Quantum kinetic relativistic theory of linearized waves in magnetized plasmas

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Abstract

In this work we have studied linear wave propagation in magnetized plasmas using a fully relativistic kinetic equation of spin-1/2 particles in the long scale approximation. The linearized kinetic equation is very long and complicated, hence we worked with restricted geometries in order to simplify the calculations. The dispersion relation of the relativistic model was calculated and compared with a dispersion relation from a previous work at the semi-relativistic limit.

Moreover, a new mode was discovered that survives in the zero temperature limit. The origin of the mode in the kinetic equation was discussed and derived from a non-relativistic kinetic equation from a previous work.

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1 Introduction

Like solids, liquids and gases, plasma is a state of matter consisting of ions, electrons and neutral atoms. Plasmas can be found in stars, fusion reactors and lightning [1]. At high temperatures and low densities of plasmas, plasmas can be described using classical mechanics. However for high densities and/or low temperatures of the plasmas, the quantum mechanical effects become more important [2].

The interest in quantum plasmas has increased lately due to the the different applications it offers, like quantum wells and astrophysics [3]. Recently, a fully relativistic kinetic theory describing spin-1/2 particles has been developed [4], which will be studied in this work. This model is based on the Dirac equation and hence it takes into account the magnetic dipole moment of the spin-1/2 particles. The model uses the Foldy-Wouthuysen transformation [5] that separates particles and anti-particles using an expansion in $\hbar$. Moreover the model takes into account all orders of v/c, i.e. it is fully relativistic, compared to the semi-relativistic model [6]. However the model is only valid in the long-scale limit and hence it dose not include any particle-dispersive effects.

The aim of this work is to analyze the linear wave propagation in magnetized plasmas using the fully relativistic model mentioned above. Since this model was recently published, it might offer some new interesting results that have not been discovered yet. However the fully relativistic kinetic equation is very long, thus we consider in this work two restricted geometries in order to simplify things. Firstly, we consider a longitudinal wave propagation parallel to an external static homogeneous magnetic field. In the second case, we study transverse waves propagating parallel to the magnetic field. In both cases, we calculate the dispersion relation. In the first case we compare the obtained dispersion relation with a dispersion relation from a previous work [6] in the semi-relativistic limit. In the second case, a new mode was discovered which survives in the zero temperature limit. However the new mode comes from a non-relativistic term in the fully relativistic kinetic equation. We derive then the new mode from a non-relativistic dispersion relation in a previous work [3].

2 Vlasov equation

The distribution function of a plasma f_s in the classical regime for species s is described by the Vlasov equation [2] (collisions are neglected)

$$\frac{\partial f_s}{\partial t} + \mathbf{v} \cdot \nabla_x f_s + \frac{q_s}{m_s}\Big[\mathbf{E}(\mathbf{x},t) + \mathbf{v} \times \mathbf{B}(\mathbf{x},t)\Big] \cdot \nabla_v f_s = 0, \tag{1}$$

where q_s and m_s are the charge and mass of the species s respectively. Another way of deriving the Vlasov equation is using the fact that the total derivative of f_s along the path of one of the particles is zero (collisions are neglected), this is related to the particle conservation [2]. Using the chain rule, the Vlasov equation is given by

$$\frac{d}{dt}f_s(\mathbf{x},\mathbf{v},t) = \frac{\partial f_s}{\partial t} + \frac{\partial f_s}{\partial \mathbf{x}}\cdot\frac{d\mathbf{x}}{dt} + \frac{\partial f_s}{\partial \mathbf{v}}\cdot\frac{d\mathbf{v}}{dt} = 0. \tag{2}$$

Since $d\mathbf{x}/dt = \mathbf{v}$ and $d\mathbf{v}/dt = (q_s/m_s)\Big[\mathbf{E}(\mathbf{x},t) + \mathbf{v}\times\mathbf{B}(\mathbf{x},t)\Big]$ is the Lorentz force divided by the mass, hence we have the Vlasov equation.

The classical relativistic Vlasov equation is obtained by using $f_s = f_s(\mathbf{x},\mathbf{p},t)$, where $\mathbf{p} = m\gamma\mathbf{v}$ is the relativistic momentum, here $\gamma = 1/\sqrt{1-v^2/c^2}$. Equation (2) in this regime gives the classical relativistic Vlasov [7]

$$\frac{df_s}{dt} = \frac{\partial f_s}{\partial t} + \mathbf{v}\cdot\nabla_x f_s + q_s(\mathbf{E}+\mathbf{v}\times\mathbf{B})\cdot\nabla_p f_s = 0. \tag{3}$$

Including the quantum mechanical effects, the distribution function now depends on the spin $f = f(\mathbf{x},\mathbf{v},\mathbf{s},t)$. We drop the subscript s in f later on in the rapport to simplify the expressions. In the long scale-limit, the non-relativistic quantum mechanical evolution equation for the distribution function $f(\mathbf{r},\mathbf{v},\mathbf{s},t)$ is [8]

$$\frac{\partial f}{\partial t}+\mathbf{v}\cdot\nabla_x f+\left[\frac{q}{m}\Big(\mathbf{E}+\mathbf{v}\times\mathbf{B}\Big)+\frac{\mu_B}{m}\nabla_x\Big(\mathbf{s}\cdot\mathbf{B}+\mathbf{B}\cdot\nabla_s\Big)\right]\cdot\nabla_v f+\frac{2\mu_B}{\hbar}\Big(\mathbf{s}\times\mathbf{B}\Big)\cdot\nabla_s f = 0, \tag{4}$$

where $\mu_B = q\hbar/2m$ is the magnetic moment. Note that we use units where $c = 1$ in this work. The new terms that arise in the kinetic equation when the quantum effects are considered are

$$\frac{\mu_B}{m}\nabla_x\Big(\mathbf{s}\cdot\mathbf{B} + \mathbf{B}\cdot\nabla_s\Big)$$

$$\frac{2\mu_B}{\hbar}\Big(\mathbf{s}\times\mathbf{B}\Big),$$

the first term is the magnetic dipole force while the second one is the spin precession. Going one step further by considering the quantum relativistic kinetic equation, we consider in this work the fully relativistic evolution equation for the distribution function $f(\mathbf{x},\mathbf{p},\mathbf{s},t)$ in the long scale-length limit [4]

$$\begin{aligned}
&\frac{\partial f}{\partial t} + \left\{\frac{\mathbf{p}}{\epsilon} - \mu_B m\nabla_p\left[\frac{1}{\epsilon}\left(\mathbf{B} - \frac{\mathbf{p}\ \times\mathbf{E}}{\epsilon+m}\right)\cdot(\mathbf{s}+\nabla_s)\right]\right\}\cdot\nabla_x f \\
&\qquad + q\left(\mathbf{E} + \left\{\frac{\mathbf{p}}{\epsilon} - \mu_B m\nabla_p\left[\frac{1}{\epsilon}\Big(\mathbf{B} - \frac{\mathbf{p}\ \times\mathbf{E}}{\epsilon+m}\Big)\cdot(\mathbf{s}+\nabla_s)\right]\right\}\times\mathbf{B}\right)\cdot\nabla_p f \\
&+\frac{\mu_B m}{\epsilon}\nabla_x\left[\left(\mathbf{B} - \frac{\mathbf{p}\times\mathbf{E})}{\epsilon+m}\right)\cdot(\mathbf{s}+\nabla_s)\right]\cdot\nabla_p f + \frac{2\mu_B m}{\hbar\epsilon}\left[\mathbf{s}\times\left(\mathbf{B} - \frac{\mathbf{p}\times\mathbf{E}}{\epsilon+m}\right)\right]\cdot\nabla_s f,
\end{aligned} \tag{5}$$

where $\epsilon = \sqrt{m^2 + p^2}$. Equation (5) is a Vlasov-like equation where the velocity is non-trivially related to the momentum variable, the whole curly bracket in the second term of the equation above is the velocity. The evolution equation above gives us information about how the particles move in the fields. However, in order to have a closed description we need information about how the motion of the particles are connected to the fields, this is obtained by using Maxwell's equations

$$\nabla \cdot \mathbf{E} = \rho_f - \nabla \cdot \mathbf{P} \tag{6}$$
$$\nabla \cdot \mathbf{B} = 0 \tag{7}$$
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{8}$$
$$\nabla \times \mathbf{B} = \mathbf{J}_f + \frac{\partial \mathbf{P}}{\partial t} + \nabla \times \mathbf{M}, \tag{9}$$

where $\mathbf{P}$ and $\mathbf{M}$ are the polarization and magnetization and $\mathbf{J}_f$ and ρ_f are the free current and charge densities. For the total charge density ρ, we have the following expression

$$\rho = q \int d\Omega f - \nabla \cdot \mathbf{P}, \tag{10}$$

where the polarization $\mathbf{P}$ is

$$\mathbf{P} = -3\mu_B \int d\Omega \, \frac{m\mathbf{s} \times \mathbf{p}}{\epsilon(\epsilon + m)} f. \tag{11}$$

The total current density is $\mathbf{J} = \mathbf{J}_f + \mathbf{J}_p + \mathbf{J}_M$, where $\mathbf{J}_f$ is the free current density and $\mathbf{J}_p$ and $\mathbf{J}_M$ are the polarization and magnetization current densities

$$\mathbf{J}_f = 3q \int d\Omega \left\{ \frac{\mathbf{p}}{\epsilon} - \mu_B m \nabla_p \left[\frac{1}{\epsilon} \left(\mathbf{B} - \frac{\mathbf{p} \times \mathbf{E}}{\epsilon + m} \right) \cdot \mathbf{s} \right] \right\} f \tag{12}$$
$$\mathbf{J}_p = \frac{\partial \mathbf{P}}{\partial t} \tag{13}$$
$$\mathbf{J}_M = \nabla \times \mathbf{M} = 3\mu_B \nabla \times \int d\Omega \frac{m}{\epsilon} \mathbf{s} f. \tag{14}$$

3 Linearized theory

In this section we will analyze equation (5) by calculating the dispersion relation in the linearized theory to see whether this quantum relativistic model predicts any new effects. In the linearizing of equation (5), we separate the variables into equilibrium quantities (using 0 as subindex) and perturbed quantities (using 1 as subindex)

$$f = f_0 + f_1 \tag{15}$$
$$\mathbf{E} = \mathbf{E}_1 \tag{16}$$
$$\mathbf{B} = B_0 \mathbf{e}_z + \mathbf{B}_1, \tag{17}$$

where B_0 is a constant. Note that we assumed that there is no background $\mathbf{E}$-field, i.e. $\mathbf{E}_0 = \mathbf{0}$. Moreover we assume that the equilibrium distribution function is $f_0 = f_0(\mathbf{p}^2, \theta_s)$, where θ_s is the spin angle. The evolution equation for the distribution function up to first order in linearized theory (we remove products of the perturbed quantities) becomes

$$
\begin{aligned}
&\frac{\partial f_1}{\partial t} + \frac{\mathbf{p}}{\epsilon}\cdot\nabla_x f_1 - \mu_B m\Big[\mathbf{B}_0\cdot(\mathbf{s}+\nabla_s)\Big]\nabla_p\Big(\frac{1}{\epsilon}\Big)\cdot\nabla_x f_1 + \frac{q}{\epsilon}(\mathbf{p}\times\mathbf{B}_0)\cdot\nabla_p f_1 \\
&\quad - q\mu_B m\Big[\mathbf{B}_0\cdot(\mathbf{s}+\nabla_s)\Big]\Big(\nabla_p\frac{1}{\epsilon}\times\mathbf{B}_0\Big)\cdot\nabla_p f_1 + \frac{2\mu_B m}{\hbar\epsilon}(\mathbf{s}\times\mathbf{B}_0)\cdot\nabla_s f_1 = \\
&\qquad = -q\mathbf{E}\cdot\nabla_p f_0 - \frac{q}{\epsilon}\Big[\mathbf{p}\times(\mathbf{B}_0+\mathbf{B}_1)\Big]\cdot\nabla_p f_0 \\
&\quad + q\mu_B m\left[\Big(\mathbf{B}_0+\mathbf{B}_1-\frac{\mathbf{p}\times\mathbf{E}}{\epsilon+m}\Big)\cdot(\mathbf{s}+\nabla_s)\right]\Big(\nabla_p\frac{1}{\epsilon}\times\mathbf{B}_0\Big)\cdot\nabla_p f_0 \\
&+ q\mu_B m\Big[\mathbf{B}_0\cdot(\mathbf{s}+\nabla_s)\Big]\Big(\nabla_p\frac{1}{\epsilon}\times\mathbf{B}_1\Big)\cdot\nabla_p f_0 - \frac{q\mu_B m}{\epsilon}\left\{\nabla_p\left[\frac{\mathbf{p}\times\mathbf{E}}{\epsilon+m}\cdot(\mathbf{s}+\nabla_S)\right]\times\mathbf{B}_0\right\}\cdot\nabla_p f_0 \\
&\quad - \frac{\mu_B m}{\epsilon}\nabla_x\Big[\mathbf{B}_1\cdot(\mathbf{s}+\nabla_s)\Big]\cdot\nabla_p f_0 + \frac{\mu_B m}{\epsilon(\epsilon+m)}\nabla_x\Big[(\mathbf{p}\times\mathbf{E})\cdot(\mathbf{s}+\nabla_s)\Big]\cdot\nabla_p f_0 \\
&\qquad - \frac{2\mu_B m}{\hbar\epsilon}\left[\mathbf{s}\times\Big(\mathbf{B}_0+\mathbf{B}_1-\frac{\mathbf{p}\times\mathbf{E}}{\epsilon+m}\Big)\right]\cdot\nabla_s f_0.
\end{aligned}
\tag{18}
$$

Due to the complexity of this equation we will consider restricted geometries in this work. We consider two geometries, in the first case we have longitudinal wave propagation parallel to the magnetic field $\mathbf{B}_0$, while in the other case we consider transverse wave propagation, but with $\mathbf{B}_0$ parallel to $\mathbf{E}$.

3.1 Longitudinal waves

We define the wave vector $\mathbf{k} = k_\perp \mathbf{e}_x + k_z \mathbf{e}_z$ without loss of generality. In this geometry we consider $\mathbf{k} = k_z \mathbf{e}_z$, $\mathbf{B}_1 = 0$ and $\mathbf{E}_1 = E_1 \exp(ikz - i\omega t)\mathbf{e}_z$, where ω is the frequency of the E-field. The fully relativistic evolution equation, equation (5), becomes in this geometry

$$
\begin{aligned}
&\frac{\partial f_1}{\partial t} + \frac{\mathbf{p}}{m} \cdot \nabla_x f_1 - \mu_B m \Big[\mathbf{B}_0 \cdot (\mathbf{s} + \nabla_s)\Big] \nabla_p \Big(\frac{1}{\epsilon}\Big) \cdot \nabla_x f_1 + \frac{q}{m}(\mathbf{p} \times \mathbf{B}_0) \cdot \nabla_p f_1 \\
&\quad - q\mu_B m \Big[\mathbf{B}_0 \cdot (\mathbf{s} + \nabla_s)\Big] \Big(\nabla_p \frac{1}{\epsilon} \times \mathbf{B}_0\Big) \cdot \nabla_p f_1 + \frac{2\mu_B}{\hbar}(\mathbf{s} \times \mathbf{B}_0) \cdot \nabla_s f_1 = \\
&= -q\mathbf{E} \cdot \nabla_p f_0 - \frac{q}{m}\Big(\mathbf{p} \times \mathbf{B}_0\Big) \cdot \nabla_p f_0 + q\mu_B m \left[\left(\mathbf{B}_0 - \frac{\mathbf{p} \times \mathbf{E}}{2m}\right) \cdot (\mathbf{s} + \nabla_s)\right] \times \\
&\quad \Big(\nabla_p \frac{1}{\epsilon} \times \mathbf{B}_0\Big) \cdot \nabla_p f_0 - q\mu_B \left\{\nabla_p \left[\frac{\mathbf{p} \times \mathbf{E}}{2m} \cdot (\mathbf{s} + \nabla_S)\right] \times \mathbf{B}_0\right\} \cdot \nabla_p f_0 \\
&+ \frac{\mu_B}{2m} \nabla_x \Big[(\mathbf{p} \times \mathbf{E}) \cdot (\mathbf{s} + \nabla_s)\Big] \cdot \nabla_p f_0 - \frac{2\mu_B}{\hbar}\left[\mathbf{s} \times \left(\mathbf{B}_0 - \frac{\mathbf{p} \times \mathbf{E}}{2m}\right)\right] \cdot \nabla_s f_0.
\end{aligned}
\tag{19}
$$

Note that we assumed that the velocities of the particles are relatively low that we can approximate γ to 1 in $\epsilon = m\gamma$ in equation (5), for $\nabla_p(\frac{1}{\epsilon})$ we evaluated the derivative first then took the limit $\epsilon \to m$. We use also a plane-wave ansatz of the perturbed quantities $f_1(\mathbf{x}, \mathbf{p}, \mathbf{s}, t) = \widetilde{f}_1(\mathbf{p}, \mathbf{s}) \exp[i(k_z z - wt)]$. For the momentum $\mathbf{p}$, we express it in cylindrical coordinates: $\mathbf{p} = p_\perp \cos\varphi_p \mathbf{e}_x + p_\perp \sin\varphi_p \mathbf{e}_y + p_z \mathbf{e}_z$ while the spin $\mathbf{s}$ is expressed in spherical coordinates: $\mathbf{s} = \sin\theta_s \cos\varphi_s \mathbf{e}_x + \sin\theta_s \sin\varphi_s \mathbf{e}_y + \cos\theta_s \mathbf{e}_z$. We expand $\widetilde{f}_1(p, s)$ in eigenfunctions of the right-hand side operators in the same way as in reference [3]

$$
\widetilde{f}_1(\mathbf{p}, \mathbf{s}) = \sum_{n=-\infty}^{\infty} \sum_{n'=-\infty}^{\infty} g_{nn'}(p_\perp, p_z, \theta_s) \psi_n(\varphi_p, p_\perp) \frac{1}{\sqrt{2\pi}} \exp(in'\varphi_s), \tag{20}
$$

where

$$
\begin{aligned}
\psi_n(\varphi_p, p_\perp) &= \frac{1}{\sqrt{2\pi}} \exp\left[i\Big(n\varphi_p - \frac{k_\perp p_\perp}{m\omega_{ce}} \sin\varphi_p\Big)\right] \\
&= \frac{1}{\sqrt{2\pi}} \sum_{l=-\infty}^{\infty} J_l\Big(\frac{k_\perp p_\perp}{m\omega_{ce}}\Big) e^{i(n-l)\varphi_p},
\end{aligned}
\tag{21}
$$

where J_l is the Bessel function of first kind and $\omega_{ce} = qB_0/m$ is the electron cyclotron frequency. Since we are considering only longitudinal waves ($k_\perp = 0$), then

$$
\psi_n(\varphi_p, p_\perp) = \frac{1}{\sqrt{2\pi}} \exp(in\varphi_p) \tag{22}
$$

We use the following relations to simplify the calculations

$$\frac{q}{m}(\mathbf{p}\times\mathbf{B}_0)\cdot\nabla_p f_1 = -\omega_{ce}\frac{\partial f_1}{\partial\varphi_p} \tag{23a}$$

$$\frac{2\mu_B m}{m\hbar}(\mathbf{s}\times\mathbf{B}_0)\cdot\nabla_s f_1 = -\omega_{cg}\frac{\partial f_1}{\partial\varphi_s}, \tag{23b}$$

where $\omega_{cg} = 2\mu_B B_0/\hbar$ is the spin precession frequency. With these relations and the plane-wave ansatz, we can express the left hand side of equation (19) as

$$\Bigg\{ - i\omega + \frac{ip_z k_z}{m}\left[1 + \frac{\mu_B B_0}{m}(\cos\theta_s - \sin\theta_s\frac{\partial}{\partial\theta_s})\right] \\ - \omega_{ce}\left[1 + \frac{\mu_B B_0}{m}\Big(\cos\theta_s - \sin\theta_s\frac{\partial}{\partial\theta_s}\Big)\right]\frac{\partial}{\partial\varphi_p} - \omega_{cg}\frac{\partial}{\partial\varphi_s}\Bigg\} f_1 = RHS, \tag{24}$$

where RHS is the right hand side of equation (19). Equation (24) is hard to solve analytically. However, if we rearrange the terms in equation (24) as

$$\left[-i\omega + \frac{ip_z k_z}{m} - \omega_{ce}\frac{\partial}{\partial\varphi_p} - \omega_{cg}\frac{\partial}{\partial\varphi_s}\right] f_1 \\ = RHS - \frac{\mu_B B_0}{m}\Big(\cos\theta_s - \sin\theta_s\frac{\partial}{\partial\theta_s}\Big)\Big(\frac{ip_z k_z}{m} - \omega_{ce}\Big) f_1. \tag{25}$$

To solve this equation using the summation of eigenfunctions in equation (20), we assume that $\mu_B B_0/m \ll 1$, hence the second term in the right-hand side of equation (25) is small relative the left-hand side of the same equation. We can then make an expansion of the perturbed distribution function f_1 up to first order using perturbation theory

$$f_1 = f_{10} + f_{11}, \tag{26}$$

where f_{10} and f_{11} are the zeroth and first order terms in the perturbation expansion respectively. The zeroth order term in the perturbation expansion is given by

$$\left[-i\omega + \frac{ip_z k_z}{m} - \omega_{ce}\frac{\partial}{\partial\varphi_p} - \omega_{cg}\frac{\partial}{\partial\varphi_s}\right] f_{10} = RHS. \tag{27}$$

While the first order term in the perturbation expansion is given by

$$\left[-i\omega + \frac{ip_z k_z}{m} - \omega_{ce}\frac{\partial}{\partial\varphi_p} - \omega_{cg}\frac{\partial}{\partial\varphi_s}\right] f_{11} \\ = -\frac{\mu_B B_0}{m}\Big(\cos\theta_s - \sin\theta_s\frac{\partial}{\partial\theta_s}\Big)\Big(\frac{ip_z k_z}{m} - \omega_{ce}\frac{\partial}{\partial\varphi_p}\Big) f_{10}. \tag{28}$$

In solving the zeroth order term in perturbation, we use the eigenfunction expansion equation (20). Equation (27) is now

$$\sum_{n=-\infty}^{\infty}\sum_{n'=-\infty}^{\infty} -i\left[\omega - \frac{ip_z k_z}{m} + \omega_{ce} n + \omega_{cg} n'\right] g_{nn'} \frac{1}{2\pi} e^{i(n\varphi_p + n'\varphi_s)} e^{i(k_z z - \omega t)}$$
$$= -qE\frac{\partial f_0}{\partial p_z} - \frac{k_z \mu_B E p_\perp}{4m}\left(\sin\theta_s + \cos\theta_s \frac{\partial}{\partial \theta_S}\right)\left(e^{i(\varphi_s-\varphi_p)} - e^{-i(\varphi_s-\varphi_p)}\right)\frac{\partial f_0}{\partial p_z}$$
$$+ \frac{\mu_B E p_\perp}{2\hbar m}\left(e^{i(\varphi_s-\varphi_p)} + e^{-i(\varphi_s-\varphi_p)}\right)\frac{\partial f_0}{\partial \theta_s}$$
$$- \frac{q\mu_B B_0 E}{4m}\left(e^{i(\varphi_s-\varphi_p)} + e^{-i(\varphi_s-\varphi_p)}\right)\left(\sin\theta_s + \cos\theta_s \frac{\partial}{\partial \theta_S}\right)\frac{\partial f_0}{\partial p_\perp}. \quad (29)$$

Multiplying both sides by $e^{-i(l\varphi_p + l'\varphi_s)}/2\pi$ for arbitrary values of l and l' and integrating over φ_s and φ_p, the resulting equation is

$$f_{10} = \frac{-iqE}{\omega - k_z p_z/m}\frac{\partial f_0}{\partial p_z} - \frac{ik_z \mu_B E p_\perp}{4m}\left(\frac{e^{i(\varphi_s-\varphi_p)}}{\omega + \Delta\omega_c - k_z p_z/m} - \frac{e^{i(\varphi_p-\varphi_s)}}{\omega - \Delta\omega_c - k_z p_z/m}\right)\times$$
$$\left(\sin\theta_s + \cos\theta_s \frac{\partial}{\partial \theta_S}\right)\frac{\partial f_0}{\partial p_z} + \frac{i\mu_B E p_\perp}{2\hbar m}\left(\frac{e^{i(\varphi_s-\varphi_p)}}{\omega + \Delta\omega_c - k_z p_z/m} + \frac{e^{i(\varphi_p-\varphi_s)}}{\omega - \Delta\omega_c - k_z p_z/m}\right)\frac{\partial f_0}{\partial \theta_s}$$
$$- \frac{iq\mu_B B_0 E}{4m}\left(\frac{e^{i(\varphi_s-\varphi_p)}}{\omega + \Delta\omega_c - k_z p_z/m} + \frac{e^{i(\varphi_p-\varphi_s)}}{\omega - \Delta\omega_c - k_z p_z/m}\right)\left(\sin\theta_s + \cos\theta_s \frac{\partial}{\partial \theta_S}\right)\frac{\partial f_0}{\partial p_\perp}, \quad (30)$$

where $\Delta\omega_c = \omega_{cg} - \omega_{ce}$. We have now the zeroth order of f_1 in the perturbation expansion. Next step is to use the expression for f_{10}, equation (30), to calculate f_{11} in equation (28). Using the same procedure of calculation as for f_{10}, we get the following expression for f_1

$$f_1 = \frac{1}{2\pi}\left[g_{00} + g_{1-1}e^{i(\varphi_p-\varphi_s)} + g_{-11}e^{i(\varphi_s-\varphi_p)}\right]. \quad (31)$$

where

$$g_{00} = -\frac{iqE}{\omega - k_z p_z/m}\left[1 + \frac{k_z p_z \mu_B B_0}{m^2(\omega - k_z p_z/m)}\left(\cos\theta_s - \sin\theta_s \frac{\partial}{\partial \theta_S}\right)\right]\frac{\partial f_0}{\partial p_z}, \quad (32)$$

and

$$g_{\pm1\mp1} = \frac{1}{\omega \mp \Delta\omega_c - k_z p_z/m}\Bigg\{\Big(\sin\theta_s + \cos\theta_s\frac{\partial}{\partial\theta_S}\Big)\Big(\pm\frac{ik_z\mu_B E p_\perp}{4m}\frac{\partial f_0}{\partial p_z}$$

$$-\frac{iq\mu_B B_0 E}{4m}\frac{\partial f_0}{\partial p_\perp}\Big) \pm \frac{\mu_B B_0(k_z p_z/m \mp \omega_c)}{m(\omega \mp \Delta\omega_c - k_z p_z/m)}\times$$

$$\cos\theta_s\left(\cos\theta_s\frac{\partial}{\partial\theta_S} - \sin\theta_s\frac{\partial^2}{\partial\theta_s^2}\right)\left(\frac{ik_z\mu_B E p_\perp}{4m}\frac{\partial f_0}{\partial p_z} \mp \frac{iq\mu_B B_0 E}{4m}\frac{\partial f_0}{\partial p_\perp}\right)$$

$$+\frac{i\mu_B E p_\perp}{2m\hbar}\left(1 + \frac{\mu_B B_0(k_z p_z/m \mp \omega_c)}{m(\omega \mp \Delta\omega_c - k_z p_z/m)}\Big(\cos\theta_s - \sin\theta_s\frac{\partial}{\partial\theta_s}\Big)\right)\frac{\partial f_0}{\partial\theta_s}\Bigg\}. \tag{33}$$

Equation (31) gives the first order approximation of the relativistic distribution function. Comparing equation (31) with equation (20), one can see that n and n' in the eigenfunction expansion gave no contribution except for $n = -1, 0, 1$ and $n' = -1, 0, 1$. In g_{00}, the first term comes from f_{10} while the second term is from f_{11}. In g_{1-1} and g_{-11}, the terms that come from f_{11} are those with square of $(\omega - \Delta\omega_c - k_z p_z/m)$ and $(\omega + \Delta\omega_c - k_z p_z/m)$ in the denominators respectively, the other terms are from f_{10}.

To calculate the total current density $\mathbf{J}$, we use equations (12)–(14) where we only use the perturbed distribution function f_1 in the integrals. For the longitudinal waves, the current density has only the z-component, one can easily check it by using Ampère's law. Thus the magnetization current density is zero in our case. Calculating firstly the free current density to first order in linearized theory

$$J_f = q\int d\Omega\frac{p_z}{m}\left[1 + \frac{\mu_B B_0}{m}\Big(\cos\theta_s - \sin\theta_s\frac{\partial}{\partial\theta_S}\Big)\right]\frac{g_{00}}{2\pi}. \tag{34}$$

For the background distribution function $f_0(\mathbf{p}^2, \theta_s)$, we assume that it has a Maxwellian distribution and a spin-dependent part

$$f_0(\mathbf{p}^2, \theta_s) = e^{-(\epsilon - m)/m^2 v_t^2}\left[e^{\mu_B B_0/k_B T}(1 + \cos\theta_s) + e^{-\mu_B B_0/k_B T}(1 - \cos\theta_s)\right], \tag{35}$$

where T is the temperature and k_B is the Boltzmann constant. For more information about the thermodynamical equilibrium in the background distribution function see reference [8]. Note that even though the derivation of the background distribution function in reference [8] was non-relativistic, the result should work in our case. We normalize the distribution function by dividing it with the normalizing factor $N_m = \int d\Omega f_0(\mathbf{p}^2, s)$. In the integration over $\mathbf{p}$, we make a first order Taylor expansion of ϵ; the argument of the exponential part of $f_0(\mathbf{p}^2, s)$ becomes: $(\epsilon - m)/m^2 v_{\text{th}}^2 \approx -p^2/m^2 v_{\text{th}}^2$, where

v_{th} is the thermal velocity. The normalized distribution function $\hat{f}_0(\mathbf{p}^2, s)$ is now

$$\hat{f}_0(\mathbf{p}^2, s) = \frac{1}{N_m} e^{-p^2/m^2 v_{th}^2} \left[e^{\mu_B B_0/k_B T}(1+\cos\theta_s) + e^{-\mu_B B_0/k_B T}(1-\cos\theta_s) \right], \tag{36}$$

where $N_m = 8\pi(\pi m^2 v_{th}^2)^{3/2}\cosh(\mu_B B_0/k_B T)$ is the normalization factor. We make a Taylor expansion of the denominators in equation (32), $(\omega - k_z p_z/m)$, otherwise the integration over p_z would be very hard to be solved analytically. The result we have for the free current density is

$$J_f = \frac{iq^2 E n}{\omega m}\left[1 + \frac{3k_z^2 v_{\text{th}}^2}{2\omega^2}\left(1 + \frac{\mu_B^2 B_0^2}{m^2}\right) + \frac{\mu_B B_0}{m}\left(1 + \frac{3k_z^2 v_{\text{th}}^2}{2\omega^2}\right)\tanh\left(\frac{\mu_B B_0}{k_B T}\right)\right]. \tag{37}$$

Similarly, the polarization current density is calculated to

$$\begin{aligned} J_p = \frac{i\mu_B^2 \omega E n}{4m}&\left[\frac{v_{\text{th}}^2 k_z^2}{2}\left(\frac{1}{\omega_+^2} + \frac{1}{\omega_-^2}\right) + \frac{3v_{\text{th}}^4 k_z^4}{4}\left(\frac{1}{\omega_+^4} + \frac{1}{\omega_-^4}\right)\right. \\ &\left. + \frac{qB_0}{m}\left(\frac{1}{\omega_+} - \frac{1}{\omega_-}\right) + \frac{qB_0 v_{\text{th}}^2 k_z^2}{2m}\left(\frac{1}{\omega_+^3} - \frac{1}{\omega_-^3}\right)\right] \\ - \frac{i\mu_B^2 \omega E n}{2\hbar}&\left[\frac{v_{\text{th}}^2}{2}\left(\frac{1}{\omega_+} - \frac{1}{\omega_-}\right) + \frac{v_{\text{th}}^4 k_z^2}{4}\left(\frac{1}{\omega_+^3} - \frac{1}{\omega_-^3}\right)\right]\tanh\left(\frac{\mu_B B_0}{k_B T}\right), \end{aligned} \tag{38}$$

where $\omega_\pm = \omega \pm \Delta\omega_c$, this notation is used to simplify the calculations. Ampère's law gives us the relation

$$-i\omega E = -4\pi J. \tag{39}$$

Using equation (39) together with the expressions of the polarization current density, equation (38), and the free current density, equation (37), we have then the dispersion relation

$$\begin{aligned} \omega^2\Bigg\{1 - \frac{\hbar^2\omega_p^2}{32m^2}&\left[v_{\text{th}}^2 k_z^2\left(\frac{1}{\omega_+^2} + \frac{1}{\omega_-^2}\right) + \frac{3v_{\text{th}}^4 k_z^4}{2}\left(\frac{1}{\omega_+^4} + \frac{1}{\omega_-^4}\right)\right] \\ - \frac{\omega_p^2 \hbar\mu_B B_0}{8m^2}&\left[\left(\frac{1}{\omega_+} - \frac{1}{\omega_-}\right) + \frac{v_{\text{th}}^2 k_z^2}{2}\left(\frac{1}{\omega_+^3} - \frac{1}{\omega_-^3}\right)\right] \\ + \frac{\hbar\omega_p^2 v_{th}^2}{16m}&\left[\left(\frac{1}{\omega_+} - \frac{1}{\omega_-}\right) + \frac{v_{\text{th}}^2 k_z^2}{2}\left(\frac{1}{\omega_+^3} - \frac{1}{\omega_-^3}\right)\right]\tanh\left(\frac{\mu_B B_0}{k_B T}\right)\Bigg\} \\ = \omega_p^2\left[1 + \frac{3k_z^2 v_{\text{th}}^2}{2\omega^2}\left(1 + \frac{\mu_B^2 B_0^2}{m^2}\right) + \frac{\mu_B B_0}{m}\right.&\left.\left(1 + \frac{3k_z^2 v_{\text{th}}^2}{2\omega^2}\right)\tanh\left(\frac{\mu_B B_0}{k_B T}\right)\right]. \end{aligned} \tag{40}$$

Equation (40) includes the spin effects of the electrons and is a generalization of the dispersion relation for the classical longitudinal waves in magnetized plasmas. If one neglects the spin effects, equation (40) becomes

$$\omega^2 = \omega_p^2 + \frac{3k^2 v_{\mathrm{th}}^2}{2}, \tag{41}$$

which is the classical dispersion relation for Langmuir waves.

3.1.1 Semi-relativistic limit

In this section we compare the dispersion relation, equation (40), with the dispersion relation in the semi-relativistic limit in reference [6]. In the semi-relativistic limit, $\mu_B B_0/m$ is small enough that it can be neglected. Thus the dispersion relation, equation (40), becomes in this limit

$$\begin{aligned} \omega^2 \Bigg\{ 1 - \frac{\hbar^2 \omega_p^2}{32m^2} \Bigg[v_{\mathrm{th}}^2 k_z^2 \Big(\frac{1}{\omega_+^2} + \frac{1}{\omega_-^2} \Big) + \frac{3v_{\mathrm{th}}^4 k_z^4}{2} \Big(\frac{1}{\omega_+^4} + \frac{1}{\omega_-^4} \Big) \Bigg] \\ + \frac{\hbar \omega_p^2 v_{\mathrm{th}}^2}{16m} \Bigg[\Big(\frac{1}{\omega_+} - \frac{1}{\omega_-} \Big) + \frac{v_{\mathrm{th}}^2 k_z^2}{2} \Big(\frac{1}{\omega_+^3} - \frac{1}{\omega_-^3} \Big) \Bigg] \tanh(\frac{\mu_B B_0}{k_B T}) \Bigg\} \\ = \omega_p^2 \Bigg(1 + \frac{3k_z^2 v_{\mathrm{th}}^2}{2\omega^2} \Bigg). \end{aligned} \tag{42}$$

This dispersion relation is identical with the one obtained in reference [3] except for an overall $-4\pi^2$ that is missing in the polarization current. Moreover one neglected terms with $1/\omega_+$ since that term is small compared with $1/\omega_-$ and the Landau damping-term appearing in reference [3] is neglected in our dispersion relation.

3.2 Transverse waves

In this geometry we have $\mathbf{k} = k_\perp \mathbf{e}_x$, $\mathbf{B}_0 = B_0 \mathbf{e}_z$, $\mathbf{E} = E_1 \exp(ikz - i\omega t)\mathbf{e}_z$ and $\mathbf{B}_1 = B_1 \exp(ikz - i\omega t)\mathbf{e}_y$. As in the longitudinal case, we use the plane-wave ansatz $f_1(\mathbf{x}, \mathbf{p}, \mathbf{s}, t) = \widetilde{f}_1(\mathbf{p}, \mathbf{s}) \exp[i(k_\perp x - \omega t)]$. The momentum $\mathbf{p}$ is expressed in cylindrical coordinates while the spin $\mathbf{s}$ is expressed in spherical coordinates. $\widetilde{f}_1(\mathbf{p}, \mathbf{s})$ is expanded in the eigenfunctions of the right hand side operators using equation (20), but this time we need to use the full equation, equation (21), for $\psi_n(\varphi_p, p_\perp)$. We use here the same assumption that we have low velocities of the particles as in the longitudinal case. With the relations in equation (23) and the plane-wave ansatz, we can express the left hand side LHS of equation (18) as

$$\Big\{ -i\omega + \frac{ip_\perp k_\perp}{m}\cos\varphi_p \left[1 + \frac{\mu_B B_0}{m}(\cos\theta_s - \sin\theta_s \frac{\partial}{\partial\theta_s})\right]$$
$$- \omega_{ce}\left[1 + \frac{\mu_B B_0}{m}(\cos\theta_s - \sin\theta_s\frac{\partial}{\partial\theta_s})\right]\frac{\partial}{\partial\varphi_p} - \omega_{cg}\frac{\partial}{\partial\varphi_s}\Big\} f_1 = RHS. \quad (43)$$

We have again a complicated equation and need to use the perturbation theory to obtain an analytical solution. We use equation (26) where we assume that $\mu_B B_0/m \ll 1$ and expand in the perturbation theory up to first order, the zeroth order term in perturbation is

$$\left[-i\omega + \frac{ip_\perp k_\perp}{m}\cos\varphi_p - \omega_{ce}\frac{\partial}{\partial\varphi_p} - \omega_{cg}\frac{\partial}{\partial\varphi_s}\right] f_{10} = RHS, \quad (44)$$

while the first order term in perturbation is

$$\left[-i\omega + \frac{ip_\perp k_\perp}{m}\cos\varphi_p - \omega_{ce}\frac{\partial}{\partial\varphi_p} - \omega_{cg}\frac{\partial}{\partial\varphi_s}\right] f_{11}$$
$$= -\frac{\mu_B B_0}{m}\Big(\cos\theta_s - \sin\theta_s\frac{\partial}{\partial\theta_s}\Big)\Big(\frac{ip_\perp k_\perp}{m}\cos\varphi_p - \omega_{ce}\frac{\partial}{\partial\varphi_p}\Big) f_{10}. \quad (45)$$

Using the same procedure as in the longitudinal case, the distribution function to the zeroth order f_{10} is

$$f_{10} = \sum_{\rho,\beta=-\infty}^{\infty} J_\beta\Big(\frac{k_\perp p_\perp}{m\omega_{ce}}\Big) e^{i\varphi_p(\beta-\rho)}\Big[A + \Big(B + C\Big)e^{\pm i\varphi_s}\Big], \quad (46)$$

where

$$A = -iqE\frac{\partial f_0}{\partial p_z}\frac{J_\rho}{\omega - \omega_{ce}\rho} \quad (47)$$

$$B = \sum_{+,-} -i\Big(\sin\theta_s + \cos\theta_s\frac{\partial}{\partial\theta_S}\Big)\frac{\partial f_0}{\partial p_\perp}\Bigg[\frac{qE\mu_B B_0}{4m(\omega - \omega_{ce}\rho \pm \omega_{cg}\rho)} J_{\rho\mp1}$$
$$\pm \frac{\mu_B B_1 k_\perp}{4(\omega - \omega_{ce}\rho \pm \omega_{cg}\rho)}(J_{\rho\pm1} + J_{\rho\mp1}) - \frac{\mu_B E k_\perp p_\perp}{8m(\omega - \omega_{ce}\rho \pm \omega_{cg}\rho)}(\pm J_\rho \mp J_{\rho\mp2})\Bigg] \quad (48)$$

$$C = \sum_{+,-} i\frac{\mu_B}{\hbar}\frac{\partial f_0}{\partial\theta_s}\left[B_1\frac{J_\rho}{(\omega - \omega_{ce}\rho \pm \omega_{cg}\rho)} + \frac{Ep_\perp}{2m}\frac{J_{\rho\mp1}}{(\omega - \omega_{ce}\rho \pm \omega_{cg}\rho)}\right], \quad (49)$$

where J_ρ is the Bessel function of first kind where we omitted the argument of the Bessel functions $k_\perp p_\perp/m\omega_{ce}$ to simplify the expression. The first order term of the distribution function f_{11} is

$$f_{11}=\frac{\mu_B B_0\omega_{ce}}{m}\Big(\cos\theta_s-\sin\theta_s\frac{\partial}{\partial\theta_S}\Big)\sum_{\tau,\beta,\alpha,\rho=-\infty}^{\infty}e^{i\varphi_p(\beta-\rho)}\rho J_\beta J_\alpha J_{\rho+\alpha-\tau}\times$$
$$\sum_{+,-}\left[\frac{A}{(\omega-\omega_{ce}\rho)}+\frac{B+C}{(\omega-\omega_{ce}\rho\pm\omega_{cg}\rho)}e^{\pm i\varphi_s}\right],\quad (50)$$

where the argument of all Bessel functions is $k_\perp p_\perp/m\omega_{ce}$, but we omit it to simplify the expression. As in the longitudinal case, we assume that the background distribution function f_0 is given by equation (36). The free current density $\mathbf{J}_f$ is calculated using equation (12). We carried out the integration over the spin angles, p_z and φ_p and deduce

$$J_f=\frac{iE\omega_p^2}{4\pi mk_BT}\Bigg\{\Big(1+\frac{\mu_B B_0}{m}\tanh\frac{\mu_B B_0}{k_BT}\Big)\sum_{\beta=-\infty}^{\infty}\frac{1}{\omega-\omega_{ce}\beta}\int_0^\infty dp_\perp p_\perp e^{-p_\perp^2/2mk_BT}J_\beta^2$$
$$+\frac{\mu_B B_0\omega_{ce}}{m}\Big(\tanh\frac{\mu_B B_0}{k_BT}+\frac{\mu_B B_0}{m}\Big)\sum_{\tau,\beta,\alpha=-\infty}^{\infty}\frac{\beta}{(\omega-\omega_{ce}\tau)(\omega-\omega_{ce}\beta)}\times$$
$$\int_0^\infty dp_\perp p_\perp e^{-p_\perp^2/2mk_BT}J_\beta J_\alpha J_\tau J_{\beta+\alpha-\tau}\Bigg\}.\quad (51)$$

Similarly, the polarization and magnetization current density, J_p and J_m, are calculated to

$$J_p=\frac{i\omega\omega_p^2\hbar E}{16\pi m^3k_BT}\int_0^\infty dp_\perp p_\perp^2 e^{-p_\perp^2/2mk_BT}\sum_{\beta=-\infty}^{\infty}\sum_{+,-}J_\beta\Bigg\{\frac{p_\perp}{k_BT}\Bigg[\pm\frac{\mu_B B_0}{4m}\frac{J_\beta}{(\omega-\omega_{ce}\beta\pm\omega_{cg})}$$
$$-\frac{\hbar k_\perp^2}{8m\omega(\omega-\omega_{ce}\beta\pm\omega_{cg})}\Big(J_\beta+J_{\beta\pm2}\Big)-\frac{\hbar k_\perp p_\perp}{16m^2(\omega-\omega_{ce}\beta\pm\omega_{cg})}\Big(J_{\beta\pm1}-J_{\beta\mp1}\Big)\Bigg]$$
$$\mp\frac{1}{2}\tanh\Big(\frac{\mu_B B_0}{k_BT}\Big)\Bigg[-\frac{k_\perp}{\omega}\frac{J_{\beta\pm1}}{(\omega-\omega_{ce}\beta\pm\omega_{cg})}+\frac{p_\perp}{2m}\frac{J_\beta}{(\omega-\omega_{ce}\beta\pm\omega_{cg})}\Bigg]\Bigg\}\quad (52)$$

$$J_M=\frac{i\omega_p^2\hbar k_\perp E}{8\pi mk_BT}\int_0^\infty dp_\perp p_\perp e^{-p_\perp^2/2mk_BT}\sum_{\beta=-\infty}^{\infty}\sum_{+,-}J_\beta\Bigg\{-\frac{p_\perp}{mk_BT}\Bigg[\pm\frac{\mu_B B_0}{4m}\frac{J_{\beta\mp1}}{(\omega-\omega_{ce}\beta\pm\omega_{cg})}$$
$$-\frac{\hbar k_\perp^2}{8m\omega(\omega-\omega_{ce}\beta\pm\omega_{cg})}\Big(J_{\beta\pm1}+J_{\beta\mp1}\Big)-\frac{\hbar k_\perp p_\perp}{16m^2(\omega-\omega_{ce}\beta\pm\omega_{cg})}\Big(J_\beta-J_{\beta\mp2}\Big)\Bigg]$$
$$\pm\frac{1}{2m}\tanh\Big(\frac{\mu_B B_0}{k_BT}\Big)\Bigg[-\frac{k_\perp}{\omega}\frac{J_\beta}{(\omega-\omega_{ce}\beta\pm\omega_{cg})}+\frac{p_\perp}{2m}\frac{J_{\beta\mp1}}{(\omega-\omega_{ce}\beta\pm\omega_{cg})}\Bigg]\Bigg\}.\quad (53)$$

Note that the first order term f_{11} survives the integration only in the free current density. To obtain the dispersion relation, we use Ampère's law where we get the relation

$$i\omega E\left(1-\frac{k_\perp^2}{\omega^2}\right)=4\pi J. \tag{54}$$

The dispersion relation for the transverse propagation is

$$\begin{aligned}
&\omega^2\Bigg(1-\frac{k_\perp^2}{\omega^2}-\frac{\omega_p^2\hbar}{4m^3k_BT}\int_0^\infty dp_\perp p_\perp^2 e^{-p_\perp^2/2mk_BT}\sum_{\beta=-\infty}^{\infty}\sum_{+,-}J_\beta\Bigg\{\frac{p_\perp}{k_BT}\times\\
&\qquad\Bigg[\pm\frac{\mu_B B_0}{4m}\frac{J_\beta}{(\omega-\omega_{ce}\beta\pm\omega_{cg})}-\frac{\hbar k_\perp^2}{8m\omega(\omega-\omega_{ce}\beta\pm\omega_{cg})}\Big(J_\beta+J_{\beta\pm2}\Big)\\
&-\frac{\hbar k_\perp p_\perp}{16m^2(\omega-\omega_{ce}\beta\pm\omega_{cg})}\Big(J_{\beta\pm1}-J_{\beta\mp1}\Big)\Bigg]\mp\frac{1}{2}\tanh\Big(\frac{\mu_B B_0}{k_BT}\Big)\Bigg[-\frac{k_\perp}{\omega}\frac{J_{\beta\pm1}}{(\omega-\omega_{ce}\beta\pm\omega_{cg})}\\
&+\frac{p_\perp}{2m}\frac{J_\beta}{(\omega-\omega_{ce}\beta\pm\omega_{cg})}\Bigg]\Bigg\}+\frac{\omega_p^2\hbar k_\perp}{2m^2k_BT\omega}\int_0^\infty dp_\perp p_\perp e^{-p_\perp^2/2mk_BT}\sum_{+,-}\sum_{\beta=-\infty}^{\infty}J_\beta\times\\
&\quad\Bigg\{\frac{p_\perp}{k_BT}\Bigg[\pm\frac{\mu_B B_0}{4m}\frac{J_{\beta\mp1}}{(\omega-\omega_{ce}\beta\pm\omega_{cg})}-\frac{\hbar k_\perp^2}{8m\omega(\omega-\omega_{ce}\beta\pm\omega_{cg})}\Big(J_{\beta\pm1}+J_{\beta\mp1}\Big)\\
&\qquad\qquad\qquad-\frac{\hbar k_\perp p_\perp}{16m^2(\omega-\omega_{ce}\beta\pm\omega_{cg})}\Big(J_\beta-J_{\beta\mp2}\Big)\Bigg]\\
&\quad\mp\frac{1}{2}\tanh\Big(\frac{\mu_B B_0}{k_BT}\Big)\Bigg[-\frac{k_\perp}{\omega}\frac{J_\beta}{(\omega-\omega_{ce}\beta\pm\omega_{cg})}+\frac{p_\perp}{2m}\frac{J_{\beta\mp1}}{(\omega-\omega_{ce}\beta\pm\omega_{cg})}\Bigg]\Bigg\}\Bigg)\\
&=\frac{\omega_p^2}{mk_BT}\Bigg\{\Big(1+\frac{\mu_B B_0}{m}\tanh\frac{\mu_B B_0}{k_BT}\Big)\sum_{\beta=-\infty}^{\infty}\frac{1}{1-\frac{\omega_{ce}}{\omega}\beta}\int_0^\infty dp_\perp p_\perp e^{-p_\perp^2/2mk_BT}J_\beta^2\\
&\quad+\frac{\mu_B B_0\omega_{ce}}{m\omega}\Big(\tanh\frac{\mu_B B_0}{k_BT}+\frac{\mu_B B_0}{m}\Big)\sum_{\tau,\beta\alpha=-\infty}^{\infty}\frac{\beta}{(1-\frac{\omega_{ce}}{\omega}\tau)(1-\frac{\omega_{ce}}{\omega}\beta)}\times\\
&\qquad\qquad\int_0^\infty dp_\perp p_\perp e^{-p_\perp^2/2mk_BT}J_\beta J_\alpha J_\tau J_{\beta+\alpha-\tau}\Bigg\}.
\end{aligned}\tag{55}$$

We can compare this dispersion relation with the one for the longitudinal case, equation (40). We let $k_\perp$ and k_z be zero and the two dispersion relations should match. In this limit, the Bessel function becomes $J_\beta=\delta_{\beta 0}$. Equation

(55) is then

$$\omega^2\left[1-\frac{\omega_p^2\hbar\mu_B B_0}{8m^2}\Big(\frac{1}{\omega-\omega_{ce}+\omega_{cg}}-\frac{1}{\omega-\omega_{ce}-\omega_{cg}}\Big)+\frac{\omega_p^2\hbar v_{th}^2}{16m}\tanh\Big(\frac{\mu_B B_0}{k_B T}\Big)\times\right.$$
$$\left.\Big(\frac{1}{\omega-\omega_{ce}+\omega_{cg}}-\frac{1}{\omega-\omega_{ce}-\omega_{cg}}\Big)\right]=\omega_p^2\left(1+\frac{\mu_B B_0}{m}\tanh\Big(\frac{\mu_B B_0}{k_B T}\Big)\right). \tag{56}$$

Equation (40) has the same expression in this limit. Thus we have both dispersion relations in the two geometries to approach the same result in the very high frequency limit as expected.

3.2.1 Long scale limit

The integrals in equation (55) have no simple analytical solutions. We make here an approximation of the argument of the Bessel functions in the integral $k_\perp p_\perp/m\omega_{ce} \ll 1$. This approximation enable us to make a Taylor expansion of the Bessel functions, we expand the Bessel function up to the second order in $k_\perp p_\perp/m\omega_{ce}$. The terms that we kept from the summation over the Bessel functions are J_0, $J_{\pm1}$ and J_{-2}, which become after the Taylor expansion

$$\begin{aligned} J_0 &= 1-\frac{k_\perp^2 p_\perp^2}{4m^2\omega_{ce}^2} \\ J_{\pm1} &= \pm\frac{k_\perp p_\perp}{2m\omega_{ce}} \\ J_{-2} &= \frac{k_\perp^2 p_\perp^2}{8m^2\omega_{ce}^2}. \end{aligned} \tag{57}$$

After the integration over $p_\perp$, equation (55) becomes

$$
\begin{aligned}
&\omega^2\Bigg(1+\sum_{+,-}\frac{\omega_p^2\hbar}{8m^2}\Bigg\{\mu_B B_0\Bigg[\mp\frac{1}{\omega\pm\omega_{cg}}\mp\frac{k_\perp^2}{\omega_{ce}^2}\frac{v_{th}^2}{2}\Bigg(\frac{1}{\omega-\omega_{ce}\pm\omega_{cg}}+\frac{1}{\omega+\omega_{ce}\pm\omega_{cg}}\\
&\qquad-\frac{2}{\omega\pm\omega_{cg}}\Bigg)+\frac{k_\perp^2}{\omega\omega_{ce}}\Bigg(\frac{1}{\omega\mp\omega_{ce}\pm\omega_{cg}}-\frac{1}{\omega\pm\omega_{cg}}\Bigg)\Bigg]\\
&+\frac{\hbar k_\perp^4 v_{\rm th}^2}{4\omega\omega_{ce}^2}\Bigg[\frac{1}{2(\omega+2\omega_{ce}\pm\omega_{cg})}-\frac{1}{\omega\mp\omega_{ce}\pm\omega_{cg}}+\frac{2}{\omega\pm\omega_{cg}}-\frac{1}{\omega\pm\omega_{ce}\pm\omega_{cg}}+\frac{1}{2(\omega\pm\omega_{cg})}\Bigg]\\
&\qquad+\frac{\hbar k_\perp^2}{4\omega_{ce}}\Bigg[\pm\frac{v_{\rm th}^2}{2}\Bigg(\frac{2}{\omega\pm\omega_{cg}}-\frac{1}{\omega\pm\omega_{ce}\pm\omega_{cg}}-\frac{1}{\omega\mp\omega_{ce}\pm\omega_{cg}}\Bigg)\\
&-\frac{k_\perp^2}{\omega^2}\Bigg(\frac{1}{\omega-\omega_{ce}\pm\omega_{cg}}-\frac{1}{\omega+\omega_{ce}\pm\omega_{cg}}\Bigg)\Bigg]\Bigg\}+\sum_{+,-}\frac{\omega_p^2\hbar}{4m}\tanh\Big(\frac{\mu_B B_0}{k_B T}\Big)\Bigg\{\pm\Bigg(\frac{v_{\rm th}^2}{4}+\frac{k_\perp^2}{\omega^2}\Bigg)\times\\
&\qquad\Bigg[\frac{1}{\omega\pm\omega_{cg}}+\frac{k_\perp^2}{\omega_{ce}^2}\frac{v_{\rm th}^2}{2}\Bigg(\frac{1}{\omega-\omega_{ce}\pm\omega_{cg}}+\frac{1}{\omega+\omega_{ce}\pm\omega_{cg}}-\frac{2}{\omega\pm\omega_{cg}}\Bigg)\Bigg]\\
&\qquad\qquad+\frac{k_\perp^2 v_{\rm th}^2}{4\omega\omega_{ce}}\Bigg(\frac{1}{\omega\pm\omega_{ce}\pm\omega_{cg}}-\frac{1}{\omega\mp\omega_{ce}\pm\omega_{cg}}\Bigg)\Bigg\}\Bigg)\\
&\quad=k_\perp^2+\sum_{+,-}\omega_p^2\Bigg[1-\frac{k_\perp^2}{\omega_{ce}^2}\frac{v_{\rm th}^2}{2}\Bigg(1-\frac{1}{2(1\pm\frac{\omega_{ce}}{\omega})}\Bigg)\pm\frac{\mu_B^2B_0^2k_\perp^2v_{\rm th}^2}{4m^2\omega\omega_{ce}}\frac{1}{(1\mp\frac{\omega_{ce}}{\omega})^2}\Bigg]\\
&+\sum_{+,-}\frac{\omega_p^2\mu_B B_0}{m}\tanh\Big(\frac{\mu_B B_0}{k_B T}\Big)\Bigg[1-\frac{k_\perp^2}{\omega_{ce}^2}\frac{v_{\rm th}^2}{2}\Bigg(1-\frac{1}{2(1\pm\frac{\omega_{ce}}{\omega})}\Bigg)\pm\frac{k_\perp^2v_{\rm th}^2}{\omega\omega_{ce}}\frac{1}{(1\mp\frac{\omega_{ce}}{\omega})^2}\Bigg].
\end{aligned}
\tag{58}
$$

In the limit of low temperatures, we let $T\to 0$, the dispersion relation becomes then

$$
\begin{aligned}
&\omega^2\Bigg(1+\sum_{+,-}\frac{\omega_p^2\hbar}{8m^2}\Bigg\{\mu_B B_0\Bigg[\mp\frac{1}{\omega\pm\omega_{cg}}+\frac{k_\perp^2}{\omega\omega_{ce}}\Bigg(\frac{1}{\omega\mp\omega_{ce}\pm\omega_{cg}}-\frac{1}{\omega\pm\omega_{cg}}\Bigg)\Bigg]\\
&-\frac{\hbar k_\perp^4}{4\omega^2\omega_{ce}}\Bigg(\frac{1}{\omega-\omega_{ce}\pm\omega_{cg}}-\frac{1}{\omega+\omega_{ce}\pm\omega_{cg}}\Bigg)\Bigg\}\sum_{+,-}\pm\frac{\omega_p^2\hbar k_\perp^2}{4m\omega^2}\frac{\tanh(\mu_B B_0/k_B T)}{\omega\pm\omega_{cg}}\Bigg)\\
&\qquad=k_\perp^2+\omega_p^2\Bigg(1+\frac{\mu_B B_0}{m}\tanh(\frac{\mu_B B_0}{k_B T})\Bigg).
\end{aligned}
\tag{59}
$$

If the background **B**-field ($\mathbf{B}_0$) is weak, i.e. $\mu_B B_0/m\ll 1$, the dispersion relation becomes even simpler

$$
\omega^2=k_\perp^2+\omega_p^2+\sum_{+,-}\frac{\omega_p^2\hbar^2k_\perp^4}{32m^2\omega_{ce}}\Bigg(\frac{1}{\omega-\omega_{ce}\pm\omega_{cg}}-\frac{1}{\omega+\omega_{ce}\pm\omega_{cg}}\Bigg).
\tag{60}
$$

3.2.2 Non-relativistic limit

Equation (60) offers an interesting result since the quantum mechanical term on the right hand side of the equation is non-zero in the zero-temperature limit. This term has its origin in the $(\frac{\mu_B m}{\epsilon}\nabla_x\Big[\mathbf{B}_1\cdot(\mathbf{s}+\nabla_s)\Big]\cdot\nabla_p f_0)$-term in the fully relativistic kinetic equation (5), which is not a relativistic term. Thus it means that the new interesting term in the dispersion relation can be found in a non-relativistic dispersion relation of quantum plasmas.

Considering σ_{zz} of the conductivity tensor in reference [3], using the same geometry as in our case and making the same expansion of the Bessel functions, the dispersion relation of the non-relativistic kinetic equation is then

$$\omega^2 = k_\perp^2 + \omega_p^2 - \sum_{+,-}\frac{\omega_p^2\hbar^2 k_\perp^4}{16m^2\omega_{ce}}\left(\frac{1}{\omega-\omega_{ce}\pm\omega_{cg}} - \frac{1}{\omega+\omega_{ce}\pm\omega_{cg}}\right). \tag{61}$$

3.2.3 Short analysis

The dispersion relation equation (60) can be analyzed further by considering the limit of high frequency, $\omega \gg \omega_{ce}+\omega_{cg}$. Taking the first order Taylor expansion of the nominators in equation (60), we get

$$\omega^2 = k_\perp^2 + \omega_p^2 + \frac{\omega_p^2\hbar^2 k_\perp^2}{16m^2\omega^2}. \tag{62}$$

This is a new wave-mode. Even though we considered the long scale approximation where the argument of the Bessel functions is assumed to be small, this mode can be found for relatively small $k_\perp p_\perp/m\omega_{ce}$, which makes the approximation still valid.

A further analysis can be done to get a more general dispersion relation than equation (62) that takes into account more terms in the Bessel functions in equation (55).

4 Summary

In this work we have used in linearized theory, the fully relativistic kinetic equation of spin-1/2 particles in the long scale approximation, equation (5). Since equation (5) is fairly complicated, we chose to restrict the geometry. In the first case we chose to consider longitudinal propagation parallel to the background B-field, while in the second case we considered transverse propagation that is perpendicular to the background B-field. In both cases, we considered magnetized plasmas with a homogeneous background distribution function, equation (36). In solving the perturbed distribution function f_1, we used perturbation theory, equation (26), and solved up to first order in perturbation in both cases. However the first order term in perturbation f_{11} contributed only in the free current J_f in the transverse case.

In the longitudinal case, the major result was the dispersion relation equation (40), which in the limit of weak background B-field matches the dispersion relation in the semi-relativistic case in reference [6] (we disregarded here the overall factor $-4\pi^2$ and the Landau damping-term). In the transverse case, the major result was the dispersion relation, equation (55), which has no simple analytical solution thus we made the long scale approximation. In this approximation we assumed that the argument of the Bessel functions is small, $k_\perp p_\perp / m\omega_{ce} \ll 1$, thus we could make a second order Taylor expansion around $k_\perp p_\perp / m\omega_{ce} = 0$. In this approximation we got the dispersion relation equation (60). A more detailed analysis of equation (55) might be necessary to do in order to obtain a more general dispersion relation. However the calculations we have done that resulted in equation (60) have shown that the low frequent waves might be strongly affected by the quantum effects that follow from the magnetic moments of the electrons.

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References

[1] Princeton Plasma Physics laboratory, A collaborative national center for fusion & plasma physics (2018).

[2] J. Zamanian, Umea Universitet. diva2:511354 (2012)

[3] J. Lundin and G. Brodin, Phys. Rev. E 82, 056407 (2010) .

[4] R. Ekman, F. A. Asenjo, and J. Zamanian, Phys. Rev. E 96, 023207 (2017).

[5] L. L. Foldy and S. A. Wothuysen, Phys. Rev. 78, 29 (1950).

[6] F. A. Asenjo, J. Zamanian, M. Marklund, G. Brodin, and P. Johansson, New J. Phys. 14, 073042 (2012).

[7] H. S. Thomas, *Waves in Plasmas*, American institute of physics (1992).

[8] J. Zamanian, M. Marklund, and G. Brodin, New J. Phys. 12, 043019 (2010).