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Cut Finite Element Methods on Parametric Multipatch Surfaces

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ISBN 978-91-7855-019-7
ISSN 1653-0810
Typeset by the author using LaTeX
Printed by: UmU Print Service, Umeå University, Umeå, 2019

List of papers

This thesis consists of an introduction to the area and following papers:

Paper I. T. Jonsson, M. G. Larson, and K. Larsson. Cut finite element methods for elliptic problems on multipatch parametric surfaces. *Comput. Methods Appl. Mech. Engrg.*, 324:366–394, 2017.

Paper II. P. Hansbo, T. Jonsson, M. G. Larson, and K. Larsson. A Nitsche method for elliptic problems on composite surfaces. *Comput. Methods Appl. Mech. Engrg.*, 326:505–525, 2017.

Paper III. T. Jonsson, M. G. Larson, and K. Larsson. Graded Parametric CutFEM and CutIGA for Elliptic Boundary Value Problems in Domains with Corners. *arXiv e-prints*, Dec. 2018.

Acknowledgments

First of all I would like to thank my main advisor Professor Mats G. Larson for introducing me to the world of finite elements and providing me with his great knowledge and ideas during the work of this thesis.

I also want to give sincere gratitude towards my assistant advisor Dr Karl Larsson for his support, encouragement, and always having time for me.

Finally, I would like to thank Linnea and my family for all the love and support during this thesis.

Tobias Jonsson
Umeå, January 2019

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Chapter 1

Introduction

Partial differential equations are widely used to model real-world phenomena appearing in our everyday life. Examples are fluid flows, elasticity, heat diffusion, and many more. Typically it is not possible to find an exact solution to these problems, instead we rely on numerical methods to find approximated solutions. Since the solution is not exact, estimates of both the error magnitude and convergence rate is of high importance.

In this thesis we develop new Cut Finite Element methods (CutFEM) which is a framework for finite element methods (FEM). The main idea in FEM is to partition the computational domain into simpler geometric objects, called elements. They can be triangles, quadrilaterals, or tetrahedrons depending on the dimension. In contrast CutFEM allows the physical domain to cut through the grid of elements, which simplifies the mesh generation. In engineering the mesh creation is a time consuming and expensive task. There is an increasing interest for robust methods to deal with complex and/or evolving geometries where the standard methods are too cumbersome and expensive.

Traditionally, numerical methods have focused on techniques for obtaining approximations of the physical equations rather than the geometry. We choose to work with problems on surfaces, since they require accurate geometric description. Some applications are the modeling of lubrication, and stresses on elastic membranes.

A common approach to deal with surface geometry is by using computer-aided design programs (CAD software), which divides the surface into several trimmed patches. Each patch has a local representation and a transformation map from a reference domain onto the surface. An illustration of this parametric setting is the Earth's surface which can be represented by two angular coordinates (parameters) longitude and latitude. The land mass of Europe could be one patch, which would have a corresponding flat representation in a reference domain.

The outline of this thesis is as follows: In Section 2 we present the background and theory for the cut finite element method; in Section 3 the parametric setting is described and we show how to use it in weak formulation; and finally in Section 4 we give a brief summary of each paper.

1.1 Thesis objectives

The main objectives for this thesis have been:

- Develop methods which combines the parametric surface setting widely used in CAD together with the Cut Finite Element method.
- Implement and provide numerical examples.
- Derive a priori error estimates for the given methods.

1.2 Main results

- Developed a method for computations on multipatch parametric surfaces which is based on a fictitious domain method (CutFEM). This approach does not require the construction of conforming meshes for the trimmed patches. (Paper I)
- By adding a stabilization term, which extends the coerciveness to cut finite elements spaces, we can perform the standard error analysis independent of how the trim curves cut the mesh. (Paper I)
- We have developed a quadrature routine for integrating cut elements trimmed by a piecewise smooth curves. (Paper I)
- A new Nitsche method for a diffusion problem which can handle interfaces where more than two surfaces meet. The method is derived

based on a Kirchhoff condition where the sum of all co-normal fluxes are zero at the interface. (Paper II)

- We have developed a cut finite element method for elliptic problems with nonconvex corner singularities. From known information about the singularity a suitable map was created that grades the computational mesh towards the singularity. (Paper III)

- Proved a priori error estimates by using bounds on the derivatives in the reference domain in terms of a weighted norm in the physical domain. (Paper III)

Keywords: cut finite element method, Nitsche method, a priori error estimation, parametric map, interface problem

Chapter 2

The Cut Finite Element Method

Standard Finite Element Method. The main idea behind the finite element method (FEM) is to discretize the computational domain into simpler geometric objects, called elements. They can be triangles, quadrilaterals, or tetrahedrons depending on the dimension. A set of basis functions, typically polynomials, is derived from the elements to approximate the solution. We derive the method by rewriting the problem in weak form, which is obtained by multiplying with a test function and integrating by parts over the domain. Solution is then found by using variational methods in a finite dimensional subspace V_h of the infinite Hilbert space V and we are left to solve a linear system of equations. We refer to [2,3,11,14] for further reading.

Cut Finite Element Method. In engineering the mesh creation is a time consuming and expensive task. There is an increasing interest for robust methods to deal with complex and/or evolving geometries. The main motivation for the Cut finite element method (CutFEM) is to make the mesh discretization less dependent of the underlying geometry constraints [5]. In this approach the geometry is allowed to arbitrarily cut through the elements. We embed the domain of interest in a fixed background mesh with simple structure. Elements cut by the boundary could be arbitrarily small and cause badly conditioned system matrix and some stabilization is needed.

2.1 Model Problem

We would like to show the different approaches by considering the following problem: find u such that

$$-\Delta u = f \quad \text{in } \Omega \tag{2.1}$$

$$u = g \quad \text{on } \partial\Omega \tag{2.2}$$

where Ω is a bounded domain in $\mathbb{R}^2$ which is shown in Figure 2.1, with its boundary $\partial\Omega$, $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the Laplacian and f is a given function.

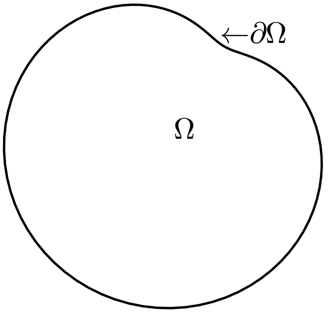


Figure 2.1: The computational domain Ω and its boundary $\partial\Omega$.

Weak Form. To solve the equation we start by rewriting the problem into its weak form, which is a variational equation. The weak formulation is useful for solving partial differential equations, since it holds in an average sense over the domain instead of absolutely satisfy the equation in each point.

The weak form is obtained by multiplying both sides of (2.1) with a test function v with $v = g$[1] on $\partial\Omega$ and integrating by parts over Ω.

[1]Here the boundary condition is strongly imposed, later we will present a method to weakly impose it.

$$\int_\Omega fv\,dx = -\int_\Omega \Delta uv\,dx \tag{2.3}$$

$$= \int_\Omega \nabla u \cdot \nabla v\,dx - \int_{\partial\Omega} n \cdot \nabla uv\,ds \tag{2.4}$$

$$= \int_\Omega \nabla u \cdot \nabla v\,dx \tag{2.5}$$

by assuming $v = 0$ on $\partial\Omega$. For the integrals above to exist we demand that both v and its gradient ∇v to behave nicely. This leads us to introducing the Sobolev spaces of order k with their corresponding inner product as

$$(v, w)_{H^k(\Omega)} = \sum_{|\alpha|\leq k} (D^\alpha v, D^\alpha w)_{L^2(\Omega)}, \quad \|v\|^2_{H^k(\Omega)} = (v, v)_{H^k(\Omega)} \tag{2.6}$$

where $L^2(\Omega) = H^0(\Omega)$ and $(v, w)_{L^2(\Omega)} = \int_\Omega vw\,dx$ is the $L^2(\Omega)$ inner product. By considering test functions in a subset of $H^1(\Omega)$ we gain control over both the function v and its gradient ∇v. The appropriate test space V_0 and trial space V_g are thus

$$V_0 = \{v \in H^1(\Omega) : v = 0 \text{ on } \partial\Omega\} \tag{2.7}$$

$$V_g = \{v \in H^1(\Omega) : v = g \text{ on } \partial\Omega\} \tag{2.8}$$

By defining the bilinear form $a(u, v) = (\nabla u, \nabla v)_{L^2(\Omega)}$ and the linear functional $l(v) = (f, v)_{L^2(\Omega)}$ we can write the problem in weak form: Given a function f, find $u \in V_g$ such that

$$a(u, v) = l(v) \quad \forall v \in V_0 \tag{2.9}$$

This setting can be used to analyse a larger class of similar problems with the same techniques.

Lax-Milgram Lemma. It is important to show that the solution of a given problem both exists and is unique. This can be done using Lax-Milgram lemma if the abstract forms $a(u, v), l(v)$ are continuous and the bilinear form $a(u, v)$ also is coercive. Those requirements can be written as

$$|a(u,v)| \lesssim \|u\|_{H^s} \|v\|_{H^s} \quad \text{(Continuous)} \tag{2.10}$$
$$a(v,v) \gtrsim \|v\|_{H^s}^2 \quad \text{(Coercive)} \tag{2.11}$$
$$|l(v)| \lesssim \|v\|_{H^s} \quad \text{(Continuous)} \tag{2.12}$$

where $s = 1$ for the model problem and $a \lesssim b$ means that there exists a positive constant C such that $a \leq Cb$.

2.2 Finite Element Approximation

In general it is not possible to find a analytical solution u to (2.1) and we thus tries seek a solution u_h in a finite subspace $V_h \subset V$. The space is typically chosen to consist of piecewise polynomials of a given order p, since they are easy to manipulate. By replacing the space V with V_h, in weak form (2.9), the finite element method reads: find $u_h \in V_h$ such that

$$a(u_h, v) = l(v) \quad \forall v \in V_h \tag{2.13}$$

The finite subspace V_h is spanned by a set of N linearly independent basis functions $\{\phi_i(x)\}_{i=1}^N$ which we can form the approximate solution as the linear combination:

$$u_h = \sum_{i=1}^N \xi_i \phi_i(x) \tag{2.14}$$

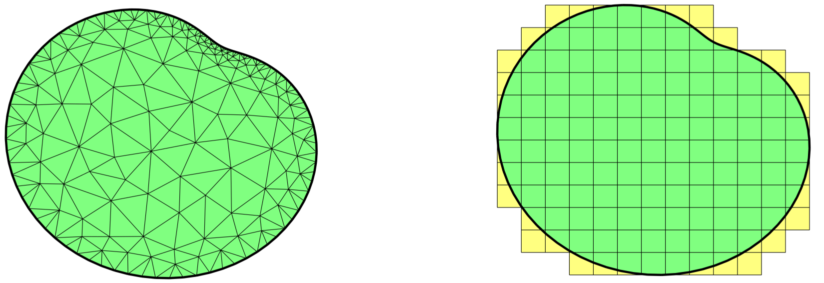
where ξ_i is the unknown coefficients and represent the degrees of freedom for the solution u_h. Inserting the linear combination of u_h in the variational form give us a $N \times N$ linear system of equations as:

$$\sum_{i=1}^N a(\phi_i, \phi_j)\xi_i = l(\phi_j) \quad \text{for } j = 1, \dots, N \tag{2.15}$$

where the test function v is chosen to be $\phi_j(x)$. This can also be written with matrices as

$$AU = F \tag{2.16}$$

with components $A_{ij} = a(\phi_i, \phi_j)$, $U_i = \xi_i$, and $F_j = l(\phi_j)$.



(a) A standard triangle mesh for the domain Ω.

(b) A quadrilateral cut mesh for the domain Ω.

Figure 2.2

2.3 The Mesh

To obtain a finite discretization we immerse the domain in a background mesh, which is chosen to be structured and simple to work with. A common choice in 2 dimensions is quadrilaterals and we show the differences between a cut mesh and a regular mesh in Figure 2.2. We let $\mathcal{K}_{h,0}$ be a quasi-uniform structured mesh consisting of elements K with mesh parameter $0 < h \leq h_0$. This partition of Ω is also where the approximate solution u_h is represented. We denote $\mathcal{K}_h$ as the elements in the background mesh which intersects with Ω. This restriction is called the active mesh and is written as

$$\mathcal{K}_h = \{K \in \mathcal{K}_{h,0} \colon K \cap \Omega \neq \varnothing\} \tag{2.17}$$

where only the elements affecting the geometry are remaining. Most of the time it will include cut elements in the vicinity of the boundary. We illustrate the active and background mesh in Figure 2.3a.

2.4 Nitsche's Method

The condition in (2.1) is called the Dirichlet boundary condition and is usually point-wise imposed. This is done by seeking the solution u in a solution space V_g which only includes functions that satisfy $u = g$ on the boundary.

In this thesis we will instead use Nitsche's method [15] to impose Dirichlet boundary conditions weakly. The condition $u = g$ on $\partial\Omega$ is enforced weakly by penalizing the jump $u - g$ as

$$\beta(\mu(u-g), v)_{\partial\Omega} = \beta \int_{\partial\Omega} \mu(u-g)v \, ds \tag{2.18}$$

where $\beta > 0$ is a penalty parameter and μ acts as a weight function. Nitsche proved that a good choice is $\mu \propto h^{-1}$, where h is the mesh size. This is added to the system $a(u,v) = l(v)$, but the weak form still lacks symmetry due to the consistency term $(n \cdot \nabla u, v)_{L^2(\partial\Omega)}$. To retrieve symmetry $(n \cdot \nabla v, (u-g))_{L^2(\partial\Omega)}$ is added and the final weak form is

$$\begin{aligned} a(u,v) = (\nabla u, \nabla v)_{L^2(\Omega)} - (n \cdot \nabla u, v)_{L^2(\partial\Omega)} - \ (n \cdot \nabla v, u)_{L^2(\partial\Omega)} \\ + \ \beta h^{-1}(u,v)_{L^2(\partial\Omega)} \\ l(v) = (f,v)_{L^2(\Omega)} - (n \cdot \nabla v, g)_{L^2(\Omega)} + \beta h^{-1}(g,v)_{L^2(\Omega)} \end{aligned} \tag{2.19}$$

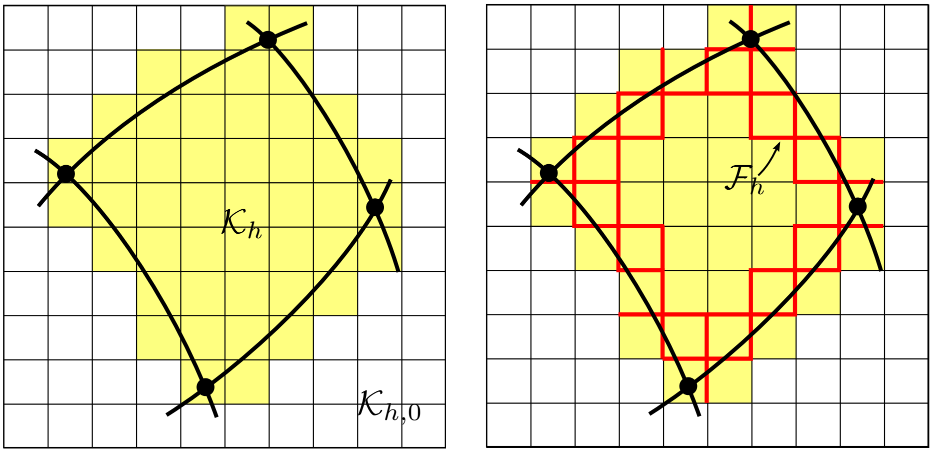
Interface Conditions. By the same merits it is possible to impose interface conditions with Nitsche's method [9].

2.5 Stabilization

Cut elements can be arbitrary small which causes numerical instability in the solution. This is manifested in a very high condition number and loss of accuracy for the finite element solution u_h. By adding a stabilization term called Ghost penalty we regain the control over elements that are being partially "outside" of the domain [4]. This recovers the properties of standard finite element methods by extending the coercivity to the whole domain. The stabilization added takes the form

$$s_h(v,w) = \sum_{F \in \mathcal{F}_h} s_{h,F}(v,w), \quad s_{h,F} = \sum_{k=1}^{p} \gamma_k \left([D_n^k v], [D_n^k w] \right)_F \tag{2.20}$$

where $\mathcal{F}_h$ is the set of interior faces F in the active mesh $\mathcal{K}_h$ such that its element is cut by the boundary $\partial\Omega$, $\gamma_k > 0$ is a positive constant and



(a) The background grid $\mathcal{K}_{h,0}$ with the active mesh $\mathcal{K}_h$ in yellow.

(b) The set of interior faces $\mathcal{F}_h$ shown in red.

Figure 2.3

$[D_n^k v]$ is the jump in the face normal derivative of order k. The stabilized bilinear form then have the form

$$A_h(v, w) = a_h(v, w) + s_h(v, w) \tag{2.21}$$

and we illustrate the set of interior faces $\mathcal{F}_h$ in Figure 2.3b.

2.6 Error Estimation

By computing the finite element solution u_h we only obtain an approximation of the exact solution u. It is therefore necessary to analyze the error $e = u - u_h$ to quantify the quality and accuracy of u_h. There are two types of error estimates, namely a priori error and a posteriori error. From the former estimate we obtain an upper bound for the error in terms of the solution u and usually the mesh size parameter h. The estimates are of interest since they can be used to find the order of convergence of the specific finite element method. By convergence we mean that the error $u - u_h$ in a given norm goes to zero as the mesh size h get refined. Usually the dependence is of polynomial order p for the mesh size h, i.e of the form:

$$\|u - u_h\| \leq ch^p \tag{2.22}$$

for some norm $\|\cdot\|$ and $c = c(\Omega)$ is a constant. We will derive and use a priori error estimates in this thesis to test and verify our methods.

The latter estimate give instead an upper bound in terms of the numerical solution u_h and the mesh size h. The posteriori error estimate is useful when dealing with adaptive finite element methods. In this thesis we will not use posteriori error estimates and we refer to [1, 8] for further reading.

Chapter 3

Partial Differential Equations on Surfaces

Partial differential equations on surfaces arises in several applications such as transport phenomena and elastic membranes. Surfaces are usually represented in two ways, either explicitly or implicitly. In this thesis we will use the former surface description, also known as parametric. Below we give a brief overview between the two approaches.

Explicit A surface, or more precise a two-dimensional manifold, can be thought of as a subset of $\mathbb{R}^3$ which locally looks like $\mathbb{R}^2$. The surface of Earth is a sphere but appears to be flat for an observer standing on the ground. This motivating example gives an idea of how we could represent surfaces as subsets of Euclidean space. Each point on a surface has a neighbourhood which is homemorphic to an open subset of $\mathbb{R}^2$ which we call the reference surface. In other words there exists a bijective continuous map between a flat reference surface $\subset \mathbb{R}^2$ and a possible curved surface $\subset \mathbb{R}^3$. In general there exist no global map for the whole domain, instead the manifold is divided into patches (also called charts), that may overlap. The collection of patches and their corresponding bijection functions are called an Atlas. To describe the patches explicitly we require functions of two variables since the map is between two surfaces. This explicit framework can be modelled such that each surface coordinates are given by a continuous functions, e.g $x = g_1(s,t), y = g_2(s,t), z = g_3(s,t)$.

Implicit An implicit surface is defined as all coordinates (x, y, z) which solves $F(x, y, z) = 0$. It is to be noted that it is not possible, most of the times, to solve for the coordinates explicitly. To give an example, the implicit equation for the sphere is given by

$$x^2 + y^2 + z^2 - R^2 = 0 \tag{3.1}$$

where R is the radius. Due to the lack of explicit coordinate representation there are no direct ways to generate the surface points. However we know from $F(x, y, z) = 0$ whether or not a point is on the surface. The implicit equation is also called the level set function and is usually modelled such that

$$F(x, y, z) > 0, \quad \text{above/inside the surface} \tag{3.2}$$
$$F(x, y, z) = 0, \quad \text{on the surface} \tag{3.3}$$
$$F(x, y, z) < 0, \quad \text{below/outside the surface} \tag{3.4}$$

3.1 Surface Problems

Here we list some real world problems that arise on surfaces.

Poisson Problem. The Poisson equation could be used to model the behaviour of electric and fluid potentials, and the heat distribution for the stationary heat equation. On surfaces the Laplace operator Δ is replaced with the Laplace-Beltrami operator Δ_Ω which is defined on Riemannian manifolds. Let Ω be a smooth connected surface immersed in $\mathbb{R}^d$ and $f \in C^2(\Omega)$, then the Laplace-Beltrami problem is: find u

$$-\Delta_\Omega u = f \quad \text{in}\, \Omega \tag{3.5}$$

where $\Delta_\Omega = \nabla_\Omega \cdot \nabla_\Omega u$ and ∇_Ω the tangential gradient operator. By adding a time derivative we obtain **Linear diffusion** on a surface:

$$\frac{\partial u}{\partial t} - D\Delta_\Omega u = f \quad \text{in}\, \Omega \tag{3.6}$$

where D is a diffusion constant. The diffusion equation models the random movement of particles in a material due to the particles stochastic nature. By adding the physical phenomena advection to the diffusion equation we obtain the **advection-diffusion equation:**

$$\frac{\partial u}{\partial t} - D\Delta_\Omega u + \nabla_\Omega \cdot (vu) = f \quad \text{in}\, \Omega \tag{3.7}$$

where v is the velocity field of the quantity u. Advection is the bulk movement of the regarded substance, and an example would be some material dissolved in the fluid with water flow v. The advection-diffusion equation can be used to model the transport of surfactants along fluid-fluid interfaces, grain boundary motion due to diffusion, and dealloying metals from surface dissolution [7].

Cahn-Hilliard Equation. The Cain-Hilliard equation is a fourth order PDE which describes the phase separation process of two fluid components. Applications can be found in areas such as planet formation, foam modeling, interfacial flows, and phase separations in membranes [6].

$$\frac{\partial u}{\partial t} = D\Delta_\Omega(u^3 - u - \gamma\Delta_\Omega u) \tag{3.8}$$

$$\tag{3.9}$$

where u is the concentration of the fluid, D the diffusion coefficient, and γ is related to the length of transition between fluid regions.

3.2 Patches and CAD

CAD is an acronym for Computer-Aided Design and a widely used tool by designers and engineers. It emerged in the 1960s from the US company International Business Machines Corporation (IBM) which replaced and enhanced manual drafting. Today the technology has expanded to be a vital part of the work flow for the development of new products. The design process is eased since the drawn objects can be rotated to any angle, become transparent, easily altered and reused. The drawn objects can be curves, surfaces and solids in both two-dimensional and three-dimensional space. In engineering, computer aided-design is often used in conjunction together with Computer-aided engineering (CAE) which incorporates tools like finite element analysis (FEA) or computational fluid dynamics (CFD). An example of a 3D geometry in a CAD program is shown below in Figure 3.1.

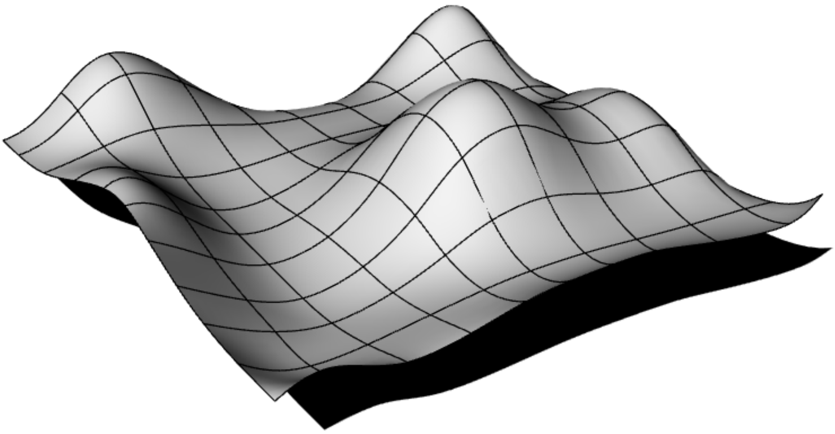


Figure 3.1: Example surface created in a CAD program

In engineering a surface geometry is modelled in CAD as a partition into trimmed patches. The patches are defined with maps F from a reference domain, which is non-curved. We define Ω to be a piecewise smooth surface immersed in $\mathbb{R}^d$, $d \geq 2$. The geometry is assumed to have a smooth boundary with possible sharp edges, but no corners. We define $\mathcal{O} = \{\Omega_i,\, i \in \mathcal{I}_\Omega\}$ the set of subdomains Ω_i called patches that makes up the whole surface. Each patch boundary $\Gamma_i = \partial\Omega_i$ is described by smooth curves and the interfaces between patches in $\mathcal{O}$ is denoted as Γ_{ij}. For each patch Ω_i we have a smooth injective map defined as $F\colon \widehat{\Omega}_i \to \Omega_i$, where $\widehat{\Omega}_i \subset I^2 = [0,1]^2$ is the reference domain. We illustrate a surface with patches and interfaces in Figure 3.2.

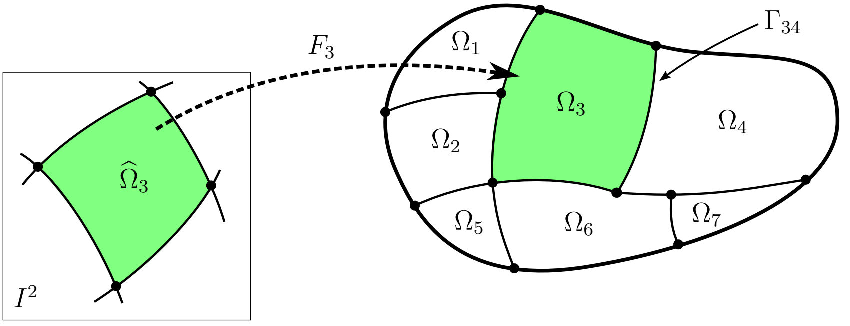


Figure 3.2: Representation of the surface via patchwise parametrizations with a bijective map F.

3.3 Parametric Description

In our setting, each patch Ω_i of the geometry Ω is a parametric surface which is defined through its corresponding map $F_i\colon \widehat{\Omega} \to \Omega_i$. For each point $x \in \Omega_i$ we have the set of tangent vectors on the surface, i.e the tangent space defined as

$$T_x(\Omega_i) = \text{span}\{\widehat{\partial}_1 F_i, \widehat{\partial}_2 F_i\} \tag{3.10}$$

where $\widehat{\partial}_k$ is the partial derivative in the $\widehat{e}_k$ direction. The map $F = (F^1, F^2, F^3)\colon \widehat{\Omega}_i \to \Omega_i$, can be written more explicit as

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} F^1(\widehat{x}_1, \widehat{x}_2) \\ F^2(\widehat{x}_1, \widehat{x}_2) \\ F^3(\widehat{x}_1, \widehat{x}_2) \end{bmatrix} \tag{3.11}$$

where $(x_1, x_2, x_3) \in \Omega_i$ and $(\widehat{x}_1, \widehat{x}_2) \in \widehat{\Omega}_i$. Any tangent vector in $T_x(\Omega_i)$ can be represented in terms of the local derivatives as

$$a = \widehat{a}_1 \widehat{\partial}_1 F + \widehat{a}_2 \widehat{\partial}_2 F = \begin{bmatrix} \widehat{\partial}_1 F^1 & \widehat{\partial}_2 F^1 \\ \widehat{\partial}_1 F^2 & \widehat{\partial}_2 F^2 \\ \widehat{\partial}_1 F^3 & \widehat{\partial}_2 F^3 \end{bmatrix} \begin{bmatrix} \widehat{a}_1 \\ \widehat{a}_2 \end{bmatrix} = \widehat{D}F\widehat{a} \tag{3.12}$$

since the derivatives spans the space.

Metric. A metric is a function that measures distance between pair of points in a set X. By using a specific metric, namely a metric tensor, we can for instance obtain distances along curves through integration. By equipping $T_x(\Omega_i)$ with the usual Euclidean inner product we define the induced inner product $\widehat{g}$ on $T_{\widehat{x}}(\widehat{\Omega})$ as

$$\widehat{g}(\widehat{a}, \widehat{b}) = a \cdot b \tag{3.13}$$

If we express a and b, above in (3.13), in the basis $\{\widehat{\partial}_1, \widehat{\partial}_2\}$ we can write

$$\begin{aligned} \widehat{g}(\widehat{a}, \widehat{b}) &= a \cdot b && (3.14)\\ &= \left(\widehat{D}F\widehat{a}\right)^T \widehat{D}F\widehat{b} && (3.15)\\ &= \widehat{a}^T \widehat{G}\widehat{b} && (3.16) \end{aligned}$$

where $\widehat{G}$ is the metric tensor with components $\widehat{g}_{kl} = \left(\widehat{\partial}_k F\right)^T \left(\widehat{\partial}_l F\right)$. For a given $a \in T_x(\Omega_i)$ we can use the identities

$$a \cdot b = \left(\widehat{a}^T \widehat{G}\right)\widehat{b} \tag{3.17}$$

$$a \cdot b = a^T \widehat{D}F\widehat{b} = \left(\widehat{D}F^T a\right)^T \widehat{b} \tag{3.18}$$

to obtain the corresponding $\widehat{a}$ as follows

$$\widehat{a} = \widehat{G}^{-1}\left(\widehat{D}F_i^T\right)^T a \tag{3.19}$$

The metric induces a norm as

$$\|\widehat{v}\|_{\widehat{g}}^2 = \widehat{g}(\widehat{v}, \widehat{v}) \tag{3.20}$$

with we can see that the mapping $T_{\widehat{x}}(\widehat{\Omega}) \ni \widehat{a} \mapsto a \in T_x(\Omega)$ is an isometry, i.e $\|a\|_{\mathbb{R}^2} = \|\widehat{a}\|_{\widehat{g}}$ and $\theta = \widehat{\theta}$.

The gradient. The tangential gradient $\nabla u \in T_x(\Omega_i)$ can be expressed in terms of the basis $\{\widehat{\partial}_1 F_i, \widehat{\partial}_2 F_i\}$ induced from the map F as

$$\nabla u = \widehat{D} F_i \widehat{\nabla u} \tag{3.21}$$

where $\widehat{\nabla u}$ is the local coordinates and can be obtain by taking the tangential gradient of $\widehat{u}$ in reference coordinates $\widehat{\nabla} = \widehat{\partial}_1 \widehat{e}_1 + \widehat{\partial}_2 \widehat{e}_2$ and applying the chain rule as

$$\widehat{\nabla} \widehat{u} = \widehat{\nabla}(u \circ F_i) = (\widehat{D} F_i)^T \nabla u = \widehat{G} \widehat{\nabla u} \tag{3.22}$$

and thus the gradient coordinates in the reference basis is given by $\widehat{\nabla u} = \widehat{G}^{-1} \widehat{\nabla} \widehat{u}$

Integration. In the weak form (2.9) there are both surface integrals and line integrals. They are formulated over a possible curved surface $\Omega \subset \mathbb{R}^3$ which could be complex to integrate over. This is resolved by utilizing the map structure $F^{-1} : \Omega \to \widehat{\Omega}$ and integrating over the reference domain $\widehat{\Omega} \subset \mathbb{R}^2$ instead. The integral over $\omega \subset \Omega$ can be written over $\widehat{\omega} \subset \widehat{\Omega}$ as

$$\int_\omega f(x)\, dx = \int_{\widehat{\omega}} \widehat{f}\, |G|^{1/2}\, d\widehat{x} \tag{3.23}$$

where $\widehat{f} = f \circ F(\widehat{x}_1, \widehat{x}_2)$, $d\widehat{x} = d\widehat{x}_1 d\widehat{x}_2$, and $|G| = |\det(G)|$. Further let $\widehat{C}$ be a curve in $\widehat{\Omega}$ parametrized by $\widehat{\gamma} : [0, l] \to \widehat{\Omega}$ then a line integral over the curve $C = F \circ \widehat{C}$ is given by

$$\int_C f(x) dC = \int_0^l \widehat{f} \circ \widehat{\gamma}(t) \Big| \frac{d}{dt}(F \circ \widehat{\gamma})(t) \Big|\, dt \tag{3.24}$$

$$= \int_0^l \widehat{f} \circ \widehat{\gamma}(t) \|\widehat{\gamma}'\|_{\widehat{g}(t)} dt \tag{3.25}$$

where $\widehat{\gamma}' = \frac{d\widehat{\gamma}}{ds}$ is the unit tangent vector to the curve $\widehat{C}$.

The Method in Reference Coordinates. By using the parametric description we can reformulate the weak terms from (2.19) in reference coordinates and some examples are:

$$(f, w)_\Omega = \int_{\widehat{\Omega}} \widehat{f}\widehat{w}|\widehat{G}|^{1/2}\, d\widehat{x} \tag{3.26}$$

$$(\nabla v, \nabla w)_\Omega = \int_{\widehat{\Omega}} \widehat{g}(\widehat{G}^{-1}\widehat{\nabla}\widehat{v}, \widehat{G}^{-1}\widehat{\nabla} w)|\widehat{G}|^{1/2}\, d\widehat{x} \tag{3.27}$$

$$(n \cdot \nabla v, w)_{L^2(\partial\Omega)} = \int_{\partial\widehat{\Omega}} \widehat{g}(\widehat{n}, \widehat{G}^{-1}\widehat{\nabla}\widehat{v})\widehat{w}\|\widehat{\gamma}'\|_{\widehat{g}}\, d\widehat{\gamma} \tag{3.28}$$

where $\widehat{g}(u, v)$ is the metric (3.16) and

$$n = \frac{\widehat{G}^{-1}\widehat{\nu}}{\|\widehat{G}^{-1}\widehat{\nu}\|_g} \tag{3.29}$$

and $\widehat{\nu}$ is the unit normal to $\widehat{\gamma}'$ with respect to the Euclidean inner product.

Chapter 4

Summary of Papers

Paper I. *Cut Finite Element Methods for Elliptic Problems on Multipatch Parametric Surfaces* [12]: In this paper we develop a general framework for the Laplace-Beltrami operator on a patchwise parametric surface. On each patch there is a bijective map with a corresponding reference domain. We use the cut finite element method together with Nitsche's method to enforce continuity over the interfaces between patches. A stabilization term is added to handle cut elements that might cause numerical errors. Each patch map imposes a Riemannian metric, which we utilize to compute quantities in the simpler reference domain. To be able to integrate the terms accurately we have developed a quadrature formula which deals with cut elements. We show that the method is stable and we derive optimal order a priori estimates in the energy and L^2 norms and also present several numerical examples confirming our theoretical results.

Paper II. *A Nitsche Method for Elliptic Problems on Composite Surfaces* [10]: We present a Nitsche method for diffusion problems on geometries which consist of an arrangement of surfaces. In this setting interfaces could have multiple (more than two) surfaces meeting. The interface could be any smooth curve, including triple points and sharp edges. This method avoids defining any co-normal to each interface, but instead the corresponding co-normal to each surface is used in combination with the conservation law called Kirchhoff condition. Over each interface the sum of all co-normal fluxes should be zero. In this paper we show that this formulation is equivalent with the standard Nitsche interface method in flat geometries. We show different examples with matching meshes, non-matching meshes and cut meshes.

Paper III. *Graded Parametric CutFEM and CutIGA for Elliptic Boundary Value Problems in Domains with Corners* [13]: In this paper we have developed a cut finite element method for elliptic problems with corner singularities. We weakly enforce Dirichlet condition on the boundary and stabilize the cut elements using Ghost penalty. Our approach is to use an appropriate radial map that grades the finite element mesh towards the corner and this counter-acts the singularity in the solution. We prove that the method is stable by pulling back to reference domain and using the additional stability obtained from Ghost penalty. Optimal a priori error estimates in L^2 and energy norm are shown by utilizing a bound on derivatives of order k in the reference domain in terms of a weighted norm in the physical domain which is bounded for the singular solution.

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