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On avoiding and completing colorings

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Abstract

This thesis consists of the following papers.

I Carl Johan Casselgren, Klas Markström, Lan Anh Pham, *Restricted extension of sparse partial edge colorings of hypercubes*, submitted (2017).

II Carl Johan Casselgren, Klas Markström, Lan Anh Pham, *Edge precoloring extension of hypercubes*, submitted (2017).

III Lan Anh Pham, *On restricted colorings of* (d, s)*-edge colorable graphs*, submitted (2019).

IV Carl Johan Casselgren, Klas Markström, Lan Anh Pham, *Latin cubes with forbidden entries*, The electronic journal of combinatorics, 26 (2019), 18 pp.

V Carl Johan Casselgren, Lan Anh Pham, *Latin cubes of even order with forbidden entries*, submitted (2019).

These papers are all related to the problem of avoiding and completing an edge precoloring of a graph. In more detail, given a graph G and a partial proper edge precoloring φ of G and a list assignment L for every non-colored edge of G, can we extend φ to a proper edge coloring of G which avoids L?

In Paper I, G is the d-dimensional hypercube graph Q_d, a partial proper edge precoloring φ and a list assignment L must satisfy certain sparsity conditions. Paper II still deals with the hypercube graph Q_d, but the list assignment L for every edge of Q_d is an empty set and φ must be a partial proper edge precoloring of at most $d-1$ edges. In Paper III, G is a (d, s)-edge colorable graph; that is G has a proper d-edge coloring, where every edge is contained in at least $s-1$ 2-colored 4-cycles, L must satisfy certain sparsity conditions and we do not have a partial proper edge precoloring φ on edges of G. The problem in Paper III is also considered in Paper IV and Paper V, but here G can be seen as the complete 3-uniform 3-partite hypergraph $K^3_{n,n,n}$, where n is a power of two in paper IV and n is an even number in paper V.

Contents

1 Introduction

A graph is a collection of points (called vertices) and lines between some pairs of vertices (called edges). Graph theory is the branch of mathematics that studies graphs, which was initiated by Leonhard Euler in 1763 when he tried to solve the "Königsberg bridges" problem. One of the most famous problem types in the field of graph theory is graph coloring, which can be used to formulate a large number of practical problems, e.g., scheduling, timetable and register allocation. The study of graph coloring started in 1852 when Francis Guthrie observed that only four colors were needed to color a map of the counties of England in order to ensure that no neighboring counties had the same color. If we form a graph by representing each county with a vertex and drawing an edge between any pair of vertices if the corresponding counties share a border on the map, then the problem of coloring a map translates into a graph coloring problem. To be more specific, a coloring of a graph is an assignment of colors to the vertices (or edges) in such a way that any two adjacent vertices (or edges) receive different colors.

In this thesis, we consider the following edge coloring problem: Given a list of acceptable colors to every edge, is it possible to find a coloring so that each edge gets a color from its list of allowed colors and no two adjacent edges share the same color? A complete bipartite graph $K_{p,q}$ is a graph in which the set of vertices can be decomposed into two disjoint sets with p and q vertices such that no two vertices within the same set are adjacent and every pair of vertices from the different sets are adjacent. Dinitz conjectured, and Galvin proved [1], that if each edge of a complete bipartite graph $K_{n,n}$ is given a list of n colors, then there is a proper edge coloring of $K_{n,n}$ with support in the lists. Motivated by the Dinitz's problem, Häggkvist [2] introduced the notion of *βn-array*, which corresponds to the list assignment L of forbidden colors for $E(K_{n,n})$, such that each edge e of $K_{n,n}$ is assigned a list $L(e)$ of at most βn forbidden colors from $\{1, \dots, n\}$, and at every vertex v each color is forbidden on at most βn edges adjacent to v; we call such a list assignment for $K_{n,n}$ *β-sparse*. Häggkvist conjectured that there exists a fixed $\beta > 0$, in fact also that $\beta = \frac{1}{3}$, such that for every positive integer n, every β-sparse list assignment for $K_{n,n}$, there is a proper n-edge coloring φ of $K_{n,n}$ that *avoids* the list assignment L, i.e., $\varphi(e) \notin L(e)$ for every edge e of $K_{n,n}$. That such a $\beta > 0$ exists was proved by Andrén in her PhD thesis [3]. We demonstrate that a similar result holds for (d, s)-edge colorable graphs, where a (d, s)-edge colorable graph is a graph that has a proper d-edge coloring, where every edge is contained in at least $s-1$ 2-colored 4-cycles, and the complete 3-uniform 3-partite hypergraph $K^3_{n,n,n}$(where n is a power of two and n is an even number) in three last papers.

In the second paper we first prove that every proper partial edge coloring of at most $d-1$ edges of the d-dimensional hypercube graph Q_d can be extended to a proper d-edge coloring of Q_d. A similar edge precoloring extension problem appeared already in 1960, when Evans [4] stated his now classic conjecture that for every positive integer n, if $n-1$ edges in $K_{n,n}$ have been (properly) colored, then this partial coloring can be extended to a proper n-edge-coloring of $K_{n,n}$. This conjecture was solved for large n by Häggkvist [5] and later for all n by Smetaniuk [6], and independently by Andersen and Hilton [7]. Moreover, Andersen and Hilton [7] characterized which $n \times n$ partial Latin squares with exactly n non-empty cells are extendable, we establish an analogue result for hypercube graphs by proving that a proper precoloring φ of at most d edges in Q_d is always extendable unless the precoloring φ satisfies some specific conditions.

Generalizing Evans' problem, Daykin and Häggkvist [8] proved several results on extending partial edge colorings of $K_{n,n}$, and they also conjectured that much denser partial colorings can be extended, as long as the colored edges are spread out in a specific sense: a partial n-edge coloring of $K_{n,n}$ is ϵ-*dense* if there are at most ϵn colored edges from $\{1, \ldots, n\}$ at any vertex and each color in $\{1, \ldots, n\}$ is used at most ϵn times in the partial coloring. Daykin and Häggkvist [8] conjectured that for every positive integer n, every $\frac{1}{4}$-dense partial proper n-edge coloring can be extended to a proper n-edge coloring of $K_{n,n}$ and proved a version of the conjecture for $\epsilon = o(1)$ (as $n \to \infty$) and n divisible by 16. Andrén et al. [9] proved that there are constants $\alpha > 0$ and $\beta > 0$, such that for every positive integer n, every α-dense partial edge coloring of $K_{n,n}$ can be extended to a proper n-edge-coloring avoiding any given β-sparse list assignment L, provided that no edge e is precolored by a color that appears in $L(e)$.

The aim of the first paper is to investigate this type of problem for the family of hypercube graphs. We show that for a given partial proper d-edge coloring of the d-dimensional hypercube graph Q_d, and lists of forbidden colors for the non-colored edges of Q_d, if both the partial coloring and the color lists satisfy certain sparsity conditions, then it is possible to extend the partial coloring to a proper d-edge coloring avoiding colors from the lists. Our basic idea is similar to the one in [9], but the proof contains more technical details since compared to the complete bipartite graph $K_{n,n}$, the hypercube graph Q_d is sparser. Both the results in [9] and the earlier papers on complete bipartite graphs bound the global density of precolored edges, and one can prove a result for Q_d using such global constraints. However, due to the much lower vertex degrees in Q_d such a direct analogue of the earlier results is quite weak, with the proportion of precolored edges going to 0 with d. This is not merely a technical problem since one can construct non-extendable precolorings using a vanishing fraction of the edges. So, instead we shall bound the number of precolored edges appearing in neighborhoods of given size.

2 Summary of papers

2.1 Paper I: Restricted extension of sparse partial edge colorings of hypercubes

The *d-hypercube graph*, also called the d-cube graph and commonly denoted Q_d, is the graph whose vertices are the 2^d strings $a_1...a_d$ where $a_i = 0$ or 1 and two vertices are adjacent iff the strings differ in exactly one coordinate. Every vertex of the d-hypercube graph is incident to d edges and the total number of edges in Q_d is $2^{d-1}d$. A *dimensional matching* M of Q_d is a perfect matching of Q_d such that $Q_d - M$ is isomorphic to two copies of Q_{d-1}, evidently there are precisely d dimensional matchings in Q_d. The *distance* between two edges e and e' is the number of edges in a shortest path between an endpoint of e and an endpoint of e', the *t-neighborhood* of an edge e is the graph induced by all edges of distance at most t from e. An edge precoloring of Q_d with colors $1, \dots, d$ is called *α-dense* if

(i) there are at most αd precolored edges at each vertex;

(ii) for every 27-neighborhood W of an edge e of Q_d, there are at most αd precolored edges with color i in W, $i = 1, \dots, d$;

(iii) for every 27-neighborhood W, and every dimensional matching M, at most αd edges of M are precolored in W.

A list assignment L for $E(Q_d)$ is *β-sparse* if the list of each edge is a (possibly empty) subset of $\{1, \dots, d\}$, and

(i) $|L(e)| \leq \beta d$ for each edge $e \in E(Q_d)$;

(ii) for every vertex $v \in V(Q_d)$, each color in $\{1, \dots, d\}$ occurs in at most βd lists of edges incident to v;

(iii) for every 27-neighborhood W, and every dimensional matching M, any color appears at most βd times in lists of edges of M contained in W.

In this paper, we prove the following theorem: *There are constants $\alpha > 0$ and $\beta > 0$ such that for every positive integer d, if φ is an α-dense d-edge precoloring of Q_d, L is a β-sparse list assignment for Q_d, and $\varphi(e) \notin L(e)$ for every edge $e \in E(Q_d)$, then there is a proper d-edge coloring of Q_d which agrees with φ on any precolored edge and which avoids L.*

For the proof we shall start with the *standard* d-edge coloring h of Q_d where all edges of the ith dimensional matching in Q_d are colored i, $i = 1, \dots, d$. If h is a proper d-edge coloring of Q_d which agrees with φ on every precolored edge and which avoids L, then we are done. Otherwise, there will be some *unsatisfied edges* (e.g. an edge e such that $h(e) \neq \varphi(e)$ or $h(e) \in L(e)$). We will find a permutation ρ of the elements of the set $\{1, \dots, d\}$ such that in the proper d-edge-coloring h' obtained by applying ρ to the colors used in h, each vertex of Q_d and each dimensional matching in Q_d contain "sufficiently few" unsatisfied edges. Next, we define a suitable new edge precoloring φ' such that for every edge e of Q_d, $\varphi'(e) = \varphi(e)$ and $\varphi'(e) \notin L(e)$ and the number of unsatisfied edges (e.g. $\varphi'(e) \neq h'(e)$) is sufficiently few. Finally, for each edge e of Q_d such that $h'(e) \neq \varphi'(e)$, we construct a subset of edges of Q_d such that performing a series of swaps on the coloring of these subsets yields a proper d-edge coloring of Q_d which is an extension of φ' (and thus φ), and which avoids L.

2.2 Paper II: Edge precoloring extension of hypercubes

We begin this paper by giving a short proof for the theorem: *Let $d \geq 2$ be a positive integer. If φ is a proper precoloring of at most $d-1$ edges of the hypercube Q_d, then φ can be extended to a proper d-edge coloring of Q_d.* The remaining part of this paper characterizes which partial edge colorings of Q_d with precisely d precolored edges are extendable to proper d-edge colorings of Q_d. We denote by $\mathcal{C}$ the set of all the colorings of Q_d, $d \geq 1$, satisfying any of the following conditions:

- there is an uncolored edge uv in Q_d such that u is incident with edges of $k \leq d$ distinct colors and v is incident to $d-k$ edges colored with $d-k$ other distinct colors (so uv is adjacent to edges of d distinct colors);
- there is a vertex u that is incident with edges of $d-k$ distinct colors $c_1, \dots, c_{d-k}$, and k vertices $v_1, \dots, v_k$ such that for $i = 1, \dots, k$, uv_i is uncolored but v_i is incident with an edge colored $c \notin \{c_1, \dots, c_{d-k}\}$;
- there is a vertex u such that every edge incident with u is uncolored but there is a color c satisfying that every edge incident with u is adjacent to another edge colored c;

- $d = 3$ and the precolored three edges use three different colors and is a subset of a dimensional matching.

Clearly, if φ is a precoloring of Q_d with exactly d precolored edges and $\varphi \in \mathcal{C}$, then φ is not extendable. We show that if φ is a proper d-edge precoloring of Q_d with exactly d precolored edges and $\varphi \notin \mathcal{C}$, then φ is extendable to a proper d-edge coloring of Q_d. The proof is quite long but the idea is not complicated. Using induction method, we consider the 3-dimensional hypercube graph Q_3 for the base case, the induction step is solved based on the observation that if M is a dimensional matching in Q_d and H_1 and H_2 are the components of $Q_d - M$, then H_1 and H_2 are both isomorphic to Q_{d-1}.

In addition to the main results, we also consider the problem of completing an edge precoloring of a hypercube graph in some other cases. We prove that an edge precoloring of Q_d is extendable if the precolored edges of Q_d form an induced matching all edges of which lie in two dimensional matchings or if the precolored edges of Q_d lie in two hypercube graphs contained in Q_d that satisfy some extra conditions.

2.3 Paper III: On restricted colorings of (d,s)-edge colorable graphs

Given a complete bipartite graph G and a list assignment L of forbidden colors for the edges of G satisfying certain sparsity conditions, Andrén [3] proves that there is a proper edge coloring of G that avoids L. This problem is also solved for a hypercube graph in Paper I. Hence, it is natural to ask if a similar result holds for other graphs as well. Paper III gives us an answer to this question for so-called (d,s)-edge colorable graph.

A cycle is 2-*colored* if its edges are properly colored by two distinct colors. A *d-regular graph* is a graph where each vertex is incident to d edges. A *(d,s)-edge colorable graph* G is a d-regular graph that admits a proper d-edge coloring h in which every edge of G is in at least $s-1$ 2-colored 4-cycles. The d-edge coloring h is called *standard coloring*. A *standard matching* of G is a maximum set of edges of G all of which have the same color in the standard coloring. Throughout, we shall assume that the standard coloring h for the edges of G uses the set of colors $\{1,\dots,d\}$. Next, similarly to Paper I, using the colors $\{1,\dots,d\}$ we define a list assignment for the edges of a (d,s)-edge colorable graph.

A list assignment L for a (d,s)-edge colorable graph G is *β-sparse* if the list of each edge is a (possibly empty) subset of $\{1,\ldots,d\}$, and

(i) $|L(e)| \leq \beta s$ for each edge $e \in G$;

(ii) for every vertex $v \in V(G)$, each color in $\{1,\ldots,d\}$ occurs in at most βs lists of edges incident to v;

(iii) for every 6-neighborhood W, and every standard matching M, any color appears at most βs times in lists of edges of M contained in W.

Our main theorem of this paper is: *Let G be a (d,s)-edge colorable graph of order n and L be a β-sparse list assignment for G. If $s \geq 11$ and $\beta \leq 2^{-11}sd^{-1}(2n)^{-2^9ds^{-2}}$, then there is a proper d-edge coloring of G which avoids L.*

We prove the main theorem by using the basic technique from Paper I. Note that the d-dimensional hypercube graph Q_d is a (d,d)-edge colorable graph, thus this theorem generalizes the result in Paper I. Finally, in the remaining part of this paper, we examine some other graphs that belong to the family of (d,s)-edge colorable graphs.

2.4 Paper IV: Latin cubes with forbidden entries

A *Latin cube* L of order n on the symbols $\{1,\ldots,n\}$ is an $n\times n\times n$ cube (n rows, n columns, n files) such that each symbol in $\{1,\ldots,n\}$ appears exactly once in each row, column and file. An $n\times n\times n$ cube where each cell contains a subset of the symbols in the set $\{1,\ldots,n\}$ is called an *(m,m,m,m)-cube (of order n)* if the following conditions are satisfied:

(a) No cell contains a set with more than m symbols.

(b) Each symbol occurs at most m times in each row.

(c) Each symbol occurs at most m times in each column.

(d) Each symbol occurs at most m times in each file.

Let $A(i,j,k)$ denote the set of symbols in the cell (i,j,k) of A, let $L(i,j,k)$ denote the symbol in position (i,j,k) of L. We say that those cells (i,j,k) of L where $L(i,j,k) \in A(i,j,k)$ are *conflict cells of L with*

A (or simply *conflicts* of L). A Latin cube L of order n *avoids* A if L does not contain any conflict cells; if there is such a Latin cube, then A is *avoidable*. A *subcube of order 2* (or just *subcube*) in L is a set of eight cells $\{(i_1, j_1, k_1), (i_1, j_2, k_1), (i_2, j_1, k_1), (i_2, j_2, k_1), (i_1, j_1, k_2), (i_1, j_2, k_2), (i_2, j_1, k_2), (i_2, j_2, k_2)\}$ in L such that

$$L(i_1, j_1, k_1) = L(i_2, j_2, k_1) = L(i_1, j_2, k_2) = L(i_2, j_1, k_2) = x_1$$

and

$$L(i_1, j_2, k_1) = L(i_2, j_1, k_1) = L(i_1, j_1, k_2) = L(i_2, j_2, k_2) = x_2.$$

An *allowed subcube* of L is a subcube $\mathcal{C}$ in L such that swapping the two symbols x_1, x_2 of $\mathcal{C}$ yields a Latin cube where non of the cells of $\mathcal{C}$ is a conflict. Given an integer t, let a_i $(1 \leq i \leq 2^t)$ be the ith smallest element of $\mathbb{Z}_2^t$. The *Boolean Latin cube* B of order $n = 2^t$ on the symbols $\{1, \ldots, n\}$ is an $n \times n \times n$ Latin cube such that $B(i, j, k) = x$ with $a_x = a_i + a_j + a_k$ (addition in $\mathbb{Z}_2^t$) for all $1 \leq i, j, k \leq n$.

This paper answers the following question: If every cell of a cube is assigned a list of symbols satisfying certain sparsity conditions, is there any Latin cube avoiding these lists? Our main result is the following: *There is a positive constant γ such that if $t \geq 30$ and $m \leq \gamma 2^t$, then any (m, m, m, m)-cube A of order 2^t is avoidable.*

The basic proof strategy is similar to the one in [10]; however, due to the extra dimension in a Latin cube, our arguments are considerably more involved and somewhat technical. Our starting point in the proof is the Boolean Latin cube; we permute its row layers, column layers, file layers and symbols so that the resulting Latin cube does not have too many conflicts with a given (m, m, m, m)-cube A. After that, we find a set of allowed subcubes such that each conflict belongs to one of them, with no two of the subcubes intersecting, and swap on those subcubes.

Let us consider the complete 3-uniform 3-partite hypergraph $K^3_{n,n,n}$ in which its vertices can be partitioned into three sets of size n such that e is an edge of $K^3_{n,n,n}$ iff e contains exactly one vertex from each set. If we view $K^3_{n,n,n}$ as a cube C and an edge of $K^3_{n,n,n}$ as a cell in C, then from another perspective, the problem of avoiding an (m, m, m, m)-cube of order n is actually the problem of avoiding list assignment of forbidden colors for every edge of the complete 3-uniform 3-partite hypergraph $K^3_{n,n,n}$.

2.5 Paper V: Latin cubes of even order with forbidden entries

In this paper we consider the problem of avoiding (m, m, m, m)-cubes of even order; generalizing the main result in Paper IV as following: *There is a positive constant γ such that if $t \geq 2^{30}$ and $m \leq \gamma 2t$, then any (m, m, m, m)-cube A of order $2t$ is avoidable.*

Since the Boolean Latin cube of order n only exists when n is a power of two, our starting point in the proof is another Latin cube called the starting Latin cube. It is defined as follows:

The *starting Latin cube* L of order $n = 2t$ on the symbols $\{1, \dots, n\}$ is an $n \times n \times n$ Latin cube such that

$$L(i,j,k) = \begin{cases} j - i + k \mod t & \text{for} \quad i, j, k \leq t, \\ j - i + k \mod t & \text{for} \quad i, k > t, j \leq t, \\ i - j + k \mod t & \text{for} \quad i, j > t, k \leq t, \\ i - j + k \mod t & \text{for} \quad k, j > t, i \leq t, \\ (j - i + k \mod t) + t & \text{for} \quad i, k \leq t, j > t, \\ (j - i + k \mod t) + t & \text{for} \quad i, j, k > t, \\ (i - j + k \mod t) + t & \text{for} \quad k, j \leq t, i > t, \\ (i - j + k \mod t) + t & \text{for} \quad i, j \leq t, k > t. \end{cases}$$

Because the starting Latin cube does not have as much symmetry and useful properties as the Boolean Latin cube, to prove this result we need to significantly extend the machinery from Paper IV.

3 Concluding remarks and further problems

The proof of the main theorem in Paper I holds when $\alpha = (2^{-3-1/\gamma}\gamma)/e$ and $\beta = (2^{-2-1/\gamma}\gamma)/e$ with $\gamma = 2^{-21}$. We show an upper bound on the values of α and β ; namely that $\alpha + \beta < \frac{1}{2}$. An interesting open question is what the optimal values for α and β are.

In Paper II we have obtained analogues for hypercubes of some classic results on completing partial Latin squares; in general we believe it might be possible to obtain similar results for $(K_{n,n})^d$. Here, G^d denotes the dth power of the Cartesian product of G with itself.

In Paper III, our proof relies heavily on the fact that every edge is contained in a large number of 2-colored 4-cycles. It would be interesting to investigate if a similar result holds for graphs containing a certain amount of 2-colored $2c$-cycles ($c \in \mathbb{N}$, $c > 2$).

Paper IV and Paper V show that if n is a power of two or n is an even number then there exists a Latin cube of order n avoiding any given (m, m, m, m)-cube of order n, where $m \leq \gamma n$, $n \geq \kappa$ for some constants γ, κ. Note that the problem of finding a Latin square of order n avoiding any given (m, m, m)-array of order n ($n \in \mathbb{N}$) was solved previously. Thus, a natural direction for future research is to consider similar problems for Latin cubes of higher dimension, and also for Latin cubes of odd order.

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