An introduction to relativistic electrodynamics

Part II: Calculus with complex 4-vectors

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The conventional way of introducing relativity when teaching electrodynamics is to leave Gibbs’ vector calculus for a more general tensor calculus. This sudden change of formalism can be quite problematic for the students and we therefore in this two-part paper consider alternate approaches. The algebra \( C^{2 \times 2} \) of 2-by-2 complex matrices (sometimes presented in the form of Clifford algebra or complex quaternions) may be used for spinor related formulations of special relativity and electrodynamics. In this Part II we use this algebraic structure but with notations that fits in with the formalism of Part I. Each observer \( e_0 \) defines a product on the space of complex 4-vectors \( M^C \) so that \( M^C \) becomes an algebra isomorphic to \( C^{2 \times 2} \) with \( e_0 \) as algebra unit. The spacetime geometric equations of Part I become complex (spinor related) equations where the antisymmetric 4-dyadics have been replaced by complex 3-vectors, i.e., by elements in \( \mathbb{C}^4 \). For example, instead of the electromagnetic dyadic field \( F \) we now have the complex field variable \( f = E/c + iB \). Some linear algebra together with the formalism of Gibbs’ vector calculus (trivially allowing for complex 3-vectors) is sufficient for dealing with the equations in their complex form.

I. INTRODUCTION

This is the second part of a two-part paper where we take the Minkowski vectorspace as a starting point. In Part I we use 4-vectors and 4-dyadics, i.e., elements in \( \mathbb{M} \) and \( \mathbb{M}^{\otimes 2} \), to write Lorentz transformations and equations of electrodynamics in a spacetime geometric form without choosing observer or coordinates. In this Part II we use complex 4-vectors (i.e., elements in the complexified Minkowski space \( \mathbb{C}^4 \)). An observer dependent product between complex 4-vectors is defined. The space \( \mathbb{C}^4 \) then becomes an algebra isomorphic to the algebra \( C^{2 \times 2} \) of complex 2-by-2 matrices.

It is well known that complex 2-by-2 matrices (sometimes presented in the form of complex quaternions or Clifford algebra) may be used for compact formulations of special relativity and electrodynamics. In order to introduce coordinates we in Part I use a normal basis \( e = [e_0 \, e_1 \, e_2 \, e_3] \) and represent a 4-vector \( x = x^\mu e_\mu \) by a 4-by-1 real matrix \( x_e = [x^0 \cdots x^3]^T \). Here and elsewhere in this paper the sum over \( \mu = 0, 1, 2, 3 \) is implied. An alternative is to represent \( x \) by a 2-by-2 matrix

\[
\begin{bmatrix}
  x^0 + x^3 & x^1 - i x^2 \\
  x^1 + i x^2 & x^0 - x^3
\end{bmatrix}
\]

\( x_{\sigma} \equiv x^\mu \sigma_\mu \) for \( \sigma \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \sigma_1 \equiv \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \sigma_2 \equiv \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \sigma_3 \equiv \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \) (1)

where

\[
\sigma_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},
\sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}
\]

Thus \( \sigma_j \) for \( j = 1, 2, 3 \) are the Pauli matrices. The 4-vectors are in this way associated with Hermitian matrices. We now have available the algebraic structures of square matrices, i.e., matrix multiplication, determinants and traces. These are useful for special relativity. Consider for example the determinant

\[
\det x_{\sigma} = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 = -x \cdot x\]

which is a quantity of obvious interest in special relativity. Furthermore, consider a complex 2-by-2 matrix \( M \) with determinant one. We may use \( M \) to define an (active) proper orthochronous Lorentz transformation \( x \to y \) by the matrix formula

\[
y_{\sigma} = M x_{\sigma} M^\dagger \]

Here the right hand side is the product of three 2-by-2 matrices and the dagger stands for Hermitian conjugation. From equation (4) it follows that the Hermitian property of \( x_{\sigma} \) implies that also \( y_{\sigma} \) is Hermitian and thus associated with a 4-vector \( y \). If we take the determinant of both sides in equation (4) we easily find \( x \cdot x = y \cdot y \). Thus the transformation is a Lorentz transformation. It may be proven that we have defined a proper orthochronous Lorentz transformation and that all such transformations may be obtained in this way. It may also be observed that the equation (4) is related to spinor theory.

Given a normal basis \( e \) we represent by equation (1) the Minkowski space \( \mathbb{M} \) by the space of all Hermitian matrices in \( \mathbb{C}^{2 \times 2} \). We then, in an obvious way, associate all of \( \mathbb{C}^{2 \times 2} \) with the complexified Minkowski space \( \mathbb{M}^C = \mathbb{M} + i \mathbb{M} \) and define a product on \( \mathbb{M}^C \) corresponding to usual matrix multiplication. In this way \( \mathbb{M}^C \) becomes an algebra isomorphic to \( \mathbb{C}^{2 \times 2} \) with the observer \( e_0 \) as algebra unit. We use this algebraic structure and rewrite some basic equations from Part I in terms of complex 4-vectors. The equations obtained are said to be in complex (spinor related) form.
This paper is organized as follows. In Section II we define the observer dependent product on the complex Minkowski space. In Theorem II.1 a simple identity is given involving this algebra product and a transformation from the space of anti-symmetric 4-dyadics to the space of complex 3-vectors. This identity is used for the transformation of equations in Part I to their complex form. The proper orthochronous Lorentz transformations are considered in Section III. They may be expressed in terms of exponentials of square matrices; in the standard tensor formalism there appear 4-by-4 real matrices and in the “spinor form” there are 2-by-2 complex matrices. We use Theorem II.1 and find a comparatively simple derivation of the exact relation between these results. In Section IV we consider relativistic electrodynamics. The complex versions of the equation for the worldline of a charged particle and Maxwell’s equations are obtained with very little algebra from the equations in Part I, again by the use of Theorem II.1. Also the stress-energy conservation theorem is derived within the complex formalism. Section V is a discussion.

II. THE COMPLEX MINKOWSKI SPACE AND THE ALGEBRA OF COMPLEX 2-BY-2 MATRICES

Like in Part I we take the Minkowski vectorspace \( \mathbb{M} \) as a starting point but instead of using 4-dyadics to represent electromagnetic fields and Lorentz transformations we now use complex 4-vectors, i.e., elements in \( \mathbb{M}^\mathbb{C} = \mathbb{M} + i\mathbb{M} \). Any complex 4-vector may be written \( C = A + iB \) where \( A, B \in \mathbb{M} \). We denote “complex conjugation” with a superscript star on complex 4-vectors defined by

\[
(A + iB)^\ast = A - iB
\]

and use the notation

\[
\text{Re } C = (C + C^\ast)/2
\]

The superscript star notation is also used on complex numbers as the usual complex conjugation. If \( C \in \mathbb{M}^\mathbb{C} \) and \( z \in \mathbb{C} \) then

\[
(zC)^\ast = z^\ast C^\ast
\]

The spatial part according to observer \( e_o \) of \( C \) is denoted \( \bar{C} \), i.e.,

\[
\bar{C} = C + (e_0 \cdot C)e_0
\]

We define \( e = [e_0 e_1 e_2 e_3] \) to be a normal basis for \( \mathbb{M}^\mathbb{C} \) if it is a normal basis for \( \mathbb{M} \). Thus, the 4-vectors in a normal basis are always real, i.e., \( \text{Re } e_\mu = e_\mu \) and equivalently \( e_\mu^\ast = e_\mu \).

Let \( A, B \in \mathbb{M}^\mathbb{C} \) and let \( e \) be a normal basis. We define a bijection from \( \mathbb{M}^\mathbb{C} \) to the space of \( \mathbb{C}^{2 \times 2} \) by

\[
A = A^\mu e_\mu \rightarrow A_{e_\sigma} = A^\mu \sigma_{\mu}
\]

\[
= \begin{bmatrix}
A^0 + A^3 & A^1 - iA^2 \\
A^1 + iA^2 & A^0 - A^3
\end{bmatrix}
\]

This is similar to equation (1) but we now allow for complex coordinates \( A^\mu \). We define a product “\( \otimes \)” between complex 4-vectors by requiring

\[
(A \otimes B)_{e_\sigma} = A_{e_\sigma} B_{e_\sigma}
\]

where on the right hand side we have the product of two matrices. This results in the observer \( e_0 \) dependent product

\[
A \otimes B = (A^0 B^0 + \vec{A} \cdot \vec{B}) e_0 + A^0 \vec{B} \otimes e_0 - B^0 \vec{A} \otimes e_0 + i\vec{A} \times \vec{B}
\]

where the cross product is defined in Section III E of Part I such that

\[
\vec{A} \times \vec{B} = (A^2 B^3 - A^3 B^2) e_1 + (A^3 B^1 - A^1 B^3) e_2 + (A^1 B^2 - A^2 B^1) e_3
\]

To prove the equivalence between equation (10) and (11) we may use the following properties

\[
A \otimes e_0 = e_0 \otimes A = A
\]

\[
e_1 \otimes e_1 = e_2 \otimes e_2 = e_3 \otimes e_3 = e_0
\]

\[
e_1 \otimes e_2 = i e_3, \quad e_2 \otimes e_3 = i e_1, \quad e_3 \otimes e_1 = i e_2
\]

and the corresponding relations for the \( \sigma \)-matrices

\[
A_{e_\sigma} \sigma_{\sigma} = \sigma_0 A_{e_\sigma} = A_{e_\sigma}
\]

\[
\sigma_1 \sigma_1 = \sigma_2 \sigma_2 = \sigma_3 \sigma_3 = \sigma_0
\]

\[
\sigma_1 \sigma_2 = i \sigma_3, \quad \sigma_2 \sigma_3 = i \sigma_1, \quad \sigma_3 \sigma_1 = i \sigma_2
\]

It follows easily that \( \mathbb{M}^\mathbb{C} \) with product “\( \otimes \)” is an algebra isomorphic to the algebra \( \mathbb{C}^{2 \times 2} \) of complex 2-by-2 matrices. The isomorphism is defined by equation (9), i.e., by

\[
e_\mu \mapsto \sigma_\mu, \quad \mu = 0, 1, 2, 3
\]

An important property that now follows is the associativity

\[
(A \otimes B) \otimes C = A \otimes (B \otimes C)
\]

For the complex conjugate \( A^\ast \) we get

\[
A^\ast = A^{\ast \mu} e_\mu \in \mathbb{M}^\mathbb{C}
\]

\[
(A^\ast)_{e_\sigma} = \begin{bmatrix}
A^0 + A^3 & A^1 - iA^2 \\
A^1 + iA^2 & A^0 - A^3
\end{bmatrix} = (A_{e_\sigma})^\ast
\]
Note that the complex conjugate on the complex 4-vector corresponds to the Hermitian conjugate on the associated matrix. We also have

\[(A \otimes B)^\dagger = B^\dagger \otimes A^\dagger\]  

(22)

that follows from the matrix relation \[(A_{\alpha\beta}B_{\alpha\beta})^\dagger = B^\dagger_{\alpha\beta}A^\dagger_{\alpha\beta}\]  

(23)

or by direct application of equation (11).

We define the observer dependent operation “tilde” on \[\bar{A} = A^0 e_0 + \tilde{A} \in \mathbb{M}^C\] by

\[\bar{A} = A^0 e_0 - \tilde{A}\]  

(24)

which has the following useful property

\[\bar{A} \otimes A = A \otimes \bar{A} = -(A \cdot A) e_0\]  

(25)

It is also easy to show that

\[(A \otimes B)^\dagger = \bar{B} \otimes \bar{A}\]  

(26)

The exponential of \[A \in \mathbb{M}^C\] is defined

\[e^A = e_0 + \frac{A \otimes A}{1!} + \frac{A \otimes A \otimes A}{2!} + \cdots\]  

(27)

and by use of the isomorphism (10) we get

\[(e^A)_{\alpha\beta} = e^{A_{\alpha\beta}}\]  

(28)

where the exponential of a 2-by-2 matrix appears on the right hand side.

Anti-symmetric 4-dyadics are important in Part I. They appear when proper orthochronous Lorentz transformations are written in exponential form and also in electrodynamics because the electromagnetic field dyadic is anti-symmetric. Such dyadics will here in Part II be replaced by complex 3-vectors. The simple identity (30) below turns out to be useful for transforming equations in Part I to their complex form.

**Theorem II.1.** We denote elements in the 3D space \(e_0^3\) of observer \(e_0\) by \(a\) and \(b\).

(a) An observer \(e_0\) defines a linear bijection from \(\mathbb{M}^C\) to \(e_0^3 = e_0^1 + i e_0^2\) by

\[e_0 \wedge a + [b \times] \rightarrow (a - ib)\]  

(29)

(b) The anti-symmetric 4-dyadics \(G = e_0 \wedge a + [b \times]\) and \(sG = e_0 \wedge b - [a \times]\) are related to the complex 3-vector \(a - ib\) by the identity

\[(G - i s) \otimes C = (a - ib) \otimes C\]  

(30)

for all \(C \in \mathbb{M}\).

(c) The real part of equation (30) is

\[(e_0 \wedge a + [b \times]) \otimes C = \text{Re}[(a - ib) \otimes C]\]  

(31)

for all \(C \in \mathbb{M}\).

**Proof.** (a) This follows by use of Theorem V.1 in Part I.

(b) Substitute \(C = C^0 e_0 + \tilde{C}\) in equation (30) and use definition (11). Then the equality (30) follows by straightforward vector algebra. (c) This is trivial.

**III. PROPER ORTHOCHRONOUS LORENTZ TRANSFORMATIONS**

Equation (1) determines proper orthochronous Lorentz transformations \(M \rightarrow \tilde{M}\) in terms of coordinates. We now formulate a spatially coordinate-free version of this construction. The time coordinate is already chosen since the observer \(e_0\) is assumed to be given but we refer to no coordinates on the 3D space \(e_0^3\).

**Theorem III.1.** If \(m \in \mathbb{M}^C\) such that \(m \cdot m = -1\) then a proper orthochronous Lorentz transformation \(x \rightarrow y\) is defined by

\[y = m \circ x \circ m^*\]  

(32)

Each proper Lorentz transformation is obtained in this way. We note that \(m\) and \(-m\) give rise to the same Lorentz transformation.

**Proof.** We may transfer the problem into a matrix equation by means of the isomorphism (1). Then the equation (32) becomes \(y_{\alpha\beta} = m_{\alpha\beta} x_{\alpha\beta} m_{\alpha\beta}\) with \(\det m_{\alpha\beta} = 1\). This is equation (31) with \(M = m_{\alpha\beta}\). We may also prove that equation (32) defines a Lorentz transformation, i.e., \(x \cdot x = y \cdot y\), without matrix algebra. By use of the equations (20), (22), (25), (26) and (32) we find

\[-(y \cdot y) e_0 = y \circ \tilde{y}\]  

\[= \left(m \circ x \circ m^*\right) \circ \left(m \circ x \circ m^*\right)^\dagger\]  

\[= \left(m \circ x \circ m^*\right) \circ \left(m^* \circ \tilde{x} \circ \tilde{m}\right)\]  

\[= m \circ x \circ m^* \circ \tilde{x} \circ \tilde{m}\]  

\[= m \circ x \circ \tilde{m} \circ \tilde{m}\]  

\[= - (x \cdot x) m \circ \tilde{m} = -(x \cdot x) e_0\]  

which completes the proof.

In order to obtain an exponential version of this theorem we need the following result.

**Theorem III.2.** If \(k \in e_0^1\) then

\[e_k^1 = \cosh (\sqrt{k \cdot k}) e_0 + \frac{\sinh (\sqrt{k \cdot k})}{\sqrt{k \cdot k}} k\]  

(34)

and \(e_k^1 \cdot e_k^1 = -1\).

**Proof.** The right hand side of the first equation is defined by the associated power series (12). Because only even elements appear we find that \(e_k^1\) only depend on \(k \cdot k\). Thus we need not define the branch of \(\sqrt{k \cdot k}\). By use of the Taylor expansion (27) with \(A = k\) and \(\kappa \otimes k = (k \cdot k) e_0\) we obtain (34) by which also \(e_k^1 \cdot e_k^1 = -1\) follows.

We now rewrite Theorem III.1 in exponential form.

**Theorem III.3.** If \(k \in e_0^1\) then

\[y = e_k^1 \circ x \circ e_k^1\]  

(35)

defines a proper orthochronous Lorentz transformation. All proper orthochronous Lorentz transformations are obtained in this way.
Proof. From Theorems III.1 and III.2 we find that this defines a proper orthochronous Lorentz transformation. In this case it is also easy and instructive to give a direct proof that only proper orthochronous Lorentz transformations are obtained. We introduce a real parameter \(0 \leq s \leq 1\) and consider the expression

\[
y = e^{sK} \otimes x \otimes e^{sK^*}
\]

(36)
defining Lorentz transformations \(y = \hat{L}_s \cdot x\). We know from Part I that \(\det \hat{L}_s = \pm 1\) and \(e_0 \cdot \hat{L}_s(e_0) \geq 1\) for any Lorentz transformation and that the proper orthochronous Lorentz transformations are determined by the conditions \(\det \hat{L}_s = 1\) and \(e_0 \cdot \hat{L}_s(e_0) \leq -1\). By continuity it follows that if some value of \(s\) correspond to a proper orthochronous Lorentz transformation, then this is true for all \(s\). But \(s = 0\) corresponds to the identity transformation and thus also \(\hat{L}_1\) is a proper orthochronous Lorentz transformation.

In Theorem III.1 of Part I a proper orthochronous Lorentz transformation is expressed as an exponential of \(K \in M^4\), i.e.,

\[
y = e^{K \cdot x}
\]

(37)

An observer \(e_0\) may write \(K = e_0 \cdot \xi + [\omega \times]\) where \(\xi, \omega \in e_0^\perp\) and the transformation becomes

\[
y = e^{e_0 \cdot \xi + [\omega \times] \cdot x}
\]

(38)

In terms of a normal basis \(e = [e_0 \ e_1 \ e_2 \ e_3]\) we write \(\xi = \xi^1 e_1 + \xi^2 e_2 + \xi^3 e_3, \omega = \omega^1 e_1 + \omega^2 e_2 + \omega^3 e_3\) and get the matrix equation (using formulas from Section IV in Part I)

\[
y_e = e^{M} x_e
\]

(39)

where

\[
M = K_{e\eta_e} = \begin{bmatrix}
0 & \xi^1 & \xi^2 & \xi^3 \\
0 & 0 & -\omega^3 & \omega^2 \\
\xi^1 & \omega^3 & 0 & -\omega^1 \\
\xi^2 & -\omega^2 & \omega^1 & 0
\end{bmatrix}
\]

(40)

The next theorem give the exact relation between the proper orthochronous Lorentz transformations expressed by equation (38) and (in spinor form) by (35).

Theorem III.4. The proper orthochronous Lorentz transformation (38) may be written

\[
y = e^{k \otimes x \otimes k^*}
\]

(41)

where \(k = \left(\xi - i\omega\right) / 2\)

Proof. Taking \(a = \xi, b = \omega\) and \(C = y\) in equation (31) we find

\[
\left(e_0 \wedge \xi + [\omega \times]\right) \cdot y = \text{Re} \left(\left(\xi - i\omega\right) \otimes y\right)
\]

(42)

Let us now introduce a real parameter \(s\) and define \(y(s)\) and \(Y(s)\) by

\[
y(s) = e^{sK \cdot x} \quad \text{and} \quad Y(s) = e^{sK \otimes x \otimes e^{sK^*}}
\]

(43)

We must prove that \(y(1) = Y(1)\). From the first equation in (43) we find

\[
y'(s) = K \cdot y(s) \equiv \left(e_0 \wedge \xi + [\omega \times]\right) \cdot y(s)
\]

(44)

By using (42) we get

\[
y'(s) = \text{Re} \left(\left(\xi - i\omega\right) \otimes y(s)\right)
\]

(45)

From equation the second equation in (43) we obtain

\[
Y'(s) = k \otimes Y(s) + Y(s) \otimes k^*
\]

\[
= k \otimes Y(s) + \left(k \otimes Y(s)\right)^*
\]

\[
= 2 \text{Re} \left(k \otimes Y(s)\right)
\]

(46)

Note that in order to get the \(k^\ast\)-term in (46) we used

\[
\frac{d}{ds}e^{sK^k} = K \otimes e^{sK^k} = e^{sK^k} \otimes K^k
\]

(47)

We substitute \(k = (\xi - i\omega) / 2\) in (46) and get

\[
Y'(s) = \text{Re} \left(\left(\xi - i\omega\right) \otimes Y(s)\right)
\]

(48)

Thus \(y(s)\) and \(Y(s)\) satisfy the same equation and since \(y(0) = Y(0) = x\) we get \(y(s) = Y(s)\) for all \(s\). The theorem follows.

The matrix form of (41) is obtained from the isomorphism \(\phi\) as

\[
y_{\sigma\sigma} = e^{k_{\sigma\sigma}} x_{\sigma\sigma} e^{k_{\sigma\sigma}} \quad \text{where} \quad k_{\sigma\sigma} = (\xi_{\sigma\sigma} - i\omega_{\sigma\sigma}) / 2
\]

(49)

Thus the problem of calculating the exponentials of 4-by-4 real matrices in equation (39) is now reduced to the calculation of exponentials of 2-by-2 complex matrices. Even better, by using the first expression of (34) in (41) we get an expression for a general proper orthochronous Lorentz transformations in closed form.

IV. RELATIVISTIC ELECTRODYNAMICS

We consider the spacetime geometric equations from Part I for the worldline equation of a charged particle in an electromagnetic dyadic field \(F\)

\[
m \frac{d^2 x}{d\tau^2} = qF(x) \cdot \frac{dx}{d\tau}
\]

(50)

and the Maxwell’s equations

\[
\nabla \cdot F = -\mu_0 J, \quad \nabla \cdot \ast F = 0
\]

(51)
We will derive the corresponding complex equations. We consider also the spacetime geometric equations from Part I for the stress-energy dyadic field

\[ T = \frac{1}{\mu_0} \left( F \cdot F + \frac{1}{4} (F \cdot F) I \right) \]  

which satisfies

\[ T^T = T, \quad \text{tr} \, T = 0, \quad \nabla \cdot T + F \cdot J = 0 \]  

The 4D nabla operator is

\[ \nabla \equiv -e_0 \partial_0 + \nabla \equiv e^\mu \partial_\mu \]  

where \( \partial_\mu = \frac{\partial}{\partial x^\mu} \), \( e^0 = -e_0 \) and \( e^j = e_j \) for \( j = 1, 2, 3 \). We get from (11) that

\[ \nabla \otimes C = -C_\mu e^\mu \partial_\mu \]  

where \( C \) is a complex 4-vector field. Here the gradient, divergence and curl for the observer \( e_0 \) appears. Let us write the equalities (30) and (31) in notations (51) for all \( C \in \mathbb{M} \).

\[ \nabla \otimes f^\mu = \mu_0 J \]  

We then substitute \( C \rightarrow \nabla \) and find that Maxwell’s equations (58) and (60) may be written

\[ \nabla \otimes f^\mu = \mu_0 J \]  

where \( J = c \rho e_0 + \vec{J} \). We may check this result directly.

The fields \( E, B \) and \( \vec{J} \) take values in the ordinary 3D space \( e_0^\mu \) of the observer. By use of equation (55) in (60) we get all of Maxwell’s equations

\[ \nabla \times B = \mu_0 \left( \vec{J} + \varepsilon_0 \frac{\partial E}{\partial t} \right), \quad \nabla \cdot E = \frac{\rho}{\varepsilon_0} \]  

and

\[ \nabla \times E = -\frac{\partial B}{\partial t}, \quad \nabla \cdot B = 0 \]  

The equations (58) and (60) are not in spacetime geometric form since they depend on the observer \( e_0 \). The complex electromagnetic field is not a spacetime geometric object but depends on the observer so in the case of two different observers \( e_\alpha \) and \( e_0 \) we get

\[ E'/c + iB' \neq E/c + iB \]  

The electromagnetic 4-dyadic is a spacetime geometric object so that

\[ c^{-1} e_\alpha \wedge E' - [B' \times] = c^{-1} e_0 \wedge E - [B \times] \]  

The transformation of the complex electromagnetic field is obtained from equation (66). We get \( f' \otimes' C = f \otimes C \) so by choosing \( C = e_\alpha \) we find

\[ E'/c + iB' = (E/c + iB) \otimes e_\alpha \]  

We next consider the stress-energy conservation. The stress-energy dyadic field \( T \) is defined by equation (52).

We will show that

\[ T \cdot C = -\frac{1}{2\mu_0} (f \otimes C \otimes f^\ast) \]  

for all \( C \in \mathbb{M} \).

By use of equation (11) we get from (60) after some algebra

\[ T = \frac{1}{2\mu_0} (f \otimes f^\ast) e_0 + \frac{i}{2\mu_0} (f \times f^\ast) \vee e_0 - \frac{1}{2\mu_0} f \cdot f^\ast I \]  

where \( I = I + e_0 \). By use of \( f = E/c + iB \) we find

\[ \frac{1}{2\mu_0} f \cdot f^\ast = \frac{\varepsilon_0}{2} E \cdot E + \frac{1}{2\mu_0} B \cdot B \equiv U, \]  

\[ \frac{i}{2\mu_0} f \times f^\ast = \frac{1}{\mu_0} E \times B/c \equiv S/c, \]  

\[ \frac{1}{2\mu_0} f \cdot f^\ast = \varepsilon_0 E \otimes E + \frac{1}{\mu_0} B \otimes B \]  

and get

\[ T = U e_{00} + (S/c) \vee e_0 - \left( \frac{\varepsilon_0}{\mu_0} E \otimes E + \frac{1}{\mu_0} B \otimes B - U I \right) \]  

This is the same stress-energy expression as we obtained in Part I. Thus, the equations (52) and (60) defines the same stress-energy tensor. Let us now derive the corresponding conservation law. Using (66) the 4-divergence of the stress-energy tensor may be written

\[ \nabla \cdot T = \delta_\mu (e^\mu \cdot T) = -\frac{1}{2\mu_0} \delta_\mu (f \otimes e^\mu \otimes f^\ast) \]  

From Maxwell’s equations (60) we get

\[ \text{Re} \left( f \otimes \nabla \otimes f^\ast \right) = \mu_0 \text{Re} (f \otimes J) \]
where we can rewrite the left hand side as

$$\text{Re}(f \otimes \nabla \otimes f^*) = \text{Re}(f \otimes e^\mu \otimes \delta_\mu f^*) = \frac{1}{2} \delta_\mu (f \otimes e^\mu \otimes f^*)$$  \hfill (72)

From (57) and (70)–(72) we now get the conservation law

$$\nabla \cdot T + F \cdot J = 0$$  \hfill (73)

V. DISCUSSION

The algebra $\mathbb{C}^{2 \times 2}$ of complex 2-by-2 matrices is useful in special relativity. We introduce this algebra structure by using the product (11) on the complex Minkowski space. The equations from Part I for a proper orthochronous Lorentz transformation (37), for the worldline of a charged particle in an electromagnetic field (50), for Maxwell’s equations (51), and for the stress energy tensor (52) are all written without coordinates or reference to any observer, i.e., as spacetime geometric equations. They have now in Part II been rewritten as complex equations (41), (58), (60) respectively (64). These have some attractive features. The representation (41) of proper orthochronous Lorentz transformations is easier to use than (37) because, in terms of coordinates, we get exponentials of complex 2-by-2 matrices rather than 4-by-4 real matrices. The worldline equation in complex form may simplify the calculation of charged particle motion.\(^{15,16}\) The complex Maxwell equations are written as one single equation and the algebra needed for deriving the conservation of stress-energy becomes more straightforward.

The complex formulation of special relativity requires only a few simple new concepts. The complex Minkowski space and the associated operation of complex conjugation should not cause any difficulties. The product (11) on $\mathbb{M}^C$ is not very mysterious since it is expressed in terms of Gibbs’ vector algebra and the product is obtained from the familiar matrix product in $\mathbb{C}^{2 \times 2}$. From a complex 4-vector $A$ we have the observer dependent operations “arrow” and “tilde” so that $\tilde{A}$ is the spatial part of $A$, i.e., a vector in $e_0^C$, and $\hat{A} = A - 2\tilde{A}$ denotes spatial reversion of $A$. Much of the algebra while dealing with the complex equations is straightforward Gibbs’ vector calculus on complex 3-vector fields.

The complex equations are not spacetime geometric because some observer $e_0$ appears explicitly in their formulation. However, these equations may be reformulated in a spacetime geometric form so that they are still in “spinor form” but without any reference to coordinates or observer. To do this we need to actually define the concept of 2-spinors. The present paper prepares for this introduction of 2-spinors.\(^{17}\) A very brief sketch of the ideas behind this development is now given. The group of proper orthochronous Lorentz transformations $L^I_4$ is closely associated with space of 4-vectors, i.e., with the real 4D vectorspace $\mathbb{M}$. It is also by equation (4) closely related to the apparently much simpler group $SL(2, \mathbb{C})$. A guiding idea now is to look for a theory founded on the group $SL(2, \mathbb{C})$ rather than on $L^I_4$. It is then natural to introduce a new kind of vectors that are elements in a 2D complex vectorspace $\mathbb{S}$ (these are first order 2-spinors and also called “spin-vectors”). We would like $\mathbb{S}$ to be associated with the group $SL(2, \mathbb{C})$ just like $\mathbb{M}$ is associated with $L^I_4$. The formalism of Part I with an axiomatic definition of $\mathbb{M}$ makes it quite simple to find a suitable axiomatic definition of $\mathbb{S}$. This is a good start towards 2-spinors but there turns out to be a problem that must be solved. We would like to construct the 4-vectors in terms of spin-vectors, however, a direct construction of the 4D real vectorspace $\mathbb{M}$ in terms of the 2D complex vectorspace $\mathbb{S}$ seems not to be possible. We need some kind of complex conjugation and the solution is to define a second space of spin-vectors $\bar{\mathbb{S}}$ that is distinct from but axiomaticaly the same as $\mathbb{S}$, together with a “complex conjugation map” $\mathbb{S} \rightarrow \bar{\mathbb{S}}$. It turns out that the vector space $\mathbb{S} \otimes \bar{\mathbb{S}}$ of second order 2-spinors then naturally becomes the 4D complex Minkowski space $\mathbb{M}^C$. We may then express equations in terms of spin-vectors rather than 4-vectors. The spacetime geometric equations of Part I may be written as spacetime geometric 2-spinor equations. The spacetime split of these 2-spinor equations becomes exactly equations found in this Part II.

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\[^2\] An “algebra” is a vector space with a product between vectors satisfying certain linearity conditions. Important examples are spaces of real or complex square matrices with matrix multiplication as product.
ical physics. They have even been called “Nature’s natural numbers” by Edmonds.


10 The representation of a proper orthochronous Lorentz transformation by use of equation (4) is related to spinor theory. This is the case even though we have not actually defined any spinors, a concept that builds on group theory. The underlying group here is the matrix group $SL(2, \mathbb{C})$ consisting of all complex 2-by-2 matrices with determinant 1. This group is by means of equation (4) closely related to the group of proper orthochronous Lorentz transformations $L^1_0$ but it is still a different group. It is obvious from equation (4) that any two matrices $\pm M \in SL(2, \mathbb{C})$ are associated with one and the same element in $L^1_0$. In fact equation (4) defines a $2 \rightarrow 1$ group homomorphism from $SL(2, \mathbb{C})$ to $L^1_0$. This homomorphism is called “the spinor map”. In the theory of 2-spinors for spacetime one in some way consider $SL(2, \mathbb{C})$ as being more fundamental than $L^1_0$.

11 H. L. Berk, K. Chaicherdsakul, and T. Udagawa, “The proper homogeneous Lorentz transformation operator $e^{i\omega S - \xi \hat{K}}$: Where’s it going, what’s the twist?” Amer. J. Phys. 69, 996–1000 (2001). In this paper the standard result for a general proper orthochronous Lorentz transform, expressed as the exponential of a real 4-by-4 matrix, is transformed to the corresponding spinor form involving exponentials of 2-by-2 complex matrices. We show in this Part II how this derivation may be much simplified.

12 The reader with some familiarity of Clifford algebras will note that this product is similar to the product in the algebra $\text{Cl}_3$.

13 The overbar is sometimes used for this operation. However, the overbar is needed for another purpose in a forthcoming paper about 2-spinors.

14 $\cosh w = 1 + \frac{w^2}{2} + \frac{w^4}{4!} + \cdots$, $\sinh w = \frac{w}{1!} + \frac{w^3}{3!} + \cdots$


17 We will address this in a forthcoming paper introducing 2-spinors.