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Quantum Kinetic Theory for Plasmas

Spin, exchange, and particle dispersive effects

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To my parents

Abstract

This thesis is about developing and studying quantum mechanical models of plasmas. Quantum effects can be important at high densities, at low temperatures, and in strong electromagnetic fields, in various laboratory and astrophysical systems. The focus is on the electron spin, the intrinsic magnetic moment; exchange interactions, a purely quantum mechanical effect arising from particles being indistinguishable; and particle dispersive effects, essentially the Heisenberg uncertainty principle. The focus is on using phase-space formulations of quantum mechanics, namely Wigner and Q-functions. These methods allow carrying over techniques from classical plasma physics and identifying quantum as opposed to classical behavior.

Two new kinetic models including the spin are presented, one fully relativistic and to first order in $\hbar$, and one semi-relativistic but to all orders in $\hbar$. Among other example calculations, for the former, conservation laws for energy, momentum, and angular momentum are derived and related to "hidden momentum" and the Abraham-Minkowski dilemma. Both models are discussed in the context of the existing literature.

A kinetic model of exchange interactions, formally similar to a collision operator, is compared to a widely used fluid description based on density functional theory, for the case of electrostatic waves. The models are found to disagree significantly.

A new, non-linear, wave damping mechanism is shown to arise from particle dispersive effects. It can be interpreted as the simultaneous absorption or emission of multiple wave quanta. This multi-plasmon damping is of particular interest for highly degenerate electrons, where it can occur on time scales comparable to or shorter than that of linear Landau damping.

Publications

This thesis is based on the following publications.

I **Relativistic kinetic equation for spin-1/2 particles in the long-scale-length approximation**
R. Ekman, F. A. Asenjo, and J. Zamanian.
Phys. Rev. E **96**, 023207 (2017).

II **Relativistic kinetic theory for spin-1/2 particles: Conservation laws, thermodynamics, and linear waves**
R. Ekman, H. Al-Naseri, J. Zamanian, and G. Brodin.
Phys. Rev. E **100**, 023201 (2019).

III **Short-scale quantum kinetic theory including spin-orbit interactions**
R. Ekman, H. Al-Naseri, J. Zamanian, and G. Brodin.
arXiv:1908.05131, submitted (2019).

IV **Nonlinear wave damping due to multi-plasmon resonances**
G. Brodin, R. Ekman, and J. Zamanian.
Plasma Phys. Control. Fusion **60**, 025009 (2017).

V **Exchange corrections in a low-temperature plasma**
R. Ekman, J. Zamanian, and G. Brodin.
Phys. Rev. E **92**, 013104 (2015).

VI **Do hydrodynamic models misestimate exchange effects? Comparison with kinetic theory for electrostatic waves**
G. Brodin, R. Ekman, and J. Zamanian.
arXiv:1809.05423, submitted (2018).

Other publications by the author, not included in the thesis

- **Quantum kinetic theories in degenerate plasmas**
  G. Brodin, R. Ekman, and J. Zamanian.
  Plasma Phys. Control. Fusion **59**, 014043 (2016).
- **Two toy models for the motion of a leaky tank car**
  R. Ekman.
  arXiv:1906.04731, preprint (2019).

Contents

Chapter 1

Introduction

The research in this thesis is concerned with developing and studying models of quantum mechanical effects in ionized gases, known as plasmas. Quantum effects are expected to be important in plasmas found in, e.g., stellar cores, white dwarfs, neutron star atmospheres [1], and other extreme astrophysical environments [2]; and laser-matter interaction with extremely intense laser systems [3]. The general conditions to see quantum effects in plasmas are strong electromagnetic fields, high density, or (comparatively) low temperature, but quantum effects can also manifest in systems that appear to be classical, i.e., non-quantum [4].

To be more concrete about which quantum mechanical effects we are interested in, this chapter briefly describes the ones studied in this thesis, and some others that are also of interest under similar conditions. This introduction is meant to be non-technical and accessible to a general audience.

In classical plasma physics, the focus is generally on collective effects where many particles act together. This requires a statistical description, and the models developed and used in this thesis can be seen as extensions and generalizations of classical models. It is also instructive to compare the behavior of quantum plasmas to classical ones. Therefore, we give a short account of classical plasma physics in Chapter 2.

The standard ("Schrödinger") mathematical formulation of quantum mechanics is quite different from the equations of classical mechanics. However, an alternative formulation using something called Wigner functions [5] produces equations similar to their classical counterparts and is especially well-suited for statistical descriptions. In Chapter 3, we outline the basics of the Wigner formalism and how to apply it to plasmas. Quantum plasmas produced by lasers or in astrophysics have particles moving at close to the speed of light and must be described relativistically. We describe the transformation [6, 7] employed in Paper I to do this, and discuss some relevant subtleties of relativistic quantum mechanics.

1.1 Spin

A quantum mechanical particle can have an intrinsic rotation (angular momentum), called its spin, despite having no constituent parts. Curiously, as demonstrated in the famous 1922 experiment of Stern and Gerlach [8], whenever the spin is measured, it takes only a finite number of discrete values, in steps of a fundamental physical constant $\hbar$, the reduced Planck constant.[1] For the work in this thesis, it is the spin of electrons that is relevant; it takes the values $-\frac{\hbar}{2}$ and $+\frac{\hbar}{2}$. Since the maximum value is $\frac{\hbar}{2}$ one says that electrons have spin $\frac{1}{2}$, and the two values are called "spin down" and "spin up" respectively.

If a particle with spin also has electric charge, it will feel an additional magnetic force and generate its own magnetic field, both due to the spin. These effects of the spin have actually been observed since antiquity: the spin is the dominant source of the permanent magnetism of iron and nickel [13].

The first mathematical model for spin was given by Pauli in 1927 [14] and was followed the next year by one compatible with Einstein's theory of relativity, by Dirac [15]. While Pauli's theory looks similar to a classical particle and is rather straight-forward to interpret, Dirac's theory is alien. How to obtain Pauli's theory from Dirac's was only realized in 1950 by Foldy and Wouthuysen [6], for the case of particle speeds much lower than the speed of light.

Foldy and Wouthuysen's work was later generalized to allow speeds up to the speed of light [7]. This gave an easier-to-understand alternative to Dirac's theory to answer questions such as what would be seen in a Stern-Gerlach-like experiment with particles close to the speed of light [16]. However, the case covered by this work is a single particle moving in given, externally generated, fields, independently of other particles. In, for example,

[1] However, since the idea of spin was invented a few years *after* the experiment [9], and the experiment used composite silver atoms rather than elementary electrons, the precise meaning of the experiment is the subject of some debate [10, 11]. An account of contemporary interpretations by Heisenberg, Einstein, and others; and of subsequent similar experiments can be found in Ref. [12].

a laser-matter experiment, on the other hand, particles move collectively in self-generated fields. Thus, in Paper I we built upon Ref. [7] to formulate a model of relativistic particles with spin in self-consistent fields. Later, in Paper II, we explored some properties and applications of the model in Paper I, for instance its conservation of energy, momentum, and angular momentum.

1.2 Particle dispersive effects

One of the most striking and well-known features of quantum mechanics, the Heisenberg uncertainty principle forbids quantum particles to have a definite position and velocity; a quantum particle is necessarily always spread out, or dispersed. Because of this spreading out, a quantum mechanical particle can "tunnel" through regions classical particles would be unable to enter due to conservation of energy.

The force on a classical charged particle is determined by the electric and magnetic fields at the particles's position, but a spread out quantum particle interacts with the fields over a volume of space. This aspect of quantum mechanics is less explicit in the Schrödinger formulation, but using Wigner functions makes it more clear, with equations that are more similar to those of classical mechanics. The Wigner formulation for a particle without spin is well-established [17, 18], and the addition of the magnetic force due to the spin was also worked out a while ago [19]. However, as a consquence of Einstein relativity, the spin also interacts with electric fields [9, 20]. In Paper III, we worked out how to include this so-called spin-orbit interaction in the Wigner formulation.

Another consquence of particle dispersion, best seen in the Wigner formulation, is that resonances in a quantum plasma are different from those in a classical plasma [21]. Actually, in the Wigner formulation, the quantum dynamics looks more like the absorption or emission of discrete wave *quanta*, called plasmons. Ref. [21] considered one-plasmon processes, i.e, involving a single wave quantum. In Paper IV we showed that there are also resonances due to multi-plasmon processes involving two or three wave quanta. While the multi-plasmon processes are less likely per particle, under the right conditions there can be many more particles participating in these than in the one-plasmon process, making the former at least as significant as the latter.

1.3 Exchange

Another feature of quantum mechanics is that particles can be indistinguishable: we cannot speak of “electron 1” having spin up and “electron 2” having spin down, only of one electron being spin up and one being spin down. In other words, all probabilities must be the same if the labels of electrons are *exchanged*. For two spin-$\frac{1}{2}$ particles like electrons, if they have the same spin, they must have different distributions in space, and if they have opposite spin, they must have the same distribution in space.[2] Since like charges repel, the former has lower energy.

Effectively, the energy depends on the spin configuration, even if no forces do. This is called an *exchange interaction* and can be much stronger than the direct magnetic interaction between spins. In fact, exchange interactions are the main mechanism keeping the spins in ferromagnets aligned [13].

Modeling exchcange interactions in a dynamic system like a plasma can be very difficult. In one popular model [22], based on density functional theory (DFT), exchange interactions look like an additional source of pressure. However, the validity of this description is unclear, especially since the model is based on time-independent theory.

An alternative model, based on the Wigner approach to quantum mechanics, also exists [23, 24]. Since this model makes fewer assumptions and is based on time-dependent theory, it is of interest to compare it to the DFT-based model. We did this in Paper V, and later more generally in Paper VI. Our work shows a significant discrepancy between the two models, around a factor of 10. The Wigner model is also able to see resonances that the DFT-based model is unable to.

1.4 Other quantum effects

The above covers the quantum effects studied in this thesis, but the list is by no means exhaustive.

In the very strong electromagnetic fields of magnetars [1] and intense laser pulses, three quantum effects are of particular interest, the subject of much theoretical research and within current [25] or near-future experimental reach [3].

[2] Technically: the state is a linear combination of products of spin states and spatial states, with one factor being symmetric and the other anti-symmetric.

Pair production At extreme field strengths, new electron-positron pairs can be created according to $E = mc^2$, either from the vacuum [26] or from a "seed" electron, which can cause a chain-reaction-like cascade [27].

Radiation reaction An accelerating, electrically charged, particle creates electromagnetic radiation, that carries energy. The particle must lose an equal amount of energy, and hence there must be a braking force on it – the equal but opposite reaction, the *radiation reaction*. This is true even in classical physics since the fundamental ingredient is just the conservation of energy. Classical radiation reaction has been studied for over a century [28–31] (see also Ref. [32], Ch. 16, and references therein), but all proposed theories break down on short time and length scales, indicating that a quantum treatment is necessary. In any case, since we know that the world is quantum and not classical, quantum electrodynamics should be used to decide which, if any, classical theory is correct [33].

While radiation reaction exists in classical physics, quantum radiation reaction has different features [31]. For example, quantum mechanically, radiation is emitted in the form of discrete photons, not continuous waves, and this has consequences for the dynamics [34, 35]. Furthermore, since a particle cannot lose more than its total kinetic energy, there will be a maximum photon energy and hence a maximum radiation frequency seen in the quantum case.

Recent papers [25, 36] have reported observing quantum aspects of radiation reaction in strong laser fields.

Vacuum polarization Classically, rays of light cross without interacting; the light from a laser pointer is not bent by the ambient light. In quantum mechanics however, light can interact with light [37, 38]. Several new phenomena arise as a result, including, in very strong magnetic fields, one photon splitting into several [39], and different polarizations of light propagating with different speeds [40]; and scattering of light by light [41]; the latter two have been observed recently.

Chapter 2

Plasma physics

A plasma is a medium that is overall electrically neutral, but contains charge carriers that can move freely. The prototypcial example is an ionized gas where the charge carriers are light electrons and heavy positively charged ions, but there is also an interest in electron-positron plasmas [42, 43] and pair-ion (X^+/X^-) plasmas [44].

Plasmas are found in a range of environments, including stars and their atmospheres, the solar wind, the interstellar and intergalactic media, lightning, and in various technological applications such as semiconductor manufacturing and neon signs. One particularly high-profile, as-yet unrealized application, is to confine a plasma at sufficiently high temperature for long enough to achieve self-sustaining nuclear fusion reactions with a net release of energy. The main thrusts of confinement research are using magnetic fields [45] and using intense lasers [46]. High-intensity laser-plasma interaction can also be used for compact electron acceleration up to GeV energies [47–50].

A distinguishing feature of plasmas is collective behavior [51]. This means that the motion of any one particle is determined by many other particles. To be more quantitative, consider a test charge q in a plasma, that will repel like charges and attract opposite charges, which are free to move. This leads to a *screened* potential,

$$\varphi = \frac{q}{4\pi\varepsilon_0 r} \exp(-r/\lambda_D) \tag{2.1}$$

where λ_D is called the Debye length and is given by, in SI units,

$$\lambda_D^{-2} = \sum_s \lambda_s^{-2} \qquad \lambda_s = \left(\frac{\varepsilon_0 k_B T_s}{n_s q_s^2} \right)^{1/2}, \tag{2.2}$$

T_s, q_s, n_s being the temperature, charge, and number density of the s:th particle species, and k_B being the Boltzmann constant [52]. It should be noted that electrons and ions often have different temperatures.

The Debye length is the length over which the electric field of a particle is significant. To have collective behavior, there should be many particles within this distance, so the *plasma parameter* should be large,

$$\Lambda_s = n_s \lambda_s^3 \gg 1. \tag{2.3}$$

An alternative interpretation can be given by noting that

$$\Lambda_s^{2/3} = n_s^{2/3} \lambda_s^2 \propto \frac{k_B T_s}{n^{1/3}}. \tag{2.4}$$

Since $n_0^{1/3}$ is the inverse typical nearest-neighbor distance, Λ_s being large means that the thermal energy is much larger than the potential energy due to the nearest neighbor. This is another way to state that the motion of any one particle is determined by a large number of other particles, i.e., collective behavior. The condition $\Lambda \gg 1$ is often very well satisfied, with Λ being on the order of 10^4 in inertial confinement fusion, 10^8 in magnetic confinement fusion, and 10^{10} in the solar wind.

2.1 Modeling plasmas

Taking the view of a plasma being an ionized gas, one can model a plasma as a fluid. For each species s, the number of particles is conserved

$$\frac{\partial n_s}{\partial t} + \nabla \cdot (n_s \mathbf{v}_s) = 0 \tag{2.5}$$

and the force density is the Lorentz force and a pressure gradient,

$$m_s n_s (\partial_t + (\mathbf{v}_s \cdot \nabla)) \mathbf{v}_s = q_s n_s (\mathbf{E} + \mathbf{v}_s \times \mathbf{B}) - \nabla P. \tag{2.6}$$

This is the simplest model; in principle, there should also be a viscosity term and an equation describing heat flux, but finding the correct forms for these requires a more detailed analysis of the microscopic physics in the plasma, more on this in Sections 2.2 and 2.4.

The electric and magnetic fields are determined by Maxwell's equations, with the charge and current densities

$$\rho = \sum_s q_s n_s \qquad \text{and} \qquad \mathbf{j} = \sum_s q_s n_s \mathbf{v}_s \tag{2.7}$$

respectively. Often, there is a background magnetic field due to a star or planet, or magnets in the laboratory, and then the plasma is said to be magnetized.

Simple examples of collective behavior can be studied by linearizing the system of equations around a homogeneous plasma with no bulk velocity. Taking the Fourier transform, $\nabla, \partial_t \mapsto i\mathbf{k}, -i\omega$, and looking for solutions with no magnetic field, Eqs. (2.5)–(2.6) and Poisson's equation are

$$i\mathbf{k}\cdot\mathbf{E} = \frac{1}{\varepsilon_0}\sum_s q_s n_{s1} \tag{2.8}$$

$$-i\omega n_{s1} + \mathbf{k}\cdot\mathbf{v}_s n_{s0} = 0 \tag{2.9}$$

$$-i\omega m n_{0s}\mathbf{v}_s = q_s n_{s0}\mathbf{E} - i\mathbf{k}P_1. \tag{2.10}$$

Since electrons are much lighter than ions, at high frequencies we can treat the ions as static. To close the system, we need to say something about the pressure gradient. If we treat the electrons as an ideal gas, then, because the frequency is high, they undergo adiabatic compression in one dimension, for which, in linear theory, $\nabla P = 3k_B T\nabla n$. This system has non-trivial solutions only if the *dispersion relation* between $\omega, \mathbf{k}$,

$$\omega^2 = \omega_{pe}^2 + 3k^2 v_{te}^2 \tag{2.11}$$

is fulfilled. Here $v_{te} = \sqrt{k_B T_e/m}$ is the electron thermal velocity, and

$$\omega_{pe}^2 = \frac{q^2 n_0}{\varepsilon_0 m_e} \tag{2.12}$$

is called the electron *plasma frequency.* The inverse plasma frequency is one of many characteristic time scales for phenomena in plasmas, and the ion and electron plasma frequencies define "low" and "high" frequency regimes, respectively.

Looking at low-frequency dynamics instead, the ion motion has to be included, too. Then, neglecting the term proportional to m_e and allowing for slightly more general equations of state $\nabla P_s = \gamma_s k_B T_s \nabla n_s$ (γ_s is the ratio of specific heats), one can derive a dispersion relation for *ion-acoustic waves*,

$$\omega^2 = k^2 \gamma_i v_{ti}^2 + \frac{k^2 \gamma_e v_{te}^2}{1 + \gamma_e k^2 \lambda_e^2}. \tag{2.13}$$

It is so named because when the wavelength is long, $k\lambda_e \ll 1$, the wave propagates with the speed of sound in an ideal gas with the ion mass, $c_s^2 = k_B(\gamma_e T_e + \gamma_i T_i)/m_i$.

These are just two examples of the wide range of waves in plasmas. Many other types of linear and non-linear waves, including solitons, exist [53, 54]. Important examples include electromagnetic waves in unmagnetized plasmas, and cyclotron waves on time scales given by the cyclotron frequency $\Omega_{cs} = q_s B/m_s$ in magnetized plasmas.

2.2 Kinetic theory

To determine the equation of state, heat flux vector, and validity of the fluid description of a plasma, a more microscopic description is needed. A plasma containing many, many particles, this description is necessarily statistical in nature, as pioneered by Ludwig Boltzmann [55].

An ensemble of many systems is described by a probability density ρ in phase space. For N particles without spin or other internal degrees of freedom, the phase space consists of Nd positions and Nd momenta, d being the number of dimensions. Then ρ evolves according to Liouville's theorem,

$$0 = \partial_t \rho + \{\rho, H\} = \partial_t \rho + \sum_{n=1}^{N} \frac{\partial \rho}{\partial \mathbf{x}^{(n)}} \cdot \frac{\partial H}{\partial \mathbf{p}^{(n)}} - \frac{\partial \rho}{\partial \mathbf{p}^{(n)}} \cdot \frac{\partial H}{\partial \mathbf{x}^{(n)}} \tag{2.14}$$

where H is the Hamiltonian of the system, $\{\cdot, \cdot\}$ is the Poisson bracket, and $\mathbf{x}^{(n)}, \mathbf{p}^{(n)}$ are the position and momentum of the n:th particle. With $2Nd+1$ independent variables, this is far too detailed to be practical, but integrating over all but k of the particles one obtains a k-particle distribution function $f^{(k)}$. All terms in the sum with $n > k$ will vanish upon integration, and for $k = 1$, the evolution equation is

$$0 = \partial_t f^{(1)} + \frac{\partial H}{\partial \mathbf{p}^{(1)}} \cdot \frac{\partial f^{(1)}}{\partial \mathbf{x}^{(1)}} - \int \frac{\partial H^{12}}{\partial \mathbf{x}^{(1)}} \cdot \frac{\partial f^{(2)}}{\partial \mathbf{p}^{(1)}} \, d\mathbf{x}^{(2)} d\mathbf{p}^{(2)} \tag{2.15}$$

where H^{12} represents the pair-wise interaction Hamiltonian. The equation for $f^{(2)}$ will likewise include $f^{(3)}$, and so on, in a hierarchy of N equations.

Since $\partial H/\partial \mathbf{p} = \dot{\mathbf{x}}$, the second term is diffusion in phase space. The third term represents the interaction of particles with all other particles, and correlations between particles.

To progress, one must truncate the hierarchy at some point. Since a plasma should be dominated by collective effects, part of the force term should contain $q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$, where $\mathbf{E}$ and $\mathbf{B}$ are the ensemble-averaged electric and magnetic fields [56]. Dropping the superscript for the 1-particle distribution function,

$$\partial_t f + \mathbf{v} \cdot \nabla_{\mathbf{x}} f + q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \nabla_{\mathbf{p}} f = G[f^{(2)}] \tag{2.16}$$

where $G[\cdot]$ is some functional, to be determined.[1] By ensemble averaged fields, we mean that they are determined by Maxwell's equations with sources

$$\rho = \sum_s q_s \int f_s \, d^3\mathbf{p} \quad \text{and} \quad \mathbf{j} = \sum_s q_s \int \mathbf{v} f_s \, d^3\mathbf{p}, \tag{2.17}$$

where we have allowed for multiple particle species. Each f_s is determined by Eq. (2.16), with $f^{(2)}$ depending on all f_s.

The right-hand side of Eq. (2.16) represents two-particle correleations, or physically, non-collective effects, termed collisions. The condition $\Lambda \gg 1$ says that the plasma cannot be too dense, and it then makes sense to neglect collisions involving more than two particles. This is accomplished by a cluster expansion,

$$f^{(2)}(1,2) = f(1)f(2) + g(1,2), \tag{2.18}$$

where g is the correlated part, and $1, 2$ are shorthand for the phase-space variables of the first and second particle. $f^{(3)}$ can be expanded similarly using f, g, and introducing a three-particle correlation function h, which is put to 0. The first two equations of the hierarchy then suffice to determine the two unknowns f and g.

As the equation for g is still complicated, one can instead assume that g is some functional of the f_s, called the collision operator. Since particles of any species can collide, the collision operator will depend on all f_s. One

[1] It should be noted that the variable $\mathbf{p}$ in Eq. (2.16) is *not* the same as the canonical momentum in Eq. (2.14), which is not gauge-invariant, but rather the gauge-invariant and observable $\mathbf{p} = m\mathbf{v} = \mathbf{p}_{\text{can}} - q\mathbf{A}$. Eq. (2.14) is written with a momentum derivative instead of a velocity derivative for consistency with the following chapter, where the reasons will be clearer, cf. Section 3.5.1.

well-established collision operator is the Boltzmann operator [55],

$$C[f_s, f_{s'}](\mathbf{p}) = \sum_{s'} \int |\mathbf{p} - \mathbf{q}| \frac{d\sigma}{d\Omega} \left(f_s(\mathbf{p}') f_{s'}(\mathbf{q}') - f_s(\mathbf{p}) f_{s'}(\mathbf{q}) \right) d^3\mathbf{q} \, d\Omega \tag{2.19}$$

where the primed variables refer to the momenta after the collision and $d\sigma/d\Omega$ is a differential crossection. When all collisions are Coulomb collisions, the Boltzmann operator is called the Landau operator, and the kinetic equation takes the form of a Fokker-Planck equation [57].

Even simpler yet would be to ignore collisions entirely. This results in the Vlasov equation [58]

$$\partial_t f_s + \mathbf{v} \cdot \nabla_\mathbf{x} + q_s(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \nabla_\mathbf{p} \cdot f_s = 0. \tag{2.20}$$

Quantum versions of the Vlasov equation, including spin, are the main focus of this thesis, although in Papers V and VI the model of exchange interactions is formally similar to a collision operator.

The Vlasov equation is valid to the extent that the plasma is dominated by collective effects and collisions are relatively negligible. Quantitatively, this means that the mean free path should be large compared to the Debye length. Now, because of the long range of Coulomb interactions the definition of the mean free path or time is somewhat arbitrary, but one useful definition is that the mean free time is the mean time between scattering through a cumulative angle of 90°, possibly due to many small-angle collisions. This corresponds to a total energy transfer comparable to the initial energy. It can be shown [52, 57] that then, the mean free time τ_e and length ℓ_e for an electron with the thermal velocity, with respect to collisions with other electrons and ions alike,[2] are of the order

$$\tau_e \sim \frac{\Lambda}{\ln \Lambda} \omega_{pe}^{-1} \;\Leftrightarrow\; \ell_e \sim \frac{\Lambda}{\ln \Lambda} \lambda_e. \tag{2.21}$$

This reaffirms that the plasma parameter Λ measures the importance of collective effects and tells us that plasmas are often nearly collisionless, and hence that the Vlasov equation is often a very good approximation.

The mean free time for ions scattering off ions is a factor $(m_i/m_e)^{1/2}$ longer than τ_e (the plasma frequency scales as $m_s^{-1/2}$) and with respect to electrons another factor $(m_i/m_e)^{1/2}$ longer (imagine table tennis balls scattering tungsten spheres the same size, the electron-proton mass ratio is almost an order of magnitude smaller still). This is why ions and electrons can have different temperatures.

[2] This is a consequence of the large mass ratio.

2.3 Landau damping

To explore the Vlasov equation, let us look for Langmuir waves. Linearizing $f = f_0 + f_1$ around a homogeneous background distribution such as a Maxwellian and taking the Fourier transform, the Vlasov equation is

$$0 = (-i\omega + i\mathbf{k} \cdot \mathbf{v}) f_1 + q\mathbf{E} \cdot \nabla_{\mathbf{p}} f_0. \tag{2.22}$$

We consider the ions as immobile and thus only need the Vlasov equation for the electrons. As before the system is closed with Poisson's equation,

$$i\mathbf{k} \cdot \mathbf{E} = \frac{q}{\varepsilon_0} \int f_1 \, d^3\mathbf{p}. \tag{2.23}$$

The dispersion relation is easily found to be, letting $\mathbf{k} = k\hat{z}$,

$$\epsilon(\omega, k) = 1 + \frac{\omega_{pe}^2}{k^2} \int \frac{g_0'(v_z)}{\omega/k - v_z} \, dv_z = 0 \tag{2.24}$$

where $g_0 = \frac{1}{n_0} \int f_0 \, dv_x dv_y$. There is, however, a large problem in that the integrand has a singularity at the phase velocity ω/k.

Vlasov [58] suggested using the principal value of the integral, but as shown by Landau [59], this is at best half correct. Treating the problem as an initial value problem to be solved with Laplace rather than Fourier transform, Landau showed that ω must have an imaginary part, and the integral should be understood as a contour integral in the complex v_z plane, with the contour passing *below* all poles.

Letting $\omega = \omega_r + i\omega_i$ and similarly for ϵ, assuming that ω_i is small and Taylor expanding around ω_r,

$$0 = \epsilon_r(\omega_r, k) + i\epsilon_i(\omega_r, k) + i\omega_i \left. \frac{\partial \epsilon_r}{\partial \omega} \right|_{(\omega_r, k)} \tag{2.25}$$

where the omitted term is quadratic in the small ω_i. In the sense of distributions, the Plemelj formula

$$\frac{1}{u - a} = \mathrm{P} \, \frac{1}{u - a} + i\pi\delta(u - a) \tag{2.26}$$

holds for contours passing below the pole (the δ has the opposite sign for contours above the pole), P being the principal value. The principal value ensures that the region near the pole contributes to the integral only on the order of $g_0''(\omega_r/k)$, so if the distribution is non-negligible only for $u \ll \omega_r/k$, it is valid to Taylor expand $(\omega_r/k - u)^{-1}$. Keeping the two first non-vanishing terms,

$$\epsilon_r(\omega_r, k) = 1 - \frac{\omega_p^2}{\omega_r^2} - \frac{3k^2 v_{te}^2 \omega_p^2}{\omega_r^4} = 0, \tag{2.27}$$

where $v_{te}^2 = \int u^2 g_0 \, du$. That ω_r/k is in the tail of the distribution implies $\omega_r/k \gg v_t$ and solving for ω_r under this assumption gives back the Langmuir dispersion relation from fluid theory,

$$\omega_r^2 = \omega_{pe}^2 + 3k^2 v_{te}^2. \tag{2.11}$$

Thus, in this limit, we recover ideal gas behavior, and we could use kinetic theory to estimate the deviations from it.

It is straightforward to solve for ω_i using Eq. (2.25),

$$\omega_i = \frac{\pi \omega_{pe}^3}{2k^2} g_0'(\omega_r/k), \tag{2.28}$$

again, valid for $\omega_r/k \gg v_t$, with the general case treated in Ref. [60]. Thus, if the background distribution has more electrons slightly slower than it has electrons slightly faster than the phase velocity ω_r/k (as is the case for a Maxwellian), the wave exhibits *Landau damping*; if the opposite holds, there is an instability: the wave amplitude grows exponentially until it reaches saturation due to non-linear effects. Landau damping is a collisionless phenomenon and is different from damping due to dissipative mechanisms, such as viscosity, in that it does not increase entropy [56, § 30].

The basic physical mechanism of Landau damping is resonant wave-particle interaction. Particles slightly slower than the phase velocity will, on average, gain energy from the wave and speed up, while the opposite applies to those slightly faster than the phase velocity [61, 62]. Many accounts of Landau damping exist in the literature [63, 64, and refs. therein].

The fluid model contains only the bulk velocity and therefore cannot resolve the resonance, that particles with different velocities interact differently with the wave. In the next chapter, we will see that in the quantum case, particle dispersive effects modify the resonance condition [21], and add new resonances as we showed in Paper IV. In Papers V and VI we also found damping due to wave-particle interaction in a kinetic model of exchange interactions; this damping is not present in a fluid model [22].

2.4 Fluid theory from kinetic theory

A systematic way of recovering fluid theory from kinetic theory is by considering the moments of the distribution function f, i.e.,

$$\langle X \rangle = \int X f \, d^3\mathbf{p} \tag{2.29}$$

for functions X on phase space. The zeroth order moment is the number density,

$$n(t, \mathbf{x}) = \int f(\mathbf{x}, \mathbf{p}, t) \, d^3\mathbf{p} \tag{2.30}$$

and taking its time derivative using the Vlasov equation, we can obtain the continuity equation

$$\partial_t n + \nabla \cdot \int \mathbf{v} f \, d^3\mathbf{p} = 0 \tag{2.31}$$

with the bulk velocity given by $n\mathbf{V} = \int \mathbf{v} f \, d^3\mathbf{p}$. If collisions are included, there may be an additional term in this equation, but a reasonable physical collision operator should not create or destroy particles; the Boltzmann operator has this property.

With a little more work, and using the continuity equation, one finds the momentum equation from $\partial_t \langle \mathbf{v} \rangle$,

$$mn(\partial_t + V_i \nabla_i) V_j = qn(E_j + \varepsilon_{ijk} v B_j) - m\nabla_i \left[n \left\langle (v_i - V_i)(v_j - V_j) \right\rangle \right] \tag{2.32}$$

where we use the summation convention for repeated indices, and ε_{ijk} is the Levi-Civita pseudotensor. This involves a new unknown, the second order moment $\langle v_i v_j \rangle$, which defines the pressure tensor. The equation for the pressure tensor will include a third order moment, and so on. This infinite hierarchy of equations arises because infinitely many numbers are needed to specify a general distribution function f. The hierarchy must be truncated somehow to yield a useful theory.

As a crude example, one can assume that the distribution is Maxwellian with position-dependent drift $\mathbf{V}$ and temperature T. In this case, the pressure tensor is that of an ideal gas, an isotropic $P = nk_B T$. Of course, an ideal gas cannot thermalize, so a more sophisticated analysis must consider relaxation of a near-Maxwellian distribution due to collisions [65–67]. This can be done for Coulomb collisions, and the resulting fluid equations, including expressions for the pressure, viscosity, heat flux, etc., are called the Braginskii equations [68]. As fluid models are not the focus of this thesis, we will not reproduce or discuss the Braginskii equations further here.

Chapter 3

Quantum kinetic theory

Although quantum effects in plasmas have been studied for a long time (e.g., Bohm and Pines in 1953 [69]), there has been an increasing interest during the past couple of decades [3, 70–72]. As mentioned in the introduction, this is motivated by neutron star atmospheres [1], other astrophysical environments [2], and intense laser-matter interactions [3], but also by applications in spintronics [73], plasmonics [74], and studies of spin dynamics in ferromagnets [75].

In this chapter, we will present a framework for describing quantum effects in plasmas, and connect it to the work presented in the Papers. Before we begin, let us first estimate under which conditions quantum effects become important.

First, the Pauli principle says that there can be at most one fermion per $\sim \hbar^3$ in phase space and spin state. Classically, at temperature T an ideal gas occupies, roughly speaking, a phase-space volume $n(mv_t)^3$ per particle, so $\hbar^3/(nv_t)^3$ measures the degree to which the Pauli principle has to be taken into account. With a more precise analysis [76, § 57], this can be expressed as

$$\frac{T_F}{T} = \frac{(3\pi^2 n)^{2/3}\hbar^2}{2mk_B T} \tag{3.1}$$

where T_F is called the Fermi temperature.

Second, the wavefunction of a particle with momentum p has a wavelength, the de Broglie wavelength, of $\hbar/p$, and we can expect this to be important at length scales $\frac{1}{k} \sim \frac{\hbar}{mv_t}$ However, we are interested in collective effects, whose importance at a length scale $1/k$ can be estimated by

$1/(k\lambda_D) \gtrsim 1$, so the parameter of interest is

$$H = \frac{\hbar}{mv_t\lambda_D} = \frac{\hbar\omega_p}{mv_t^2}. \tag{3.2}$$

Note that this is the energy of a wave quantum at ω_p compared to the typical particle kinetic energy, and for high densities we should use $v_F = \sqrt{k_BT_F/m}$ instead of v_t. Hence H scales as $n^{1/2}/T$ for low densities and $n^{-1/6}$ for high densities; for $n \approx 10^{23}\,\mathrm{cm}^{-3}$, $H \approx 1$.

This shows that quantum effects are associated with high densities and low temperatures. On the other hand, the plasma parameter, Eq. (2.4), decreases with density and increases with temperature. It is, however, defined under the assumption that the typical kinetic energy is k_BT, but at high densities it is $k_BT_F \propto n^{2/3}$, which increases faster than the nearest-neighbor energy $\propto n^{1/3}$. The discussion so far is illustrated for electrons in Fig. 3.1.

As for the spin, it is associated with an energy $\frac{g}{2}\mu_B B$, where B is the strength of the magnetic field, $\mu_B = \frac{q\hbar}{2m}$ is the Bohr magneton and g is the gyromagnetic ratio. For electrons $g \approx 2.0023$ including corrections from quantum electrodynamics, and comparing to the thermal energy,

$$\frac{\mu_B B}{k_BT} \approx \frac{0.67B/\mathrm{T}}{T/\mathrm{K}}, \tag{3.3}$$

gives an indicator of spin polarization, but the spin can also be important if $\nabla\mathbf{B}$ is large.

The various parameters characterizing quantum behavior are discussed in further detail in Ref. [77]. Finally, because of the large mass ratio, quantum effects are always much weaker for ions; while the principles apply to any species, in practice, we have electrons in mind throughout this chapter.

3.1 The Wigner function

In the previous chapter we saw how to describe plasmas, or any statistical system, in terms of a distribution function on phase space, such that $f(\mathbf{x},\mathbf{p})\,d^3\mathbf{x}\,d^3\mathbf{p}$ is the number of particles in a volume element of phase space. In quantum mechanics on the other hand, the state is a wave-function ψ of just one half of the phase space coordinates, either $\mathbf{x}$ or $\mathbf{p}$, but not both. Indeed, the Heisenberg uncertainty principle,

$$\Delta x \Delta p \geq \frac{\hbar}{2} \tag{3.4}$$

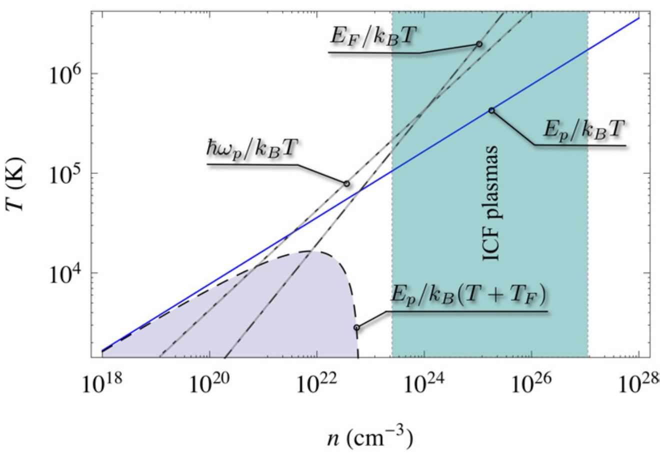


Figure 3.1: Quantum and classical plasma regimes. All parameters are for electrons and each curve shows where that parameter is equal to unity. In general, quantum effects are important below either of the lines $1 = E_F/k_BT$ and $1 = \hbar\omega_p/k_BT$. Here E_p is the nearest-neighbor potential energy and $E_p/k_B(T+T_F)$ is thus essentially the inverse plasma parameter; the blue shaded region is a regime of high collisionality. Reproduced from J. Zamanian, M. Marklund, and G. Brodin, "Scalar quantum kinetic theory for spin-1/2 particles: mean field theory", New. J. Phys. **12**, 043019 (2010).

seems to make it hopeless to talk about "the number of particles with positions *and* momenta in a certain volume element of phase space" in quantum mechanics. On the other hand, the world looks classical: Boltzmann's approach is supremely successful for describing gases, or, indeed, plasmas. One might therefore hope that there is a formulation of quantum mechanics such that a description like Boltzmann's can be recovered in the limit $\hbar \to 0$, where the Heisenberg uncertainty principle has no power.

This formulation is the Wigner formalism, named for its inventor [5]. A pedagogical introduction can be found in Ref. [78]. We will use it for plasma physics, but the Wigner formalism has numerous applications in fields such as semiconductor physics, quantum optics, quantum chemistry, and quantum computing [79].

The fundamental object is the Wigner function, the simplest definition of which is as the Fourier transform of the density matrix ρ; for a single particle

$$W(\mathbf{x},\mathbf{p},t) = \int \frac{d^3\mathbf{z}}{(2\pi\hbar)^3} \langle \mathbf{x}-\mathbf{z}/2| e^{\frac{i}{\hbar}\mathbf{p}\cdot\mathbf{z}} \rho |\mathbf{x}+\mathbf{z}/2\rangle. \tag{3.5}$$

Clearly, the Wigner function is a function on phase space, and it is real. Furthermore,

$$\int W(\mathbf{x},\mathbf{p},t)\, d^3\mathbf{p} = \langle \mathbf{x}|\rho|\mathbf{x}\rangle \tag{3.6}$$

which is the probability density at $\mathbf{x}$, according to the Born rule.

Before saying more about the properties of the Wigner function, let us see how it evolves in time. The density matrix evolves according to the von Neumann equation,

$$\partial_t \rho = \frac{1}{i\hbar}[\hat{H},\rho] \tag{3.7}$$

and the simplest case is a particle in a scalar potential V so that the Hamiltonian is

$$\hat{H} = \frac{\hat{\mathbf{p}}^2}{2m} + \hat{V}. \tag{3.8}$$

After some straightforward calculations, one finds

$$\partial_t W + \frac{\mathbf{p}}{m}\cdot(\nabla_x W) = \frac{2}{\hbar} V \sin\left(\frac{\hbar}{2}\overleftarrow{\nabla}_x \cdot \overrightarrow{\nabla}_p\right) W \tag{3.9}$$

where arrows indicate the direction an operator acts in, and the sine is defined by its power series. The right-hand side contains particle dispersive effects: the evolution of the Wigner function at a point in world-space depends on higher-order derivatives of the potential. These effects include tunneling, clearly seen in numerical solutions [80] of Eq. (3.9).

It is clear that Eq. (3.9) reduces to Liouville's theorem in the limit $\hbar \to 0$. We can make the connection to Poisson brackets more explicit. We define the *Weyl symbol* [81] of an operator $\hat{O}$ by

$$O(\mathbf{x},\mathbf{p}) = \int \frac{d^3\mathbf{z}}{(2\pi\hbar)^3}\, \langle \mathbf{x}-\mathbf{z}/2| e^{\frac{i}{\hbar}\mathbf{p}\cdot\mathbf{z}} \hat{O} |\mathbf{x}+\mathbf{z}/2\rangle. \tag{3.10}$$

The Wigner function is the Weyl symbol of the density matrix and the expectation value of any operator is given by

$$\langle \hat{O}\rangle = \mathrm{Tr}[\hat{O}\rho] = \int d^3\mathbf{x}\, d^3\mathbf{p}\, O(\mathbf{x},\mathbf{p}) W(\mathbf{x},\mathbf{p}). \tag{3.11}$$

From this we can realize that the Heisenberg principle indeed forbids the Wigner function to be arbitrarily peaked

By induction, one can show that if $\hat{O}$ is *Weyl-ordered*, i.e., expressed as a totally symmetric power series in $\hat{\mathbf{x}}$ and $\hat{\mathbf{p}}$, its Weyl symbol is the same power series of $\mathbf{x}$ and $\mathbf{p}$. The Weyl symbol of an operator product $\hat{O}\hat{Q}$ is [82, 83]

$$\hat{O}\hat{Q} \mapsto O \star Q := O \exp\left(\frac{i\hbar}{2}(\overleftarrow{\nabla}_x \cdot \overrightarrow{\nabla}_p - \overleftarrow{\nabla}_p \cdot \overrightarrow{\nabla}_x)\right) Q \tag{3.12}$$

with this operation called the Moyal or star product. The Weyl symbol of a commutator is then given by

$$\begin{aligned}[\hat{O}, \hat{Q}] \mapsto O \star Q - Q \star O &= O\, 2i \sin\left(\frac{\hbar}{2}(\overleftarrow{\nabla}_x \cdot \overrightarrow{\nabla}_p - \overleftarrow{\nabla}_p \cdot \overrightarrow{\nabla}_x)\right) Q \\ &= i\hbar\{O \overset{\star}{,} Q\} \end{aligned} \tag{3.13}$$

where $\{O \overset{\star}{,} Q\}$ is called the Moyal bracket; it defines a Lie algebra on functions on phase space. In the limit $\hbar \to 0$, the star product reduces to the product of functions, and the Moyal bracket to the Poisson bracket. With this machinery, the derivation of Eq. (3.9) is trivial.

From Eq. (3.11), it is tempting to think of the Wigner function as a probability distribution. However, while real, the Wigner function is not guaranteed to be everywhere non-negative, so it is only a *quasidistribution*, not a bona fide probability distribution. We should not expect it to be a probability distribution, given that the Wigner formalism is a complete description of quantum mechanics, including the Heisenberg uncertainty principle. Correspondingly, not every function on phase space is an admissible Wigner function. While negativity of the Wigner function can be an indicator of non-classicality, even everywhere non-negative Wigner functions can display quantum behavior [84, and refs. therein], e.g., violations of Bell's inequalities [85]. We may say that quantum mechanics "resides in" both the Wigner function and the star product.

All of the above has been for a single particle, for conceptual and notational clarity. The generalization to N particles is straight-forward: take the Weyl transform for each particle. Consequently, the N-particle Wigner function depends on Nd coordinates and Nd momenta, and its evolution equation contains a sum over all particles, as in the N-particle Liouville theorem, Eq. (2.14).

Like the N-particle distribution function $f^{(N)}$, the N-particle Wigner function can be reduced to an $N-k$ particle Wigner function by integrating out the coordinates and momenta of k particles. In this way, we can follow the steps from Liouville's theorem to a quantum Boltzmann equation, or, when neglecting collisions, a quantum Vlasov equation.

However, the cluster expansion is slightly different in quantum mechanics since the *state* should be totally symmetric (anti-symmetric) for bosons (fermions). We will discuss this further in Section 3.6.

3.2 Gauge invariance

For plasma physics it is essential to include the magnetic field, and hence we must introduce the gauge potential $\mathbf{A}$ and modify the Hamiltonian to

$$\hat{H} = \frac{(\hat{\mathbf{p}} - q\hat{\mathbf{A}})^2}{2m} + q\varphi. \tag{3.14}$$

While one can now compute $\{H \stackrel{\star}{,} \rho\}$, the resulting equation will depend explicitly on $\mathbf{A}$. This would force a choice of gauge before performing any calculations. Moreover, under a gauge transformation

$$|\psi\rangle \mapsto e^{iq\lambda/\hbar}|\psi\rangle \qquad \mathbf{A} \mapsto \mathbf{A} + \nabla\lambda \tag{3.15}$$

the Wigner function is not invariant, and thus, not even the initial state of a system can be specified without reference to a particular gauge. Contrastingly, when there is no gauge potential, the Wigner function, being the expectation value of a Hermitian operator, is directly measurable [86].

An alternative, originally proposed by Stratonovich [17], is to use that the operator

$$\hat{\boldsymbol{\pi}} := \hat{\mathbf{p}} - q\hat{\mathbf{A}}. \tag{3.16}$$

is gauge invariant. Hence, define a Stratonovich-Weyl transform by

$$O(\mathbf{x}, \boldsymbol{\pi}) = \int \frac{d^3\mathbf{u}}{(2\pi\hbar)^3}\, e^{\frac{i}{\hbar}\mathbf{u}\cdot\left(\boldsymbol{\pi} + q\int_{-1/2}^{1/2}\mathbf{A}(\mathbf{x}+\tau\mathbf{u})\,d\tau\right)} \langle \mathbf{x} - \mathbf{u}/2 \mid \hat{O} \mid \mathbf{x} + \mathbf{u}/2\rangle. \tag{3.17}$$

The Stratonovich-Weyl transform of the density matrix is our sought-after gauge-invariant Wigner function, and similarly to before, the Stratonovich-Weyl symbol of a totally symmetric power series in $\hat{\mathbf{x}}$ and $\hat{\boldsymbol{\pi}}$ (Stratonovich-Weyl ordering) is the same power series of of $\mathbf{x}$ and $\boldsymbol{\pi}$.

The Stratonovich-Weyl transform can be written as [18]

$$O = \mathrm{Tr}[\hat{W}\hat{O}] = [\mathrm{Tr}(\mathcal{F}\hat{T})\hat{O}] \tag{3.18}$$

where $\mathcal{F}$ is the Fourier transform from $(\mathbf{u}, \mathbf{v})$ to $(\boldsymbol{\pi}, \mathbf{x})$ and

$$\begin{aligned}\hat{T}(\mathbf{u}, \mathbf{v}) &= \exp\left[\frac{i}{\hbar}\left(\mathbf{u}\cdot\hat{\boldsymbol{\pi}} + \mathbf{v}\cdot\hat{\mathbf{x}}\right)\right] \\ &= e^{-\frac{i}{2\hbar}}\exp\left(\frac{i}{\hbar}\mathbf{u}\cdot\hat{\boldsymbol{\pi}}\right)\exp\left(\frac{i}{\hbar}\mathbf{v}\cdot\hat{\mathbf{x}}\right).\end{aligned} \tag{3.19}$$

The second expression here is an example of a Baker-Campbell-Hausdorff formula, and various other forms of $\hat{T}$, corresponding to different operator orderings, exist, see Eqs. (2.25)–(2.27) in Ref. [18]. The gauge-invariant Wigner function is $W = \langle\hat{W}\rangle$, the expectation value of $\hat{W}$, and using the Heisenberg picture finding the kinetic equation comes down to evaluating $[\hat{T}, \hat{H}]$, eventually resulting in [17, 18]

$$\partial_t W + \frac{\tilde{\boldsymbol{\pi}}}{m}\cdot\nabla_x W + q(\tilde{\mathbf{E}} + \frac{\tilde{\boldsymbol{\pi}}}{m}\times\tilde{\mathbf{B}})\cdot\nabla_\pi W = 0. \tag{3.20}$$

Here we have introduced the notation

$$\tilde{\mathbf{E}} = \mathbf{E}\int_{-1/2}^{1/2}\cos(\hbar\tau\overleftarrow{\nabla}_x\cdot\overrightarrow{\nabla}_\pi)\,d\tau \tag{3.21}$$

$$\tilde{\mathbf{B}} = \mathbf{B}\int_{-1/2}^{1/2}\cos(\hbar\tau\overleftarrow{\nabla}_x\cdot\overrightarrow{\nabla}_\pi)\,d\tau \tag{3.22}$$

$$\tilde{\boldsymbol{\pi}} = \boldsymbol{\pi} + q\hbar\mathbf{B}\int_{-1/2}^{1/2}\tau\sin(\hbar\tau\overleftarrow{\nabla}_x\cdot\overrightarrow{\nabla}_\pi)\,d\tau\times\nabla_\pi. \tag{3.23}$$

First, we can see that in the limit $\hbar\to 0$ we recover the Vlasov equation. Second, the objects $\tilde{\mathbf{E}}$ and $\tilde{\mathbf{B}}$ contain particle dispersive effects; $\tilde{\mathbf{E}}$ appears already in Eq. (3.9) and $\tilde{\mathbf{B}}$ comes from the gauge potential in a similar fashion. Third, the object $\tilde{\boldsymbol{\pi}}$ appears due to the components of $\hat{\boldsymbol{\pi}}$ not commuting,

$$[\hat{\pi}_i, \hat{\pi}_j] = i\hbar q\varepsilon_{ijk}\hat{B}_k, \tag{3.24}$$

where ε_{ijk} is the Levi-Civita pseudotensor. This commutator makes the derivation of Eq. (3.20) significantly more involved than that of Eq. (3.9).

For a closed system of equations, we add Maxwell's equations with charge and current densities

$$\rho = q\int d^3\boldsymbol{\pi}\,W \quad\text{and}\quad \mathbf{j} = q\int d^3\boldsymbol{\pi}\,\frac{\boldsymbol{\pi}}{m}W, \tag{3.25}$$

analogously to the classical theory, Eq. (2.17).

There is a star product for Stratonovich-Weyl symbols [87], which can be used to derive Eq. (3.20), though the expression for it is more complicated than Eq. (3.12). The complication is due to the commutator Eq. (3.24): the exponential of derivatives in Eq. (3.12) accounts for Weyl-ordering a product of $\hat{\mathbf{x}}$ and $\hat{\mathbf{p}}$ using the commutator $[\hat{x}_i, \hat{p}_j] = i\hbar\delta_{ij}$, but when Stratonovich-Weyl-ordering according to Eq. (3.24), $\mathbf{B}$ and its derivatives will appear. Here we will only note that to lowest order in $\hbar$, the corresponding Moyal bracket agrees with the Poisson bracket, as written in the non-canonical coordinates $(\mathbf{x}, \boldsymbol{\pi})$.

3.3 Multi-plasmon damping

The difference between Eq. (3.9) and the Vlasov equation can be illustrated by linear electrostatic waves [21], as in Section 2.3. With a plane wave Ansatz,

$$W(\mathbf{x}, \mathbf{p}, t) = W_0(\mathbf{p}) + W_1(\mathbf{p})e^{-i\omega t + \mathbf{k}\cdot\mathbf{x}} \tag{3.26}$$

$$\phi = \phi_1 e^{-i\omega t + \mathbf{k}\cdot\mathbf{x}} \tag{3.27}$$

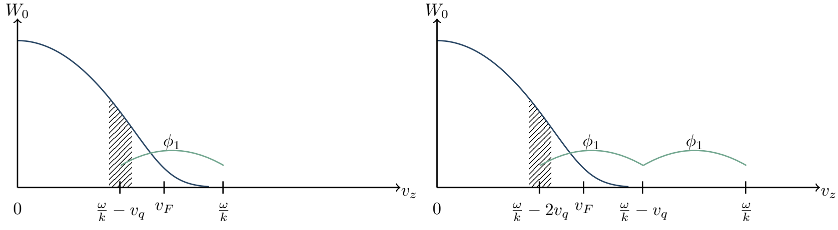
we have $\nabla_x \mapsto i\mathbf{k}$, making the linearization of Eq. (3.9) read

$$\begin{aligned} -i\omega W_1 + i\frac{\mathbf{p}}{m}\cdot\mathbf{k}W_1 &= \frac{2q}{\hbar}\phi_1 \sin\left(\frac{\hbar}{2} i\mathbf{k}\cdot\nabla_p\right) W_0 \\ &= \frac{q}{\hbar}\phi_1\left(e^{\hbar\mathbf{k}\cdot\nabla_p/2} - e^{-\hbar\mathbf{k}\cdot\nabla_p/2}\right) W_0. \end{aligned} \tag{3.28}$$

Since the derivative is the generator of translations, the right-hand side is a finite difference operator, and writing out arguments

$$i(-\omega + \mathbf{p}\cdot\mathbf{k}/m)W_1(\mathbf{p}) = \frac{q\phi}{h}\Big(W_0(\mathbf{p} + \hbar\mathbf{k}/2) - W_0(\mathbf{p} - \hbar\mathbf{k}/2)\Big). \tag{3.29}$$

Where the Vlasov equation describes particles undergoing smooth acceleration, Eq. (3.29) describes particles gaining or losing momentum $\hbar\mathbf{p}/2$ in discrete scattering events. Put another way, the electrons can be thought of as absorbing or emitting wave quanta ("plasmons"), and Eq. (3.29) is to the Vlasov equation as Compton scattering is to smooth acceleration in an electric field.



(a) One-plasmon resonance (b) Two-plasmon resonance

Figure 3.2: The potential ϕ_1 couples the Wigner function at v to the Wigner function at $v - v_q$, leading to resonances at $\omega/k - nv_q$, $n = 1, 2, \ldots$. Which process is more important depends on the number of particles near the respective resonances, shaded. The background distribution is a Fermi-Dirac distribution at $T = 0.2T_F$, integrated over the two transverse directions.

Poisson's equation, after a change of variables in the integral, leads to the dispersion relation

$$0 = 1 + \frac{q^2}{\hbar k^2 \varepsilon_0} \int d^3\mathbf{p} \left[\frac{W_0(\mathbf{p})}{\omega - \mathbf{k} \cdot (\mathbf{p}/m - \mathbf{v}_q)} - \frac{W_0(\mathbf{p})}{\omega - \mathbf{k} \cdot (\mathbf{p}/m + \mathbf{v}_q)} \right] \quad (3.30)$$

where $\mathbf{v}_q = \hbar\mathbf{k}/2m$. Evidently, just like the classical Eq. (2.24), the integrand in Eq. (3.30) has poles that should be treated according to Landau's prescription, but they are shifted from the phase velocity by $\pm\mathbf{v}_q$. This means that the Landau damping rate can differ significantly between the classical and the quantum theory. For example, if W_0 is a highly degenerate Fermi-Dirac distribution, v_q is of the same order as the Fermi velocity v_F, and $\omega/k \gtrsim v_F$, then W_0 is negligible near ω/k but significant near $\omega/k - v_q$, as illustrated in Fig. 3.2a. (The linear damping rate can be found for a Fermi-Dirac background at any temperature and any $\mathbf{k}$ [21, 88].)

Drawing upon the analogy with Compton scattering, one can imagine processes with simultaneous absorption or emission of several plasmons, with resonances at $\omega/k \pm nv_q$ where n is the number of plasmons involved. Even if the probability for a multi-plasmon process is smaller than that of a one-plasmon process, under the right circumstances many more particles can participate in the former. This is illustrated in Fig. 3.2b where the one-plasmon resonance is outside the Fermi surface, but the two-plasmon resonance is inside.

Multi-plasmon processes are inherently non-linear, but a semi-analytical treatment is feasible using an insight previously applied to the classical theory [89]: for small amplitudes non-linear effects are important only in a small part of velocity space, near the resonances. In Paper IV we carried out a thorough analysis of weakly non-linear two- and three-plasmon damping. We showed that the wave field decays like $1/(1+t/t_0)^{1/2}$, where, up to $T \lesssim 0.2T_F$, the damping time t_0 can be comparable to or shorter than the linear damping time calculated in Ref. [88].

3.4 Spin

The Weyl transform as defined in Eq. (3.10) involves only the spatial degrees of freeedom, but for a complete description of quantum mechanics we also need to treat spin degrees of freedom. Strictly speaking, we do not need to do anything about the spin, as we can let the Wigner function be matrix-valued and keep spin operators as matrices. In the case of spin 1/2, the density matrix is 2×2 and Hermitian and can be written in the basis $\{1, \boldsymbol{\sigma}\}$; this is a little more natural as the components form a scalar corresponding to the number density and a vector corresponding to the spin density, that evolve according to a set of coupled equations [90].[1]

However, it is more in line with the Wigner formalism to ask if there is a way to define a quasidistribution on the phase space of spin, the 2-sphere, analogous to the Wigner function. One choice is the Q-function [91] for spin s, defined by

$$Q(\mathbf{s}) = \frac{2s+1}{4\pi}\langle \mathbf{s}|\rho|\mathbf{s}\rangle \tag{3.31}$$

where $|\mathbf{s}\rangle$ is a spin coherent state [92, 93], defined as

$$|\mathbf{s}\rangle = e^{i\varphi\tau_3}e^{i\theta\tau_2}|s\rangle \tag{3.32}$$

where τ_i is a generator of rotations, φ, θ are angles on the sphere, and $|s\rangle$ is an eigenstate of τ_3 with maximal eigenvalue. The Q-function is obviously non-negative, but it cannot be interpreted as a probability density because $Q(\mathbf{s}_1)$ and $Q(\mathbf{s}_2)$ do not represent disjoint events, due to the non-orthogonality of coherent states. Nevertheless, like the Wigner function, it can be used to calculate expectation values etc. Many other phase space quasidistributions exist [94] and recently a direct generalization of the Wigner function to spin has been found [95–97]. In this thesis we will, however, only use the Q-function.

[1] For higher spins, additional tensor components would be required.

Specializing to spin 1/2, the Q-function can be expressed as

$$Q(\mathbf{s}) = \frac{1}{4\pi}\,\mathrm{Tr}[(1+\mathbf{s}\cdot\boldsymbol{\sigma})\rho] = \mathrm{Tr}[\hat{S}\rho], \tag{3.33}$$

and we have

$$\langle\boldsymbol{\sigma}\rangle = 3\int \mathbf{s}Q(\mathbf{s})\,d^2s. \tag{3.34}$$

Using the Wigner function for the spatial degrees of freedom and the Q-function for the spin we obtain a quasidistribution f on the total phase space,

$$f = \mathrm{Tr}[\hat{S}\hat{W}\rho], \tag{3.35}$$

with time evolution again determined by evaluating $[\hat{S},\hat{H}]$ and $[\hat{T},\hat{H}]$. For the Pauli Hamiltonian

$$\hat{H} = \frac{\hat{\boldsymbol{\pi}}^2}{2m} + q\varphi - \mu_B\boldsymbol{\sigma}\cdot\mathbf{B} \tag{3.36}$$

this results in the kinetic equation [19]

$$\begin{aligned}\partial_t f + \frac{\tilde{\boldsymbol{\pi}}}{m}\cdot\nabla_x f + q(\tilde{\mathbf{E}} + \frac{\tilde{\boldsymbol{\pi}}}{m}\times\tilde{\mathbf{B}})\cdot\nabla_\pi f \\ + \mu_B[(\mathbf{s}+\nabla_s)\cdot\tilde{\mathbf{B}}](\overleftarrow{\nabla}_x\cdot\overrightarrow{\nabla}_\pi)f + \frac{2\mu_B}{\hbar}\big[\mathbf{s}\times(\tilde{\mathbf{B}}+\Delta\tilde{\mathbf{B}})\big]\cdot\nabla_s f = 0.\end{aligned} \tag{3.37}$$

The new terms compared to Eq. (3.20), on the second line, are the Stern-Gerlach force and the torque on the spin, respectively, and the new object $\Delta\tilde{\mathbf{B}}$ is given by

$$\Delta\tilde{\mathbf{B}} = \mathbf{B}\int_{-1/2}^{1/2}\tau\sin(\tau\hbar\overleftarrow{\nabla}_x\cdot\overrightarrow{\nabla}_\pi)\,d\tau\,(\hbar\overleftarrow{\nabla}_x\cdot\overrightarrow{\nabla}_\pi). \tag{3.38}$$

It should be noted that other than the ∇_s in the magnetic dipole force, the $\hbar\to 0$ limit fits precisely with Liouville's theorem using the Poisson bracket on S^2 (see Ref. [98], Ch. 8.). In fact, the ∇_s-term can be demonstrated to be responsible for the splitting into two of Stern and Gerlach's beam [99].

3.5 Relativistic considerations

For intense laser-plasma interactions [3] or many forms of astrophysical environments [2] a relativistic treatment is needed. In the former case, it has been shown that electrons can spin-polarize on time scales of a few laser cycles [100–102], although these simulations are not self-consistent but based on QED cross-sections. Searching for a self-consistent kinetic description raises the question of the correct relativistic Hamiltonian; in a sense, it is that of Dirac [15],

$$\hat{H} = mc^2\beta + c\boldsymbol{\alpha}\cdot\boldsymbol{\pi} + q\varphi \tag{3.39}$$

where β and $\boldsymbol{\alpha}$ are matrices fulfilling the anti-commutation relations of the Dirac algebra,

$$\beta\boldsymbol{\alpha} = -\boldsymbol{\alpha}\beta \qquad \alpha_i\alpha_j = -\alpha_j\alpha_i + 2\delta_{ij}. \tag{3.40}$$

However, this has some issues. As was noted by Dirac, the smallest matrices that can satisfy Eq. (3.40) are 4×4. Furthermore, in the free-field case, the Dirac Hamiltonian has both positive and negative energy states. The latter can be interpreted as anti-particles, explaining why there are twice as many states, but typically, both particle and anti-particle states have all components non-zero. The Dirac Hamiltonian also bears little resemblance to the classical relativistic expression for the energy, $E = \sqrt{\mathbf{p}^2 + m^2}$, and the velocity operator is $c\alpha_i$ with eigenvalues $\pm c$, even for a massive particle (so-called "Zitterbewegung" [103]). Finally, it is not obvious how to take the non-relativistic limit to obtain the Pauli Hamiltonian.

All these issues are related and can, to an extent, be overcome. Note that quantum mechanics is only defined up to unitary transformations, so Eq. (3.39) is not the unique answer to the question of the correct relativistic Hamiltonian. Foldy and Wouthuysen [6] found the transformation that block-diagonalizes the Hamiltonian and other observables (but see Ref. [104], and refs. therein). For a free particle, this is exact, but in the presence of non-zero fields, it is a series expansion in E/mc^2, and to order v^2/c^2 the positive-energy block is the Pauli Hamiltonian with spin-orbit interaction including the Thomas precession and the Darwin term,

$$\hat{H} = mc^2 + \frac{\hat{\boldsymbol{\pi}}^2}{2m} - \frac{\hat{\boldsymbol{\pi}}^4}{8m^3c^2} - \mu_B\left(\hat{\mathbf{B}} + \frac{\hat{\mathbf{E}}\times\hat{\boldsymbol{\pi}}}{2mc^2}\right)\cdot\boldsymbol{\sigma} - \frac{q\hbar^2}{8m^2c^2}(\nabla\cdot\hat{\mathbf{E}}), \tag{3.41}$$

where the cross product should be understood to be in Hermitian order.

From this, it is rather straight-forward to find the kinetic equation to lowest order in $\hbar$ [105]. In Paper III the all-orders kinetic equation was found by evaluating the commutator $[\hat{T}, \hat{H}]$; this calculation is more lengthy, owing to Eq. (3.24) and the spin-orbit interaction containing both position and momentum operators. An alternative approach is to use the star product for Stratonovich-Weyl symbols [87], as in Ref. [106].

Instead of the energy, one can take as the expansion parameter $k\lambda$, where k is the wavenumber of the fields and $\lambda = \hbar/p$ is the typical de Broglie wavelength [7], allowing for $v \approx c$ and strong fields. This expansion has been carried out only to lowest order, meaning a kinetic equation can also only be valid to first order in $\hbar$. This model is the subject of Papers I and II.

An all-orders theory based on this approach would anyway not be valid, since for $p \geq mc$ the limit set is the Compton wavelength, and at that scale physics should be described by quantum *field* theory, which inter alia allows pair production. Decoupling positive and negative energy states means neglecting pair production, so there is also a limit to how strong fields can be allowed, set by the rest-frame electric field compared to the QED critical field $E_{\text{cr}} = m^2c^3/qh$ [3]. The applicability of this model is discussed further in Papers I and II.

3.5.1 Currents and "hidden momentum"

For self-consistency we also need the charge and current densities. First, to order v^2/c^2, the polarization and magnetization densities are found to be [105, 107]

$$\mathbf{P} = -3\mu_B \int \frac{\mathbf{s} \times \boldsymbol{\pi}}{2mc} f\, d\Omega - \frac{q\hbar^2}{8m^2c^2} \nabla \int f\, d\Omega \tag{3.42}$$

$$\mathbf{M} = 3\mu_B \int \mathbf{s}\, d\Omega, \tag{3.43}$$

the second term in $\mathbf{P}$ arising from the Darwin term. The *free* charge density is easy, since the number density is given by $n = \int f\, d\Omega$, but the continuity equation then takes the form

$$\partial_t n + \nabla \cdot \int \left(\frac{\boldsymbol{\pi}}{m} - \frac{(\boldsymbol{\pi} \cdot \boldsymbol{\pi})\boldsymbol{\pi}}{2m^3c^2} + 3\mu_B \frac{\mathbf{E} \times \mathbf{s}}{2mc^2} \right) f\, d\Omega, \tag{3.44}$$

i.e., the would-be velocity is not proportional to $\boldsymbol{\pi}$, as one might naively expect. Actually, from Eq. (3.41), one sees that this is the Stratonovich-Weyl symbol of the velocity operator $\hat{\mathbf{v}} = \frac{i}{\hbar}[\hat{H}, \hat{\mathbf{x}}]$. The $\mathbf{E} \times \mathbf{s}$ term is called "hidden momentum" and is common to systems with a magnetic moment, and there is an extensive literature on it [108–110; 111, and refs. therein]. The fully relativistic generalizations of the sources, found in Paper I, account for $\mathbf{s}$ being the rest-frame spin [32, Sec. 11.11; 112], which is contracted in the lab frame, and treat the kinetic energy and Thomas precession with no assumption of $|\boldsymbol{\pi}| \ll mc$.

The hidden momentum is related to that the operator $\hat{\mathbf{x}}$ in the Foldy-Wouthuysen representation is not the same observable as in the Dirac representation [6, 113]; in the Foldy-Wouthuysen representation it is the "mean" or Newton-Wigner position operator [114], which evidently is free of Zitterbewegung.

3.6 Exchange

As in classical mechanics, we may want to reduce the number of degrees of freedom. In quantum mechanics this is done by taking a partial trace of the density matrix,

$$\rho_p = \mathrm{Tr}_{p+1,\ldots}(\rho_n), \tag{3.45}$$

defining the reduced p-particle density matrix in terms of the n-particle density matrix. However, a many-particle state must be totally symmetric (anti-symmetric) under permutations of bosons (fermions). Consequently, a cluster expansion like

$$\rho_2 \overset{!}{=} \rho_1 \otimes \rho_1 + g_2 \tag{3.46}$$

is not valid because $\rho_1 \otimes \rho_1$ does not have the correct symmetry. The correct cluster expansion is [115]

$$\rho_n = \hat{\Lambda}_n \left(\sum_{p=1}^{n-1} \rho_{n-p} \otimes g_p \right) \hat{\Lambda}_n + g_n \tag{3.47}$$

where $\hat{\Lambda}_n$ is the operator that (anti-)symmetrizes an n-particle state, $\otimes$ is the tensor product, and g_n contains n-body correlations and has the correct symmetry.

We now specialize to fermions, such as electrons, and consider only the simplest case, 2-particle density matrices. Then

$$\hat{\Lambda}_2 = \hat{\Lambda} = 1 - \hat{P}_{12} \tag{3.48}$$

where $\hat{P}_{12} : |a\rangle|b\rangle \mapsto |b\rangle|a\rangle$. We will also only treat the non-relativistic case. The Hamiltonian for N electrons interacting electrostatically, and with an external potential φ is

$$\hat{H} = \sum_{i=1}^{N} \left(\frac{\hat{\mathbf{p}}_i{}^2}{2m} + \varphi(\hat{\mathbf{x}})_i \right) + \frac{q^2}{4\pi\varepsilon_0} \sum_{i \neq j} \frac{1}{|\hat{\mathbf{x}}_i - \hat{\mathbf{x}}_j|}. \tag{3.49}$$

Neglecting correlations g_2, the evolution equation for ρ_1 is

$$i\hbar\partial_t\rho_1 = [\hat{h}_1, \rho_1] + \mathrm{Tr}_2[\hat{V}, \hat{\Lambda}(\rho_1 \otimes \rho_1)\hat{\Lambda}] = [\hat{h}_1, \rho_1] + [\hat{V}_1, \rho_1] \tag{3.50}$$

where $\hat{h}_1 = \hat{\mathbf{p}}^2/2m + q\varphi$ is the one-particle Hamiltonian and $\hat{V}_1$ is the Hartree-Fock potential operator,

$$\hat{V}_1 = 2\,\mathrm{Tr}_2[\hat{V}\hat{\Lambda}(I \otimes \rho_1)] \tag{3.51}$$

with I the identity operator. The meaning of $\hat{V}_1$ can be understood by considering a 2-particle state $|\psi\rangle = \frac{1}{\sqrt{2}}(|a\rangle|b\rangle - |b\rangle|a\rangle)$.[2] Then

$$\langle\hat{V}_1\rangle = \frac{q^2}{4\pi\varepsilon_0}\int \frac{1}{|\mathbf{x}-\mathbf{y}|}\left(|a(\mathbf{x})|^2|b(\mathbf{y})|^2 - b^\dagger(\mathbf{x})a(\mathbf{x})a^\dagger(\mathbf{y})b(\mathbf{y})\right)\, d^3\mathbf{x}d^3\mathbf{y} \tag{3.52}$$

where the first term is the electrostatic energy between a particle in the state $|a\rangle$ and a particle in the state $|b\rangle$, and the second term is called the *exchange* term or exchange interaction. The exchange term has no classical analog.

Taking the Weyl transform of Eq. (3.50) results in

$$\partial_t W + \frac{\mathbf{p}}{m}\cdot\nabla_x W + q\mathbf{E}\cdot\nabla_p W = C[W] \tag{3.53}$$

where W is understood to be matrix-valued, and $C[\cdot]$ is a quadratic functional [23, 24]. Thus in the Wigner formalism exchange effects appear similarly to a collision operator, although we emphasize that exchange is distinct from collisions – the exchange operator appears even though we have ignored correlations g_2. Modifications of the Boltzmann collision operator for Fermi-Dirac or Bose-Einstein statistics exist [116, 117].

The exchange functional $C[\cdot]$ is non-linear and non-local in phase space, making calculations onerous. Calculations with collision operators are also difficult, but often fluid models, where collisions have been reduced to transport coefficients, are applicable. Thus exchange effects can be included in a fluid model [22] as a potential or pressure,

$$V_x = \frac{0.985(3\pi^2)^{2/3}}{4\pi}\frac{\hbar\omega_c^2}{mv_F^2}\left(\frac{n}{n_0}\right)^{1/3}, \tag{3.54}$$

where the numerical coefficient is determined by density functional theory.

[2]We suppress the spin degrees of freedom for clarity.

The validity of Eq. (3.54) is not obvious, in particular because the coefficient comes from a time-independent calculation. In Papers V and VI we compared the models by looking at exchange contributions to the dispersion relations for Langmuir and ion-acoustic waves. We find that the models disagree by up to an order of magnitude, due to wave-particle interaction that the fluid model cannot resolve. This wave-particle interaction also casuses damping that cannot be seen in the fluid model.

Summary of papers

Paper I – Relativistic kinetic equation for spin-1/2 particles in the long-scale-length approximation
We present a fully relativistic self-consistent kinetic theory for spin-1/2 particles in the long scale length limit. This generalizes previous work [105] from $O(v^2/c^2)$ to $v \approx c$. The model is based on separating electrons and positrons in the Dirac equation by means of a Foldy-Wouthuysen transformation, and expressions for the charge and current densities are derived using a Lagrangian method. We discuss the model in the context of "hidden momentum" and the Abraham-Minkowski dilemma.
My contribution was to derive the sources and the conservation of energy, and to clarify the meaning of the velocity operator and the connection to the hidden momentum. I also wrote most of the paper.

Paper II – Relativistic kinetic theory for spin-1/2 particles: Conservation laws, thermodynamics, and linear waves
Following up on Paper I, we present the full set of conservation laws (energy, momentum, angular momentum), the background magnetization at a given temperature, and an example of linear wave propagation. We discuss in some detail the applicability of the model to scenarios in astrophysics and intense laser-plasma interactions.
My contribution was to derive the conservation laws and write the corresponding part of the paper. I also took part in analyzing the applicability of the model, and helped write the other sections of the paper.

Paper III – Short-scale quantum kinetic theory including spin-orbit interactions
We extend a previous quantum kinetic theory with spin-orbit interaction [105] to include short-scale particle dispersive effects to all orders

in $\hbar$. As an example calculation, we study two linear wave modes and compare the Landau damping due to the electron spin to that due to particle motion for various densities, background magnetic field strengths, and wavenumbers. We find that spin-orbit interaction is the dominant damping mechanism in a large region of the parameter space. We compare our model to a similar one in the literature [106].
I performed all the calculations involved in deriving the model and wrote the corresponding part of the paper. I also participated in discussing the model, especially the connection to Ref. [106], writing other parts of the paper, and contributed to code used for the figure.

Paper IV – Nonlinear wave damping due to multi-plasmon resonances
We demonstrate the existence of a new class of short-wavelength nonlinear resonances in plasmas. These resonances can be intepreted as the simultaneous absorption of multiple wave quanta, and we term them multi-plasmon resonances. Because multi-plasmon and single-plasmon linear resonances are offset in velocity space, the number of resonant particles can differ between them. This effect is most pronounced at high degeneracy and we show with numerical calculations that up to $T \lesssim 0.2T_F$ the multi-plasmon damping rate is comparable to the linear damping rate, even for modest amplitudes.
I performed all calculations leading up to the final set of evolution equations and wrote code to solve these equations and prepare the figures. I also participated in analyzing the results, especially the comparison to the linear theory, and helped in writing the paper.

Paper V – Exchange corrections in a low-temperature plasma
A widely used quantum fluid model [22] includes exchange effects as a potential or pressure term, with a coefficient found from density functional theory, but the validity of the assumptions of the model is unclear. As a check, we find the dispersion relations for Langmuir waves in the high-frequency limit and ion-acoustic waves in a degenerate background, using a Wigner quantum kinetic model of exchange. We find that the kinetic model gives a much larger value for the exchange contribution. There is also damping due to wave-particle interaction that the fluid model cannot resolve.
I derived the main result, Eq. (9) and participated in discussing the results.

Paper VI – Do hydrodynamic models misestimate exchange effects? Comparison with kinetic theory for electrostatic waves

We extend the results of Paper V to ion-acoustic waves without the assumption of quasineutrality, and to Langmuir waves at any wavenumber. We again find that there is significant disagreement between the kinetic and fluid models, both in magnitude but also in the dependence on the wavenumber.

I wrote code to evaluate the exchange contribution to the dispersion relation for a range of wavenumbers and prepare the figures. I also participated in analyzing the results and in writing the paper.

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