as strong monotonicity. Technically we say that, for a choice set in \(\mathbb{R}^n\) such as the previous quantities, then \(\succ\) satisfies strong monotonicity if and only if, for a relation \(x \succ y, x \neq y\) implies \(x \succ y\). For more on the economical aspects of this please see Chapter 7 of [27].

We shall now get a bit more specific, or perhaps a bit less, dependent on the reader’s point of view. In the previous example of bread, wine and leisure we assumed that Alice could make her choice of bread, wine and leisure on a subset of \(\mathbb{R}^3\). But that is not something we have to do. The next few paragraphs will show how the model works at a higher level of abstraction. A level at which it is not necessary to assume that \(Y \subset \mathbb{R}^n\). Instead we shall work with ordered topological spaces. I will assume that the reader is already familiar with the basics of topology. However, we will not use that much general topology and the treatment will again be quite informal. With this said let us formulate what we wish to study.

Let \(Y\) be a topological space (the topology is not important at this point) together with the binary relation \(\succ\). We will write this pair as \((Y, \succ)\). Now, \((Y, \succ)\) is where agents make their consumption choices. What we wish to do is to express the order on \(Y\) in terms of the order on another topological space, \(X\). To simplify the exposition we will denote the ordering on a set \(Y\) by \(\succ_Y\), and the order on a set \(X\) by \(\succ_X\). A preference relation, \(\succ\), does naturally define equivalence classes on its choice space \(Y\) by its symmetric part \(\sim\). Because of this we could equally well work with \(Y\) and \(\succ_Y\) or the quotient space \(Y/\sim\) with \(\succ_Y\). In the future we will often not mention this and use \(Y\) for both spaces.

We will now begin to look directly at maps \(f: (Y, \succ_Y) \rightarrow (X, \succ_X)\) and in particular maps \(f: (Y, \succ) \rightarrow (\mathbb{R}, >)\). The first step will be to show that there is a map \(f\) such that the orders on the two spaces are isomorphic.

**Definition 2.4.** Let \((Y, \succ_Y)\) and \((X, \succ_X)\) be ordered sets with \(y', y'' \in Y\), and let \(f: Y \rightarrow X\) be a map. If \(y' \succ_Y y'' \iff f(y') \succ_X f(y'')\), then we say that \(f\) is an order isomorphism.
To get an order isomorphism we will have to assume two things about the order: that it is a total order and it is order separable. We define these in the following two definitions. A total order is actually what most people would already assume that a order is. For instance the standard order $\geq$ on $\mathbb{R}$ is a total order.

**Definition 2.5.** A binary relation $\succsim$ is a **total order** if it is a preference relation, as by Definition 2.2, and is also antisymmetric, i.e. for all $y_1, y_2 \in Y$ then $(y_1 \succsim y_2 \text{ and } y_2 \succsim y_1) \Rightarrow y_1 \sim y_2$.

Reflexivity and antisymmetry are quite straightforward and do not carry any deeper economical meaning. On the other hand, transitivity two do have some economic implications. First of all, transitivity removes cyclic behaviour. If for instance Alice exchanges A for B and B for C, then she is unwilling to trade C for A again. Thereby it will be no possibility for a looping behaviour. This is in most cases a quite rational assumption. Perhaps there are persons who would do so but that would break the minimum level of rationality often assumed in economics.

**Definition 2.6.** Let $X$ be a set ordered by the preference relation $\succsim$. If there exists a countable set $Z \subset X$ such that, for all $x, y \in X \setminus Z$, $x \succ y$ implies that there exists a $z \in Z$ such that $x \succ z \succ y$, then we say that $(X, \succsim)$ is **order separable** (in the sense of Birkhoff).

There are a number of equivalent formulation of Definition 2.6 (see Proposition 1.4.4 of [7]). But the core of order separability is that there are no open gaps in the order on $X$. That is, on $\mathbb{R}$ with the standard topology a preference relation $\succsim$ is Birkhoff separable if there is no open set, $(a, b)$, such that $a \succsim b$. With the specific formulation of Birkhoff separability we have that there exists a dense, with respect to $\succsim$, subset $Z \subset X$. This can be used to explicitly construct a order isomorphism $f : X \to \mathbb{R}$. We shall partially show how this construction is carried out. While we will leave out some details the general idea will remain the same and the interested reader could look at Chapter 1.6 of [7].

**Example 2.7** (An explicit order isomorphism $f : X \to \mathbb{R}$). First consider the piecewise map $r : X \times X \to \{0, 1\}$ defined by

$$r(x, y) = \begin{cases} 1 & \text{if } x \succ y \\ 0 & \text{otherwise} \end{cases}$$

Now, since by definition $Z$ is a countable set ordered by $\succ$ we can create the sum

$$f(x) = \sum_{n=1}^{\infty} 2^{-n} r(z_n, x),$$

where $\{z_n\}_{n=1}^{\infty} = Z$. Certainly, $f(x)$ is convergent and $f(x) \in [0, 1]$. Since $Z$ is dense in $X$ it follows that if $x \succ y$, then there is a $z_n \in Z$ for some $n$ such that $x \succ z_n \succ y$. Hence $f(x) > f(y)$.

Similarly, if $f(x) > f(y)$, then

$$\sum_{n=1}^{\infty} 2^{-n} r(z_n, x) > \sum_{n=1}^{\infty} 2^{-n} r(z_n, y),$$

which can only happen if $r(z_n, x) > r(z_n, y)$ for some $n$. But that is to say that there is a $z_n \in Z$ such that $x \succ z_n \succ y$. Since $x$ and $y$ are separated it follows that $x \succ y$. 

Given the ease by with an order isomorphism can be constructed one could be lead to believe that the implications of it are quite trivial. This is sadly true and in
almost all settings additional assumptions are placed on \( f \). One useful assumption is that of continuity. For if the order isomorphism is continuous then the intermediate value theorem applies. Furthermore, the “level curves” of a continuous map are also continuous.

Continuity is often a minimum requirement on the maps an economist studies. Thankfully, this is not a severe assumption on the preference relation. All that is required is second countability, see e.g. [21], of the choice space in addition to the assumptions already made. See [7] for a number of related proofs, including a constructive proof similar to that of the plain order isomorphism.

So far we have not touched upon a very important part of order isomorphisms: uniqueness of representation. To this end let \( f_1 : Y \to \mathbb{R} \) and \( f_2 : Y \to \mathbb{R} \) be two order isomorphisms. If there is a continuous and monotonically increasing function \( g : \mathbb{R} \to \mathbb{R} \), then \( g \) must be an order preserving mapping on \( \mathbb{R} \). Since \( g \) is an order isomorphism on \( \mathbb{R} \) such that \( g(f_1) = f_2 \), then the orders induced by \( f_1 \) and \( f_2 \) must be the same. Therefore order isomorphisms are only unique up to monotone and continuous transformations. This view is within economics often called “ordinal utility”. Note that order isomorphisms are often known as utility functions in economics. However, this view is seemingly unable to explain the list of common isomorphisms in Example 2.8, as all these should be equivalent under the ordinal assumption.

**Example 2.8.** The following are common economical examples of order isomorphisms.

1. **Cobb-Douglas:** \( f(x_1, x_2) = ax_1^\alpha x_2^{1-\alpha} \). A very common and easy function that can be used in a variety of situations where there are two distinct goods.

2. **Constant elasticity of substitution:** \( f(x_1, x_2) = a(x_1^{\rho} + x_2^{\rho})^{-\rho} \). A generalization of the Cobb-Douglas function. For an account of elasticity, please see [27].

3. **The Khaneman-Tversky utility function:** \( f(x) = \lambda_x x^\alpha \), where \( \lambda_x |x| < 0 \) is smaller than \( \lambda_x |x| \geq 0 \). A common, and revolutionary, function used in conjunction with decision under risk. Please see [19] for further information.

**2.1.1. Differentiability.** The difference between the isomorphisms of Example 2.8 lies in differentiability. One cannot underestimate the usefulness of this property within economics. While continuity is important, many results and theories of modern economics relies heavily on differentiability. For instance, the optimization problems involved in the behaviour of agents is almost exclusively done using derivatives. We will work with two examples that illustrates the use of the derivative. The first one is a simple, bare bones, approach to finding the optimal consumption of goods with a budget restriction. And the second is a slightly more intricate one, which describes consumption within a larger economy.

**Example 2.9.** Let Bob be an agent in an economy with two goods, which are easily quantifiable and infinitely divisible. That is, Bob’s choice space \( Y \) is a subset of \( \mathbb{R}^2 \). Furthermore, assume that Bob is endowed at the start with a vector \( \omega \) of 2 goods. Certainly, the endowment of a good could be zero and possibly even negative. Let the best trade opportunity in this economy be one unit of good one for two units
of good two. If Bob has an initial wealth of $\omega = (2, 3)$, then the possible points Bob can obtain are on the line between $\omega = (3.5, 0)$ and $(0, 7)$, we call this the budget constraint.

Say that Bob’s preferences are represented by a Cobb-Douglas of the form $f(x_1, x_2) = x_1^{0.2} x_2^{0.8}$. Since the slope of the budget constraint is $-2$, it follows that the optimal consumption bundle is when

$$\frac{\partial x_1}{\partial x_2} = -2.$$

This example is economically quite simple. In an economy one would expect that the individual, Bob, would interact with other individuals and firms. The step we will take in the next example is to look at a more involved market example.

**Example 2.10.** Let $X \subset \mathbb{R}$ be a consumption set we wish to study and let $Y \subset \mathbb{R}$ be a consumption set we are not interested in. For instance, if we want to study the consumption of cars, then this is our set $X$ and $Y$ would be a representation of our other consumption, such as food and housing. Also, let $f: X \to \mathbb{R}$ be an order isomorphism and let the order on $Y$ be represented by id$_Y$. Thus, the combined order isomorphism $g: X \times Y \to \mathbb{R}$ is given by $g(x, y) = f(x) + y$. Now, assume that all individuals are equivalent and endowed with goods at a value of $\omega$. In this setting the consumption of, $y$, must be the difference between the endowment, $\omega$, and cost of $x$, $c(x)$. Hence the problem can be formulated as

$$\max_{x,y} (u(x) + y)$$

subject to $y = \omega - c(x)$.

This can be rewritten as

$$\max_x (u(x) + \omega - c(x)),$$

which has the first-order condition

$$\frac{\partial f}{\partial x} = c'(x).$$

This gives an indication as to how optimization is used in economics.

As can be discerned from the previous examples it is not mathematically difficult to work with this type of economics. The challenge, in this case, is to properly define differentiability in a satisfactory way. One could introduce differentiability by assuming that $X$ is a subset of $\mathbb{R}$ and then only look at differentiable functions as the order isomorphisms. Another possibility is to view $Y$ as a topological manifold and introduce a differentiable structure on it. We will take the former path but the reader may, if so inclined, find an explanation of the latter method in Chapter 8 of [7].

What we will do is to use hyperplanes as done by [13]. To simplify we assume that $Y = \mathbb{R}_+^n$, which is the positive cone of $\mathbb{R}^n$. This is a common assumption but it may seem severe. However, much of what economics study can be made to fit nicely into the positive part of $\mathbb{R}^n$. All goods, for instance, are traded in positive quantities as are financial commodities and time. If we also assume that things such as health is quantifiable then there is really no loss in generality to assume that $Y = \mathbb{R}_+^n$. By using this assumption we can make the following definition.
Figure 1. Order isomorphisms with some normals represented as arrows

**Definition 2.11.** Let $S \subset \mathbb{R}^n$ be of codimension 1 then we say that $S$ is a hyperplane.

Since $\succ$ is satisfies monotonicity and is continuous it follows that the level curves, or indifference surfaces, are continuous surfaces in $\mathbb{R}^n$ of codimension 1. We take a moment here and consider what we are doing. Our order isomorphic mapping of $\mathbb{R}^n_+$ to $\mathbb{R}$ will take a path beginning at $0 \in \mathbb{R}^n_+$ and map this to $\mathbb{R}$. This will take one dimension of $\mathbb{R}^n_+$ while the remaining points will belong to the equivalence classes of $\sim$. Transitivity and strong monotonicity of $\succ$ will assure that the equivalence classes do not intersect and are uniformly of dimension $n-1$ (except possibly for 0).

Now, we will need normals of the hypersurfaces. By definition, a normal on the equivalence classes is the direction of the biggest increase of $f: \mathbb{R}^n_+ \to \mathbb{R}$. This is apparent if we assume that $f$ is differentiable and then use the gradient as the normal. Economically the normal gives the preferred change in consumption, given an initial consumption bundle. In Figure 1 we can see the normals of one-dimensional hypersurfaces.

**Definition 2.12.** Let $f: \mathbb{R}^n_+ \to \mathbb{R}$ be a differentiable map and let $\nabla$ be the standard gradient on $\mathbb{R}^n$. We say that for any point $x \in \mathbb{R}^n_+$ the vector $\nabla f(x)$, in the direction of increasing $f$, is the normal at $x$. We shall often call the normals preferences.

As we going for the ordinal approach to isomorphisms we are not interested in the magnitude of the normal. But unlike the continuous case we do attach importance to the shape of the hyperplanes. Consequentially the direction of the normal does matter. This is what explains the difference between the isomorphisms of Example 2.8.

2.2. The Chichilnisky-Heal setting. One of the biggest issues in economics is that of aggregation: is there any way to combine preferences held by the individuals of a group into a single preference for the entire group. As an example we have elections, which were mentioned in the introduction and studied by Arrow. This is also solved by Arrow: under some plausible restrictions there is no way to consistently aggregate individual preferences into a group preference. While this result is a great one it not founded on the same base as the usual economics we
have discussed here. Instead it is a discrete model. In this section we will work with the Chichilnisky-Heal framework, which sets out to answer the same question but based on the theory we have discussed so far.

We will now explore the Chichilnisky-Heal model and state the main theorem of this essay: Theorem 2.16. Fundamental to the model are the preferences, or normals, of Definition 2.12. To an individual with an order isomorphism \( f : \mathbb{R}^+_n \to \mathbb{R} \) the normal at a point \( x \in \mathbb{R}^+_n \) represents the change to \( x \) that gives the largest increase in \( \mathbb{R} \).

**Definition 2.13.** Let \( P = \{ p \mid p \text{ is a preference} \} \). Then a social choice rule (for \( n \) individuals) is a map

\[
\phi : \prod_{i=1}^n P \to P.
\]

In addition we say that \( \prod_{i=1}^n P \) is the space of profiles.

The social choice rule thus takes preferences \( \{ p_i \}_{i=1}^n \in \prod P \) and map them into a single preference \( p \in P \). Less formally the social choice rule takes the collection of order preserving maps of each of the individuals and produce a new, aggregated, order preserving map.

We would like \( \phi \) to behave nicely with respect to the following conditions. First of all, \( \phi \) should not discriminate between individuals, each one should have an equal amount of power in the final preference. Also, we would like the final preference not to be arbitrary in the following sense: if all preferences are \( P \) then the resulting preference should also be \( P \). However, this is not to say that if \( y \succ_i y' \) for all \( i \), \( 1 \leq i \leq n \), implies \( \phi(y) \succ \phi(y') \).

Lastly, we shall require that the social choice map is continuous with respect to the product topology it inherits from \( P \). Of all requirements this last one is the furthest from the usual assumptions. Still, it is mathematically, the most familiar while the other two require some defining.

**Definition 2.14.** Let \( P \) be a set of preferences, let \( p_i \in P \) and let \( \phi \) be a social choice rule. We say that \( \phi \) respects unanimity (or is unanimous) if the following holds

\[
\phi(p_i, \ldots, p_i) = p_i.
\]

Both anonymity and unanimity can be motivated by what can be considered “fair” in a choice situation. However, continuity is not as readily motivated from just the act of choosing. Instead one would have to rely on the usefulness of continuity in other applications, such as discussed in the previous section. Finally, as continuity is the defining feature of the idea by Chichilnisky and Heal, as compared to Arrow’s, one could argue that this is the most interesting property of the model.
This is nearly all we need to state the main theorem of the essay. Notably contractible spaces will not be defined even though they are instrumental in the theorem. Instead we shall define them in the next section. For now one may think of a contractible space as a space, which can be deformed into a single point. While this is enough to understand the theorem we shall see that the devil, as per usual, is in the details. Therefore, in order to prove Theorem 2.16 we must work through some algebraic topology, which we do after a brief discussion of the theorem.

**Theorem 2.16.** Let \( P \) be the space of preference profiles. A continuous function \( \phi : \prod_{i \in \mathcal{I}} P \to P \), which satisfies anonymity and unanimity, exists if and only if \( P \) is contractible.

This is a striking result. At first glance it is somewhat surprising that the solution to an economical problem lies within algebraic topology, a subject rarely used in economy. If we were to dig deeper it might be expected that contractibility is sufficient for the existence of a well behaved social choice rule. Intuitively if \( P \) is deformable to a point then all \( p \in P \) are, in a sense, equivalent to each other. Then their order does not matter and therefore \( \phi \) satisfies anonymity. By similar reasoning would it be impossible for the social choice rule not to be unanimous since all preferences are equal. Finally, continuity can also be motivated by the idea that we are only working with a single point, which implies a trivial continuity. Therefore it seems natural that contractibility is a sufficient condition for \( \phi \). But that it is necessary for \( \prod_{i=1}^{n} P \) to be contractible – that is quite remarkable since it is a somewhat severe restriction.

In order to prove Theorem 2.16 we have to develop quite a bit of algebraic topology. Specifically we are interested in extending maps \( f : A \to Y \) to a map \( f : X \to Y \), where \( A \subset X \).
3. Algebraic Topology

The aim of this section is to provide a coherent exposure to the fundamentals of algebraic topology. We will go as far as to Obstruction Theory (Section 4) in a fairly rigorous way. Still, the exposition of (for instance) Homology and Cohomology (Section 4.1) will be kept very short. Ultimately, the goal with the use of algebraic topology is to prove Theorem 4.35, which we will then use in the proof of the main theorem (Theorem 2.16). As a note of caution: we will use the shorthand notation $fg$ for the composition of two maps throughout the essay.

The reader is expected to know the basics of general topology. However, we will state three definitions for reference and as a reminder.

**Definition 3.1.** Let $X$ be a topological space and let $\{X_j\}_{j \in J}$ be a sequence of topological spaces, for some indexing set $J$. Given functions $g_j : X_j \to X$, we say that the finest topology on $X$ such that each $g_j$ is continuous is the *coinduced topology* (on $X$ by $g_j$).

**Definition 3.2.** Let $X$ be a topological space and let $X' \subset X$. We say that the *quotient space* is $X/X'$ together with the topology coinduced by the projection map $X \to X'$.

**Definition 3.3.** Let $X$ be a topological space and let

$$\mathscr{A} = \{ A | A \text{ is a subspace of } X \}$$

be a collection of subspaces of $X$. If the topology on $X$ is coinduced by the inclusion maps $i : A \hookrightarrow X$, then we say that $X$ has a topology *coherent with $\mathscr{A}$*. This is also known as the *weak topology*.

Historically Homotopy has its roots in the study of integration on $\mathbb{C}$ by Cauchy [8]. But the field is also partially rooted in the connectedness of spaces studied by Riemann [22]. Naturally, neither of them had access to what we now know as homotopy. To lay down the basics of algebraic topology we will begin with going back to the roots and the deformation of paths. In Section 3.2 we will move into more modern methods as we look at category theory, which perhaps is the defining feature of modern algebraic topology. Chronologically, category theory is a quite recent addition to mathematics. It was only in the mid twentieth century when the field was explored by Samuel Eilenberg and Saunders MacLane. The homotopy introduction will then be ended by looking at $H$-groups in Section 3.3, which will be important as we move into Section 3.5.

3.1. Deformation of Paths. Homotopy is the study of deformations of paths, i.e. curves in topological spaces. In general, homotopy theory asks the question: how do loops behave on a space? For instance, if every loop on a space $X$ can be continuously deformed into a point, then we say that the space is contractible. If a space is not contractible, then the way in which it fails to be so conveys information about it. We will go deeper into this later. Especially into contractibility, which is defined in Definition 3.7. But first, we will make precise what kind of spaces we are going to study.

One could work with simple topological spaces: a set $X$ together with a topology on it. That would be sufficient for most of the elementary topics presented here and it is the approach of [16]. However, we will stay closer to [24] and use so called
topological pairs. An advantage is that the ordinary topological spaces are special cases of topological pairs.

**Definition 3.4.** Let $X$ be a topological space and let $A$ be a subspace of $X$. Then we say that the pair $(X, A)$ is a topological pair. The special pair $(X, \emptyset)$ is commonly denoted $X$ rather than $(X, \emptyset)$.

We can now use topological pairs to define the continuous deformation of a map. Since the interval $[0, 1]$ will be used frequently throughout the essay, we will use $I = [0, 1]$. Furthermore, we use $\dot{I}$ to denote the set $\{0, 1\}$.

**Definition 3.5.** Let $(X, A)$ and $(Y, B)$ be topological pairs and let $X'$ be a subspace of $X$. Furthermore let $f_0$ and $f_1$ be functions $(X, A) \to (Y, B)$ such that $f_0|_{X'} = f_1|_{X'}$. If there exists a continuous map $F : (X, A) \times I \to (Y, B)$ such that $F(x, 0) = f_0(x)$, $F(x, 1) = f_1(x)$ for all $x \in X$, and $F(x, t) = f_1(x) = f_2(x)$ for all $x \in X'$ and all $t \in I$, then we say that $f_0$ is homotopic to $f_1$ relative to $X'$.

In Definition 3.5 the map $F : (X, A) \times I \to (Y, B)$ is so central to the point that it has its own name. We say that any map such as $F$ is a homotopy relative to $X'$ from $f_0$ to $f_1$, and write $F : f_0 \simeq f_1$ rel $X'$. The special case $X' = \emptyset$ would imply that nowhere on $X$ is $f_1 = f_0$. On the other hand if we have a homotopy relative to $X$ for $f_0$ and $f_1$, then the maps $f_0$ and $f_1$ must be equivalent everywhere on $X$. In fact, for $X'' \subset X'$ and a homotopy $F : f_0 \simeq f_1$ rel $X'$, there is a homotopy $G : f_0 \simeq f_1$ rel $X''$. Figure 2 shows two maps that are homotopic relative to their endpoints.

When it comes to the spaces we have the convex spaces as a special case. Regardless of how the space is otherwise defined: a convex space will always possess a native homotopy $F$. We prove this in Corollary 3.6 and it is illustrated in Figure 2.

**Corollary 3.6.** Let $X$ be an arbitrary space and let $Y$ be a convex space. For any two functions $f_0, f_1 : X \to Y$ there is a natural homotopy $F(x, t) = tf_1 + (1 - t)f_0$.

**Proof.** Because, if $f_1(x)$ and $f_2(x)$ are in $Y$, then for $t \in I$, $tf_1 + (1 - t)f_0$ must be in the $Y$. Also, $F(x, 0) = f_0(x)$ and $F(x, 1) = f_1(x)$. \[\Box\]

\[\text{Figure 2. The continuous deformation of } f \text{ to } f' \text{ by } F(x, t) = tf_1 + (1 - t)f_0. \text{ The functions } f \text{ and } f' \text{ are homotopic relative to their endpoints.}\]
A path is said to be contractible if it is homotopic to a point, that is: if it is continuously deformable to a constant map. Surely the types of maps that are contractible on a pair $(X, A)$ says something about the pair. To formalize this idea we will use identity maps a lot, so let $\text{id}_X$ will be the identity map on $X$. For a topological pair, $(X, A)$, then $\text{id}_{(X, A)} = \text{id}_X$ since $A \subset X$ so that $\text{id}_A = \text{id}_X|A$.

**Definition 3.7.** Let $(X, A)$ be a topological pair and let $c : (X, A) \to (X, A)$ be a constant map of $(X, A)$ to $(X, A)$. If there is a homotopy $F : (X, A) \times I \to (X, A)$ such that $\text{id}_X \simeq c$ under $F$, then we say that $F$ is a contraction of $(X, A)$. Furthermore we say that $(X, A)$ is contractible.

Based on the existence of a natural homotopy on every convex space we could expect them to have some special relation to contractibility, which they do.

**Proposition 3.8.** Every convex space is contractible.

**Proof.** This follows easily from Corollary 3.6 since on a convex space we can always use the homotopy $F(x, t) = \text{tid}_X + (1 - t)c$. Thus a convex space must be contractible. \hfill $\square$

We have now discussed the absolute basics of homotopy theory. It would be possible to continue from here and not mention categories but we would still implicitly rely on them. Therefore, we are going to properly introduce category theory. To put homotopy in a categorical context we will need two theorems (Theorem 3.9 and Theorem 3.11). In addition, the perhaps most useful piece of notation used in this essay is introduced in Definition 3.10.

**Theorem 3.9.** Homotopy relative to $X'$ is an equivalence relation in the set of maps from $(X, A)$ to $(Y, B)$.

**Proof.** We prove reflexivity, symmetry and transitivity.

1. Reflexivity. For $f : (X, A) \to (Y, B)$ define $F : f \simeq f$ rel $X$ by $F(x, t) = f(x)$.

2. Symmetry. Given $F : f_0 \simeq f_1$ rel $X'$, define $F' : f_1 \simeq f_0$ rel $X'$ by $F(x, t) = F(x, 1 - t)$.

3. Transitivity. Given $F : f_0 \simeq f_1$ rel $X'$ and $G : f_1 \simeq f_2$ rel $X'$, define $H : f_0 \simeq f_2$ rel $X'$ by

$$H(x, t) = \begin{cases} F(x, 2t) & 0 \leq t \leq \frac{1}{2}, \\ G(x, 2t - 1) & \frac{1}{2} \leq t \leq 1. \end{cases}$$

Note that $H$ is continuous because its restriction to each of the closed sets $X \times [0, \frac{1}{2}]$ and $X \times [\frac{1}{2}, 1]$ is continuous. \hfill $\square$

Since homotopy yields a equivalence relation on maps between two homotopy pairs one would expect there to be equivalence classes as well. Not only do we have and use such classes but they also turn out to be extremely useful. The following definition and the notation will be very important in the remainder of the essay.

**Definition 3.10.** Let $(X, A)$ and $(Y, B)$ be topological pairs and let $X'$ be a subset of $X$. Furthermore, let $F$ be a homotopy relative to $X'$ on the set of maps between $(X, A)$ and $(Y, B)$. Then the homotopy classes relative to $X'$ are the disjoint equivalence classes on the set of maps between $(X, A)$ and $(Y, B)$ induced by any $F$. We denote the collection of homotopy classes relative to $X'$ by $[X, A; Y, B]_{X'}$.  


Let us expend a few words on notation. For a map \( f : (X, A) \to (Y, B) \) we will use \([f_{X,Y}]\) to denote its class in \([X, A; Y, B]_X\). If \( X' = \emptyset \) then we write \([f] \) instead of \([f']_{X,Y} \) and \([X, A; Y, B]_X\) instead of \([X, A; Y, B]_X\).

We now continue with the final result we shall need to move into categories: that composition preserves homotopy. This is specifically needed in the very definition of a category, see (3) in Definition 3.12.

**Theorem 3.11.** Composites of homotopic maps are homotopic.

**Proof.** Let \( f_0, f_1 : (X, A) \to (Y, B) \) be homotopic relative to \( X' \) and let \( g_0, g_1 : (Y, B) \to (Z, C) \) be homotopic relative to \( Y' \), where \( f_1(X') \subset Y' \). To show that \( g_0, f_0, g_1, f_1 : (X, A) \to (Z, C) \) are homotopic relative to \( X' \), let \( F : f_0 \simeq f_1 \rel X' \) and \( G : g_0 \simeq g_1 \rel Y' \). Then the composite

\[
(X, A) \times I \xrightarrow{F} (Y, B) \xrightarrow{g_0} (Z, C)
\]

is a homotopy relative to \( X' \) from \( g_0f_0 \) to \( g_0f_1 \) and the composite

\[
(X, A) \times I \xrightarrow{f_1 \times \text{id}_I} (Y, B) \times I \xrightarrow{G} (Z, C)
\]

is a homotopy relative to \( f_1^{-1}(Y') \) from \( g_0f_1 \) to \( g_1f_1 \). Since \( X' \subset f_1^{-1}(Y') \), we have shown that \( g_0f_0 \simeq g_0f_1 \rel X' \) and \( g_0f_1 \simeq g_1f_1 \rel X' \). The result now follows since by Theorem 3.9 homotopy relative to \( X' \) is transitive. \( \square \)

### 3.2. Category Theory.

The reader is surely excited to see how categories can be used to work with homotopy, or perhaps even eager to see what a category is. One could say that a category is like a set together with a family of maps between sets. But this comparison is lacking as we shall see: a category is a far more interesting concept. What category theory gives us both in connection with homotopy and in general is the ability to solve problems in new settings. For instance, we are going to solve the extension problem, see Definition 3.18, by moving the problem into the setting of groups. This move is conducted by what is known as a functor, see Definition 3.15.

As a quick explanation of notation: we are going to use \( \text{hom} (X, Y) \) to denote the set of morphisms between the objects, as defined in the following definition. For two spaces \( X \) and \( Y \) the set \( \text{hom} (X, Y) \) would be all functions from \( X \) to \( Y \).

**Definition 3.12.** A category \( \mathcal{C} \) consists of

1. a class of objects,
2. for each ordered pair of objects \( X \) and \( Y \) in \( \mathcal{C} \) a set \( \text{hom} (X, Y) \) of morphisms with domain \( X \) and range \( Y \),
3. for every ordered triple of objects \( X, Y, Z \), a function associating to a pair of morphisms \( f : X \to Y \) and \( g : Y \to Z \) their composite

\[
gf = g \circ f : X \to Z.
\]

These satisfy **associativity**: if \( f : X \to Y \), \( g : Y \to Z \) and \( h : Z \to W \), then

\[
h(gf) = (hg)f : X \to W.
\]

As well as **identity**: for every object \( Y \) there is a morphism \( \text{id}_Y : Y \to Y \) such that, if \( f : X \to Y \), then \( \text{id}_Y f = f \) and if \( h : Z \to W \), then \( \text{id}_Y h = h \).
Given the terminology used at the end of Definition 3.12 one could be led to believe that a category is some kind of group. That is certainly not true. A group on the other hand is, however, a special kind of category with just one object with isomorphisms as morphisms on it. One should not confuse this with the category of abelian groups, \( \text{Ab} \), which has abelian groups as objects and homomorphisms between them as morphisms. Another (important) category is \( \mathcal{T} \): the category of pointed spaces. The objects of \( \mathcal{T} \) are spaces with distinguished points and maps preserving these distinguished points as morphisms. When we use spaces in the remainder of the essay these are assumed to belong to \( \mathcal{T} \).

More specifically, we say that a topological space \( X \) is a pointed space if, and only if, it has a distinguished point \( x_0 \in X \). Commonly a pointed space is denoted by \((X,x_0)\) and a pointed topological pair by \((X,A,x_0)\), where \( x_0 \in A \). Similarly, a basis point preserving map is a map \( f : (X,A,x_0) \to (Y,B,y_0) \). Pointed spaces together with basis point preserving maps do form a category, with spaces as objects and maps as morphisms. In the future we will assume that all spaces are pointed and all maps are basis point preserving. However, as said previously, we will suppress the \((X,A,x_0)\) notation in favour of the shorter \((X,A)\).

Even tough it was just stated that categories are not groups, some of the things commonly associated with the latter exists on the former as well. For instance given a morphism \( f \) in a category \( \mathcal{C} \) if there is a morphism \( g \) in \( \mathcal{C} \) such that \( g \circ f = \text{id} \), we say that \( g \) is a left inverse of \( f \). Similarly if \( f \circ g = \text{id} \), then \( g \) is a right inverse of \( f \). If \( f \) has both a left and right inverse then these are equal (see Spanier [24] Lemma 1.1) and we say that \( g \) is the inverse.

**Definition 3.13.** Let \( \mathcal{C} \) be a category with objects \( X \) and \( Y \) and a morphism \( f \). If \( f : X \to Y \) is such that there is an inverse \( g : Y \to X \), then we say that \( f \) is an equivalence, denoted \( f : X \approx Y \).

With the basics of categories in place we will now show that there is a homotopy category, as defined in Proposition 3.14.

**Proposition 3.14.** There is a homotopy category with topological pairs as objects and homotopic maps as morphisms.

**Proof.** All that has to be done is to verify the conditions of Definition 3.12. First, there is a class of objects: the topological pairs. Since the continuous functions satisfy condition two then the homotopy classes must also satisfy it. Finally, from Theorem 3.11 we know that composites of homotopic maps are homotopic. Therefore we do indeed have a homotopy category and we say that it is the homotopy category of pairs. We will use \( \mathcal{C}_0 \) to denote this category. \( \square \)

While categories may surely be exciting, there is not much use in a single category. Instead, the usefulness of category theory lies in the ability to move between categories. Because sometimes a difficult problem in one category has a quite simple solution in another. However, this requires the transition between two categories to be rigorously defined.

**Definition 3.15.** Let \( \mathcal{C} \) and \( \mathcal{D} \) be categories, let \( X \) and \( Y \) be objects in \( \mathcal{C} \). A covariant functor \( T \) then consists of two things. First, an object function which assigns to each object \( X \) in \( \mathcal{C} \) an object \( T(X) \) in \( \mathcal{D} \). Second, a morphism function assigning to each morphism \( f \) of \( \mathcal{C} \) a morphism \( T(f) : T(X) \to T(Y) \) in \( \mathcal{D} \) such that:
(1) \( T(\text{id}_X) = \text{id}_{T(X)} \)

(2) \( T(gf) = T(g)T(f) \)

If instead \( T(f) : T(X) \leftarrow T(Y) \) and \( T(gf) = T(f)T(g) \), we say that \( T \) is a contravariant functor.

To use categories and functor as tools for solving problems it is immensely helpful to have diagrams. We define these in Definition 3.16.

**Definition 3.16.** Let \( X, Y \) and \( Z \) be objects of a category and let \( f : X \rightarrow Y \), \( g : Y \rightarrow Z \) and \( h : X \rightarrow Z \) be morphisms of this category. We can represent this in a diagram where arrows indicate direction of morphisms.

\[
\begin{array}{ccc}
  Y & \xrightarrow{g} & Z \\
  \downarrow{f} \quad & & \quad \downarrow{h} \\
  X & \xrightarrow{h} & Z
\end{array}
\]

If the diagram is such that \( g \circ f = h \) then we say that the diagram commutes.

Diagrams can also be used in reference to functors. We will, for instance, use them in the definition of the natural transformation next. In this definition we will use a morphism between functors.

**Definition 3.17.** Let \( \mathcal{C} \) and \( \mathcal{D} \) be categories and let \( F : \mathcal{C} \rightarrow \mathcal{D} \) and \( G : \mathcal{C} \rightarrow \mathcal{D} \) be two contravariant (or covariant) functors. Furthermore, let \( X \) and \( Y \) be objects of \( \mathcal{C} \) and let \( f \) be a morphism on \( \mathcal{C} \). If there is a morphism, \( \Phi \), between \( F \) and \( G \) such that the following diagram exists then we say that \( \Phi \) is a natural transformation.

\[
\begin{array}{ccc}
  F(X) & \xrightarrow{\Phi_X} & G(X) \\
  \downarrow{F(f)} \quad & & \quad \downarrow{G(f)} \\
  F(Y) & \xrightarrow{\Phi_Y} & G(Y)
\end{array}
\]

If \( \Phi \) makes the diagram commute then \( \Phi \) is said to be a natural isomorphism.

The natural transformations are important within category theory. We will, however, not make much use of them and the reader is referred to Chapter 1.4 of [25] for more details. Instead, we shall look at the use of functors (and diagrams) in stating and solving problems.

**Definition 3.18.** Let \( X \) and \( Y \) be topological spaces and let \( A \) be a subspace of \( X \) such that there is a continuous function \( f : A \rightarrow Y \). Then the extension problem is the determination of whether or not \( f \) can be extended to a continuous function \( f^* : X \rightarrow Y \). Diagrammatically this is equivalent to the existence of the dashed in the following diagram.

\[
\begin{array}{ccc}
  A & \xrightarrow{f} & Y \\
  \downarrow{f^*} \quad & & \quad \downarrow{f} \\
  X
\end{array}
\]

As a brief side note: in category theory one can often take a problem or a proof and construct its dual by flipping the direction of all morphisms. For instance, the extension problem of Definition 3.18 has a dual in the lifting problem, which is the question of existence of a map \( \lambda \) making the following diagram commute.
Furthermore, if one proves something for a covariant functor, say, then that result has a dual result for a contravariant functor. This will be of importance later.

Aside from being a good example the extension problem is also important for the main theorem of the essay. In the proof of Theorem 2.16 we will show that a function exists on a subset and then extend it to a larger set. We will now use functors to derive a necessary condition for the existence of \( f^* \) in Theorem 3.19.

**Theorem 3.19.** Let \( T \) be a covariant functor from the category of topological spaces and continuous maps to a category \( \mathcal{C} \). A necessary condition that a map \( f : A \to Y \) is extendible to \( X \) is that there exist a morphism \( \varphi : T(X) \to T(Y) \) such that \( \varphi \circ T(id_X) = T(f) \).

**Proof.** Assume that \( f' : X \to Y \) is an extension of \( f \). Then \( f'i = f \). Therefore \( T(f') \circ T(i) = T(f) \), and \( T(f') \) can be taken as the morphism \( \varphi \). \[ \square \]

In Theorem 3.19 we do not limit ourselves in the choice of functor. We could use the forgetful functor, which strips the objects of some of their structure. For instance topological spaces could be stripped of their topological structures sending them to the category of sets.

As a tangential note there are functor categories where the objects are functors and the morphisms, which map functors to each other, are known as natural transformations. It is surprising, but every category can actually be embedded in a functor category by the Yoneda embedding. But this is not what we are going to use. Instead we will look at two very special functors in Example 3.20.

**Example 3.20.** Let \( X, Y \) and \( Z \) be objects of \( \mathcal{C} \) and let \( f : X \to Z \) be a morphism of \( \mathcal{C} \). Then there is a covariant functor \( \pi_Y \) such that \( \pi_Y(X) = \text{hom}(Y,X) \) and \( \pi_Y(f) = f_\#: \text{hom}(Y,X) \to \text{hom}(Y,Z) \). There is also a contravariant functor \( \pi^Y \) such that \( \pi^Y(X) = \text{hom}(X,Y) \) and \( \pi^Y(f) = f^\#: \text{hom}(Z,Y) \to \text{hom}(X,Y) \).

The functors of Example 3.20 are the fundamental blocks of the type of algebraic topology that we are going to use: the homotopy groups. We will use these functors to study the extension problem in the category of groups. However, a glaring issue is that there is nothing group-like about either \( \pi_Y \) nor \( \pi^Y \), yet. In the next section we will work out a remedy for this.

3.3. **H-groups.** To make \( \pi_Y \) take values in the category of groups and homomorphisms we will endow the topological spaces with an additional structure. This will yield new spaces known as \( H \)-spaces and \( H \)-cospaces in honour of the work done by Heinz Hopf, see for instance [17]. With some additional assumptions on the spaces we get \( H \)-groups and \( H \)-cogroups. In conjunction with these spaces we will define the loop space \( \Omega X \) and the suspension \( SX \) of a topological space \( X \). While the suspension is to us the far more important one, both will come to use in Section 3.4. Unfortunately, our treatment of what is quite an interesting subject will be brief and the reader is referred to Chapter 2 of [3] or Section 9.2 of [25] for more. For a brief historical overview please see Chapter 26 of [18].
Technically, the topological spaces will be endowed with a multiplication. With just the multiplication and nothing else we have the $H$-space. To create a proper $H$-group we shall need some additional assumptions on the multiplication.

**Definition 3.21.** Let $P$ be a topological space and let $\mu : P \times P \to P$. We say that $P$ is an $H$-space, if the constant map $c : P \to P$ is such that $\mu \circ (c, \text{id}_P) : P \to P \times P \to P$ and $\mu \circ (\text{id}_P \times c) : P \to P \times P \to P$ are both homotopic to $\text{id}_P$.

We will now impose associativity on $\mu$. To do so on a topological space rather than a group will require us to use diagrams. By associativity in a homotopy sense we mean that the following diagram is homotopy commutative, i.e. that it commutes in the homotopy category.

In the definition we are only interested in the behaviour in the homotopy category and not on $\mathcal{F}$. In other words it is just required that $\mu \circ (c, \text{id}_P) \simeq \mu \circ (\text{id}_P, c)$, rather than an equality. This should not be surprising since we look at $\pi_P$, which just contains the homotopy equivalence classes.

In addition to associativity we need inverses to create a group. Again it is in the homotopy category the work will be done. We say that a map, $\psi : P \to P$, is a *homotopy inverse* for $P$ and $\mu$ if, and only if $\mu \circ (\text{id}, \psi) \simeq c \simeq \mu \circ (\psi, \text{id})$, where $c : P \to P$ is the constant map on $P$. By adding associativity and the inverse to a $H$-space we get an $H$-group. This is formalized in the following definition.

**Definition 3.22.** Let $P$ be a $H$-space with multiplication $\mu : P \times P \to P$. If $\mu \circ (\mu \times \text{id}) \simeq \mu \circ (\text{id} \times \mu)$ and there is a homotopy inverse $\psi$ of $\mu$ on $P$, then we say that $P$ is an $H$-group.

The reader interested in reading more about topological groups is referred to [2]. For that reader it may be useful to know that our $H$-groups (as by Definition 3.22) are technically paratopological groups. That is, however, not a concern to us. We shall instead focus on a few key properties of $H$-groups. In particular, Theorem 3.23 make it clear why we are interested in those groups.

**Theorem 3.23.** If $P$ is an $H$-group, then $\pi^P$ is a contravariant functor from the category of topological spaces and continuous functions to the category of groups and homomorphisms. If $P$ is an abelian $H$-group, then $\pi^P$ takes values in the category of abelian groups.

**Proof.** See the discussion preceding Theorem 1.5.1 of [24].

The perhaps most useful example of an $H$-group is known as the loop space. If $(Y, y_0)$ is a space with distinguished point $y_0$, then the *loop space*, $\Omega Y$, of $Y$ is the set of continuous maps $\omega : (I, \bar{I}) \to (Y, y_0)$. That is, $\Omega Y$ are all loops in $Y$ based at $y_0$. We shall return to $\Omega$ later on, both when discussing suspensions and...
in Section 4 while addressing obstructions. Now we will instead move to a perhaps even more interesting space.

As said earlier: many things in category theory have useful duals. In this case, there is a dual construction to $H$-groups, which dualizes Theorem 3.23. To arrive at that result we need to define a new kind of multiplication on topological spaces. Let $Q$ be a topological space then we want a multiplication akin to $\nu : Q \to Q \times Q$.

However, since $Q \times Q$ is not a pointed space such a multiplication will not work. Instead we shall introduce the wedge, which produces a pointed product of spaces.

**Definition 3.24.** Let $X$ and $Y$ be pointed spaces with basis points $x_0$ and $y_0$, respectively. Then the wedge $X \vee Y$ is defined as the quotient space of the disjoint union of $X$ and $Y$ with $x_0$ identified with $y_0$. Specifically

$$X \vee Y = (X \sqcup Y)/(x_0 \sim y_0).$$

**Definition 3.25.** Let $Q$ be a pointed topological space and let $\nu : Q \to Q \vee Q$ be a continuous multiplication. Then we say that $\pi_Q$ is an $H$ cogroup if for a constant map $c : Q \to Q$ both $\nu \circ (c, \text{id}_Q) : Q \to Q \vee Q \to Q$ and $\nu \circ (\text{id}_Q, c) : Q \to Q \vee Q \to Q$ are homotopic to $\text{id}_Q$. And also the square

$$\begin{array}{ccc}
Q & \xrightarrow{\nu} & Q \vee Q \\
\downarrow & & \downarrow \text{id}_Q \times \nu \\
Q \vee Q & \xrightarrow{\nu \circ \text{id}_Q} & Q \vee Q \vee Q
\end{array}$$

commutes homotopically.

With the $H$-cogroups we have a dual result to that of Theorem 3.23.

**Theorem 3.26.** If $Q$ is an $H$ cogroup, $\pi_Q$ is a covariant functor from the homotopy category of pointed spaces with values in the category of groups and homomorphisms. If $Q$ is an abelian group then this functor takes values in the category of abelian groups and homomorphisms.

**Proof.** Please see the discussion preceding Theorem 1.5.5 of [24].

Similar to how loop spaces are the quintessential example of an $H$-group we have suspensions as a prominent example of an $H$-cogroup.

**Definition 3.27.** Let $X$ be a pointed space with $x_0$ as basis point. Then the reduced suspension of $X$, $SX$ is the quotient space

$$(X \times I)/(X \times \{0\})/(\cup \{x_0\} \times I) \cup (X \times \{1\})$$

Thus, the suspension is an extrusion of the space $X$ with the top and bottom collapsed to a point. Given this, our definition may seem strange. One would perhaps expect the suspension to be $(X \times I)/(X \times \{0\} \cup X \times \{1\})$ and this is actually the (non-reduced) suspension. While the latter suspension could make more sense geometrically, it does not preserve the basis point of $X$.

We can now apply what we have done so far to the extension problem. This is done on some very specific spaces: the $n + 1$ dimensional ball, $E^{n+1}$, and the $n$-dimensional sphere, $S^n$. The use of $S$ both here and in Definition 3.27 is not a mistake. One can view the $n$-dimensional sphere as $n$ successive suspensions of the unit interval. We will go deeper into this in Definition 3.41. For now we will look
at the extension of a map \( f : S^n \to Y \) to a map \( f' : E^{n+1} \), which is a special case of what we want to achieve. This is Theorem 1.3.12 of [24].

**Example 3.28.** Let \( p_0 \) be any point of \( S^n \) and let \( f : S^n \to Y \), then \( f \) can be continuously extended to \( E^{n+1} \) if and only if \( f \) is null homotopic, i.e. homotopic to the constant map. We first show that being null homotopic implies that \( f \) can be extended to \( E^{n+1} \). Let \( F : f \simeq c \), where \( c \) is the constant map of \( S^n \) to \( y_0 \in Y \). Define an extension \( f' \) of \( f \) to \( E^{n+1} \) by

\[
f'(x) = \begin{cases} 
  y_0 & 0 \leq ||x|| \leq \frac{1}{2} \\
  F(x/||x||, 2 - 2||x||) & \frac{1}{2} \leq ||x|| \leq 1
\end{cases}
\]

Since \( F(x, 1) = y_0 \) for all \( x \in S^n \), the map \( f' \) is well-defined. Furthermore, \( f' \) is continuous because its restrictions to each of the closed sets \( \{x \in E^{n+1} | 0 \leq ||x|| \leq \frac{1}{2}\} \) and \( \{x \in E^{n+1} | \frac{1}{2} \leq ||x|| \leq 1\} \) are continuous. Since \( F(x, 0) = f(x) \) for \( x \in S^n \), \( f|_{S^n} = f \) and \( f' \) is a continuous extension of \( f \) to \( E^{n+1} \).

It now remains to show that if the map \( f \) can be extended to \( E^n \), then \( f \simeq c \). If \( f \) has the continuous extension \( f' : E^{n+1} \to Y \), define \( F : S^n \times I \to Y \) by

\[
F(x, t) = f'(1-t)x + tp_0.
\]

Then \( F(x, 0) = f'(x) = f(x) \) and \( F(x, 1) = f'(p_0) \) for \( x \in S^n \). Since \( F(p_0, t) = f'(p_0) \) for \( t \in I \), \( F \) is a homotopy relative to \( p_0 \) from \( f \) to the constant map to \( f'(p_0) \). This shows that \( f \) is null homotopic relative to \( p_0 \), which in turn implies that \( f \) is null homotopic. \( \square \)

### 3.4. Exact Sequences of Sets of Homotopy Classes

We will now get deeper into algebraic topology. While the goal is to be thorough, the amount of material to be covered is vast. The layout is structured in the following way. The present section looks into the algebraic property of exactness, with the goal of creating an exact sequence of spaces. This is then combined with \( H \)-spaces in Section 3.5 to define the higher homotopy groups. On its own an exact sequence is easy to define, we do this in definitions 3.31 and 3.32. However, more work is necessary before exact sequences can be used on topological spaces.

In the category of groups and homomorphisms we have the concepts of kernels and images. For two groups \( A \) and \( B \) and a morphism \( f : A \to B \), we say that the kernel of \( f \) are precisely the elements of \( A \) which maps to \( 0 \in B \) under \( f \) and we denote those by \( \ker f \). In other words \( \ker f = f^{-1}\{0\} \). The image are the elements of \( B \) given by \( f(A) \). We will denote the image of \( f \) by \( \text{Im } f \). The most important kernel and image, to us, is covered in Example 3.29.

**Example 3.29.** Let \( (X, A), (X', A') \) and \( (Y, B) \) be topological pairs, and let \( 0 \in [X, A; Y, B] \) be the class of null homotopic maps \( (X, A) \to (Y, B) \). For a map, \( f : (X', A') \to (X, A) \), we say that the kernel of the induced map \( f^# \), \( \ker f^# \), is the homotopy classes of maps \( g : (X', A') \to (Y, B) \) such that \( f^# g = 0 \). \( \square \)

Much in the same spirit we can define the image in a homotopy sense.

**Definition 3.30.** Let \( (X, A), (X', A') \) and \( (Y, B) \) be topological pairs and let \( f \) be a map \( (X', A') \to (X, A) \). The image of the induced map \( f^# \), \( \text{Im } f^# \), is the homotopy classes of compositions \( f^# \circ g \), where \( [g] \in [X', A'; Y, B] \).
In abstract algebra we say that a series of the form

\[ 0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0, \]

where \( f \) and \( g \) are morphisms and \( A, B \) and \( C \) are groups, is a short sequence. Furthermore, a short sequence is said to be exact if \( \ker g = \text{Im } f \). We will now use the image and kernel of induced maps to create a corresponding definition for topological spaces.

**Definition 3.31.** Let \((X', A'), (X, A), (X'', A'')\) and \((Y, B)\) be topological pairs. A short exact sequence of (pairs and maps) is a three-term sequence of pairs and maps

\[ \cdots \to (X', A') \xrightarrow{f} (X, A) \xrightarrow{g} (X'', A'') \to \cdots \]

(possibly terminating at either end) such that for any pair \((Y, B)\) the associated sequence of pointed sets

\[ \cdots \to [Y, B; X', A'] \xrightarrow{f_#} [Y, B; X, A] \xrightarrow{g_#} [Y, B; X'', A''] \to \cdots \]

is exact.

A coexact three-term sequence of pairs and maps is a sequence

\[ \cdots \to (X', A') \xleftarrow{f} (X, A) \xrightarrow{g} (X'', A'') \to \cdots \]

such that for any pair \((Y, B)\) the associated sequence of pointed sets

\[ \cdots \to [X'', A''; Y, B] \xrightarrow{g_#} [X, A; Y, B] \xleftarrow{f_#} [X', A'; Y, B] \to \cdots \]

is exact.

However, short exact sequences are of limited interest. In order to prove anything interesting the sequences has to be longer than three spaces.

**Definition 3.32.** Let there be a sequence

\[ \cdots \to (X_{n+1}, A_{n+1}) \xrightarrow{f_n} (X_n, A_n) \xrightarrow{f_{n-1}} (X_{n-1}, A_{n-1}) \to \cdots \]

of topological pairs and functions, possibly terminating with a 0 at both, one or neither end. If \( \ker f_k \subset \text{Im } f_{k+1} \) for \( k + 1 \leq n \), then we say that the sequence is exact at \( k \). Should the sequence be exact at every \( n \), then we say that it is a long exact sequence.

The previous definition connects spaces with exactness in the homotopy category. We will now work to rewrite the sequence

\[ (X', A') \xrightarrow{f} (X, A) \xrightarrow{g} (X'', A'') \]

in terms of \( H \)-cogroups. The first step is to introduce the reduced cone, which is the lower half of a suspension. From this the mapping cone is created, and in Theorem 3.35 the coexactness of a sequence including the mapping cone is proven.

**Definition 3.33.** Let \( X \) be a pointed space with basis point \( x_0 \), then the absolute reduced cone is the quotient space

\[ CX = X \times I/(X \times \{0\} \cup I \times \{x_0\}). \]

Similarly for a topological pair \((X, A)\) the relative reduced cone is the quotient space \( C(X, A) = (CX, CA) \).
Just as the case were with the suspension, the reduced cone is a construction to preserve the basis point. Indeed, to see that $(X \times I)/((X \times \{0\}) \cup (I \times \{x_0\}))$ is a structure that preserves the basis point consider the following. Let $X$ be the disk $E^2$, then $X \times I$ is a regular cylinder. Identifying the entire top of the cylinder with a point, i.e. $X \times I/(X \times 0)$, yields a regular cone. Comparing this to the reduced cone it is clear that the latter is a regular cone with the line above $x_0$ collapsed to a single point.

**Definition 3.34.** Let $X'$ and $X$ be pointed spaces with basis points $x_0'$ and $x_0$, respectively. Furthermore, let $f$ be a map between $X'$ and $X$, then the **mapping cone** of $f$, $C_f$ is defined to be the space $CX' \vee X$ with the identification $(x', 1) \simeq f(x')$ for all $x' \in X'$.

We will now show that there is a coexact sequence $(X', A') \xrightarrow{f} (X, A) \rightarrow (C_f, C_f')$, with $f'$ and $f''$ defined later. By definition this is true if
\[ [C_f, C_f'; Y, B] \rightarrow [X, A; Y, B] \rightarrow [X', A'; Y, B] \]
is exact. We utilise this in the next theorem. Later on, the reverse implication, that coexactness in $\mathcal{C}$ implies exactness in the homotopy category, is used to prove Theorem 3.48. But at that point the sequence will have been heavily modified, while keeping its coexactness.

**Theorem 3.35.** For any map $f : (X', A') \rightarrow (X, A)$ the sequence
\[ (X', A') \xrightarrow{f} (X, A) \xrightarrow{i} (C_f, C_f') \]
is coexact.

**Proof.** Let $(Y, B)$ be arbitrary (with $B$ not necessarily closed in $Y$) and consider the sequence
\[ [C_f, C_f'; Y, B] \xrightarrow{i^\#} [X, A; Y, B] \xrightarrow{f^\#} [X', A'; Y, B]. \]
We now show that $\text{Im } i^\# \subset \ker f^\#$. The composite $i \circ f : (X', A') \rightarrow (C_f, C_f')$ equals the composite
\[ (X', A') \rightarrow C(X', A') \rightarrow C(X', A') \vee (X, A) \xrightarrow{k} (C_f, C_f'), \]
where $k$ is the canonical map to the quotient. However, the inclusion map $(X', A') \rightarrow (X', A')$ is null homotopic (see Lemma 7.1.1 of [24]). Therefore, $i \circ f$ is null homotopic, and so $\text{Im } (f^\# \circ i^\#) = 0$, proving that $\text{Im } i^\# \subset \ker f^\#$.

Assume that $g : (X, A) \rightarrow (Y, B)$ is such that $f^\#[g] = 0$ (that is, $g \circ f$ is null homotopic). There is a map $G' : C(X', A') \vee (X, A) \rightarrow (Y, B)$ such that $G'|_{C(X', A')} = G$ and $G'|_{(X, A)} = g$ (see Lemma 7.1.1 of [24]). Since
\[ G'[x', 1] = G[x', 1] = g(f(x')) = G'(f(x')) \quad \text{for all } x' \in X', \]
there is a map $h : (C_f, C_f') \rightarrow (Y, B)$ such that $G' = h \circ k$. Then $h|_{(X, A)} = g$, showing that $h \circ i = g$ or $[g] = i^\# [h]$. Therefore $\ker f^\# \subset \text{Im } i^\#$. \qed

Theorem 3.35 is quite interesting as it places no condition on the maps and spaces that goes into it. Hence, we can take a map $j : (C_f, C_f') \rightarrow (C_{f'}, C_{f'})$ and Theorem 3.35 then assures that the sequence
\[ (C_f, C_f') \xrightarrow{j} (C_{f'}, C_{f'}) \xrightarrow{k} (C_{f'}, C_{f'}) \]
Figure 3. The left figure depicts the composition of \( C_f \) with the identification of \( CX' \) and \( X \) by \( f \). Similarly, the centre figure shows the same relation between \( CX \) and \( C_f \) by \( i \) for \( C_i \). From this it is easy to see that \( C_i/CX = C_f/X \), which is illustrated in the right figure.

is coexact. By extension, the following sequence must be coexact as well.

\[
(X', A') \xrightarrow{f} (X, A) \xrightarrow{i} (C_f', C_{f''}) \xrightarrow{j} (C_i', C_{i''}) \xrightarrow{k} (C_j', C_{j''}).
\]

Equation 3 is a fundamental part in the definition of the homotopy groups (in Definition 3.43), especially in showing that the homotopy sequence of pairs is exact. In the end we like the corresponding homotopy sequence to be groups. As we know, \( \pi Q \) takes on values in the category of groups whenever \( Q \) is an H-cogroup, which the suspension is. We would therefore like to transform the cones into suspensions.

The following part will show how to transform a mapping cone into a suspension by collapsing its base. To do this we will use a collapsing map, which is a map \( k : (Y, B) \to (Y, B)/Y' \) for a topological pair \((Y, B)\) and a subspace \(Y'\) of \(Y\). First, we will show that the collapsing map \( k \) is a homotopy equivalence. In Figure 3 a reduced mapping and collapsed mapping cones can be seen.

To help the reader navigate the remainder of the section and all the way to Theorem 4.11, a schematic of the connections within the theory is included in Figure 4.

**Lemma 3.36.** Let \((Y, B)\) be a pair and let \(Y'\) be a closed subset of \(Y\). Assume that there is a homotopy \( H : (Y, B) \times I \to (Y', B) \) such that:

1. \( H(y, 0) = y \) for \( y \in Y \),
2. \( H(Y' \times I) \subset Y' \),
3. \( H(Y' \times 1) = y_0 \).

Then the collapsing map, \( k : (Y, B) \to (Y, B)/Y' \), is a homotopy equivalence.

**Proof.** Define a map, \( f : (Y, B)/Y' \to (Y, B) \), by the equation

\[
f(k(y)) = H(y, 1), \quad y \in Y,
\]

which is well defined, because \( H(Y' \times 1) = y_0 \). We show that \( f \) is a homotopy inverse of \( k \). By definition of \( f \), we see that \( H \) is a homotopy from \( 1_{(Y, B)} \) to \( f \circ k \). On the other hand, because \( H(Y' \times 1) \subset Y' \), there is a homotopy

\[
H' : (Y, B)/Y' \to (Y, B)/Y',
\]

such that \( H'(k(y), t) = k(H(y, t)) \) for \( y \in Y \) and \( t \in I \). Then, \( k(f(k(y))) = k(H(y, 1)) = H'(k(y), 1), \quad y \in Y. \)
Therefore, $H'$ is a homotopy from the identity map of $(Y, B)/Y'$ to $k \circ f$, and $f$ is a homotopy inverse of $k$. Since $k$ has a homotopy inverse, it must be a homotopy equivalence. \hfill \Box

We will now extend the result of Lemma 3.36 to a map $k : (C'_\varphi, C''_\varphi) \to (C'_\varphi, C''_\varphi)/CX$. This is done in two steps: first we show the existence of a well behaved homotopy $F : C(X, A) \times I \to (Y, B)$, and then use it to find the extended $k$. 

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{dependencies.png}
\caption{A schematic of the interdependencies between the lemmas, corollaries and theorems necessary to prove Theorem 4.11.}
\end{figure}
Lemma 3.37. Given a map \( f : (X, A) \to (Y, B) \) and a homotopy \( G : (X, A) \times I \to (Y, B) \) of \( f \mid_{(X,A)} \). Then, there is a homotopy \( F : (X, A) \times I \to (Y, B) \) of \( f \) such that \( F \mid_{(X,A)} = G \).

Proof. An explicit formula for \( F \) is
\[
F([x,t], t') = \begin{cases} 
  f(x, t(1 + t')) & t(1 + t') \leq 1, \\
  G(x, t(1 + t') - 1) & 1 \leq t(1 + t').
\end{cases}
\]

\( \square \)

Theorem 3.38. Let \( f : (X', A') \to (X, A) \) be a map and let \( i : (X, A) \to (C_f, C_{f'}) \). Then, \( CX \subset C_f \), \((C_{f'}, C_{f'})/CX = (C_f, C_{f'})/X\), and the collapsing map
\[
k : (C_{f'}, C_{f'}) \to (C_{f'}, C_{f'})/CX
\]
is a homotopy equivalence.

Proof. \( C_{f'} \) is the quotient space of \( CX' \cup CX \) with the identifications \([x', 1] = [f(x'), 1]\) for all \( x' \in X' \), hence \( CX \subset C_{f'} \). It follows that
\[
C_{f'}/CX = CF'/CF' \cap CX = CF'/X.
\]
Similarly, \( C_{f''}/CA = CF''/A \), and because \( C_{f''} \cap CX = CA \)
\[
(C_{f'}, C_{f''})/CX = (C_{f'}, C_{f''})/X.
\]
This proves the first two parts of the corollary. An illustration of this can be seen in Figure 3 where the absolute space case is depicted.

To prove the remaining statement: that \( k : (C_{f'}, C_{f''}) \to (C_{f'}, C_{f''})/CX \) is a homotopy equivalence; we make the following definitions. Let \( F : (X, A) \times I \to C(X, A) \) be the contraction defined by \( F([x,t], t) = [x,(1-t')t] \), and let \( g : (X', A') \to (C_{f'}, C_{f''}) \) be the composite
\[
C(X', A') \subset C(X', A') \cup C(X, A) \to (C_{f'}, C_{f''}),
\]
where the second map is the canonical map. The composite
\[
(X', A') \times I \xrightarrow{f \times 1} (X, A) \times I \subset C(X, A) \times I \xrightarrow{F} C(X, A) \subset (C_{f'}, C_{f''})
\]
is a homotopy \( G : (X', A') \times I \to (C_{f'}, C_{f''}) \) such that \( G(x', 0) = [f(x'), 1] = g[x', 1] \).
By Lemma 3.37, there is a homotopy \( F' : (X', A') \times I \to (C_{f'}, C_{f''}) \) such that \( F'(X', A') \times I = G \) and \( F'([x', t], 0) = g[x', t] \). Then, there is a homotopy
\[
H : (C_{f'}, C_{f''}) \times I \to (C_{f'}, C_{f''}),
\]
defined by the equations
\[
H([x', t], t') = F'([x', t], t') \quad x' \in X'; t, t' \in I,
H([x, t], t') = F([x, t], t') \quad x \in X; t, t' \in I,
\]
which is well-defined since \( F'([x', 1], t') = G(x', t') = F([f(x'), 1], t') \). Then \( H \) satisfies (1), (2) and (3) of Lemma 3.36 with \( (Y, B) = (C_{f'}, C_{f''}) \) and \( Y' = CX \).
Therefore \( k : (C_{f'}, C_{f''}) \to (C_{f'}, C_{f''})/CX \) is a homotopy equivalence. \( \square \)

Consulting the right figure of Figure 3, it is obvious by inspection that \( C_{f'}/CX \) and \( C_{f'}/X \) are equivalent but also suspensions. Formally, as \( C_{f'} = CX' \cup X \), with the identification \( x' \in X' \times 0 = f(x') \), it follows that \( C_{f'}/X = SX' \). By the same reasoning \( C_{f'}/CX = SX' \). Finally, since \( A' \subset X' \) the two previous observations hold for \( A' \) as well. This shows that \( (C_{f'}, C_{f''})/CX = (C_{f'}, C_{f''})/X = S(X', A') \) in
the homotopy category. Another implication is that $k : (C_{i'}, C_{i''}) \to (C_{i'}, C_{i''})/CX$ is a homotopy equivalence between $C_{i'}, C_{i''}$ and $S(X', A')$.

All of the preceding discussion is indicative of the way forward. The natural path is to replace both of the latter mapping cones of Equation 3 with their homotopy equivalent suspensions. This then yields a sequence of $H$-cogroups on which we can work. Formally, we have the following lemma.

**Lemma 3.39.** For any map $f : (X', A') \to (X, A)$ the sequence

$$(X', A') \xrightarrow{ι} (X, A) \xrightarrow{g} (C_{j'}, C_{j''}) \xrightarrow{\delta} S(X', A') \xrightarrow{Sf} S(X, A)$$

is coexact.

**Proof.** By Corollary 3.38, there is a homotopy equivalence

$$k' : C_{i'}, C_{i''} \to (C_{i'}, C_{i''})/X = S(X', A').$$

Defining $k = k' \circ j$, for $j : (C_{i'}, C_{i''}) \to (C_{i'}, C_{i''})$, yields a collapsing map

$$(C_{i'}, C_{i''}) \xrightarrow{j} (C_{i'}, C_{i''}) \xrightarrow{k'} S(X', A').$$

Also by Corollary 3.38, there is a homotopy equivalence

$$k'' : (C_{j'}, C_{j''}) \to (C_{j'}, C_{j''})/CC_{i'} = (C_{i'}, C_{i''})/C_{i'} = S(X, A).$$

Defining the collapsing map $(C_{i'}, C_{i''}) \to (C_{i'}, C_{i''})/C_{j'}$ as the composition $l \circ k'$:

$$(C_{i'}, C_{i''}) \xrightarrow{l} (C_{j'}, C_{j''}) \xrightarrow{k''} S(X', A')$$

gives the following diagram.

$$\begin{array}{ccc}
(C_{i'}, C_{i''}) & \xrightarrow{k'} & S(X', A') \\
\downarrow{k'} & & \downarrow{g} \\
S(X', A') & \xrightarrow{g} & S(X, A)
\end{array}$$

Here $g : S(X', A') \to S(X, A)$ is given by $g([x', t]) = [f(x'), 1 - t]$.

Defining a homotopy $H : (C_{i'}, C_{i''}) \times I \to S(X, A)$ by

$$H([x', t], t') = [f(x'), 1 - tt'] \quad x' \in X', t, t' \in I,$n

$$H([x, t], t') = [x, (1 - t')t] \quad x \in X; t, t' \in I,$n

shows that $k \simeq g \circ k'$. Therefore there is a homotopy-commutative diagram,

$$\begin{array}{ccc}
(C_{i'}, C_{i''}) & \xrightarrow{j} & (C_{i'}, C_{i''}) \\
\downarrow{k'} & & \downarrow{k''} \\
S(X', A') & \xrightarrow{g} & S(X, A)
\end{array}$$

in which $k'$ and $k''$ are homotopy equivalences. Since Equation 3 is coexact and both $k'$ and $k''$ are homotopy equivalences, the sequence

$$(X', A') \xrightarrow{ι} (X, A) \xrightarrow{g} (C_{j'}, C_{j''}) \xrightarrow{\delta} S(X', A') \xrightarrow{Sf} S(X, A)$$

must be coexact as well. \qed

Now, Lemma 3.39 hints at the possibility to, from any exact sequence, create exact sequences of suspensions. This possibility will give us a way to create an arbitrary number of sequences. In the proof we shall use the fact that exactness of a sequence in $(X, A)$ is a property that holds against any $(Y, B)$. It is therefore feasible to try to use the dual property of $\Omega$ and $S$. 
Lemma 3.40. If the sequence

$$(X', A') \xrightarrow{f} (X, A) \xrightarrow{g} (X'', A'')$$

is coexact, so is the suspended sequence

$$S(X', A') \xrightarrow{Sf} S(X, A) \xrightarrow{Sg} S(X'', A'').$$

Proof. For any pair $(Y, B)$ let $\Omega(Y, B) = (\Omega Y, \Omega B)$. By Theorem 2.8 of [24], there is a commutative diagram in which the vertical maps are equivalences of pointed sets.

Hence, $\text{Im}(Sg)^\# = \ker(Sf)^\#$ in the top sequence is equivalent to $\text{Im} g^\# = \ker f^\#$ in the bottom sequence. \hfill \Box

Therefore, if we suspend $(X', A') \xrightarrow{f} (X, A) \xrightarrow{i} (C_f', C_f'')$, we get a coexact sequence

$$S(X', A') \xrightarrow{Sf} S(X, A) \xrightarrow{Si} S(C_f', C_f'').$$

However, from Lemma 3.39, we know that

$$(X', A') \xrightarrow{f} (X, A) \xrightarrow{i} (C_f', C_f'') \xrightarrow{k} S(X', A') \xrightarrow{Sf} S(X, A)$$

is coexact. By an argument similar to that preceding Equation (3), we can guess that, since $\text{Im} k = \ker Sf$, the two sequences can be joined into a longer coexact sequence. Continuing this process for more spaces would then result in an arbitrarily long coexact sequence. This is proven formally in the following theorem, Theorem 3.42. But as we are going to make use of iterated suspensions, i.e. suspensions of suspensions of suspensions and so on, we make the following definition to simplify the notation.

Definition 3.41. If $(X, A)$ is a topological pair, then the $n$-th suspension of $(X, A)$, $S^n(X, A)$, is recursively defined by

$$S^n(X, A) = S(S^{n-1}(X, A)) \quad n \geq 1$$

with

$$S^0(X, A) = (X, A).$$

With this notation and the previous lemmas, the next theorem is trivial.

Theorem 3.42. For any map $f : (X', A') \rightarrow (X, A)$ the sequence

$$(X', A') \xrightarrow{f} (X, A) \xrightarrow{i} \cdots \xrightarrow{S^n f} S^n(X, A) \xrightarrow{S^n i} S^n(C_f', C_f'') \xrightarrow{S^n k} \cdots \xrightarrow{S^n k} S^{n+1}(X', A') \xrightarrow{S^{n+1}f} \cdots$$

is coexact.
Figure 5. Homotopy functors built as the equivalences of maps from $n$-dimensional spheres to $(X, A)$.

Proof. From Lemma 3.39 and Lemma 3.40, for $n \geq 0$ there is a coexact sequence

$$(5) \quad S^n(X', A') \xrightarrow{S^n f} S^n(X, A) \xrightarrow{S^n i} S^n(C_{f'}, C_{f''}) \xrightarrow{S^n k} S^{n+1} \cdots \xrightarrow{S^n k} (X', A') \xrightarrow{S^{n+1} f} S^{n+1}(X, A).$$

Since every three-term subsequence of the sequence in the theorem is contained in one of these five-term coexact sequences, the result follows. □

3.5. Higher Homotopy Groups. This section will be all about two things. We first define the relative homotopy groups and using the results of the previous section we show in Theorem 3.48 that the homotopy sequence of pairs is exact. This will be important for the next topic of the section: $n$-connectedness. Our big result of this subsection is Corollary 3.50, which shows that $(E^n, S^{n-1})$, where $E^n$ is the $n$-dimensional ball, is $n$-connected. This last result holds an important application to CW-complexes, which are treated in the next section.

Definition 3.43. Let $(X, A)$ be a pair with base point $x_0 \in A$. For $n \geq 1$ the $n$-th relative homotopy group, denoted by $\pi_n(X, A, x_0)$, is defined to be equal to $[S^{n-1}(I, \hat{I}); X, A]$. The absolute homotopy group, $\pi_n(X, A, x_0)$, is similarly defined as $[S^{n-1}(I); X]$.

In other words, the $n$-th homotopy group of $(X, A)$, $\pi_n(X, A, x_0)$ is the equivalence class of maps from the $n$th suspension of $(I, \hat{I})$ to the pair $(X, A)$. The suspension $S^n(I, \hat{I})$ is the topological pair of the $n$-dimensional ball and the $n$-dimensional sphere as a subspace. See Figure 5 for a rough depiction.

Since $\pi_n(X, A)$ has a group structure it must therefore have a trivial element. Following the standard practice of abstract algebra we shall denote this element by 0. In the case of the absolute homotopy group, $\pi_n(X)$, a map $\alpha : S^n \to X$ represents the trivial element of $\pi_n(X)$ for $n \geq 1$ if, and only if, $\alpha$ can be continuously extended to $E^{n+1}$. We are going to prove this for the topological space $(X, A)$ in Theorem 3.44. In order to do that a minor, but technical, rewriting
of the suspension map must be undertaken. The point is to show that $S^n(I, \hat{I})$ is homeomorphic to the relative hypercube: $(I^n, \hat{I}^n, z_0)$.

First, $S^{n-1}(I, \hat{I})$ is homeomorphic to $(I \times I^{n-1}, \hat{I} \times I^{n-1})/(I \times \hat{I}^{n-1} \cup 0 \times I^{n-1})$. Because of this homeomorphism it is true that the homotopy classes of

$$(I^n, \hat{I}^n, 0 \times I^{n-1})/(I \times \hat{I}^{n-1} \cup 0 \times I^{n-1}) \to (X, A, x_0)$$

are isomorphic to $\pi_n(X, A, x_0)$ for all $n \geq 1$. Furthermore, for $z \in (I \times \hat{I}^{n-1} \cup 0 \times I^{n-1})$, we have

$$(I^n, \hat{I}^n, z) \subset (I \times \hat{I}^{n-1} \cup 0 \times I^{n-1}).$$

If we let $z_0 = \prod^n 0$ then since $(I \times \hat{I}^{n-1} \cup 0 \times I^{n-1})$ is contractible we have

$$(I^n, \hat{I}^n, z_0) \subset (I \times \hat{I}^{n-1} \cup 0 \times I^{n-1}).$$

Hence,

$$(I^n, \hat{I}^n, z_0) = (I \times \hat{I}^{n-1} \cup 0 \times I^{n-1}).$$

Therefore the suspension $S^n(I, \hat{I})$ is homeomorphic to the set $(I^n, \hat{I}^n, z_0)$.

It is known that the $n$-cube $I^n$ is homeomorphic to the ball $E^n$ and that the border of the $n$-cube, $\hat{I}^n$, is homeomorphic to the sphere $S^n$. Assuming that the homeomorphism sends $z_0$ to $p_0 \in S^n$. Then for $n \geq 1$, $\pi_n(X, A, x_0)$ is in a one to one correspondence with the equivalence classes of

$$(E^n, S^n, p_0) \to (I^n, \hat{I}^n, z_0).$$

It is by using $(E^n, S^{n-1}, p_0)$, in place of the suspension, we will approach the classification of maps representing 0 in $\pi_n$. The proof itself is fairly straightforward, involving only the construction and checking of homotopies.

**Theorem 3.44.** Given a map $\alpha : (E^n, S^{n-1}, p_0) \to (X, A, x_0)$, then $[\alpha] = 0$ in $\pi_n(X, A, x_0)$ if, and only if, $\alpha$ is homotopic relative to $S^{n-1}$ to some map of $E^n$ to $A$.

**Proof.** Assume $[\alpha] = 0$ in $\pi_n(X, A, x_0)$. Then there is a homotopy

$$H : (E^n, S^{n-1}, p_0) \times I \to (X, A, x_0)$$

from $\alpha$ to the constant map $E^n \to x_0$. The issue here is to find a homotopy relative to $S^{n-1}$. However, a homotopy $H'$ relative to $S^{n-1}$ from $\alpha$ to some map $E^n$ to $A$ can be constructed from $H$ as follows

$$H'(z, t) = \begin{cases} H \left( \frac{z}{1-t/2} \right), & 0 \leq \|z\| \leq 1 - \frac{t}{2}, \\ H \left( \frac{z}{\|z\|}, 2 - 2\|z\| \right), & 1 - \frac{t}{2} \leq \|z\| \leq 1. \end{cases}$$

Conversely, if $\alpha$ is homotopic relative to $S^{n-1}$ to some map $\alpha'$, such that $\alpha'(E^n) \subset A$, then $[\alpha] = [\alpha']$ in $\pi_n(X, A, x_0)$, and it suffices to show that $[\alpha'] = 0$ in $\pi_n(X, A, x_0)$. A homotopy $H : (E^n, S^{n-1}, p_0) \times I \to (X, A, x_0)$ from $\alpha'$ to the constant map $c : E^n \to x_0$ is defined by

$$H(z, t) = \alpha'((1-t)z + tp_0).$$
Consider the case where every map \( \alpha : (E^n, S^{n-1}, p_0) \to (X, A) \) satisfies the condition of Theorem 3.44. This would mean that every map is the zero element of \( \pi_n(X, A) \). As an example consider the case when \( n = 1 \). Then we would look at homotopies from the bounded interval relative to the endpoints. The requirement is then that every map \( \alpha(I, \hat{I}, p_0) \to (X, A, x_0) \) is homotopic relative to \( \hat{I} \) to some map \( \hat{I} \to A \). In other words it has to be possible to connect each pair of points in \((X, A)\) by a path. This property is what we will call \( n \)-connectedness. It is defined as follows.

**Definition 3.45.** A pair \((X, A)\) is said to be \( n \)-connected for \( n \geq 0 \) if, for \( 0 \leq k \leq n \), every map \( \alpha : (E^k, S^{k-1}) \to (X, A) \) is homotopic relative to \( S^{k-1} \) to some map of \( E^k \) to \( A \).

By Theorem 3.44 and Definition 3.45, an \( n \)-connected space has trivial homotopy groups up to \( \pi_n \). Conversely, a space with trivial homotopy groups to dimension \( n \) is obviously \( n \)-connected. For future reference we state this as a corollary.

**Corollary 3.46.** A pair \((X, A)\) is \( n \)-connected for \( n \geq 0 \) if and only if every path component of \( X \) intersects \( A \), and for every point \( a \in A \) and every \( 1 \leq k \leq n \), \( \pi_k(X, A, a) = 0 \).

As we proceed, we make use of a map \( \partial : \pi(X, A, x_0) \to \pi_{n-1}(A, x_0) \), defined by 
\[
\partial[\alpha] = [\alpha|S^{n-1}(I)],
\]
for \( \alpha : S^{n-1}(I, \hat{I}) \to (X, A) \).

**Definition 3.47.** Let \((X, A)\) be a pair of pointed spaces with inclusion maps \( i : A \hookrightarrow X \) and \( j : (X, \{x_0\}) \to (X, A) \). Then the homotopy sequence of \((X, A)\) is the sequence of pointed sets 
\[
\cdots \to \pi_{n+1}(X, A) \xrightarrow{\partial} \pi_n(A) \xrightarrow{i_*} \pi_n(X) \xrightarrow{j_*} \pi_n(X, A) \xrightarrow{\partial} \cdots \xrightarrow{j_*} \pi_0(X).
\]

Recall that in Theorem 3.42 we proved that 
\[
(X', A') \xrightarrow{j} (X, A) \xrightarrow{i} \cdots \xrightarrow{S^n f} S^n(X, A) \xrightarrow{S^n i} S^n(C_{f'}, C_{f''}) \xrightarrow{k} \cdots \\
\cdots \xrightarrow{S^n k} S^{n+1}(X', A') \xrightarrow{S^{n+1} f} \cdots
\]
is coexact. By using this, one can show that the homotopy sequence is exact. This exactness will play an important part when we are to show that \((E^n, S^n)\) is \( n \)-connected.

**Theorem 3.48.** The homotopy sequence of a pair is exact.

**Proof.** Let \( f : (\hat{I}, \{0\}) \subset (\hat{I}, \hat{I}) \) and let \( f' : \hat{I} \subset \hat{I} \) and \( f'' : \{0\} \to \hat{I} \). By Theorem 3.42, there is a coexact sequence 
\[
(\hat{I}, \{0\}) \xrightarrow{f} (\hat{I}, \hat{I}) \xrightarrow{f'} (C_{f'}, C_{f''}) \xrightarrow{k} S(\hat{I}, \{0\}) \xrightarrow{S f} S(\hat{I}, \hat{I}) \to \cdots
\]
Let \( g : (C_{f'}, C_{f''}) \to (I, \hat{I}) \) be the homeomorphism defined by \( g([0, t]) = 0 \) and \( g([1, t]) = t \). Then the composite \( g \circ i \) is the inclusion map \( i' : (\hat{I}, \hat{I}) \to (I, \hat{I}) \), and the composite \( k \circ g^{-1} \) equals the composite 
\[
(I, \hat{I}) \xrightarrow{k'} (I/\hat{I}), \xrightarrow{h} (S(\hat{I}), \{0\})
\]
where $k'$ is the collapsing map and $h$ is the homeomorphism used in identifying $\pi_n(X, \{x_0\})$ with $\pi_n(X)$. Therefore, there is a coexact sequence

$$\left(I, \{0\}\right) \xrightarrow{f} \left(I, I\right) \xrightarrow{\partial} \left(I, I\right) \xrightarrow{h \circ k'} S(I, \{0\}) \xrightarrow{Sf} \cdots,$$

This yields an exact sequence

$$\cdots \rightarrow \pi_{n+1}(X, A) \xrightarrow{(S^n f)^\#} \pi_n(A) \xrightarrow{(S^n f)^\#} \pi_n(X) \xrightarrow{(S^{n-1}(h \circ k'))^\#} \cdots$$

$$\cdots \rightarrow \pi_n(X, A) \rightarrow \cdots \rightarrow \pi_0(X).$$

The proof is completed by the trivial verification that $(S^n, i'^\#) = \partial, (S^n f)^\# = i_\#$, and $(S^{n-1}(h \circ k'))^\# = j_\#$. □

Now, since the homotopy sequence of $(X, A)$ is exact it follows by definition that $\text{Im} \partial = \ker i_\#$. Should it be the case that $\pi_k(A) = 0$, for some $k$, then $\ker \partial = \pi_{k+1}(X, A)$. This follows since the kernel of $\partial : \pi_{k+1}(X, A) \rightarrow \pi_k(A)$ must be the entire space $\pi_{k+1}(X, A)$, if $\partial$ maps everything to 0. Furthermore, if $\pi_{k+1}(X) = 0$, then $\text{Im} j_\#$ must necessarily be 0. Since if this weren’t the case, then $j$ could not possibly be an inclusion. Given the preceding reasoning it is clear that if both $\pi_{k+1}(X)$ and $\pi_k(A)$ are trivial, then $\pi_{k+1}(X, A) = 0$. Referring back to Corollary 3.46 one can see a connection between what was just discussed and $n$-connectedness. It is this connection that provides a method to prove the connectedness of $(E^{n+1}, S^n)$. Of course, given that $S^n$ is $(n-1)$-connected for $n \geq 1$.

**Theorem 3.49.** $S^n$ is $(n-1)$-connected for $n \geq 1$.

**Proof.** For a proof please see Theorem 3.4.1 of [24]. □

**Corollary 3.50.** For $n \geq 0$, $(E^{n+1}, S^n)$ is $n$-connected.

**Proof.** For $n \geq 0$, $E^{n+1}$ is path connected and $S^n$ is nonempty; therefore every path component of $E^{n+1}$ meets $S^n$. If $x \in S^n$, then $\pi_k(E^{n+1}, x) = 0$ for $0 \leq k$, because $E^{n+1}$ is contractible. By Theorem 3.49, $\pi_k(S^n, x) = 0$ if $0 \leq k < n$. It follows from Theorem 3.48 that $\pi_k(E^{n+1}, S^n, x) = 0$ for $1 \leq k \leq n$. The result follows from Corollary 3.46. □

Thus we have a pair that we know is $n$-connected. Now, the idea is to use this pair to construct a more general space which is $n$-connected. The most appealing way to do is that is to intruduce the CW-complex.

### 3.6. CW-Complexes

In this section we introduce what J. H. C. Whitehead in [28] named “closure finite complexes with weak topology” or CW-complexes for short. These spaces satisfy a particular kind of $n$-connectedness, which we address in Lemma 3.66. The remainder of the section will be quite technical. While this result is important in our context it is still only scratching the surface of the theory of CW-complexes. The reader who wants to explore the versatility of CW-complexes may read Section 10.5 and Section 10.6 of [20], for instance. A nice exposition on the topological aspects of the closure finite topology can be found in [29, p.89].

Before working with CW-complexes, it is necessary to talk about their construction. A CW-complex is “built” by adding together topological spaces of successively
higher dimensions. These spaces are known as \( n \)-cells. Formally we have the following definition.

**Definition 3.51.** Let \( X \) be a topological space. If \( X \) is homeomorphic to the \( n \)-dimensional ball, \( E^n \), we say that \( X \) is an \( n \)-cell and denote it by \( e^n \).

Homomorphically there is just one of each \( n \)-dimensional cell. For instance, a point is the \( 0 \)-cell, \( e^0 \), and the unit interval \( I \) is the \( 1 \)-cell, \( e^1 \). If we attach both ends of the \( 1 \)-cell to the \( 0 \)-cell then we get the pointed circle \( S^1 \). In a similar way we could use cells to construct many of the topological spaces that we normally use. For instance, \( S^2 \) can be constructed by attaching the edge of a \( 2 \)-cell to a \( 0 \)-cell.

However, the manner in which cells are attached is important. We therefore make the following definition regarding the attachment process.

**Definition 3.52.** Let \( A \) be a topological space and let \( \{e^n_j\} \) be a sequence of \( n \)-cells. Then we say that \( X \) is obtained from \( A \) by adjoining \( n \)-cells if:

1. for each \( j \), \( e^n_j \subset X \);
2. if \( \dot{e}^n_j = e^n_j \cap A \), then for \( j \neq i \), \( e^n_i - \dot{e}^n_i \) and \( e^n_j - \dot{e}^n_j \) are disjoint;
3. \( X \) has a topology coherent with \( \{A, e^n_j\} \) and \( X = A \cup u_j e^n_j \);
4. for each \( j \) there is a map \( f_j : (E^n, S^{n-1}) \rightarrow (e^n_j, \dot{e}^n_j) \)

such that \( f_j(E^n) = e^n_j \), \( f_j \) maps \( E^n - S^{n-1} \) homeomorphically onto \( e^n_j - \dot{e}^n_j \), and \( e^n_j \) has the topology coinduced by \( f_j \) and the inclusion map \( \dot{e}^n_j \hookrightarrow e^n_j \).

We say that a map such as in (4) is a **characteristic map**.

This process of constructing \( X \) from \( A \) will constitute our definition of a CW-complex. While this will suffice it does not clarify either the “weak topology” or the “closure finiteness”. Unfortunately, we shall not spend time on those aspects of CW-complexes. After the definition we shall instead work with another property of the complexes.

**Definition 3.53.** A space \( X \) obtained from \( A \) by adjoining \( n \)-cells is a **relative CW-complex**. Since it is a topological pair we will often just write it as \( (X, A) \).

This process of creating spaces is remarkably powerful. Some of its power stems from the possibility to continuously retract the complex back to \( A \). In fact, all the results, which we are to use, are consequences of this. However, there are a lot to be said about CW-complexes and an interested reader could consult [12] or [1]. We will move on to retraction instead.

**Definition 3.54.** Let \( A \) be a subspace of the topological space \( X \) with the inclusion map \( i : A \rightarrow X \). If there is a continuous map \( r : X \rightarrow A \) such that \( ri = \text{id}_A \), then we say that \( A \) is a **retract** of \( X \).

In Section 5 we will return to the retract. To continue with the CW-complexes we will define the strong deformation retract.

**Definition 3.55.** If there is a retraction \( r \) of \( X \) to \( A \) such that if \( i : A \rightarrow X \), then \( \text{id}_X \simeq ir \) rel \( A \), then we say that \( A \) is a **strong deformation retract**.
The notion of a retract is very similar to this strong deformation retract. The difference between these two concepts is the order in which the inclusion and retraction are applied. Specifically, the latter definition requires the retraction of \( i : A \to X \) to be homotopic to \( \text{id}_X \) relative to \( A \). While the former definition required \( i \circ r \) to be homotopic to \( \text{id}_A \). Formally, a strong deformation retract is a left homotopy inverse while a retract is a right homotopy inverse.

For every CW-complex \((X, A)\) the space \( A \) is a strong deformation retract of \( X \). This will follow from the way that \((X, A)\) is constructed. To prove it one must first look into the structure of spaces created by adjoining \( n \)-cells. Such a study could well constitute its own section and for that the reader may wish to read Chapter 3 of [24]. Our exposition will, as usual, be brief.

We begin by defining a simplicial complex. From our perspective this is a abstract form of a CW-complex, almost combinatorial in its nature. For instance, it is not apparent that there is a non trivial topology on a simplicial complex. However, we shall later construct one.

**Definition 3.56.** Let \( K \) be a set of vertices \( \{v\} \) together with a collection \( \{s\} \) of subsets of \( \{s\} \). If \( \{s\} \) contains all singletons of \( K \) and all nonempty subsets of a set in \( \{s\} \) is again in \( \{s\} \), then we say that \( K \) is a simplicial complex. Furthermore, we say that each \( s \in \{s\} \) is a simplex.

The dimension (of a simplex) is defined as the number of vertices that are inside the simplex. Building on that, the dimension of a simplicial complex is the largest dimension of simplexes contained within it. It should be apparent that this way of thinking of dimension corresponds well with what we usually think that it should be.

Let us now discuss the faces of a simplex. Let \( s' \) be a subset of \( s \), then we say that \( s' \) is a face of \( s \). A word of warning: the faces of a simplex does not necessarily correspond with what a face is usually thought to be. In particular, faces are not only on what would geometrically be “the outside” of the simplex but on all combination of vertices. Furthermore, if \( s' \neq s \), then we say that \( s' \) is a proper face of \( s \).

Since they are not defined with a topology, it follows that simplicial complex are not topological spaces but sets. However, there exists a covariant functor that will take simplicial complexes into the category of topological spaces. We will not describe this in any detail and the interested reader is referred to page 111 in [24]. In broad strokes, one would begin with a family of functions of the form \( \alpha : K \to I \).

If we assume that each function \( \alpha \) satisfies the following two conditions:

1. for any \( \alpha \), \( \{v \in K | \alpha(v) \neq 0\} \) is a simplex of \( K \);
2. for any \( \alpha \), \( \sum_{v \in K} \alpha(v) = 1 \).

Then, we denote this set by \( |K| \). One can then use \( |K| \) to construct a metric topology, denoted by \( |K|_d \).

It is also possible to define a similar structure on the simplexes of \( K \). Let \( s \) be a simplex of \( K \), then we adopt the following notation

\[ |s| = \{ \alpha \in |K| : \alpha(v) \neq 0 \Rightarrow v \in s \}. \]

From \( |K|_d, |s| \) inherits a topology, denoted \( |s|_d \). In turn, by the relation between \( s \) and \( K \), this creates a weak topology on \( |K| \). For details please, again, see [24], we can now use this in the following lemma.
Lemma 3.57. For any simplex \( s, |s| \times 0 \cup |s| \times I \) is a strong deformation retract of \(|s| \times I\).

Proof. For a proof, please see Lemma 3.2.3 of [24] \(\square\)

For a simplicial complex there is an analogous construction to the subspace of a space, known as a subcomplex.

Definition 3.58. Let \( K \) and \( L \) be simplicial complexes. If each simplex \( s \) of \( L \) is also a simplex of \( K \), then we say that \( L \) is a subcomplex of \( K \).

As a sidenote, this provides a way to relate a topological pair \((X, A)\) to a simplicial complex, specifically to a simplicial pair \((K, L)\). Technically, one would denote this by \(((K, L), f)\), where \( f \) is a homeomorphism \(((|K|, |L|) \rightarrow (X, A))\). We say that \(((K, L), f)\) is a triangulation and that \((X, A)\) is a polyhedral pair, if it has a triangulation. This will be used more in Section 4.1, when we look at homology and cohomology.

At the moment we will use the subcomplex \(L\) of \(K\) to give a generalization of Lemma 3.57.

Lemma 3.59. For any subcomplex \(L \subset K\) the subspace \(|K| \times 0 \cup |L| \times I\) is a strong deformation retract of \(|K| \times I\).

Proof. Please see Lemma 3.2.4 of [24] \(\square\)

Now, Lemma 3.59 can be used to prove the existence of a similar strong deformation retract on a CW-complex, \((X, A)\). The key is that Lemma 3.59 provides a strong deformation retract \(D: (E^n \times I) \times I \rightarrow E^n \times I\).

Lemma 3.60. If \(X\) is obtained from \(A\) by adjoining \(n\)-cells, then \(X \times 0 \cup A \times I\) is a strong deformation retract of \(X \times I\).

Proof. For each \(n\)-cell \(e^n_j\) of \(X - A\) let

\[
f_j : (E^n, S^{n-1}) \rightarrow (e^n_j, e^n_j)
\]

be a characteristic map. Let \(D: (E^n \times I) \times I \rightarrow E^n \times I\) be a strong deformation retraction of \(E^n \times I\) to \(E^n \times 0 \cup S^{n-1} \times I\) (which exists, by Lemma 3.59). There is a well-defined map \(D_j : (e^n \times I) \times I \rightarrow e^n_j \times I\) characterized by the equation

\[
D_j((f_j(z), t), t') = (f_j \times 1)(D(z, t, t')) \quad z \in E^n; t, t' \in I.
\]

Then there is a map \(D' : (X \times I) \times I \rightarrow X \times I\) such that \(D'|_{(e_j \times I) \times I} = D_j\) and \(D'(a, t, t) = (a, t)\) for \(a \in A\), and \(D'\) is a strong deformation retraction of \(X \times I\) to \(X \times 0 \cup A \times I\) \(\square\).

We will now look at an interesting topic in algebraic topology: cofibrations. Later, in Section 4 we will work with fibrations, which are closer to the fibers that one would normally meet in for instance differential geometry.

Definition 3.61. Let \(f : X' \rightarrow X\) be a map and let \(Y\) be an arbitrary space. For maps \(g : X \rightarrow Y\) and \(G : X' \times I \rightarrow Y\) such that \(g(f(x')) = G(x', 0)\) for \(x \in X\). If there is a map \(F : X \times I \rightarrow Y\) such that \(F(x, 0) = g(x)\) for \(x \in X\) and \(F'(f(x'), t) = G(x', t)\) for \(x' \in X'\) and \(t \in I\), then \(f\) is a cofibration. Consider the following diagram.
Clearly $X \times 1 \cup A \times I \subset X \times I$ and also $A \times I \subset X \times 0 \cup A \times I$. Therefore, given maps $g : X \to Y$ and $G : A \times I \to Y$ such that $G(x, 0) = g(x)$ for $x \in A$, there is a simple way to find a map $F : X \times I \to Y$. Let $f : X \times 0 \cup A \times I \to Y$ be defined by $f(x, 0) = g(x)$ for $x \in X$ and $f(x, t) = G(x, t)$ for $x \in A$ and $t \in I$. Then, by Lemma 3.60, there is a continuous map $F : X \times 0 \cup A \times I \to X \times I \to Y$. This is displayed in the following diagram and stated in the next corollary.

![Diagram](image)

**Corollary 3.62.** If $X$ is obtained from $A$ by adjoining $n$-cells, then the inclusion map $i : A \hookrightarrow X$ is a cofibration.

Given Corollary 3.62 one may expect that the inclusion map $i : A \to X$ for a relative CW-complex should be a cofibration. This is indeed true and almost an immediate consequence of the corollary. Since if $X$ cannot be obtained from $A$ by adding just cells of dimension $n$, then by the definition of CW-complexes one can attach cells of successively higher dimension. At each step one then applies Corollary 3.62.

**Theorem 3.63.** If $(X, A)$ is a relative CW-complex, then the inclusion map $A \hookrightarrow X$ is a cofibration.

**Proof.** This follows from Corollary 3.62, using induction. \qed

Because of Theorem 3.63, there is a simple way to extend homotopies on a CW-complex $(X, A)$. For any map $f : X \to Y$, if there is a homotopy $F : A \times I \to Y$, then there is a homotopy $F' : X \times I \to Y$. We will use this to prove that there is a homotopy, relative to $A$, from $(X, A)$ to any $n$-connected space $(Y, B)$ in Theorem 3.65. However, in order to do that a technical lemma is necessary.

**Lemma 3.64.** Let $X$ be obtained from $A$ by adjoining $n$-cells and let $(Y, B)$ be a pair such that $\pi_n(Y, B, b) = 0$ if $n \geq 1$, for all $b \in B$. Furthermore let $(Y, B)$ be such that every point of $Y$ can be joined to $B$ by a path if $n = 0$. Then, any map from $(X, A)$ to $(Y, B)$ is homotopic relative to $A$ to a map from $X$ to $B$.

**Proof.** See Lemma 7.6.3 of [24]. \qed

Similar to how we had to generalize Corollary 3.62 to Theorem 3.63 the same has to be done here. It is still possible to rely on an iterative process to show that each stage taken in the construction of the CW-complex satisfies Lemma 3.64. However, the proof will be trickier if the number of dimensions of the CW-complex can be
Theorem 3.65. Let \((X, A)\) be a relative CW-complex, with dimension \((X - A) \leq n\), and let \((Y, B)\) be \(n\)-connected. Then any map from \((X, A)\) to \((Y, B)\) is homotopic, relative to \(A\), to a map from \(X\) to \(B\).

Proof. This follows, using induction, from Corollary 3.46, Lemma 3.64 and Theorem 3.63.

The preceding proof leaves us in a position to prove that a CW-complex is \((n-1)\)-connected. It would actually be possible to prove a stronger result: that \((X, (X, A))\) is \((n-1)\)-connected. But again, this is not necessary. Instead we shall use the fact that \((X, A)\) is \((n-1)\)-connected and therefore \(\pi_q(X, A) = 0\) for \(q < n\) to prove an important result in Section 4.

Lemma 3.66. For \(n \geq 1\) if \(X\) is obtained from \(A\) by adjoining \(n\)-cells, then \((X, A)\) is \((n-1)\)-connected.

Proof. For \(k \leq n-1\), let \(f : (E^k, S^{k-1}) \to (X, A)\) be a map. We will now create two spaces. Because \(f(E^k)\) is compact, there exist a finite number, say \(e_1, \ldots, e_m\), of \(n\)-cells of \(X - A\) such that \(f(E^k) \subset A \cup_{i=1}^m e_i\). For \(1 \leq i \leq m\), let \(x_i\) be a point of \(e_i - e_i\). From this we can create the spaces \(Y = A \cup_{i=1}^m (e_i - x_i)\) and \(e_i - e_i\). Both of these spaces intersect \(f(E^k)\) in a set open in \(f(E^k)\) for \(1 \leq i \leq m\).

There is a simplicial triangulation of \(E^k\), say \(K\), such that (identifying \(|K|\) with \(E^k\)) for every simplex \(s \in K\) either \(f(|s|) \subset Y\) or, for some \(1 \leq i \leq m\), \(f(|s|) \subset e_i - e_i\). Let \(A'\) be the subpolyhedron of \(E^k\) which is the space of all simplices \(s \in K\) such that \(f(|s|) \subset Y\), and for \(1 \leq i \leq m\), let \(B_i\) be the subpolyhedron which is the space of all simplices \(s \in K\) such that \(f(|s|) \subset e_i - e_i\). Then \(S^{n-1} \subset A'\) and \(E^k = A' \cup_{i=1}^m B_i\). If \(i \neq j\), then \(B_i - A'\) is disjoint from \(B_j - A'\). Let \(\tilde{B}_i = B_i \cap A'\) and observe that \((B_i, \tilde{B}_i)\) is a relative CW-complex, with \(\dim(B_i - \tilde{B}_i) \leq k \leq n - 1\).

For \(1 \leq i \leq m\) the pair \((e_i - \tilde{e}_i), (e_i - \tilde{e}_i) - x_i\) is homeomorphic to \((E^n - S^{n-1}, (E^n - S^{n-1}) - 0)\) and has the same homotopy groups as \((E^n, S^{n-1})\). By Corollary 3.50, \((E^n, S^{n-1})\) is \((n-1)\)-connected. It follows from Theorem 3.65 that \(f(B_i - \tilde{B}_i)\) is homotopic relative to \(B_i\) to a map from \(B_i\) to \((e_i - \tilde{e}_i) - x_i\). Because \(B_i - \tilde{B}_i\) is disjoint from \(B_j - \tilde{B}_j\) for \(i \neq j\), these homotopies fit together to define a homotopy relative to \(A'\) of \(f\) to some map \(f'\) such that \(f'(E^k) \subset Y\). Clearly, \(A\) is a strong deformation retract of \(Y\). Therefore \(f'\) is homotopic relative to \(S^{k-1}\) to a map \(f'' : (E^n, S^{n-1}) \to (X, A)\) such that \(f''(E^k) \subset A\). Then \(f \simeq f' \simeq f''\), all homotopies relative to \(S^{k-1}\). Therefore \((X, A)\) is \((n-1)\)-connected.

Theorem 3.67. For \(n\) finite or infinite let \(f : X \to Y\) be an \(n\)-equivalence and let \((P, Q)\) be a relative CW-complex with the \(\dim(P - Q) \leq n\). Given maps \(g : Q \to X\) and \(h : P \to Y\) such that \(h|Q = f \circ g\), there exists a map \(g' : P \to X\) such that \(g'|Q = g\) and \(f \circ g' \simeq h\) relative to \(Q\).

Proof. Let \(Z_f\) ve the mapping cylinder of \(f\), with inclusion maps \(i : X \hookrightarrow Z_f\) and \(j : Y \hookrightarrow Z_f\), and retraction \(r : Z_f \to Y\) a homotopy inverse of \(j\). Then, in the following diagram
a homotopy \( i \circ g \simeq j \circ h \mid_Q \) can be found whose composite with \( r \) is constant. By Theorem 3.63, there is a map \( h' : P \to Z_f \) such that \( h'|_Q = i \circ g \), and such that \( r \circ h' \simeq r \circ j \circ h \) relative to \( Q \). We regard \( h' \) as a map from \((P, Q)\) to \((Z_f, X)\). Since \((Z_f, X)\) is \( n \)-connected and dim \((P - Q)\) \( \leq n \), it follows from Theorem 3.65 that \( h' \) is homotopic relative to \( Q \) to some map \( g' : P \to X \). Then \( g'|_Q = g \) and
\[
f \circ g' = r \circ i \circ g' \simeq r \circ j \circ h = h
\]
all the homotopies being relative to \( Q \). Hence \( g' \) has the desired properties. \( \square \)

3.7. Homotopy Functors. The functor \( \pi_n \) is sometimes known as the homotopy functor. However, there is in actuality a wide class of functors, called the homotopy functors. For example we do not only have \( \pi_n \) but also the cohomology functor \( H^n \), which we will look at in Section 4.1. Regardless of the kind of functor they all have to respect equivalence between homotopic maps. We shall now define this equivalence in a category theoretic sense.

Definition 3.68. Let \( \mathcal{C} \) be a category with objects \( X, Y, Z \), and \( Z' \) as well as morphisms \( f_0 : X \to Y \) and \( f_1 : X \to Y \), an coequalizer of \( f_0 \) and \( f_1 \) is a morphism \( j : X \to Z \) in \( \mathcal{C} \) such that:
\[
(1) \quad j \circ f_0 = j \circ f_1,
\]
\[
(2) \quad \text{if } j' : X \to Z' \text{ is a morphism in } \mathcal{C} \text{ such that } j' \circ f_0 = j' \circ f_1, \text{ there is a morphism } g : Z \to Z' \text{ such that } j' = g \circ j.
\]
The coequalizer can be represented in a diagram in the following way.

\[
\begin{array}{ccc}
X & \xrightarrow{f_0} & Y \\
\downarrow{f_1} & & \downarrow{g} \\
& \xrightarrow{j'} Z' \\
\end{array}
\]

As a remark for reference, the map \( j \) is in [24] called the equalizer. We have, however, gone with the greater literature (such as [25]) in naming it the coequalizer. We now show that there are equalizers on \( \mathcal{C}_0 \).

Lemma 3.69. Let \( \{Y_n\}_{n \geq 0} \) be objects of \( \mathcal{C}_0 \) that are subspaces of a space \( Y \) in \( \mathcal{C}_0 \) such that \( Y_n \hookrightarrow Y_{n+1} \) is a cofibration for all \( n \geq 0 \), \( Y = \bigcup_n Y_n \), and \( Y \) has the topology coherent with \( \{Y_n\} \). Let \( i_n : Y_n \hookrightarrow Y_{n+1}, \text{id}_n : Y_n \to Y_n \), and \( j_n : Y_n \hookrightarrow Y \) be the inclusion maps. Then, the homotopy class \( \{[j_n]\} : \bigvee Y_n \to Y \) is an equalizer in \( \mathcal{C}_0 \) of the homotopy classes
\[
\bigvee i_n : \bigvee Y_n \to \bigvee Y_n \quad \text{and} \quad \bigvee \text{id}_n : \bigvee Y_n \to \bigvee Y_n.
\]

Proof. Since \( j_{n+1} \circ i_n = j_n \circ \text{id}_n \), it follows that \( \{j_n\} \circ \bigvee i_n = \{j_n\} \circ \bigvee \text{id}_n \). Given a map \( j' : \bigvee Y_n \to Z' \) such that \( j' \circ \bigvee i_n \simeq j' \circ \bigvee \text{id}_n \), let \( j'_n : Y_n \to Z' \) be defined by \( j'_n = j'|_{Y_n} \). Then, \( j_{n+1} \circ i_n \simeq j'_n \) and using the fact that \( Y_n \hookrightarrow Y_{n+1} \) is a cofibration and by induction on \( n \), there is a sequence of maps \( g_n : Y_n \to Z' \) such that \( g_n \simeq j'_n \) and \( g_{n+1} \circ i = g_n \). Let \( g : Y \to Z' \) be the map such that \( g|_{Y_n} = g_n \). Then, \( g \circ j \simeq j' \), completing the proof. \( \square \)
Definition 3.70. A homotopy functor is a contravariant functor $H$ from $\mathcal{C}_0$ to $\mathcal{T}_*$ such that the following two conditions hold.

1. If $[j] : X \to Z$ is an coequalizer of $[f_0], [f_1] : A \to X$ and if $u \in H(X)$ is such that $H([f_0])|v(u) = H([f_1])|v(u)$, there is a $v \in H(Z)$ such that $H([j])|v = u$.

2. If $\{X_\lambda\}_\lambda$ is an indexed family of objects in $\mathcal{C}_0$ and $i_\lambda : X_\lambda \hookrightarrow \bigvee X_\lambda$, there is an equivalence

\[
\{H[i_\lambda]\}_\lambda : H \left( \bigvee X_\lambda \right) \approx \prod H(X_\lambda).
\]

All homotopy functors have something known as coefficient groups. The $q$-th coefficient group is the functor taken on the $q$-th suspension of $I$, $S^q$. For the homotopy functor $\pi^Y$ this means that $\pi_q(Y)$ is the $q$-th coefficient group. In general the $q$-th coefficient group is $H(S^q)$. It is natural to wonder whether two homotopy functors share the same group structure on their coefficient groups. And even more interesting: is $H(S^q) \approx \pi_q(Y)$? This would surely depend in some way on the space $Y$ and in the next definition we formalize this dependence.

Definition 3.71. An element $u \in H(Y)$ is said to be $n$-universal for $H$, where $n \geq 1$, if the homomorphism

\[
T_u : \pi^Y(S^n) \to H(S^q),
\]

where $T_u$ is defined by

\[
T_u([f]) = H([f])(u), \quad [f] \in \pi^Y,
\]

is an isomorphism for $1 \leq q < n$ and an epimorphism for $q = n$. If $u$ is $n$-universal for all $n \geq 1$ we say that $Y$ is a classifying space for $H$.

Thus, if $Y$ is a classifying space for $H$ then all coefficient groups of $H$ and $\pi^Y$ are isomorphic.

The use we will make of the classifying spaces is somewhat technical in its nature. Recall that the ultimate goal is to extend a map on a subset of $P$ onto $P$ in its entirety. In our case we could perhaps do this using only the functor $\pi_q(P)$ since the space and function we try to extend is particularly simple. Nonetheless, we shall try and do this more generally, which results in a problem: $\pi_q(Y)$ is seldom easy to calculate. It would therefore be convenient to use another functor and this is what classifying spaces will allow us to do.

We now continue with the classifying spaces and prove Theorem 3.74. This is done over three theorems in an inductive fashion. First, in Lemma 3.72, we will show that we can in a sense extend an element of $H(Y)$ to a larger set by adding 1-cells to $Y$. We then build on this in Lemma 3.73 and allow attachment of $(n+1)$-cells.

As a small note on notation, we shall, given an inclusion $i : X \to X'$ and $u \in H(X')$, write $u|X$ instead of $H(i)(u)$. This is also the equivalent to a restriction in this case.

Lemma 3.72. Let $H$ be a homotopy functor, $Y$ an object in $\mathcal{C}_0$, and $u \in H(Y)$. There exist an object $Y'$ in $\mathcal{C}_0$, obtained from $Y$ by attaching 1-cells, and a 1-universal element $u' \in H(Y')$ such that $u'|Y = u$.

Proof. For each $\lambda \in H(S^1)$ let $S^1_\lambda$ be a 1-sphere and define $Y = Y \bigvee \left( \bigvee \lambda^1 S^1_\lambda \right)$. Then $Y'$ is an object of $\mathcal{C}_0$ obtained from $Y$ by attaching 1-cells. If $g_\lambda$ is the composite
$S^1 \xrightarrow{\cong} S^1 \subset Y'$, it follows from condition (2) of Definition 3.70 that there is an element $u' \in H(Y')$ such that $u'|Y = u$ and $H(g\lambda)(u') = \lambda$ for $\lambda \in H(S^1)$. Since $T_u(\{g\lambda\}) = \lambda$, $T_u(\{[S^1; Y']\}) = H(S^1)$, and $u'$ is 1-universal.

□

Lemma 3.73. Let $H$ be a homotopy functor and $u \in H(Y)$ an $n$-universal element for $H$, with $n \geq 1$. There exist an object $Y'$ in $\mathcal{C}_0$, obtained from $Y$ by attaching $(n + 1)$-cells, and an $(n + 1)$-universal element $u' \in H(Y')$ such that $u'|Y = u$.

Proof. For each $\lambda \in H(S^{n+1})$ let $S^{n+1}_\lambda$ be an $(n + 1)$-sphere, and for each map $\alpha : S^n \to Y$ such that $H(\alpha)(u) = 0$ attach an $(n + 1)$-cell $e_{\alpha}^{n+1}$ to $Y$ by $\alpha$. Let $Y'$ be the space obtained from $Y \vee \bigvee \lambda S^{n+1}_\lambda$ by attaching the $(n + 1)$-cells $\{e_{\alpha}^{n+1}\}$. Then $Y'$ is an object of $\mathcal{C}_0$ obtained from $Y$ by attaching $(n + 1)$-cells.

If $g : S^n \to Y \vee \bigvee \lambda S^{n+1}_\lambda$ is the composite

$$S^{n+1} \xrightarrow{\cong} S^{n+1}_\lambda \to Y \vee \left( \bigvee_\lambda S^{n+1}_\lambda \right),$$

it follows from (2) of Definition 3.70 that there is an element $\bar{u} \in H\left(Y \vee \left( \bigvee \lambda S^{n+1}_\lambda \right)\right)$ such that $\bar{u}|Y = u$ and $H(\alpha)\bar{u} = 0$. Let $S^n_\alpha$ be an $n$-sphere and let $f_0 : \bigvee \alpha S^n_\alpha \to Y \vee \left( \bigvee \lambda S^{n+1}_\lambda \right)$ be the constant map. Furthermore, let the map $f_1 : \bigvee \alpha S^n_\alpha \to Y \vee \left( \bigvee \lambda S^{n+1}_\lambda \right)$ be such that $S^n_\alpha$ is mapped by $\alpha$. Then the inclusion

$$j : Y \vee \left( \bigvee_\lambda S^{n+1}_\lambda \right) \to Y$$

is a map such that $[j]$ is an equalizer of $[f_0]$ and $[f_1]$. Since $H(f_0)\bar{u} = 0 = H(f_1)\bar{u}$, by condition (1) of Definition 3.70, there is an element $u' \in H(Y')$ such that $H(j)u' = \bar{u}$. Thus, $u'|Y = u$ and to complete the proof we need only show that $u'$ is $(n + 1)$-universal.

There is a commutative diagram

$$\begin{array}{ccc}
\pi_{q+1}(Y', Y) & \xrightarrow{\partial} & \pi_q(Y) \\
& \alpha & \downarrow i_# \\
& H(S^n) & \xrightarrow{T_{u'}} \pi_q(Y', Y)
\end{array}$$

with the top row exact. Since $Y'$ is obtained from $Y$ by attaching $(n + 1)$-cells, it follows from Lemma 3.66 that $\pi_q(Y', Y) = 0$ for $q \leq n$. Therefore, $i_#$ is an isomorphism for $q < n$ and an epimorphism for $q = n$. Since $u$ is $n$-universal, $T_u$ is an isomorphism for $q < n$ and an epimorphism for $q = n$. Furthermore, if $a \in [S^n; Y]$ is in the kernel of $T_u$, then $a$ is represented by a map $\alpha : S^n \to Y$ and

$$a = [\alpha] \in \partial(\pi_{n+1}(e_{\alpha}^{n+1}, e_{\alpha}^{n+1})) \subset \partial(\pi_{n+1}(Y', Y)) = \ker i_#.$$

Therefore, for $q = n$, $\ker T_{u'} = \ker i_#$, and so $T_{u'}$ is an isomorphism from $\pi_n(Y')$ to $H(S^n)$. For any $\lambda \in H(S^{n+1})$ the map $j \circ g_\lambda : S^{n+1} \to Y'$ has the property that

$$T_{u'}(\{j \circ g_\lambda\}) = H(g_\lambda)(\bar{u}) = \lambda,$$

showing that $T_{u'}$ is an epimorphism for $q = n + 1$, and so $u'$ is $(n + 1)$-universal. □

Theorem 3.74. Let $H$ be a nonempty functor let $Y$ be an object in $\mathcal{C}_0$, and let $u \in H(Y)$. Then, there is a classifying space $Y'$ for $H$ containing $Y$ such that
(\(Y', Y\)) is a relative CW-complex and a universal element \(u' \in H(Y')\) such that \(u'|Y = u\).

Proof. Using Lemma 3.72 and Lemma 3.73, we construct, by induction over \(n\), a sequence of objects \(\{Y_n\}_{n \geq 0}\) in \(C_0\) and elements \(u_n \in H(Y^n)\) such that:

1. \(Y_0 = Y\) and \(u_0 = u\);
2. \(Y_{n+1}\) is obtained from \(Y_n\) by attaching \((n + 1)\)-cells, where \(n \geq 0\);
3. \(u_{n+1}|Y_n = u_n\).
4. \(u_n\) is \(n\)-universal for \(n \geq 1\).

It follows from (2) above that \(Y' = \cup Y_n\), topologized with the topology coherent with \(\{Y_n\}\), is a path-connected pointed space containing \(Y\) such that \((Y', Y)\) is a relative CW-complex. By Lemma 3.69, the homotopy class \([\cup Y_n] : \bigvee Y_n \to Y'\) is an equalizer of the homotopy classes \([\bigvee i_n] : \bigvee Y_n \to \bigvee Y_n\) and \([\bigvee 1_n] : \bigvee Y_n \to \bigvee Y_n\). By condition (2) of Definition 3.70, there is an element \(\bar{u} \in H(\bigvee Y_n)\) such that \(\bar{u}|Y_n = u_n\). It follows from (3) above that \(H(\bigvee i_n)(\bar{u}) = H(\bigvee 1_n)(\bar{u})\), and by condition (1) of Definition 3.70 there is an element \(u' \in H(Y')\) such that \(H(\{j_n\})(u') = \bar{u}\) (that is, \(u'|Y_n = u_n\) for \(n \geq 0\)). Then \(u'|Y = u\), and it remains to show that \(u'\) is universal.

By definition of \(Y'\) and \(u'\), there is a commutative diagram for \(q \geq 1\)

\[
\begin{array}{ccc}
\lim_{\to} \{\pi_q(Y_n)\} & \approx & \pi_q(Y') \\
T_{u_n} \downarrow & & \downarrow T_{u'} \\
H(S^q) & \leftarrow & H(S^q)
\end{array}
\]

Since \(u_n\) is universal, \(T_{u_n}\) is an isomorphism for \(n > q\), and so the left-hand map is an isomorphism. Therefore \(T_{u'}\) is also an isomorphism, and \(u'\) is universal. \(\square\)

This is what we need on classifying spaces and homotopy functors. In the next section we will use classifying spaces and Theorem 3.74 in particular to prove our first result regarding extensions of functions in Theorem 4.11.
4. Obstruction Theory

Obstruction theory is a method for extending maps, exactly what we are looking for. To use it we will first look at Eilenberg-MacLane spaces in Section 4.1 and this will also give our first real extension result. This will rely on the important topic of homology and cohomology, which we address in the same section. Following this, we will then use a Moore-Postnikov factorization (defined in Section 4.2) to generalize the results of Section 4.1. Finally, in Section 4.3 the generalization is finalized and we prove the extension problem that motivated the use of algebraic topology in the first place.

4.1. Eilenberg-Maclane Spaces and Cohomology. In this section we will finally be able to use algebraic topology to state when it is possible to extend maps, for certain spaces. Our big result of the section, Theorem 4.11, does this on a certain kind of space known as a Eilenberg-MacLane space, which we will discuss shortly. This is the fundamental part of the extension we do by use of obstruction theory in the next section. We will give two constructions of cohomology, one by Strom [25] and one by Spanier [24]. The argument for introducing two constructions is that the former is explanatory, short and fits nicely with the flow of the essay. While the latter is necessary for much of the remainder of the essay. Unfortunately, the construction due to [24] is quite involved and beyond the scope of this essay. Therefore, please see the first construction as an explanation and perhaps as context for the Eilenberg-MacLane spaces, which we are going to introduce.

As said before: the higher homotopy groups of nearly all but the most trivial of spaces are hard to compute. There is, however, one family of spaces, or perhaps more accurately CW-complexes, that are particularly nice: the Eilenberg-MacLane spaces.

Definition 4.1. Let \( \pi \) be a group, and let \( n \geq 1 \) be an integer. A space of type \((\pi, n)\) is a path-connected pointed space \( Y \) such that \( \pi_q(Y, y_0) = 0 \) for \( q \neq n \) and \( \pi_n(Y, y_0) \) is isomorphic to \( \pi \). We will often say that \( Y \) is a \( K(\pi, n) \) space.

Being a \( K(\pi, n) \) space is quite a restrictive condition on a space, which not all spaces satisfy. For instance, \( S^1 \) is a \( K(\mathbb{Z}, 1) \) space while no other sphere is an Eilenberg-MacLane space. However, the limitations on \( K(\pi, n) \) spaces are well outweighed by their usefulness when it comes to mapping other spaces into Eilenberg-MacLane spaces. The tool we will use is cohomology, which is the homotopy classes of maps from a space \( X \) into a \( K(\pi, n) \) space. Our presentation of this will mostly follow [25].

Definition 4.2. Let \( X \) be a topological space, then the \( n \)-th cohomology group with coefficients in \( \pi \), \( H^n(X; \pi) \), is defined as

\[
H^n(X; \pi) = [X, K(\pi, n)].
\]

For this definition to make sense we would need to know that there actually is a \( K(\pi, n) \) space for an arbitrary group \( \pi \). This is not something we will prove explicitly but instead we state restate a part of Theorem 17.49 of [25].

Proposition 4.3. For every abelian group \( \pi \) and every \( n \geq 1 \), there is an Eilenberg-MacLane space \( K(\pi, n) \), which is a topological manifold.
Naturally, we are just interested in the existence part of Proposition 4.3. The fact that the $K(\pi,n)$ space is a manifold is just included as a curiosity for the reader.

Proof. For a proof please see Theorem 17.49 of [25].

Another property, which we will not prove, is that $H^n(\cdot;G)$ is a contravariant homotopy functor. Instead we restate (a) of Theorem 21.2 in [25]. Note that $\text{Ab}$ is the category of abelian groups.

Proposition 4.4. For any $X$ in $T_*$ and any group $\pi$ the $n$-th cohomology group $H^n(X;\pi)$ is a contravariant homotopy functor $T_* \rightarrow \text{Ab}$.

Proposition 4.3 and Proposition 4.4 assures that there actually is a $K(\pi,n)$ space for any $\pi$ and that there is a homotopy functor $H^n$ to the category of abelian groups. Since $H^n$ is contravariant, an element $v$ of $H^n(X;\pi)$ is a map $v : K(\pi,n) \rightarrow X$, unique up to homotopy.

We will now give a minimalized version of Spaniers construction of cohomology. It is not intended to be formally correct or particularly explanatory, either of which would require a treatment beyond the scope of this essay. For a formal treatment please see Chapter 5.1 of [24].

Now, let $\Delta^n$ be a simplex with n dimensions and let $X$ be a topological space. Then we say that the continuous mapping $\sigma_n : \Delta^n \rightarrow X$ is the singular $n$-simplex. The boundary of $\sigma_n(\Delta^n)$, often written as $\partial_n \sigma_n(\Delta^n)$, is the formal sum of the $(n-1)$-simplexes that form the boundary of $\Delta^n$. For a topological space, $X$, there is a functor $\Delta$, which sends $X$ to its singular complex $\Delta(X)$, see Theorem 4.1.5 of [24]. By looking at different dimensions of the singular complex of a space, $X$, we can form a chain of singular simplices where the boundary map $\partial_n : \Delta^n \rightarrow \Delta^{n-1}$ moves downwards through the dimensions. We call this chain, $C$, together with the boundary map a singular chain complex and write it as $\{C,\partial\}$.

Linked to these chain complexes are the cochain complexes. These are chain complexes with a coboundary map, which is the boundary map reversed. This map assigns to a $(n-1)$-complex a $n$-complex. We denote the coboundary map by $\delta$. Adopting the common notation, $C^q$ for the $q$-th complex in a chain complex we can define

$$\delta^q = C^q \rightarrow C^{q+1}.$$ 

This can then be used to give an alternative definition of $H^n(X)$ as $\ker \delta^{n-1} / \text{Im} \delta^n$ where $\delta$ is defined on the cochain complex of $X$. To get the cohomology with coefficients in $\pi$ we define $H^q(C;\pi)$, where $C$ is a chain complex of some space $X$, by

$$H^q(C;\pi) = H^q(\text{hom}(C,\pi)).$$

Finally, the cohomology group $H^n(X,A;\pi)$ is then given by the cohomology group $H^n(\text{hom}(\Delta(X)/\Delta(A),\pi))$, since $\Delta(X)/\Delta(A)$ forms a chain complex. Importantly, the map $\delta^n$ is then a morphism from $H^n(X,A;\pi)$ to $H^{n+1}(X,A;\pi)$.

It is difficult to convey the full gravity of cohomology while keeping the presentation to a working minimum. The reader who is interested to learn more could look at Chapter 3 of [16], Chapter 21 of [25] or the latter half of Chapter 5 of [24]. We shall instead briefly work through another important aspect of algebraic topology: homology.
In short, homology is the dual to cohomology and closely related to $\pi_n$. We shall need two things from homology: the n-characteristic classes of a cohomology group and the Hurewicz isomorphism theorem. The latter connects homology and homotopy on the first nontrivial coefficient groups on a space. We will again follow [25] in the definition of homology.

**Definition 4.5.** A homology $h_*$ is a covariant homotopy functor, $h_* : T_* \to ABG^*$, such that:

1. there is a natural isomorphism $h_* \circ S \xrightarrow{\cong} S \circ h_*$;
2. if $A \to B \to C$ is a cofiber sequence, then the sequence
   $$h_*(A) \to h_*(B) \to h_*(C)$$
   is exact.

We will use $H_n(X, A)$ to denote $h_*\left(S^n(I), S^n(\hat{I}); X, A\right)$.

An important aspect of homology is that the first nontrivial groups of $H_n(X, A)$ and $\pi_n(X, A)$ are isomorphic. This is a famous result known as the Hurewicz isomorphism theorem. We will not go into the details of the proof. That would require a far deeper understanding of homology that what is given here. Instead we restate Theorem 7.5.4 of [24], which is the relative Hurewicz Isomorphism Theorem.

**Theorem 4.6.** Let $x_0 \in A \hookrightarrow X$ and assume that $A$ and $X$ are path connected. If there is an $n \geq 2$ such that $\pi_q(X, A; x_0) = 0$ for $q < n$, then $H_q(X, A) = 0$ for $q < n$ and $\varphi'$ is an isomorphism

$$\varphi' : \pi_n'(X, A; x_0) \approx H_n(X, A).$$

Conversely, if $A$ and $X$ are simply connected and there is an $n \geq 2$ such that $H_q(X, A) = 0$ for $q < n$ and $\varphi$ is an isomorphism

$$\varphi : \pi_n(X, A; x_0) \approx H_n(X, A).$$

**Proof.** For a proof, please see Theorem 7.5.4 of [24]. ∎

We will need another important concept, involving homotopy, homology and cohomology: the n-characteristic cohomology classes. This is yet again something that will not elaborated upon. In particular, we will use a map $h : H^q(C; \pi) \to \text{hom}(H_q(C), \pi)$ defined by

$$(h\{f\})\{\sum c_i\} = \sum f(c_i)$$

for $\{f\} \in H^q$ and $\sum c_i \in H_q(C)$. Note that this is a simplification in order to avoid the introduction of more homology and cohomology. A more fleshed out theory together with the surrounding theory can be found on page 224 of [24].

**Definition 4.7.** Let $(X, A)$ be a pointed pair, which is path connected. We say that a cohomology class $v \in H^n(X, A; \pi)$ is n-characteristic for $(X, A)$ if it satisfies either of the following:

1. $n = 1$ and $\iota_\#(\pi_1(A))$ is a normal subgroup of $\pi_1(X)$ whose quotient group is mapped isomorphically onto $\pi$ by the composite
   $$\pi_1(X)/\iota_\#(\pi_1(A)) \xrightarrow{\psi} H_1(X)/\iota_\#(H_1(A)) \xrightarrow{i_\#} H_1(X, A) \xrightarrow{h(v)} \pi;$$
(2) \( n > 1 \) and the composite
\[
\pi_n(X, A) \xrightarrow{\phi} H_n(X, A) \xrightarrow{h(v)} \pi
\]
is an isomorphism.

**Lemma 4.8.** Let \( i : A \hookrightarrow X \) be a simple inclusion map between path-connected pointed spaces such that the pair \((X, A)\) is \((n-1)\)-connected, where \( n \geq 1 \). Then, there exist cohomology classes \( v \in H^n(X, A; \pi) \) which are \( n \)-characteristic for \((X, A)\), where \( \pi = \pi_1(X)/i_\#(\pi_1(A)) \) for \( n = 1 \) and \( \pi = \pi_n(X, A) \) for \( n > 1 \).

**Proof.** If \( n = 1 \), it follows from the absolute Hurewicz isomorphism theorem applied to \( A \) and to \( X \) that there are isomorphisms
\[
\pi_1(X)/i_\#(\pi_1(A)) \xrightarrow{\cong} H_1(X)/i_\#(H_1(A)) \cong H_1(X, A).
\]
By the universal-coefficient formula for cohomology, there is also an isomorphism
\[
h : H^1(X, A; \pi) \approx \text{Hom}(H_1(X, A), \pi).
\]
Hence, if \( \pi = \pi_1(X)/i_\#(\pi_1(A)) \), then there exist \( 1 \)-characteristic elements \( v \in H^1(X, A; \pi) \).

If \( n > 1 \), it follows, from the relative Hurewicz isomorphism theorem and the universal-coefficient formula for cohomology, that there are isomorphisms \( \varphi : \pi_n(X, A) \approx H_n(X, A) \) and \( h : H^n(X, A; \pi) \approx \text{Hom}(H_n(X, A), \pi) \). Therefore, if \( \pi = \pi_n(X, A) \), there are \( n \)-characteristic elements \( v \in H^n(X, A; \pi) \).

This is all the homology and cohomology necessary for our purposes. We will now continue on the track set by the Hurewicz isomorphism theorem and work with isomorphic coefficient groups of homotopy functors.

**Lemma 4.9.** Let \( F : H \to H' \) be a natural transformation between homotopy functors which induces an isomorphism of their \( q \)th coefficient groups for \( q < n \) and a surjection of their \( n \)th coefficient groups (where \( 1 \leq n \leq \infty \)). For any path-connected pointed CW-complex, \( W \), the map
\[
F(W) : H(W) \to H'(W)
\]
is a bijection if \( \dim W \leq n - 1 \) and a surjection if \( \dim W \leq n \).

**Proof.** For a proof please see Theorem 8.1.7 of [24].

Now, consider the following specialization of Lemma 4.9 to \( H^n \) and \( \pi_n \).

**Theorem 4.10.** Let \( \pi \) be an abelian group, \( Y \) a space of type \((\pi, n)\), and \( \iota \in H^n(Y, y_0; \pi) \) an \( n \)-characteristic element for \( Y \). Let \( \psi : \pi^Y \to H^n(\cdot; \pi) \) be the natural transformation, defined by \( \psi[f] = f^*\iota \), for \( [f] \in [X; Y] \). Then, \( \psi \) is a natural equivalence on the category of path-connected pointed CW-complexes.

**Proof.** By Lemma 4.9, it suffices to verify that \( \psi \) induces an isomorphism on all coefficient groups of the two homotopy functors \( \pi^Y \) and \( H^n(\cdot; \pi) \). The only non-zero coefficient groups are \( \pi_n(Y, y_0) \) and \( H^n(S^n, p_0; \pi) \), and we need only verify that
\[
\psi(S^n) : \pi_n(Y, y_0) \to H^n(S^n, p_0; \pi)
\]
is an isomorphism. If \( \nu : H^n(S^n, p_0; \pi) \approx \pi \) is defined by \( \nu(v) = h(v)(\phi[1_{S^n}]) \), see the proof of lemma 8.1.4 of [24], then \( \nu \circ \psi(S^n) = h(\iota) \circ \phi \). Because \( \iota \) is \( n \)-characteristic for \( Y \), \( \nu \circ \psi(S^n) \) is an isomorphism, and thus, so is \( \psi(S^n) \).
We can now give our first result regarding obstructions. In essence this is just Theorem 3.67 with some additional detail but the reader should not dismiss it as uninteresting. Theorem 4.11 will incorporate most of the ideas we have developed in Section 4. Furthermore, it is this theorem that we will generalize in Theorem 4.35, the final result of the section.

**Theorem 4.11.** Let $Y$ be a space of type $(\pi, n)$, with $n \geq 1$ and $\pi$ abelian, and let $\iota \in H^n(Y, y_0; \pi)$ be $n$-characteristic for $Y$. If $(X, A)$ is a relative $CW$-complex, a map, $f : A \to Y$, can be extended to $X$ if, and only if, $\delta f^*(\iota) = 0$ in $H^{n+1}(X, A; \pi)$.

**Proof.** Assume that $f = g \circ i$, where $i : A \to X$ and $g : X \to Y$. Then, $\delta f^*(\iota) = \delta i^* g^*(\iota) = 0$, because $\delta i^* = 0$. Hence, if $f$ can be extended over $X$, then $\delta f^*(\iota) = 0$.

Conversely, assume $\delta f^*(\iota) = 0$. To extend $f$ over $X$ we need only extend $f$ over each path component of $X$, and therefore there is no loss of generality in assuming $X$ to be path connected and $A$ to be non-empty. Let $Y'$ be the space obtained from the disjoint union $X \cup Y$ by identifying $a \in A$ with $f(a) \in Y$ for all $a \in A$. Then $Y$ is imbedded in $Y'$, the pair $(Y', Y)$ is a relative $CW$-complex, and there is a cellular map, $j : (X, A) \to (Y', Y)$, which induces an isomorphism, $j^* : H^*(Y', Y) \approx H^*(X, A)$, such that the following diagram commutes. Note that the map $\delta$ is the coboundary map from the construction of the cohomology groups.

$$
\begin{array}{ccc}
H^n(Y, y_0) & \overset{\delta}{\longrightarrow} & H^{n+1}(Y', Y) \\
\downarrow{f'} & & \downarrow{j^*} \\
H^n(A) & \overset{\delta}{\longrightarrow} & H^{n+1}(X, A)
\end{array}
$$

Since $\delta f^*(\iota) = 0$, it follows that $\delta(\iota) = 0$, and there is $v \in H^n(Y', y_0; \pi)$ such that $v|Y, y_0 = \iota$. Since $X$ and $Y$ are path connected and $A$ is nonempty, $Y'$ is path connected.

Let $\bar{Y} = Y' \cup I$ (that is, $y_0 \in Y'$ is identified with $0 \in I$) and let $\bar{y}_0 = 1 \in \bar{Y}$. Then $\bar{Y}$ is a path-connected space with nondegenerate base point $\bar{y}_0$. Let $r : (\bar{Y}, I) \to (Y', y_0)$ be the retraction which collapses $I$ to $y_0$ and let $\bar{v} = r^*(v)|\bar{Y}, y_0 \in H^n(\bar{Y}, \bar{y}_0; \pi)$. By Theorem 3.74, there is an imbedding of $\bar{Y}$ in a space $Y''$ which is a classifying space for the $n$th cohomology functor with coefficients $\pi$ and which has a universal element $\bar{u} \in H^n(Y'', \bar{y}_0; \pi)$ such that $\bar{u}|\bar{Y} = \bar{v}$. Then, $Y''$ is a space of type $(\pi, n)$, and there is a unique $n$-characteristic element, $u \in H^n(Y'', y_0; \pi)$, such that $u|Y'' = \bar{u}|Y''$. Then $u|Y, y_0 = \iota$, and it follows from Theorem 4.10 and the commutativity of the next diagram.

$$
\begin{array}{ccc}
[S^q, p_0; Y, y_0] & \overset{\psi_1}{\longrightarrow} & [S^q, p_0; Y'', y_0] \\
\downarrow{\psi} & & \downarrow{\psi_u} \\
H^n(S^q, p_0; \pi)
\end{array}
$$

that $Y \subset Y''$ is a weak homotopy equivalence. Since the composite $X \xrightarrow{j} Y' \subset Y''$ is an extension of the composite $X \xrightarrow{f} Y \subset Y''$, it follows from Theorem 3.67 that $f$ can be extended to a map $X \to Y$. \qed

While Theorem 4.11 is interesting we would like it to apply to a wider class of spaces. In the next theorem section we will begin this work by introducing factorizations of spaces.
4.2. The Tower of Postnikov. In Theorem 4.11 we gave a condition for extending a map $f: A \to Y$ to $X$, given that $Y$ was a $K(n, \pi)$-space. We will now work to weaken this condition. Ultimately we will require only that $Y$ is simply connected. The idea is that we will factorize $Y$ into a sequence of spaces. To do this we make the following definition.

**Definition 4.12.** Let $\{E_n\}_{n \in \mathbb{N}}$ be a sequence of topological spaces. If there is a sequence of functions $\{f_n\}_{n \in \mathbb{N}}$ such that $f_n: E_{n+1} \to E_n$, then we say that $E_1 \leftarrow E_2 \leftarrow \cdots \leftarrow E_n \leftarrow E_{n+1} \leftarrow \cdots$ is a tower. We will use $E_\infty$ to denote $\lim_{n \to \infty} E_n$.

To see how this is applied, we consider the CW-complex $(X, A)$. To extend a map $A \to Y$ to a map $X \to Y$ we need to find a map $g: X \to Y$ such that $f = g \circ i$. Let us now assume that $Y$ can be switched for a tower $\{E_n\}_{n \in \mathbb{N}}$. We will look at the procedure for doing this later. Then, extending the map $f': A \to E_\infty$ is equivalent to the existence of $g: X \to E_\infty$ of the following diagram.

$$
\begin{array}{c}
A \xrightarrow{f'} E_\infty \\
\downarrow i \quad \quad \quad \quad \quad \quad \downarrow p_{\infty} \\
X \xrightarrow{f''} E_0
\end{array}
$$

We will set aside the issue of what $f'$ and $f''$ are for the moment. Instead we note that the extension of $f'$ now is equivalent to the existence of the map $g$, of the diagram. However, we can also see that the extension of $f'$ would also solve the lifting of $f''$ and vice versa. Therefore, it is possible to solve the extension problem by inductively solving the problem of lifting $f''$ to a map $X \to E_n$.

Let us now return to the issue of how to define the maps $f'$ and $f''$. First, let $a_n: E_\infty \to E_n$ be given by $p_n \circ p_{n+1} \circ \cdots$. This is just a convenience since, in particular, $a_0: E_\infty \to E_0$ will require only one map between $E_\infty$ and $E_0$. With $i$ and $a_0$ as the two vertical maps in a square diagram we will introduce the mapping pair.

**Definition 4.13.** Let $P'$, $P''$, $Q'$ and $Q''$ be topological spaces with maps $\alpha: P'' \to P'$ and $\beta: Q'' \to Q'$. We say that a pair of maps $f'': P'' \to Q''$ and $f': P' \to Q'$ is a map pair, if the following diagram commutes.

$$
\begin{array}{c}
P'' \xrightarrow{f''} Q'' \\
\downarrow \alpha \quad \quad \downarrow \beta \\
P' \xrightarrow{f'} Q'
\end{array}
$$

To shorten the notation we will denote the map pair $(f', f'')$ by $f: \alpha \to \beta$.

If we are going to exchange $Y$ for $\{E_n\}_{n \in \mathbb{N}}$, we will need a map pair $f: i \to a_0$. We will look at this in Corollary 4.27. Right now we shall instead look closer at the properties of the $\{E_n\}_{n \in \mathbb{N}}$ side and, in particular, the maps $\{p_n\}_{n \in \mathbb{N}}$. 
Definition 4.14. Let $E$, $B$, and $X$ be spaces. Let $f : X \to E$, $p : E \to B$, and $F : X \times I \to B$ be mappings such that the following diagram commutes.

$$
\begin{array}{c}
X \times 0 \xrightarrow{f} E \\
\downarrow i \\
X \times I \xrightarrow{p} B
\end{array}
$$

If a homotopy $F' : X \times I \to E$ exists such that the following diagram commutes as well,

$$
\begin{array}{c}
X \times 0 \xrightarrow{f} E \\
\downarrow i \\
X \times I \xrightarrow{p} B
\end{array}
$$

then we say that $p$ has the homotopy lifting property with respect to $X$.

Definition 4.15. Let $E$, $B$, and $p$ be as given in Definition 4.14. If $p$ has the homotopy lifting property with respect to all spaces $X$, then we say that $p$ is a fibration.

We are interested in using fibrations to simplify the homotopy structure of $Y$. The way this work is by the use of a special kind of fibration: the principal fibration. Before we can define it we must introduce some auxiliary facts about fibrations in general. First, for two maps $f : B' \to B$ and $p : E \to B$, we define the fibered product, $E'$, as

$$
E' = \{(b', e) \in B' \times E | f(b') = p(e)\}.
$$

From the definition of $E'$ it follows that we have a map $p'(b', e) = b'$ from $E'$ to $B'$ and a map $f'(b', e) = e$ from $E'$ to $E$.

Now, by using the fibered product we can take a fibration $p : E \to B$ and induce a fibration $p' : E' \to B'$, via a map $f : B' \to B$. For details on this see [24, p.98]. We say that fibration $p' : E' \to B'$ is the fibration induced from $p$ by $f$.

There is a natural way to induce fibrations from any base point preserving map between two spaces. To do this we shall first show that every space has certain kind of fibration. Formally, let $B'$ be a space with a basis point $b_0$, then we say that the space $PB' = \{\gamma : I \to B' | \gamma(0) = b_0\}$ is the path space of $B'$. For any space $B'$ we can construct a fibration $p : PB' \to B'$ by setting $p$ as the endpoint evaluation map $p(\gamma) = \gamma(1)$.

Definition 4.16. Let $\theta$ be a basis point preserving map $\theta : B \to B'$. For any fibration, $p : PB' \to B'$, there is a fibered product $E_\theta$ and a fibration $p_\theta : E_\theta \to B$ induced by $\theta$. We say that this is the principal fibration induced by $\theta$.

We shall now look at how principal fibrations can help us to extend maps. To do this we first need to make an observation about the nature of liftings to principal fibrations. Let $p_\theta : E_\theta \to B$ be the principal fibration induced by $\theta : B \to B'$. For any space $W$ and map $f : W \to E_\theta$ we can decompose $f$ into two maps. Specifically we can decompose $f$ into $f_1 : W \to B$ and $f_2 : W \to PB'$, since $E_\theta$ is the fibered product of $B$ and $PB'$.

As $PB'$ technically is $B'^I$, then we can use the following theorem, which is stated without proof as Theorem 1.2.8 in [24].
Theorem 4.17. Let \( X \) be a locally compact Hausdorff space and \( Y \) and \( Z \) are topological spaces, a map \( g : Z \to Y \) is continuous if and only if \( E \circ (g \times \text{id}_X) : Z \times X \to Y \), where \( E \) is the evaluation map, is continuous.

Now, we can write \( PB' \) as \( B' \). Therefore, by Theorem 4.17, there is a correspondence between the map \( f_2 : W \to PB' \) and the homotopy \( F : W \times I \to B' \) from the constant map to \( \theta \circ f_1 \), given by

\[
F : W \times I \xrightarrow{f_2 \times \text{id}_I} B' \times I \xrightarrow{E} B'
\]

We now assume that there is a CW-complex \( (X, A) \) and a map pair \( f : i \to p_\theta \). Furthermore, let

\[
\theta(f) : (A \times I \cup X \times \hat{I}, X \times \{0\}) \to (B', b'_0)
\]

be a map such that \( \theta(f)(x, 0) = b'_0 \), \( \theta(f)(x, 1) = \theta(f')(x) \), and \( \theta(f)|_{(A \times I)} \) is the homotopy from the constant map \( A \to b'_0 \) to the map \( \theta \circ f' \circ i \). By the previous paragraph, this map corresponds to the lifting \( f'' \) of \( f' \circ i \). Therefore, there is a one-to-one correspondence between liftings of \( f \) and extensions of \( \theta(f) \) over \( X \times I \).

When we apply this to simplify the structure of \( Y \) we will need to assume that \( B' \) is a \( K(\pi, n) \) space, with \( n \geq 1 \) and \( \pi \) abelian. Extending a map is particularly simple. To see this assume that \( \iota \in H^n(B', b'_0, \pi) \) is \( n \)-characteristic for \( B' \) and let \( \theta : B \to B' \) be a map. Furthermore, let \( p_\bar{\theta} : E_\bar{\theta} \to B \) be the principal fibration induced by \( \theta \). For any CW-complex \( (X, A) \) then \( (X, A) \times (I, \hat{I}) \) is also a CW-complex. Given a map \( g : A \times I \cup X \times \hat{I} \to B' \), it follows from Theorem 4.11 that \( g \) can be extended over \( X \times I \), if and only if, \( \delta g^*(i) = 0 \) in \( H^{n+1}((X, A) \times (I, \hat{I})) \).

In particular, given a map pair \( f : i \to p_\bar{\theta} \), there is a lifting of \( f \) if and only if \( \delta \theta(f)^* (\iota) = 0 \).

Let \( \tau \) be an isomorphism between \( H^n(X, A; \pi) \) and \( H^{n+1}((X, A) \times (I, \hat{I})) \). For a definition please see Section 8.1 of [24]. We say that \( \delta \theta(f)^* (\iota) = (\bar{\tau} \tau)(c(f)) \) is the obstruction to lifting \( f \) and denote it by \( c(f) \). In Theorem 4.18 we will compute \( c(f) \) for a special case, which we are going to apply in the proof of Theorem 4.35.

Theorem 4.18. Let \( \iota \in H^n(B', b'_0; \pi) \) and \( \iota' \in H^{n-1}(\Omega B', \omega'_0; \pi) \) be related characteristic elements. Let \( (X, A) \) be a relative CW-complex, with inclusion map \( i : A \hookrightarrow X \). Given a map pair \( f : i \to p' \), where \( p' : \Omega B' \to b'_0 \), then \( c(f) = -\delta f''^*(\iota') \), where \( f'' : A \to \Omega B' \) is part of \( f \).

Proof. Let \( \bar{f} : (A \times I, A \times \hat{I}) \to (PB', \Omega B') \) be the map defined by \( \bar{f}(a, t)(t') = f''(a)(tt') \). Then

\[
\theta(f) : (A \times I \cup x \times \hat{I}, X \times 0) \to (B', b'_0)
\]

is the map such that \( \theta(f)|A \times I = p \circ \bar{f} \) and \( \theta(f)(X \times \hat{I}) = b'_0 \). Let \( \tilde{f} : (A \times I \cup X \times \hat{I}, X \times \hat{I}) \to (B', b'_0) \) be the map defined by \( \bar{f} \) and let \( f' : (A \times I, A \times 0) \to (\Omega B', \omega'_0) \) be the map defined by \( f \). There is then a commutative diagram, in which \( j \) and \( j' \) are appropriate inclusion maps and \( h_1 : A \to (X \times \hat{I}, X \times 0) \) is defined by \( h_1(a) = (a, 1) \).
Furthermore, \( \delta \circ \tau^{-1} \circ j^* = \tau^{-1} \circ \delta : H^n(A \times I \cup X \times \hat{I}, X \times \hat{I}) \to H^n(X, A) \). Since \( f'' = f' \circ h_1 \), then \( f''^* = h_1^* \circ j^* \), and we have
\[
(\hat{1})^{n-1} \tau^{-1}(\theta(f))^*(\iota) = \delta f''^*(\iota).
\]
By definition, the left-hand side equals \(-c(f)\).

\[\square\]

With a clear condition for both lifting and extending maps to principal fibrations we will now look at how to replace \( Y \) with a sequence of such fibrations.

**Definition 4.19.** If, for any \( n < \infty \), there is an \( N_n \) such that for \( p_q \) from the sequence of fibrations \( E_0 \xleftarrow{p_1} E_1 \xleftarrow{p_2} \cdots \) is an \( n \)-equivalence for \( q < N \), then we say that the sequence is **convergent**.

To replace \( Y \) we are not going to factorize \( Y \) directly. Instead, it is the map \( a_0 : E_\infty \to E_0 \), that we are going to factorize. However, we do not yet have any meaningful way to talk about factorizations of maps. This is rectified with the following definition.

**Definition 4.20.** Let \( f : Y' \to Y \) be a map. Then, the sequence \( \{p_q, E_q, f_q\}_{q \geq 1} \) is said to be a **convergent factorization of \( f \)** if it satisfies the following conditions:

1. for \( q > 1, p_q : E_q \to E_{q-1} \) is a fibration, and for \( q = 1, p_1 : E_1 \to Y \) is a fibration;
2. for \( q \geq 1, f_q : Y' \to E_q \) is a map \( f_q = p_{q+1} \circ f_{q+1} \) for \( q \geq 1 \), and \( f = p_1 \circ f_1 \);
3. for any \( n < \infty \) there is \( N_n \) such that \( f_q \) is an \( n \)-equivalence for \( q > N_n \).

We now show that \( a_0 : E_\infty \to E_0 \) has a convergent factorization,
\[
E_0 \xleftarrow{p_1} E_1 \xleftarrow{p_2} \cdots.
\]

**Theorem 4.21.** If \( E_0 \xleftarrow{p_1} E_1 \xleftarrow{p_2} \cdots \) is a convergent sequence of fibrations then \( \{p_q, E_q, a_q\}_{q \geq 1} \) is a convergent factorization of the map \( a_0 : E_\infty \to E_0 \).

**Proof.** Conditions (1) and (2) of Definition 4.20 are clearly satisfied. To prove that condition (3) is also satisfied, given \( 1 \leq n < \infty \), choose \( N \) so that \( p_q \) is an \((n+1)\)-equivalence if \( q \geq N \). We prove that \( a_q \) is an \( n \)-equivalence for \( q \geq N \), it suffices to prove that \( a_N \) is an \( n \)-equivalence.
Let \((P, Q)\) be a polyhedral pair such that \(\dim P \leq n\), and let \(\alpha : Q \to E_\infty\) and \(\beta_N : P \to E_N\) be maps such that \(\beta_N|_Q = \alpha_N \circ \alpha\). We now prove that there is an extension \(\beta : P \to E_\infty\) with the desired properties, we must obtain a sequence of maps \(\beta_q : P \to E_q\) such that

\[
\beta_q|_Q = \alpha_q, \quad \beta_q = p_{q+1} \circ \beta_{q+1}, \quad \beta_N = \beta_N'.
\]

Such a sequence of maps \(\{\beta_q\}\) is defined, for \(q \leq N\), by \(\beta_q = p_{q+1} \circ \cdots \circ p_N \circ \beta_N'\), and for \(q \geq N\), we use Theorem 3.67 to find a map \(\beta_N' : P \to E_{q+1}\) such that:

1. \(\beta_{q+1}'|_Q = \alpha_{q+1}\),
2. \(\beta_q \simeq p_{q+1} \circ \beta_{q+1}'\) rel \(Q\).

We use the fact that \(p_{q+1}\) is a fibration (and Theorem 7.2.6 of [24]) to alter \(\beta_{q+1}'\) by a homotopy relative to \(Q\) to obtain a map, \(\beta_{q+1} : P \to E_{q+1}\), such that \(\beta_{q+1}|_Q = \alpha_{q+1}\) and such that \(\beta_q \simeq p_{q+1} \circ \beta_{q+1}'\). Thus, the sequence, \(\{\beta_q\}\), can be found, and hence a map \(\beta : p \to E_\infty\) with the requisite properties exist.

Taking \(P\) to be a single point and \(Q\) to be empty, we see that \(a_N\) is surjective. Therefore, \(a_N\) maps \(\pi_0(E_\infty)\) injectively to \(\pi_0(E_N)\). If instead, \((P, Q) = (I, I)\), we see that \(a_N\) maps \(\pi_0(E_\infty)\) injectively to \(\pi_0(E_N)\). Then, \(a_N\) induces a one-to-one correspondence between the set of path components of \(E_\infty\) and the set of path components of \(E_N\).

Let \(e_* = (e_q) \in E_\infty\) be arbitrary and let \(1 \leq k \leq n\). Taking \((P, Q) = (S^k, z_0)\) it follows that \(a_{N, k}\) maps \(\pi_k(E_\infty, e_*)\) epimorphically to \(\pi_k(E_N, e_*)\). For \(1 \leq k < n\), taking \((P, Q) = (E^{k+1}, S^k)\), it follows that \(a_{N, k}\) maps \(\pi_k(E_\infty, e_*)\) monomorphically to \(\pi_k(E_N, e_*)\). Hence, \(a_N\) is an \(n\)-equivalence.

We have now shown that there is a convergent factorization of \(a_0\). However, we are not interested in just any factorization. We need the fibers to be principal fibrations of type \((\pi, n)\). Such a sequence of fibrations is known as a Moore-Postnikov sequence of fibrations.

**Definition 4.22.** Let \(E_0 \xleftarrow{P_1} E_1 \xleftarrow{P_2} \cdots\) be a convergent sequence of fibrations. If \(p_q : E_q \to E_{q-1}\) is a principal fibration of type \((\pi_q, n_q)\) for \(q \geq 1\), then \(E_0 \xleftarrow{P_1} E_1 \xleftarrow{P_2} \cdots\) is a Moore-Postnikov sequence of fibrations.

Similarly, given a map \(f : Y' \to Y\) a Moore-Postnikov factorization is a convergent factorization \(\{p_q, E_q, f_q\}\) such that \(E_0 \xleftarrow{P_1} E_1 \xleftarrow{P_2} \cdots\) is a Moore-Postnikov sequence.

We are interested in a particular Moore-Postnikov factorization. To define it, let \(f : Y' \to Y\) be a map. A Postnikov factorization of \(Y'\) is then a Moore-Postnikov factorization of the constant map \(Y' \to y_0\). Another factorization, which is not a Moore-Postnikov factorization but very useful, is the \(n\)-factorization.

**Definition 4.23.** Let \(f : Y' \to Y\) be a map between path-connected pointed spaces. For \(n \geq 1\) a \(n\)-factorization of \(f\) is a factorization of \(f\) as a composite \(Y' \xleftarrow{b'} Y' \xrightarrow{\pi_q} Y\) such that

1. \(E'\) is a path-connected pointed space, \(p'\) is a fibration, and \(b'\) is a lifting of \(f\);
2. \(b'_\# : \pi_q(Y') \to \pi_q(E')\) is an isomorphism for \(1 \leq q < n\) and an epimorphism for \(q = n\).
We show that the following diagram commutes up to a sign:

\[
\begin{array}{ccc}
E' & \xrightarrow{b'} & E'' \\
\downarrow{p'} & & \downarrow{p''} \\
Y & \xrightarrow{Y} & Y''
\end{array}
\]

We will use \(n\)-factorizations to show that we can replace the target space of the map we wish to extend with a Postnikov fibration. This will, however, get a bit technical. Should the reader feel tired, she, or he, could jump directly to Corollary 4.27.

**Lemma 4.24.** Let \((X,A)\) be a pointed set pair of path-connected spaces \((n-1)\)-connected for some \(n \geq 1\) and such that the inclusion map \(i : A \hookrightarrow X\) is simple. Then there is an \(n\)-factorization \(A \xrightarrow{i'} E' \xrightarrow{p'} X\) of \(i\) such that \(p'\) is a principal fibration of type \((\pi,n)\), where \(\pi = \pi_1(X)/i_#(\pi_1(A))\), if \(n = 1\), and \(\pi = \pi_n(X,A)\), if \(n > 1\).

**Proof.** By Lemma 4.8, there is a class \(\in H^n(X,A;\pi)\) which is \(n\)-characteristic for \((X,A)\). Let \(CA\) be the nonreduced cone over \(A\) and observe that \(\{X,CA\}\) is an excisive couple in \(X \cup CA\). Therefore, there is an element \(v' \in H^n(X \cup CA;\pi)\) corresponding to \(v\) under the isomorphisms

\[
H^n(X \cup CA;\pi) \xrightarrow{\sim} H^n(X,A;\pi) \rightarrow H^n(X,A;\pi).
\]

It is possible to imbed \(X \cup CA\) into a space \(X'\) of type \((\pi,n)\) having an \(n\)-characteristic element \(v'\) such that \(v'\mid X \cup CA = v'\). Let \(p' : E' \rightarrow X\) be the principal fibration induced by the inclusion \(X \hookrightarrow X'\) and let \(p'_A : E'_A \rightarrow A\) be the restriction of this fibration to \(A\). There is a section \(s : A \rightarrow E'_A\) such that \(s(a) = (a,\omega_a)\) for \(a \in A\), where \(\omega_a\) is the path from \(x_0\) to the vertex of \(CA\) followed by the path from the vertex of \(CA\) to \(a\). That is, \(\omega_a(t) = [x_0, 1-2t]\) for \(0 \leq t \leq \frac{1}{2}\) and \(\omega_a(t) = [a, 2t-1]\) for \(\frac{1}{2} \leq t \leq 1\). We define \(b' : A \rightarrow E'\) to be the composite \(A \xrightarrow{s} E'_A \xrightarrow{i_A} E'\) and we shall prove that \(A \xrightarrow{b'} E' \xrightarrow{p'} X\) is an \(n\)-factorization of \(i\).

The fiber of \(p'\), hence also of \(p'_A\), is \(\Omega X'\), and we define \(g : E'_A \rightarrow \Omega X'\) by \(g(a,\omega) = \omega \ast (s(a))^{-1}\). Then \(g|\Omega X : \Omega X' \rightarrow \Omega X'\) is homotopic to the identity map. If \(i'' : \Omega X' \hookrightarrow E'_A\) is the inclusion map, it follows from the exactness of the homotopy sequence of the fibration, \(p'_A : E' \rightarrow A\), that there is a direct-sum decomposition

\[
\pi_q(E'_A) \approx i_#\pi_q(\Omega X') \oplus S_#\pi_q(A) \quad q \geq 1.
\]

For \(q = 1\), this is a direct-product decomposition, but we shall still write it additively. We define the homomorphism, \(\lambda : \pi_q(X,A) \rightarrow \pi_{q-1}(\Omega X')\), where \(q \geq 1\), to be the composite

\[
\pi_q(X,A) \xrightarrow{\lambda} \pi_q(E'_A) \xrightarrow{\partial} \pi_{q-1}(E'_A) \xrightarrow{\partial} \pi_{q-1}(\Omega X').
\]

We show that the following diagram commutes up to a sign:
In fact, the left-hand and middle squares are easily seen to be commutative. We shall show that \( b'_\# \circ \partial = -i'_\# \circ \lambda \).

For \( q = 1 \) this is because \( \pi_0(A) = 0 \) implies that \( b'_\# \circ \partial \) is the trivial map and the fact that \( j_\# \) is surjective and \( i'_\# \circ \lambda \circ j_\# = i'_\# \circ \bar{\partial} = 0 \) implies that \( i'_\# \circ \lambda \) is also the trivial map. For \( q > 1 \), we have

\[
\alpha = i'_\# \lambda g_\# \alpha + s_\# p'_{A_\#} \alpha \quad \alpha \in \pi_{q-1}(E'_A).
\]

Since the composite \( \pi_q(E', E'_A) \xrightarrow{\partial} \pi_{q-1}(E'_A) \xrightarrow{i'_\#} \pi_{q-1}(E') \) is trivial, it follows that for \( \beta \in \pi_q(E', E'_A) \)

\[
0 = i'_\# \lambda g_\# \partial \beta + i'_\# s_\# p'_{A_\#} \partial \beta = i'_\# \lambda g_\# \partial \beta + b'_\# \partial p_\#. \]

By definition of \( \lambda \), we see that \( \lambda p_\# \beta = g_\# \partial \beta \). Therefore,

\
\[ i'_\# \lambda p_\# \beta + b'_\# \partial p_\# \beta = 0. \]

Since \( p_\# : \pi_q(E', E'_A) \approx \pi_q(X, A) \), this proves \( b'_\# \circ \partial = -i'_\# \circ \lambda \).

A straightforward verification shows that \( \lambda \) is also the composite

\[
\pi_n(X, A) \to \pi_n(X \cup CA, CA) \xrightarrow{\approx} \pi_n(X \cup CA) \to \pi_n(X') \xrightarrow{\partial} \pi_{n-1}(\Omega X').
\]

The construction of \( X' \) and \( i' \in H^n(X', \pi) \) shows that there is a commutative diagram

\[
\begin{array}{ccc}
\pi_q(X, A) & \xrightarrow{\varphi} & \pi_n(X \cup CA, CA) \\
\approx & & \approx \\
H_n(X, A) & \xrightarrow{h(v)} & H_n(X \cup CA, CA)
\end{array}
\]

\[
\begin{array}{ccc}
\pi_n(X \cup CA) & \xrightarrow{\varphi} & \pi_n(X) \\
\approx & & \approx \\
H_n(X) & \xrightarrow{h(v')} & H_n(X')
\end{array}
\]

Therefore, \( \lambda : \pi_n(X, A) \approx \pi_{n-1}(\Omega X') \).

In case \( n = 1 \), \( \partial : \pi_1(X) \to \pi_0(\Omega X') \) is surjective [because \( \pi_0(A) = 0 \)], and so \( E' \) is path-connected. If \( n > 1 \), \( E' \) is path connected because \( \pi : 0(\Omega X') = 0 \). Therefore \( E' \) is a path-connected pointed space. Since \( \pi_q(\Omega X') = 0 \) for \( q \geq n \) it follows from the exactness of the homotopy sequence of the fibration, \( p' : E' \to X \), that \( p'_\# : \pi_q(E') \to \pi_q(X) \) is an isomorphism for \( q > n \) and a monomorphism for \( q = n \).

Because \( \lambda : \pi_q(X, A) \to \pi_{q-1}(\Omega X') \) is a bijection for \( q \leq n \), with the only non-trivial case when \( q = n \), it follows from Lemma 4.5.11 of [24] and the commutativity, up to a sign, of the first diagram of the proof that \( b'_\# : \pi_q(A) \to \pi_q(E') \) is an isomorphism for \( 1 \leq q < n \) and an epimorphism for \( q = n \). Therefore \( b' \) and \( p' \) have the properties required of an \( n \)-factorization of \( i \). \( \square \)
Lemma 4.24 then readily extends to the following corollary.

**Corollary 4.25.** Let \( g : X' \rightarrow X \) be a simple map between path-connected pointed spaces such that for some \( n \geq 1 \) the map \( g_\pi : \pi_q(X') \rightarrow \pi_q(X) \) is an isomorphism for \( 1 \leq q < n-1 \) and an epimorphism for \( q = n-1 \). Then there is an \( n \)-factorization \( X' \xrightarrow{\bar{b}'} E' \xrightarrow{\bar{p}} X \) of \( g \) such that \( \bar{p} \) is a principal fibration of type \((\pi,n)\) for some abelian group \( \pi \).

**Proof.** Let \( Z \) be the reduced mapping cylinder of \( g \). That is, the mapping cylinder of \( g|_{x_0} : x_0 \rightarrow x_0 \) has been collapsed to a point. Then \((Z,X')\) is a pointed pair of path-connected spaces \((n-1)\)-connected and with simple inclusion map \( i : X' \hookrightarrow Z \).

By Lemma 4.24, there is an \( n \)-factorization \( X' \xrightarrow{\tilde{b}'} E'' \xrightarrow{\tilde{p}''} Z \) of \( i \) such that \( \tilde{p}'' \) is a principal fibration of type \((\pi,n)\). Let \( p' : E' \rightarrow X \) be the restriction of \( \tilde{p}'' \) to \( X \).

Then \( E' \hookrightarrow E'' \) is a homotopy equivalence, so there is a map \( \tilde{b}'' : X' \rightarrow E' \) such that \( \tilde{b}'' \) is homotopic to the composite \( X' \xrightarrow{\tilde{b}'} E' \xrightarrow{\tilde{p}} E'' \). The composition \( \tilde{b}' \circ \tilde{b}'' \) is easily seen to be homotopic to \( g \). By the homotopy lifting property of \( p' \), there is a map \( b' : X' \rightarrow E' \) homotopic to \( \tilde{b}'' \) such that \( p' \circ b' = g \). It is easy to verify that, \( X' \xrightarrow{b'} E' \xrightarrow{p'} X \) has the required properties. \( \square \)

The last corollary gave us a way to find a principal fibration between two path connected spaces. While the principal fibration is one of the conditions for a Moore-Postnikov fibration, it remains to check whether the remaining two are also satisfied.

**Theorem 4.26.** Let \( f : Y' \rightarrow Y \) be a simple map between path connected pointed spaces. There is a Moore-Postnikov factorization \( \{p_q, E_q, f_q\}_{q \geq 1} \) of \( f \) such that for \( n \geq 1 \) the sequence,

\[
Y' \xrightarrow{f_{n-1}} E_n \xrightarrow{p_n \cdots p_2} Y,
\]

is an \( n \)-factorization of \( f \).

**Proof.** By induction on \( q \), we prove that the existence of a sequence, \( \{p_1, E_q, f_q\}_{q \geq 1} \), such that:

1. for \( n = 1 \) the sequence \( Y' \xrightarrow{f_1} E_1 \xrightarrow{p_1} Y \) is a 1-factorization of \( f \);
2. for \( n > 1 \) the sequence \( Y' \xrightarrow{f_n} E_n \xrightarrow{p_n} E_{n-1} \) is an \( n \)-factorization of \( f_{n-1} \);
3. for \( n \geq 1 \), \( p_1 \) is a principal fibration of type \((\pi_n,n)\) for some \( \pi_n \).

Once such a sequence \( \{p_q, E_q, f_q\}_{q \geq 1} \) has been found, it is easy to verify that it is a Moore-Postnikov factorization of \( f \) with the desired property. Therefore we limit ourselves to proving the existence of such a sequence.

By Corollary 4.25, with \( n = 1 \), there is a 1-factorization \( Y' \xrightarrow{f_1} E_1 \xrightarrow{p_1} Y \) of \( f \) with \( p_1 \) a principal fibration of type \((\pi_1,1)\) for some \( \pi_1 \). This defines \( p_1, E_1 \) and \( f_1 \). Assume that \( \{p_q, E_q, f_q\} \) is defined for \( 1 \leq q < n \), where \( n > 1 \), to satisfy (1), (2) and (3) above. Again by Corollary 4.25, there is an \( n \)-factorization \( Y' \xrightarrow{f_{n-1}} E_{n-1} \xrightarrow{p_{n-1}} E_{n-2} \) of \( f_{n-1} \) such that \( p_n \) is a principal fibration of type \((\pi_n,n)\) for some \( \pi_n \). Then \( P_n, E_n \) and \( f_n \) have the desired properties. \( \square \)

This leaves us in a position to actually prove what we wanted with regards to replacing a space \( Y' \) with a Postnikov fibration.
Corollary 4.27. Let $Y'$ be a simple path-connected pointed space. Then $Y'$ has a Postnikov factorization $\{p_q, E_q, f_q\}_{q \geq 1}$ in which $\pi_q(E_n) = 0$ for $q \geq n$ and $f_n : Y' \to E_n$ is an $n$-equivalence.

Proof. If $Y'$ is a simple space, then the constant map $Y' \to y_0$ is a simple map. The result follows then from Theorem 4.26. □

4.3. The Final Obstruction. Whether the reader has comfortably absorbed the material or feels left at the deep end of the pool, we are nearly done. This section is where we make use of Postnikov factorizations to extend maps. In this section we will often use map pair and the reader might want to refer back to Definition 4.13.

Using maps and map pairs one can define a related concept: homotopy pairs. These will be used in a sequence of three theorems that ultimately, in Theorem 4.34, proves the existence of a bijection between the inclusion map and a fibration.

Definition 4.28. A homotopy pair $H : f_0 \simeq f_1$ is a commutative square,

$$
\begin{array}{ccc}
P' \times I & \xrightarrow{H''} & Q'' \\
\downarrow_{\alpha \times \text{id}_I} & & \downarrow_{\delta} \\
P'' \times I & \xrightarrow{H''} & Q'
\end{array}
$$

Before we can use homotopy pairs to prove Theorem 4.32 we need to talk about weak homotopy equivalences.

Definition 4.29. Let $f$ be a continuous map $X \to Y$. Then, if $f$ is an $n$-characteristic for all $n \geq 1$, we say that $f$ is a weak homotopy equivalence.

Let us now prove two things regarding these weak homotopy equivalences. First off, a homotopy equivalence in the usual sense is also a weak homotopy equivalence.

Proposition 4.30. A homotopy equivalence is a weak homotopy equivalence.

Proof. This follows from Corollary 7.3.15 of [24]. □

We now show that the convergent factorization of Definition 4.20 can be used to create a weak homotopy equivalence.

Corollary 4.31. Let $\{p_q, E_q, f_q\}_{q \geq 1}$ be a convergent factorization of a map $f : Y' \to Y$ and let $f' : Y' \to E_\infty$ be the map such that $a_q \circ f' = f_q$ for $q \geq 1$ and $a_0 \circ f' = f$. Then $f'$ is a weak homotopy equivalence.

Proof. For any $1 \leq n \leq \infty$ there is a $q$ such that $a_q$ and $f_q$ are both $n$-equivalences, by Theorem 4.21. Because $a_q \circ f' = f_q$, then $f'$ is also an $n$-equivalence. Since this holds for all $n$, it means that $f'$ is a weak homotopy equivalence. □

To understand Theorem 4.32 please note that a weak fibration is a fibration that has the homotopy lifting property with respect to only cubes, $\{I^n\}_{n \geq 0}$, rather than all spaces.

Theorem 4.32. Let $(X, A)$ be a relative CW-complex, with inclusion map $i : A \hookrightarrow X$, and let $p : E \to B$ be a weak fibration. Given a map $f : X \to E$ and a homotopy pair $H : i \times 1_I \to p$ consisting of a homotopy $H' : X \times I \to B$ starting at $p \circ f$ and a homotopy $H'' : A \times I \to E$ starting at $i \circ 1_I$, there is a homotopy $\tilde{H} : X \times I \to E$ starting at $\tilde{f}$ such that $H' = p \circ \tilde{H}$ and $H'' = \tilde{H} \circ (i \times 1_I)$. 
Proof. Let \( g : X \times 0 \cup A \times I \to E \) be the map defined by \( g(x, 0) = \tilde{f}(x) \) for \( x \in C \) and \( g(a, t) = H''(a, t) \) for \( a \in A \) and \( t \in I \). Then \( H' \) is an extension of \( p \circ g \), and by the standard stepwise-extension procedure over the successive skeleta of \((X, A)\) there is a map \( \tilde{H} : X \times I \to E \) such that \( p \circ \tilde{H} = H' \) and \( \tilde{H}|X \times 0 \cup A \times I = g \). Then \( \tilde{H} \) has the desired properties. \( \square \)

**Theorem 4.33.** Let \((X, A)\) be a relative CW-complex, with inclusion map \( i : A \hookrightarrow X \), and let \( g : p_1 \to p_2 \) be a weak homotopy equivalence between weak fibrations. Given a map pair \( f : i \to p_1 \) and a lifting \( \bar{h} : X \to E_2 \) of the map pair \( g \circ f \), there is a lifting \( \bar{f} : X \to E_1 \) of \( f \) such that \( g'' \circ \bar{f} \) and \( \bar{h} \) are homotopic relative to \( g \circ f \).

Proof. The proof involves two applications of Theorem 4.36 and then two applications of Theorem 4.32. We shall not make specific reference to these when they are invoked.

We have a commutative diagram

\[
\begin{array}{ccc}
A & \longrightarrow & E_1 \\
\downarrow & & \downarrow \quad g'' \\
X & \longrightarrow & E_2 \\
\end{array}
\]

in which \( g'' \) and \( g' \) are weak homotopy equivalences, and we are given a map \( \bar{h} : X \to E_2 \), such that \( h \circ i = g'' \circ f'' \) and \( p_2 \circ \bar{h} = g' \circ f' \). Then there is a map \( \tilde{f} : X \to E_1 \) such that \( \tilde{f} \circ i = f'' \) and a homotopy \( G'' : g'' \circ \tilde{f} \simeq \bar{h} \) rel \( A \).

The map \( p_1 \circ \tilde{f} \) and \( f' \) agree on \( A \) and \( p_2 \circ G'' \) is a homotopy relative to \( A \) from \( g' \circ p_1 \circ \tilde{f} = p_2 \circ g'' \circ \tilde{f} \) to \( g' \circ f' = p_2 \bar{h} \). Therefore, there is a homotopy \( F' : p_1 \circ \tilde{f} \simeq f' \) rel \( A \) and a homotopy \( H' : g' \circ F' \simeq p_2 \circ G'' \) rel \( A \times I \times X \times I \).

Let \( F'' : X \times I \to E_1 \) be a lifting of \( F' \) such that \( F''(x, 0) = \tilde{f}(x) \) for \( x \in X \) and \( F''(a, t) = f''(a) \) for \( a \in A \) and \( t \in I \). Define \( \bar{f} : X \to E_1 \) by \( \bar{f}(x) = F''(x, 1) \). We show that \( \bar{f} \) has the desired properties. It is clearly a lifting of \( f \).

The maps \( g'' \circ F'' \) and \( G'' \) are homotopies relative to \( A \) from \( g'' \circ \tilde{f} \) to \( g'' \circ \bar{f} \) and to \( \bar{h} \), respectively, and \( H' \) is a homotopy from \( p_2 \circ g'' \circ F'' \) to \( p_2 \circ G'' \) rel \( A \times I \cup X \times I \). Since there is a homeomorphism of \((X \times I \times I, A \times I \times I)\) onto itself taking \( X \times (I \times I) \) onto \( X \times I \times 0 \), there is a lifting \( H'' \) of \( H' \), which is a homotopy from \( g'' \circ F'' \) to \( G'' \) rel \( X \times 0 \cup A \times I \). Then the map \( H : X \times I \to E_2 \) defined by \( H(x, t) = H''(x, 1, t) \) is a homotopy from \( g'' \circ \bar{f} \) to \( \bar{h} \) relative to \( g \circ f \). \( \square \)

**Theorem 4.34.** Let \((X, A)\) be a relative CW-complex, with inclusion map \( i : A \hookrightarrow X \), and let \( g : p_1 \to p_2 \) be a weak homotopy equivalence between weak fibrations. Given a map pair \( f : i \to p_1 \), the map pair \( g \) induces a bijection

\[
g''_# : [X; E_1]_f \approx [X; E_2]_{g \circ f}.
\]

Proof. The fact that \( g''_# \) is surjective follows immediately from Lemma 4.33. The fact that \( g''_# \) is injective follows from application of Lemma 4.33 to the relative CW-complex \((X, A) \times (I, I)\). \( \square \)

Let us now look at how to extend a map \( f : A \to Y \) to a map from \( X \) to \( Y \). Our setting is that we have a \((n - 1)\) connected space \( Y \) and a CW-complex \((X, A)\) with a map \( f : A \to Y \). First, we replace \( Y \) with a Postnikov factorization, which respects the homotopy structure of \( Y \). Within the factorization, we can inductively lift the map, one level at a time. These lifts are particularly simple, since the
fibration by definition is a Principal fibration. Finally, as we are working with a Postnikov factorization, it follows that the lifting problem is equivalent to the extension problem. Now, let us prove the theorem.

**Theorem 4.35.** Let \( i \in H^n(Y,y_0;\pi) \) be \( n \)-characteristic for a simple \((n-1)\)-connected pointed space \( Y \) where \( n \geq 1 \), and let \((X,A)\) be a relative CW-complex such that \( H^{q+1}(X,A;\pi_q(Y,y_0)) = 0 \) for \( q > n \). A map \( f : A \to Y \) can be extended over \( X \) if, and only if, \( \delta f^*(i) = 0 \) in \( H^{n+1}(X,A;\pi) \).

**Proof.** First, let \( i : A \hookrightarrow X \). For \( Y \) we can apply Corollary 4.27 to get a Postnikov factorization \( \{p_q, E_q, \bar{f}_q\}_{q \geq 1} \) in which \( \pi_q(E_n) = 0 \) for \( q \geq n \). Since all Moore-Postnikov factorizations are convergent, then Corollary 4.31 gives a weak homotopy equivalence \( f' : Y \to E_\infty \). Let \( a_q : E_\infty \to E_q \) and in particular let \( a_0 : E_\infty \to B \).

Then, Theorem 4.34 tells us that the lifting problem for a map pair from \( i \) to \( p \) is equivalent to the one for a map pair from \( i \) to \( a_0 \). Furthermore, since \( y_0 \) is a point then the lifting problem is equivalent to the extension problem for a map \( f'' : A \to E_\infty \). Therefore we can solve the extension problem by solving the lifting one.

To solve the lifting problem we have to find a sequence of maps, \( \bar{f}_q : X \to E_q \), such that

1. each \( \bar{f}_q \) is a lifting of \( f_q \),
2. \( p_{q+1} \circ \bar{f}_{q+1} = \bar{f}_q \).

Let \( g(\bar{f}_q) \) be the map pair that forms the commutative square

\[
\begin{array}{ccc}
A & \xrightarrow{f''_{q+1}} & E_{q+1} \\
\downarrow & & \downarrow p_{q+1} \\
X & \xrightarrow{f_q} & E_q
\end{array}
\]

Then, a lifting of \( g(\bar{f}_q) \) has to be such that:

1. \( \bar{f}_1 \) is a lifting of \( f_1 \);
2. for \( q \geq 1 \), \( \bar{f}_{q+1} \) is a lifting of \( g(\bar{f}_q) \).

In particular, these has to be such that \( \bar{f}_1 : X \to E_1 \) is an extension of \( \alpha_1 \circ f'' \) and \( \bar{f}_{q+1} : X \to E_{q+1} \) for \( q \geq 1 \) is a lifting of the map pair \( g(\bar{f}_q) : i \to p_{q+1} \) consisting of

\[
\begin{array}{ccc}
A & \xrightarrow{\alpha_{q+1} \circ f''} & E_{q+1} \\
\downarrow & & \downarrow p_{q+1} \\
X & \xrightarrow{f_q} & E_q
\end{array}
\]

Because \( \{p_q, E_q, f_q\}_{q \geq 1} \) is a Postnikov factorization, then \( p_q \) is a principal fibration of type \((\pi_q(Y,y_0), q + 1)\). Hence, the obstruction to lifting \( g(\bar{f}_q) \) is an element of \( H^{q+1}(X,A;\pi_q(Y,y_0)) \approx H^{n+1}(X,A;\pi) \). Therefore, the obstruction to extending \( f'' \) is defined, and the \((q + 1)\)th obstruction is a subset of \( H^{n+1}(X,A;\pi) \). Since \( Y \) is \((n - 1)\)-connected, the lowest non-trivial obstruction lies in \( H^{n+1}(X,A;\pi) \).

Since \( i \) is \( n \)-characteristic for \( Y \), it follows from Theorem 4.18 that the obstruction in \( H^{n+1}(X,A;\pi) \) is \( \pm \delta f''^*(i) \). Therefore, the obstruction vanishes if and only if \( \delta f^*(i) = 0 \).
5. Proof of Theorem 2.16

In Section 5 we looked at order isomorphisms and asked whether it was possible to combine different isomorphisms into a single one. With the algebraic topology developed, it is finally possible to answer this with the theorem by Chichilnisky and Heal. We restate this theorem here for easier reference. After proving Theorem 2.16 we will take a look at an interesting connection between Arrow’s theorem and Theorem 2.16.

**Theorem 2.16** Let $P$ be the space of preference profiles. A continuous function $\phi : \prod_{i \in I} P \to P$, which satisfies anonymity and unanimity, exists if and only if $P$ is contractible.

The idea of the proof of sufficiency is to first define $\phi$ as the vector average on a convex set $K$, such that $P \subset K$. Technically $K$ could be defined as the convex hull of $P$. We then extend the identity map of $P$ to $K$, by use of Theorem 4.35. This extended map then gives a retract from $K$ to $\prod P$.

**Proof of sufficiency.** To prove sufficiency we will show the following. Let $K$ be a convex set such that $P \subset K$. Then the map, $\phi : \prod K \to K$, defined by

$$\phi(p_1, \ldots, p_n) = \frac{1}{n} \sum_{i=1}^{n} p_i$$

for $(p_1, \ldots, p_n) \in K$,

is a continuous social choice rule, respecting anonymity and unanimity.

We show that $\phi$ satisfies each of the properties in turn. First let $(p'_1, \ldots, p'_n)$ be a permutation of $(p_1, \ldots, p_n)$. Since addition is commutative on $K$ then $\phi(\{p_i\}_{i=1}^{n}) = \phi(\{p_j\}_{j=1}^{n})$. Hence, $\phi$ is invariant under permutations of $(p_1, \ldots, p_n)$ and therefore satisfies anonymity. Since $\phi$ is the usual vector average we have for any $(p_k, \ldots, p_k) \in K$,

$$\phi(p_k, \ldots, p_k) = p_k,$$

and therefore $\phi$ is unanimous. Finally, $\prod_{i=1}^{n} K$ inherits its product topology from the topology on $K$, see [21, p. 86]. It follows that $\frac{1}{n} \sum_{i=1}^{n} p_i$ is continuous.

The idea now is to retract the image of $\phi$ to $P$. A retraction is a space $A \subset X$ such that there is a continuous map $r$ with the property $r \circ i = \text{id}_A$. This is where the obstruction theory comes in. Since $P$ is contractible then the identity map is trivial in every cohomology group $H^n(P; \pi)$. Therefore, $\delta \text{id}_P = 0$ and $\text{id}_P : P \to P$ can be extended to a map $r : K \to P$.

Since $r|_P = \text{id}_P$ we can use $r$ to retract $K$ to $P$ and form the function

$$\phi(p_1, \ldots, p_n) = r \circ \left( \frac{1}{n} \sum_{i=1}^{n} p_i \right).$$

By the previous paragraphs, we know that averaging is a continuous social choice rule that respects anonymity and unanimity. Therefore the $\phi$ defined in Equation 6 is also a continuous, anonymous and unanimous social choice rule, completing the proof of sufficiency.

As the question of sufficiency has been settled we now turn to proving necessity. We must show that the existence of a social choice rule implies that $\pi_n(P) = 0$ for all $n \geq 1$, which implies that $P$ is contractible. To do this we will rely on some additional algebraic structure. Specifically, we will use free and torsion subgroups.
Definition 5.1. Let $G$ be an abelian group. The elements in $G$ of finite order forms a subgroup, known as the torsion subgroup.

Definition 5.2. Let $G$ be an abelian group. We say that $G$ is torsion free or simply free, if there is no nontrivial torsion subgroup.

Proposition 5.3. Each group can be divided into free and torsion groups.

Proof. This follows from the definitions. □

Proof of necessity. To prove necessity we will show the following. If the preference space $P$ admits a continuous social choice rule, satisfying anonymity and unanimity, then $P$ must be contractible.

By assumption there is a continuous map $\phi : P^k \rightarrow P$, which satisfies unanimity and anonymity. This map has a corresponding induced map $\phi^* : \pi_i(\prod P) \rightarrow \pi_i(P)$ for all $i \geq 1$. Because of the Huruwicz Isomorphism Theorem, Theorem 4.6, we know that the first nontrivial group $\pi_1(P)$ is isomorphic to the abelian group $H_1(P)$, for $i \geq 2$, with the case $i = 1$ left for later. The idea now is to show that, for each $i \geq 2$, $\pi_i(P)$ is the trivial group. Then, for all $q \geq 2$, we can split $\pi_q(P)$ into a torsion and a torsion free part. If both of these are trivial then so is $\pi_q(P)$, by Proposition 5.3.

To show that the free part of $\pi_i(P)$ is trivial we will rely on three observations. First, by our construction, the group $\pi_i(\prod P)$ is isomorphic to the product of $k$ copies of $\pi_i(P)$. Second, for any $x \in \pi_i(P)$ we have, for $(x, x, \ldots, x) \in \pi_i(\prod P)$,

$$\phi^*(x, x, \ldots, x) = x,$$

by unanimity. Finally, since both $\pi_i(P)$ and $\pi_i(\prod P)$ are groups, then they each have binary operators and identity elements. Specifically let $\bullet$ be the binary operation on $\pi_i(\prod P)$ and let $e$ be the identity element of $\pi_i(P)$, implying that $(e, \ldots, e)$ is the identity element of $\pi_q(\prod P)$. By the group structure on $\pi_i(\prod P)$ we have

$$(x, x, \ldots, x) = (x, e, \ldots, e) \bullet (e, x, \ldots, e) \bullet \ldots \bullet (e, e, \ldots, e, x).$$

Now, let $*$ be the binary operation on $\pi_i(P)$, then since $\phi^*$ is a homomorphism we have the following equation,

$$\phi^*(x, x, \ldots, x) = \phi^* \{ (x, e, \ldots, e) \bullet (e, x, e, \ldots, e) \bullet \ldots \bullet (e, e, \ldots, e, x) \}$$

$$= \phi^*(x, e, \ldots, e) \ast \phi^*(e, x, e, \ldots, e) \ast \ldots \ast \phi^*(e, e, \ldots, e).$$

As we require the social choice rule to be anonymous, then all permutations of $(x, e, \ldots, e)$ must be equivalent. Consequentially, the right hand side of Equation 7 can be rewritten as $n(x, e, \ldots, e)$, since $\pi_i(P)$ is abelian. At the same time, unanimity assures that the left hand side of Equation 7 must be equal to $x$. Therefore, we have

$$x = n\phi^*(x, e, \ldots, e) = ny,$$

for some $y$ and $n$.

Assume now that $\pi_i(P)$ has a free part. Then there must be a generator $x \in \pi_i(P)$ of that free part. However, by anonymity and unanimity $x$ must be equivalent to $n \ast y$ for some $n$ and $y$. Since $\phi$ is unanimous, then $x$ would have to be divisible by every integer, which is a contradiction if $x \neq e$. As the only possible generator is $e$ then the free part of $\pi_q(P)$ must be trivial, for all $q \geq 2$.

Given that the torsion free part of $\pi_q(P)$ is trivial, then the torsion part must be nontrivial, unless $\pi_q(P) = 0$. Now consider $x$ a generator in the torsion part of
\( \pi_i(P) \). Then \( x \in \mathbb{Z}_k \), for some \( k \). However, since \( x = n\phi^*(x,e,\ldots,e) \) for any \( k \in \mathbb{Z} \), it follows that \( \mathbb{Z}_{p_i} \) is zero for all \( p_i \), which implies that the torsion part of \( \pi_i(P) \) is trivial. Since \( \pi_i(P) \) has no torsion and no free part, it is the zero group, for all \( i \geq 2 \).

In case \( i = 1 \), and \( \pi_i(P) \) is abelian, then the group \( \pi_1(P) \) is isomorphic to \( H_1(P) \). Thus, if we can show that \( \pi_1(P) \) abelian, the rest of the proof above applies when \( i = 1 \). Therefore, we show next that the existence of the map \( \phi \) implies immediately that \( \pi_1(P) \) is abelian. Because \( \phi^* \) is a group homomorphism, we have for any two elements, \( x \) and \( y \) in \( \pi_1(P) \), and for any \( k \geq 1 \)

\[
x * y = k\phi^*(x,e,\ldots,e) * k\phi^*(e,y,e,\ldots,e) = \phi^*(kx,ky,e,\ldots,e),
\]

But by the symmetry condition on \( \phi \) we have

\[
\phi^*(kx,ky,e,\ldots,e) = \phi^*(ky,kx,e,\ldots,e) = y * x.
\]

Therefore, \( x * y = y * x \), and \( \pi_1(P) \) is abelian. This completes the proof that \( \pi_1(P) = 0 \) for all \( i \geq 1 \).

Since \( P \) is a connected CW complex and \( \pi_i(P) = 0 \) for all \( i \geq 1 \), then \( P \) must be contractible.

Economically this result is handy. If we, for instance, relate back to Section 2 where we defined the preference space \( P \), then we can see the following. By construction, \( P \) is the sphere \( S^1 \) but \( S^1 \) is not contractible. Specifically, \( \pi_1(S^1) = \mathbb{Z} \). Therefore, we can see that there is no well behaved social choice rule on \( P \), given a quite standard way of constructing the economical framework.

Furthermore, each social choice rule respecting the conditions of Theorem 2.16 must be homotopically equivalent to Equation 6. This is a direct consequence of Theorem 2.16 since it states that a space of preference profiles must be contractible if there is a social choice function on it. For a general overview of the continuous social choice the reader is recommended to take a look at [15]. We will instead look at more mathematical aspects of continuous social choice.

6. Intriguing Equivalences

Obstruction theory excluded, the proof of Theorem 2.16 is not that interesting, mathematically, and quite simple. It would be easy to take this simplicity as an indication of triviality. However, there is a very interesting connection between continuous social choice rules and Brouwer’s Fixed Point Theorem, which is stated in Theorem 6.1. Curiously, there is also a connection between Brouwer’s Fixed Point Theorem and the original social choice theorem by Arrow. We will restate two theorems by Tanaka [26] as Theorem 6.3 and Theorem 6.5, which shows this.

First of all a fixed point, \( x_0 \in X \) is such that for a function \( f : X \to X \) we have

\[
f(x_0) = x_0.
\]

We now state the Brouwer Fixed point Theorem for reference.

**Theorem 6.1** (Brouwer’s Fixed Point Theorem). Every continuous map \( f : E^n \to E^n \) has at least one point \( x_0 \in E^n \) such that \( f(x_0) = x_0 \).

**Proof.** For a proof see Corollary 2.15 of [16]; Theorem 4.7.5 of [24] or Theorem 6.6.1 of [14]. The last contains a number of equivalent results.
As recently said, the existence of a Chichilnsky social choice rule on a space connected to Brouwer’s Fixed point theorem on that space. Strangely, the two theorems actually imply each other. That is: the existence of a continuous, anonymous and unanimous social rule is equivalent to the existence of a fixed point. We restate the proof of [9], which is for two individuals. Do note that the proof is slightly edited for readability within the context of this essay.

**Theorem 6.2.** A social choice rule \( \phi : P \times P \rightarrow P \) which is continuous, anonymous and respects unanimity exists if, and only if, there exists a continuous map \( f \) from the closed unit disk, \( D \), of \( \mathbb{R}^2 \) into itself without fixed points.

**Proof.** First we shall prove that existence of a map \( \phi : P \times P \rightarrow P \) is equivalent to the existence of an extension of continuous map \( g : \partial D \rightarrow \partial D \) to a map \( f : D \rightarrow \partial D \). Then, we note that such an extension exists if and only if \( g \) is homotopic to some constant map on \( \partial D \), and that this latter result, by 6.6.1 of Dieck [14], is equivalent to Brouwer’s Fixed Point Theorem on the disk. We shall therefore have proven that the non-existence of a continuous anonymous rule \( \phi : P \times P \rightarrow P \) that respects unanimity is equivalent to the nonexistence of maps from the disk into itself without fixed points.

To begin the proof of the first part let \( \phi : P \times P \rightarrow P \) be a social choice rule. As argued in Section 2.2, \( P = S^1 \). This implies that \( P \times P = S^1 \times S^1 \), which is the torus. Let \( p_0 \) be the distinguished point of this torus. Furthermore, let \( A(p) = (p_0, p) \), \( B(p) = (p, p_0) \), and let \( \Delta \) be the diagonal map, these are depicted in the left picture of Figure 6.

Note that we can always define on \( \operatorname{Im} \Delta \cup \operatorname{Im} A \cup \operatorname{Im} B \), or \( \Delta \cup A \cup B \) for short, a social choice rule, \( \phi \). Specifically, if this social choice rule is defined by

\[
\phi / \Delta = \text{id}_\Delta, \quad \phi(p_0, p) = p_0 \quad \text{and} \quad \phi(p, p_0) = p_0, \quad \forall p \in S^1,
\]

then \( \phi \) satisfies all the conditions for a well behaved social choice rule. Therefore, on the set \( \Delta \cup A \cup B \) an adequate rule always exists. If we could extend \( \phi \) from \( \Delta \cup A \cup B \) to the interior, which is the triangle \( T \) of Figure 6. Then, we would have constructed a social choice rule \( \phi : S^1 \times S^1 \rightarrow S^1 \) satisfying the desired conditions, because \( S^1 \times S^1 \) is the union of \( T \) and the symmetric set of \( T \), i.e., \( S^1 \times S^1 = T \cup T^s \) where \( T^s = (y, x) : (y, x) \in T \). Therefore, if \( \phi \) can be continuously defined on \( T \) as an extension of the map on \( \partial T \) then since \( S^1 \times S^1 = T \cup T^s \), \( \phi : S^1 \times S^1 \rightarrow S^1 \) could be defined satisfying all conditions. Note that \( T \) and \( D \) are homeomorphic, i.e., there exists a continuous one to one map from \( T \) onto \( D \). Therefore, the social choice problem is equivalent to that of extending a map \( g : \partial D \rightarrow \partial D \) to a map \( f : D \rightarrow \partial D \). We shall now move on to the second part of the argument.

A continuous map \( g : \partial D \rightarrow \partial D \) can be extended to another continuous map \( f : D \rightarrow \partial D \) if and only if the map \( g \) can be deformed to a constant function mapping all of \( \partial D \) into a given \( x_0 \) in \( \partial D \). Furthermore, this result is equivalent to Brouwer’s Fixed Point Theorem. For a proof of this equivalence see Theorem 6.6.1 of Dieck [14].

Finally, we now show that if a continuous map \( h \) is defined on the boundary of \( T \), \( h : \partial T \rightarrow S^1 \) respecting unanimity and anonymity, then \( h \) cannot be deformed into a constant map from \( \partial T \) into a fixed point \( x_0 \) in \( S^1 \). Unanimity requires that the restriction of \( h \) to \( \Delta \), \( h/\Delta \), covers \( S^1 \) exactly once; in fact since \( \Delta \) is homeomorphic to \( S^1 \), \( h/\Delta \) can be thought of as the identity map from \( S^1 \) to \( S^1 \). Since \( h/\Delta \) wraps around \( S^1 \) in the opposite direction once, in order that \( h \) could be deformed
Figure 6. The left picture depicts the standard torus, which is recreated by joining the A and B edges together with their opposite side. The point $P$ is the distinguished point $(p_0, p_0)$. The left picture shows the same construction with all, equivalent, corners shown as one point.

to a constant map, $h/A \cup B$ must wrap around $S^1$ in the opposite direction once. However, by anonymity $h/A = h/B$. This implies that if $h/A$ wraps around $S^1$ once ($n$-times) then $h/A \cup B$ does so twice ($2n$ times). If $h/A$ does not wrap around $S^1$ at least once, neither will $h/A \cup B$, so that in all cases $h$ cannot be deformed into a constant map from $\partial T$ into $S^1$ when it is both anonymous and respects unanimity. Therefore such an $h$ cannot be extended to all of $T$.

Since we have already shown that the social choice paradox is equivalent to the impossibility of extending a map from $\partial D$ to $D$, and that the problem of extendability of maps from $\partial D$ to $D$ is equivalent to a fixed point theorem, it therefore follows that the social choice problem is equivalent to the problem of existence of fixed points of continuous maps from the disk to itself. This completes the proof. \qed

This shows that the social choice paradox as stated by Chichilnisky is equivalent to Brouwer’s Fixed Point Theorem in two dimensions. Supposedly, one can further extend this result to higher dimensions, as stated in Chichilnisky [9]. However, because of what we will look at next, two dimensions suffices.

While it might not be an surprise that Chichilnisky’s formulation of the social choice rule is equivalent to Brouwer’s Fixed Point Theorem. The following theorems prove a far more surprising equivalence: one between Brouwer’s Fixed Point Theorem and Arrow’s Theorem. Since Arrow’s model is discrete this is perhaps a more interesting connection.

This essay will use proofs from Tanaka [26], which establishes an equivalence between the theorems for $n = 2$. With this we will have proven that both the model by Chichilnisky and Heal, and by Arrow are equivalent to Brouwer’s theorem for $n = 2$. Thereby, we will also have proven that Arrow’s and Chichilnisky’s formulations are the same in the two dimensional case. There are reasons to believe that $n = 2$ is not a special case. Instead, the continuous and discrete social choice problems are probably equivalent for all $n$, see Baryshnikov [5], but this is beyond the scope of the essay.

The following theorems, Theorem 6.3 and Theorem 6.5, with Lemma 6.4 in between constitute the proof by Tanaka. The first theorem establishes an intermediate equivalence with Brouwer’s Fixed Point Theorem. It is then shown that Arrow’s theorem is equivalent to this intermediate theorem. From this the equivalence of Arrow and Brouwer follows. We state this in Theorem 6.5.

**Theorem 6.3.** The following two statements are equivalent.
(1) If there exists a continuous function from an \(n\)-dimensional ball \(D^n\) to an \((n-1)\)-dimensional sphere \(S^{n-1}\), \((n = 2)\), \(F : D^n \rightarrow S^{n-1}\), then the following function, which is obtained by restricting \(F\) to the boundary \(S^{n-1}\) of \(D^n\), \(F|S^{n-1} \times S^{n-1} \rightarrow S^{n-1}\) is homotopic to a constant mapping. Since the degree of mapping of a constant mapping is zero, the degree of mapping of \(F|S^{n-1}\) is zero.

(2) (The Brouwer fixed point theorem) Any continuous function from \(D^n\) to \(D^n\) \((n = 2)\), \(G : D^n \rightarrow D^n\), has a fixed point.

Proof. (1) \(\rightarrow\) (2) Assume that \(G\) has no fixed point. Since we always have \(v \neq G(v)\) at any point \(v\) in \(D^n\), there is a half line starting \(G(v)\) across \(v\). Let \(F(v)\) be the intersection point of this half line and the boundary of \(D^n\), which is \(S^{n-1}\). Then, we obtain the following continuous function from \(D^n\) to \(S^{n-1}\)

\[ F : D^n \rightarrow S^{n-1}. \]

In particular, we have \(F(v) = v\) for \(v = S^{n-1}\). Therefore, \(F|S^{n-1}\) is an identity mapping. But, because an identity mapping on \(S^{n-1}\) is not homotopic to any constant mapping, it is a contradiction.

(2) \(\rightarrow\) (1) We show that if there exists a continuous function \(F\) from \(D^n\) to \(S^{n-1}\), (1) of this theorem is correct whether a continuous function \(G\) from \(D^n\) to \(D^n\) has a fixed point or not. Define \(f_t(v) = F[(1-t)v], \quad (0 \leq t \leq 1)\) for any point \(v\) of \(S^{n-1}\). Then, we get a continuous function \(f_t : S^{n-1} \rightarrow S^{n-1}\). \((1-t)v\) is a point which divide \(t : 1-t\) a line segment between \(v\) and the center of \(D^n\), and it is transferred by \(F\) to a point on \(S^{n-1}\). We have \(f_0 = F|S^{n-1}\), and \(f_1 = F(0)\) is a constant mapping whose image is a point \(F(0)\). Since \(f_t\) is continuous with respect to \(t\), it is a homotopy from \(F|S^{n-1}\) to a constant mapping, and the degree of mapping of \(F|S^{n-1}\) is zero.

Lemma 6.4. Suppose that there exists a social welfare function \(F : R \times R \rightarrow S^1\) which satisfies transitivity, Pareto principle and independence of irrelevant alternatives. If \(F\) has no dictator, then the degree of mapping of \(F|_{\Delta \cup A \cup B}\) is not zero, and hence it is not homotopic to a constant mapping.

Proof. The proof is quite technical and the reader is referred to Lemma 1 of [26].

Thus, Tanaka has shown that the conditions by Arrow creates a situation satisfying the conditions of Theorem 6.3. And therefore the following result holds.

Theorem 6.5. The non-existence of social welfare function which satisfies transitivity, Pareto principle, independence of irrelevant alternatives and has no dictator (the Arrow imposibility theorem) is equivalent to the Brouwer fixed point theorem.
7. References


The author can be reached at anton.vernersson@outlook.com