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**Document distances using the Zipf distribution and a novel metric**

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## **Abstract**

A novel metric is proposed in the present report for the evaluation of the goodness-of-fit criterion between the distribution functions of two samples. We extend the usage of the proposed criterion for the case of the generalized Zipf distribution. Detailed mathematical analysis of the proposed metric, which is embodied in a hypothesis testing, is also provided.

## **Keywords**

Zipf distribution, n-gram frequencies, bhattacharyya metric

## 1 Introduction

In a plethora of natural phenomena the distribution of a characteristic under consideration is heavily skewed. For example, biological, ecological, and chemical systems sometimes tend to exhibit an exponential decaying model. Web site popularity, web access statistics, Internet traffic, population and growth of cities also comply with the same decaying model. Furthermore, many references can be found in bibliometrics, informetrics and library science. A plethora of distributions exists in the literature that are capable to model the above phenomena; with the most prevalent among them the well-know *Zipf* distribution [1, 3].

The Zipf distribution rely on an empirical law discovered by Estoup in 1916 and named after the Harvard linguistic professor G. K. Zipf (1902-1950). This distribution relates the frequency of occurrence of an event  $\alpha$  and the rank,  $m_\alpha$ , of the event when the rank is determined by the above frequency of occurrence. The relationship is the power-law function:

$$P(\alpha) \sim 1/m_\alpha^\theta \quad (1)$$

with the exponent  $\theta$  to be close to unity. The probability distribution in Eq. (1) is an instance of a power law. Zipf's law is an experimental law, not a theoretical one. The causes of Zipfian distributions in real life are a matter of some controversy. However, Zipfian distributions are commonly observed in many kinds of phenomena.

Initially the Zipf distribution was confined to the linguistic community and associated the frequency of word in a document with its rank [4, 7]. The prerequisite for the above law to be applicable in linguistics is that the size of the document to be fairly large.

## 2 Document distance

Its is generally admissible that the contextual "similarity" between documents (regardless of their size) can be based on their structural textual elements, namely the words forming these documents. The previous fact is the basic principle behind the *vector space model* (VSM) [6]. In the VSM, the available textual data are encoded into a numerical form and are represented by numerical vectors. Furthermore, it is generally agreed upon that the contextual similarity between documents exists also in their vectorial representation. Since the Zipf distribution of a document employs the frequencies of the words forming that particular document, it is justified to evaluate the contextual similarity based on the numerical encoding produced by the particular distribution.

A novel distance measure will be provided in the current chapter. In Appendix A it will be proven that the proposed distance measure is also a metric. This metric is used in order to evaluate the similarity between Zipf distributed vectors. The suggested metric can be easily proven that it is computationally superior (faster) than the Euclidean distance. For example, for two  $N_w$ -dimensional vectors, the computational cost of the suggested metric is  $N_w$  multiplications, a bit-shift operation and  $N_w$  additions compared to  $N_w$  multiplications and  $(2N_w - 1)$  addition of the Euclidean distance.

Furthermore, by exploiting the fact that the vectors under consideration are distributed according to the Zipf law enables us to extend the suggested metric towards the direction of a statistical hypothesis. The hypothesis under consideration is whether two Zipf distributed vectors, and subsequently two documents, are similar or not. For this reason a detailed distribution is provided for the proposed metric along with a detailed proof. Also, two distribution tables are supplied for the proposed metric to make the chapter self-content.

In what follows, section 2.1 provides a description of the proposed metric and section 2.2 describes the process of incorporating the Zipf distribution in the proposed metric. It also provides a detailed proof for the evaluation of the distribution associated with the proposed metric. Following, section 2.3 provides the hypothesis testing for the evaluation of the similarity between two Zipf distributed vectors.

## 2.1 Proposed metric

Let us suppose that  $\mathcal{X}^N = \{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_N\}$  is a collection of  $N_w$ -dimensional random vectors, where  $\mathbf{x}_i = (x_{i1}, x_{i2}, \dots, x_{iN_w})^T$  with cumulative probability density function  $f_{\mathbf{x}}(i)$ . Let also  $x_{im}$  denote the univariate random variable with distribution function  $f_i(m)$ , where  $f_i(m)$  corresponds to the probability of the  $m$ th element of the  $i$ th vector, that is,  $f_i(m) = P(x_{im})$ , where

$$\sum_{m=1}^{N_w} P(x_{im}) = 1. \quad (2)$$

We further assume that the probabilities in Eq. (2) follow the Zipf distribution. In order to assess whether two vectors drawn independently from the set  $\mathcal{X}^N$  are of the same ‘‘shape’’, one needs to compare their distribution functions. For this purpose a novel metric is introduced. Let  $\mathbf{x}_i$  and  $\mathbf{x}_j$  denote two vectors randomly drawn from the set  $\mathcal{X}^N$ , ( $\mathbf{x}_i, \mathbf{x}_j \in \mathcal{X}^N$ ). The hypothesis whose validity is to be tested is:

$H_0$ : The two cumulative distribution functions are ‘‘identical’’  $\Rightarrow f_{\mathbf{x}}(i) = f_{\mathbf{x}}(j)$

or equivalently

$H_0$ :  $f_i(m) \cong f_j(m)$ , for almost each  $m$ ,

against the negation of  $H_0$ . If the null hypothesis is true, the population distributions are identical and the two samples are drawn from the same population, meaning that the vectors  $\mathbf{x}_i$  and  $\mathbf{x}_j$  should be regarded as instances of the same population. Therefore, allowing for statistically neglectful sampling variations, under  $H_0$  there should be reasonable agreement between the two distributions. The proposed criterion between the  $i$ th and  $j$ th distributions, henceforth denoted by  $D_{ij}$ , is defined as:

$$D_{ij} = h(\mathbf{x}_i, \mathbf{x}_j, \mathbf{x}_i \circ \mathbf{x}_j) = (\mathbf{x}_i + \mathbf{x}_j + g(\mathbf{x}_i, \mathbf{x}_j)) (\mathbf{x}_i + \mathbf{x}_j + g(\mathbf{x}_i, \mathbf{x}_j))^T, \quad (3)$$

where the notion ( $\circ$ ) corresponds to the Hadamard product between two vectors and  $g(\mathbf{x}_i, \mathbf{x}_j)$  corresponds to the  $N_w$ -dimensional vector whose the  $k$ -th element is  $\sqrt{P(x_{ik})P(x_{jk})}$  (square root of the *Hadamard product* between the vectors  $\mathbf{x}_i$  and  $\mathbf{x}_j$ ).

From Eq. (3), the following form for the variable  $D_{ij}$ , derives:

$$D_{ij} \triangleq \sum_{m=1}^{N_w} \frac{(f_i(m) + \sqrt{f_i(m)f_j(m)})^2}{f_i(m)} \quad (4)$$

$$= \sum_{m=1}^{N_w} \frac{(f_i^2(m) + f_i(m)f_j(m) + 2f_i(m)\sqrt{f_i(m)f_j(m)})}{f_i(m)}$$

$$= \sum_{m=1}^{N_w} \left( f_i(m) + f_j(m) + 2\sqrt{f_i(m)f_j(m)} \right)$$

$$= \sum_{m=1}^{N_w} (f_i(m) + f_j(m)) + \sum_{m=1}^n \left( 2\sqrt{f_i(m)f_j(m)} \right)$$

$$= 2 + 2 \sum_{m=1}^{N_w} \left( \sqrt{f_i(m)f_j(m)} \right) \quad (5)$$

$$= 2 + 2L_{ij}^B \quad (6)$$

From Eq. (5) is evident that only the square roots of the  $x_{im}$  and  $x_{jm}$  are needed. Therefore, instead of storing the actual values for the  $x_{im}$  and  $x_{jm}$  one can only retain the square roots of them. In that way there is no need

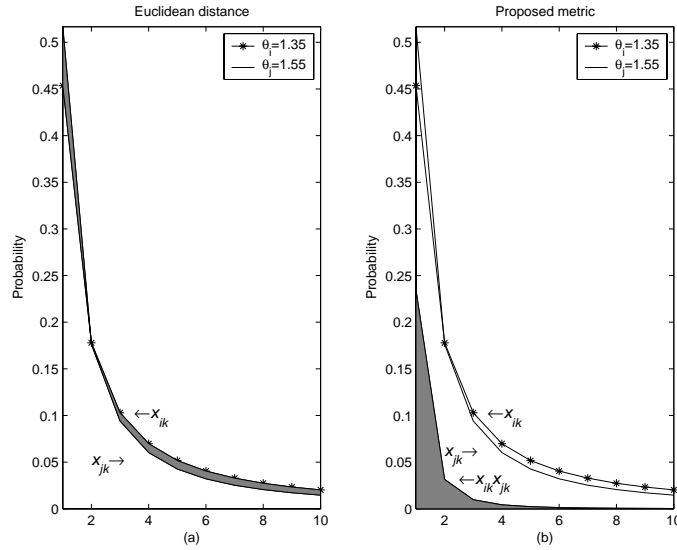


Figure 1: The divergence viewed under different metrics. The grayed area corresponds to the divergence measured from: (a) the Euclidean distance, which relies on the shaded area between the distribution functions, and (b) the proposed metric that is based on the shaded area in the bottom left side of the plot.

to evaluate the square roots each time one needs to evaluate the value of the random variable  $D_{ij}$ , thus limiting the computations cost to just  $N_w$  multiplications, a bit-shift operation (the multiplication by 2) and  $N_w$  additions. Appendix A proves that the proposed distance measure satisfies also the three properties of a metric, so it will be referred as a metric, henceforth.

From Eq. (25) is obvious that  $D_{ij} \in [2, 4]$ , where  $D_{ij}$  equals four when  $f_i(m) = f_j(m), \forall m$ . On the other hand  $D_{ij}$  equals two only in the extreme case where the distributions of the  $i$ th and  $j$ th RVs are of the following form:

$$P(x_{im}) = \begin{cases} \neq 0, & \text{when } P(x_{im}) = 0 \\ = 0, & \text{elsewhere} \end{cases} \quad \forall m \quad (7)$$

in which case the product  $f_i(m)f_j(m)$  equals to zero and therefore  $D_{ij}$  tends towards the value two. So the closer the pdf of the  $i$ th RV is to the pdf of the  $j$ th RV the larger is the value of  $L_{ij}$  and subsequently, the value of  $D_{ij}$  tends toward the value of four. So the hypothesis test mentioned earlier is transformed into:

$H_0$ :  $D_{ij}$  is statistically equal to four

$H_1$ : The negation of  $H_0$

It must be noted here that Eq. (4) resembles the *Chi-square goodness-of-fit* test proposed by Pearson but there is no other resemblance with that particular test. In fact, since Chi-square uses the maximum divergence between the distribution under considerations this might lead to unexpected results in case when the distributions differ in just two samples out of the  $N_w$  samples comprising the  $N_w$ -dimensional vectors.

Figure 1 depicts the areas used by the proposed metric and the Euclidean distance in evaluating the similarity between the distributions<sup>1</sup>.

## 2.2 The Zipf distribution and the proposed metric

In order to evaluate the hypothesis test mentioned in section 2.1 it is needed to compute the probability density function of the random variable  $D_{ij}$ . In doing so one must first determine the probability of the random variable

<sup>1</sup>The vectors used in this figure were artificially generated.

$x_{im}$ . For the case under consideration the probability of the random variable is:

$$f_i(m) = \frac{1}{m^{\theta_i} H_{N_w, \theta_i}}, \quad (8)$$

where  $\theta_i$  is a parameter dependent on the data-set under consideration and  $H_{N_w, \theta_i}$  is the so-called  $N_w$ th Harmonic number of order  $\theta_i$  which is a normalizing factor equal to:

$$H_{N_w, \theta_i} = \sum_{m=1}^{N_w} \frac{1}{m^{\theta_i}}. \quad (9)$$

Equation (8) is the well known *generalized Zipf* distribution [1].

The first step towards the computation of the distribution of the variable  $D_{ij}$  is to evaluate the distribution of the elements of the random vector  $\mathbf{z}_{ij} = (z_{ij1}, z_{ij2}, \dots, z_{ijN_w}) = \mathbf{x}_i \circ \mathbf{x}_j = (x_{i1}x_{j1}, x_{i2}x_{j2}, \dots, x_{iN_w}x_{jN_w})$ . Since for the formation of the  $m$ th element of  $\mathbf{z}_{ij}$  it is needed to multiply the corresponding  $m$ th elements in both  $\mathbf{x}_i$  and  $\mathbf{x}_j$  this leads to the following:  $P(z_{ijm}) = P(x_{im}x_{jm})$ . In the previous expression the random variable  $x_{im}$  is independent of the variable  $x_{jm}$  since they refer to two different random vectors, which leads to:  $P(z_{ijm}) = P(x_{im})P(x_{jm})$ .

For the evaluation of the probability of  $z_{ijm}$  it is needed first to determine the cdf for  $m$  a given number, where  $m \in N$ . Lets denote this distribution by  $F_{ij}(m)$ :

$$\begin{aligned} F_{ij}(m) &= P(\text{until the } m\text{th element of } \mathbf{z}_{ij}) & (10) \\ &= F_i(m) \cdot F_j(m) = \sum_{s=1}^m f_i(s) \cdot \sum_{t=1}^m f_j(t) \\ &= \sum_{s=1}^m P(x_{is}) \cdot \sum_{t=1}^m P(x_{jt}) \\ &= \sum_{s=1}^m \frac{1}{s^{\theta_i} H_{N_w, \theta_i}} \cdot \sum_{t=1}^m \frac{1}{t^{\theta_j} H_{N_w, \theta_j}} \\ &= \frac{1}{H_{N_w, \theta_i} H_{N_w, \theta_j}} \cdot \sum_{s=1}^m \frac{1}{s^{\theta_i}} \cdot \sum_{t=1}^m \frac{1}{t^{\theta_j}} & (11) \end{aligned}$$

where  $F_i(m)$  and  $F_j(m)$  are the cdfs of the  $i$ th and  $j$ th RVs respectively. The next step is to find the pdf for the random variable  $z_{ijm}$ , that is:

$$\begin{aligned} f_{ij}(m) &= P(z_{ijm}) = F_{ij}(m) - F_{ij}(m-1) \\ &= a_{ij} \left[ \sum_{s=1}^m \sum_{t=1}^m \frac{1}{s^{\theta_i}} \cdot \frac{1}{t^{\theta_j}} - \sum_{s=1}^{m-1} \sum_{t=1}^{m-1} \frac{1}{s^{\theta_i}} \cdot \frac{1}{t^{\theta_j}} \right] & (12) \end{aligned}$$

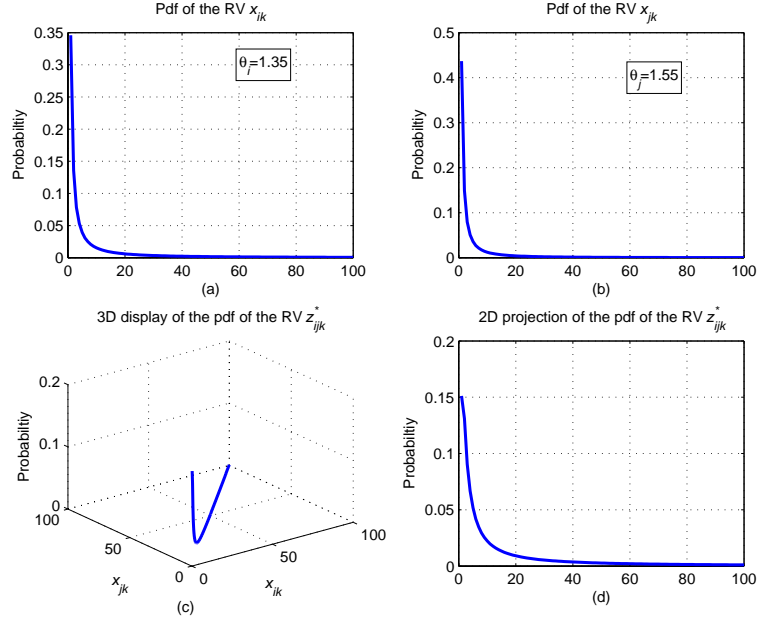


Figure 2: The probability density function for the Zipf distribution for  $N_w = 100$  and for (a)  $\theta_i = 1.35$ , (b)  $\theta_j = 1.55$ , and (c) the product  $z_{ijm}^*$ .

where  $a_{ij}$  denotes the fraction  $1 / (H_{N_w, \theta_i} H_{N_w, \theta_j})$ . From (12) derives:

$$\begin{aligned}
 f_{ij}(m) &= a_{ij} \left[ \frac{1}{m^{(\theta_i + \theta_j)}} + \frac{1}{m^{\theta_i}} \sum_{t=1}^{m-1} \frac{1}{t^{\theta_j}} + \frac{1}{m^{\theta_j}} \sum_{s=1}^{m-1} \frac{1}{s^{\theta_i}} \right] \\
 &= a_{ij} \left[ \frac{1}{m^{(\theta_i + \theta_j)}} + \frac{H_{N_w, \theta_j}}{m^{\theta_i}} F_j(m-1) + \frac{H_{N_w, \theta_i}}{m^{\theta_j}} F_i(m-1) \right] \Rightarrow \\
 f_{ij}(m) &= \begin{cases} a_{ij}, & m = 1 \\ a_{ij} \left[ \frac{1}{m^{(\theta_i + \theta_j)}} + \frac{H_{N_w, \theta_j}}{m^{\theta_i}} F_j(m-1) + \frac{H_{N_w, \theta_i}}{m^{\theta_j}} F_i(m-1) \right], & \forall m \in \{2, N_w\} \\ 0, & elsewhere \end{cases} \quad (13)
 \end{aligned}$$

Figure 2 depicts the process of obtaining the distribution of the random variable  $z_{ijm}$ .

After the computation of the pdf for  $z_{ijm}$  it is needed to compute the density function of the random variable  $\sqrt{z_{ijm}}$ . This is due to the fact that  $D_{ij}$  is a linear combination of  $\sqrt{z_{ijm}}$ . Let  $z_{ijm}^*$  denote the square root of  $z_{ijm}$ , that is,  $z_{ijm}^* = \sqrt{z_{ijm}}$ , where  $m \in \{1, 2, \dots, N_w\}$ . Since the sample space for the RV  $z_{ijm}$  is the set  $Z_1 = \{1, 2, \dots, N_w\}$ , the sample space corresponding to  $z_{ijm}^*$  is the set  $Z_2 = \{1, \sqrt{2}, \dots, \sqrt{N_w}\}$ . It must be noted here that the cardinality of the set  $Z_2$  is equal to  $N_w$  since each element of the set  $Z_2$  is the square root of the set  $Z_1$ . So  $z_{ijm}$  is a discrete RV then the RV  $z_{ijm}^*$  is of the same pdf as the RV  $z_{ijm}$  [5] and if  $f_{ij}^*(m)$  denotes the pdf of the RV  $z_{ijm}^*$ , then,  $f_{ij}^*(m) = f_{ij}(m), \forall m$ .

The final step is to evaluate the pdf of the random variable  $L_{ij} = \sum_{m=1}^{N_w} \sqrt{z_{ijm}} = \sum_{m=1}^{N_w} z_{ijm}^*$ . For a large value

of  $N_w$  and due to the central limit theorem (CLT) the pdf of the above sum tends toward the normal distribution with mean value  $\mu$  and variance  $\sigma^2$  [5]. The mean value is:

$$\begin{aligned}
 \mu &= \mathbb{E}[L_{ij}] = \mathbb{E}\left[\sum_{m=1}^{N_w} z_{ijm}^*\right] = \sum_{m=1}^{N_w} \mathbb{E}[z_{ijm}^*] \\
 &= N_w \mathbb{E}[z_{ijm}^*] = N_w \sum_{m=1}^{N_w} \sqrt{m} f_{ij}^*(m) \\
 &= N_w a_{ij} \sum_{m=2}^{N_w} \sqrt{m} \left[ \frac{1}{m^{(\theta_i+\theta_j)}} + \frac{H_{N_w, \theta_j}}{m^{\theta_i}} F_j(m-1) + \frac{H_{N_w, \theta_i}}{m^{\theta_j}} F_i(m-1) \right] \\
 &= N_w a_{ij} \sum_{m=1}^{N_w} \left[ \frac{1}{m^{(\theta_i+\theta_j)-0.5}} + \frac{H_{N_w, \theta_j}}{m^{\theta_i-0.5}} F_j(m-1) + \frac{H_{N_w, \theta_i}}{m^{\theta_j-0.5}} F_i(m-1) \right] \\
 &= N_w a_{ij} \begin{bmatrix} H_{N_w, (\theta_i+\theta_j)-0.5} + \\ H_{N_w, \theta_j} \sum_{m=1}^{N_w} \frac{F_j(m-1)}{m^{\theta_i-0.5}} + \\ H_{N_w, \theta_i} \sum_{m=1}^{N_w} \frac{F_i(m-1)}{m^{\theta_j-0.5}} \end{bmatrix} \tag{14}
 \end{aligned}$$

and the variance is:

$$\begin{aligned}
 \sigma^2 &= \mathbb{E}[(L_{ij} - \mu)^2] = \mathbb{E}[(L_{ij})^2] - \mu^2 \\
 &= \mathbb{E}\left[\left(\sum_{m=1}^{N_w} z_{ijm}^*\right)^2\right] - \mu^2 \\
 &= \mathbb{E}\left[\sum_{m=1}^{N_w} (z_{ijm}^*)^2 + 2 \sum_{\substack{m_1, m_2=1 \\ m_1 \neq m_2}}^{N_w} z_{ijm_1}^* z_{ijm_2}^*\right] - \mu^2 \\
 &= \sum_{m=1}^{N_w} \mathbb{E}[(z_{ijm}^*)^2] + 2 \sum_{\substack{m_1, m_2=1 \\ m_1 \neq m_2}}^{N_w} \mathbb{E}[z_{ijm_1}^* z_{ijm_2}^*] - \mu^2 \\
 &= N_w \mathbb{E}[(z_{ijm}^*)^2] + 2N_w(N_w - 1) \mathbb{E}[z_{ijm_1}^* z_{ijm_2}^*] - \mu^2 \tag{15}
 \end{aligned}$$

At this point, and without loss of generality, it can be regarded that the RVs  $z_{ijm_1}^*$  and  $z_{ijm_2}^*$  are independent. Having this postulate:

$$\mathbb{E}[z_{ijm_1}^* z_{ijm_2}^*] = \mathbb{E}[z_{ijm_1}^*] \mathbb{E}[z_{ijm_2}^*] \tag{16}$$



The first term on the right side of the variance equation is equal to:

$$\begin{aligned}
 E \left[ (z_{ijm})^2 \right] &= a_{ij} \sum_{m=1}^{N_w} (\sqrt{m})^2 \left[ \begin{array}{l} \frac{1}{m^{(\theta_i+\theta_j)}} + \\ \frac{H_{N_w, \theta_j}}{m^{\theta_j}} F_j(m-1) + \\ \frac{H_{N_w, \theta_i}}{m^{\theta_i}} F_i(m-1) \end{array} \right] \\
 &= a_{ij} \left[ \begin{array}{l} H_{N_w, (\theta_i+\theta_j)-1} + \\ H_{N_w, \theta_j} \sum_{m=1}^{N_w} \frac{F_j(m-1)}{m^{\theta_j-1}} + \\ H_{N_w, \theta_i} \sum_{m=1}^{N_w} \frac{F_i(m-1)}{m^{\theta_i-1}} \end{array} \right] \quad (17)
 \end{aligned}$$

whereas the second term equals to:

$$E \left[ z_{ijm_1}^* \right] E \left[ z_{ijm_2}^* \right] = \mu^2 \quad (18)$$

So the total variance of the random variable  $L_{ij}$  is:

$$\begin{aligned}
 \sigma^2 &= N_w a_{ij} \left[ H_{N_w, (\theta_i+\theta_j)-1} + H_{N_w, \theta_j} \sum_{m=1}^{N_w} \frac{F_j(m-1)}{m^{\theta_j-1}} + H_{N_w, \theta_i} \sum_{m=1}^{N_w} \frac{F_i(m-1)}{m^{\theta_i-1}} \right] \\
 &+ [2N_w(N_w-1) - 1] \mu^2 \\
 &= N_w a_{ij} \left[ H_{N_w, (\theta_i+\theta_j)-1} + H_{N_w, \theta_j} \sum_{m=1}^{N_w} \frac{F_j(m-1)}{m^{\theta_j-1}} + H_{N_w, \theta_i} \sum_{m=1}^{N_w} \frac{F_i(m-1)}{m^{\theta_i-1}} \right] \\
 &+ N_w a_{ij} [2N_w(N_w-1) - 1] \left[ \begin{array}{l} H_{N_w, (\theta_i+\theta_j)-0.5} + \\ H_{N_w, \theta_j} \sum_{m=1}^{N_w} \frac{F_j(m-1)}{m^{\theta_j-0.5}} + \\ H_{N_w, \theta_i} \sum_{m=1}^{N_w} \frac{F_i(m-1)}{m^{\theta_i-0.5}} \end{array} \right] \\
 &= N_w a_{ij} \left[ H_{N_w, (\theta_i+\theta_j)-1} + [2N_w(N_w-1) - 1] H_{N_w, (\theta_i+\theta_j)-0.5} \right] \\
 &+ 2a_{ij} N_w^2 (N_w-1) H_{N_w, \theta_j} \left[ \sum_{m=1}^{N_w} \frac{F_j(m-1)}{m^{\theta_j-1}} + \sum_{m=1}^{N_w} \frac{F_j(m-1)}{m^{\theta_j-0.5}} \right] \\
 &+ 2a_{ij} N_w^2 (N_w-1) H_{N_w, \theta_i} \left[ \sum_{m=1}^{N_w} \frac{F_i(m-1)}{m^{\theta_i-1}} + \sum_{m=1}^{N_w} \frac{F_i(m-1)}{m^{\theta_i-0.5}} \right] \quad (19)
 \end{aligned}$$

which is equal to:

$$\begin{aligned}
 \sigma^2 &= N_w a_{ij} \sum_{m=1}^{N_w} \frac{1 + [2N_w(N_w-1) - 1] m^{-0.5}}{m^{(\theta_i+\theta_j)-1}} \\
 &+ 2a_{ij} N_w^2 (N_w-1) H_{N_w, \theta_j} \sum_{m=1}^{N_w} \frac{F_j(m-1) [1 - m^{-0.5}]}{m^{\theta_j-1}} \\
 &+ 2a_{ij} N_w^2 (N_w-1) H_{N_w, \theta_i} \sum_{m=1}^{N_w} \frac{F_i(m-1) [1 - m^{-0.5}]}{m^{\theta_i-1}} \quad (20)
 \end{aligned}$$

Finally, the pdf of the RV  $D_{ij} = 2 + 2L_{ij}$  has to be computed. Given the fact that  $L_{ij}$  is normally distributed we get the following pdf for the RV  $D_{ij}$ :

$$f_{D_{ij}}(t) = \frac{f_{L_{ij}}\left(\frac{t-2}{2}\right)}{2} = \frac{1}{\sqrt{8\pi\sigma}} \exp\left\{-\frac{1}{8\sigma^2}(t-2-2\mu)^2\right\} \quad (21)$$

where  $\mu$  and  $\sigma$  are the expected value and the standard deviation of the random variable  $L_{ij}$ .

But since the random variable  $D_{ij}$  is confined in the interval  $[2, 4]$  ( $D_{ij} \in [2, 4]$ ), Eq. (21) obviously underestimates the true pdf of  $D_{ij}$ . The accurate form of the pdf is:

$$f_{D_{ij}}(t) = \begin{cases} 0, & -\infty \leq t \leq 2 \\ \frac{\exp\left\{-\frac{1}{8\sigma^2}(t-2-2\mu)^2\right\}}{\int_2^4 \exp\left\{-\frac{1}{8\sigma^2}(t-2-2\mu)^2\right\} dt}, & 2 \leq t \leq 4 \\ 0, & 4 \leq t \leq +\infty \end{cases} \quad (22)$$

which is the so-called *truncated* normal distribution [5]. Equation (22) can be simplified in the following form:

$$f_{D_{ij}}(t) = \begin{cases} \frac{\exp\left\{-\frac{1}{8\sigma^2}(t-2(1+\mu))^2\right\}}{\sqrt{2\pi\sigma} \operatorname{erf}\left(\frac{x}{\sqrt{2\sigma}} - \frac{2(1+\mu)}{\sqrt{2\sigma}}\right)\Big|_{x=2}}^4, & 2 \leq t \leq 4 \\ 0, & \text{elsewhere} \end{cases} \quad (23)$$

where  $\operatorname{erf}(a) = \frac{2}{\sqrt{\pi}} \int_0^a e^{-t^2} dt$  denotes the so-called *error function* [2].

### 2.3 Hypothesis test evaluation

To assess whether the  $i$ th and  $j$ th distributions are of the same ‘‘shape’’, the RV  $D_{ij}$  will be employed in a hypothesis test. The hypothesis is as follows:

$H_0$ : The  $i$ th and  $j$ th pdf are statistically identical which equals to  $D_{ij} \rightarrow 4$ .

$H_1$ : The  $i$ th and  $j$ th pdf are not identical.

Given a pre-defined significant level  $a$ , the rejection region for the above hypothesis test is formulated as follows:

$$\begin{aligned} a &= P(D_{ij} \leq z_a) = \int_2^{z_a} f_{D_{ij}}(t) dt \\ &= \frac{\operatorname{erf}\left(\frac{(1+\mu)}{\sqrt{2\sigma}} - \frac{z_a}{2\sqrt{2\sigma}}\right) - \operatorname{erf}\left(\frac{(1+\mu)}{\sqrt{2\sigma}} - \frac{1}{\sqrt{2\sigma}}\right)}{\operatorname{erf}\left(\frac{(1+\mu)}{\sqrt{2\sigma}} - \frac{2}{\sqrt{2\sigma}}\right) - \operatorname{erf}\left(\frac{(1+\mu)}{\sqrt{2\sigma}} - \frac{1}{\sqrt{2\sigma}}\right)}. \end{aligned} \quad (24)$$

In Eq. (24) the only unknown is the parameter  $z_\alpha$ . After evaluating the parameter  $z_\alpha$  the null hypothesis is accepted if the expression  $z_\alpha \leq D_{ij}(t)$  is true otherwise its rejected. Figure 3 depicts the distribution of the RV  $D_{ij}$  along with the support regions for the hypothesis  $H_0$  and the alternative hypothesis  $H_1$ .

Appendix B provides a brief description of the computation of the critical values for the acceptance or the rejection of the null hypothesis along with two tables with critical values computed by the proposed Eq. (27).

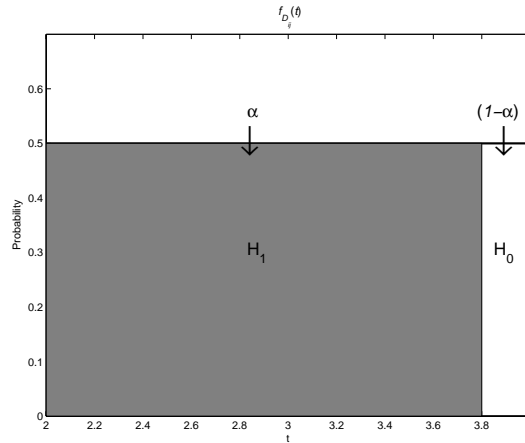


Figure 3: The support regions for the null and the alternative hypothesis for the RV  $D_{ij}$  for  $N_w = 2000$ ,  $\theta_i = 1.35$  and  $\theta_j = 1.55$  at a significant level of  $\alpha = 0.90$ . (Important remark: Although the above graph implies a uniform distribution this is not true. The slope of the line in the graph approaches zero but still is significantly different than this value.)

### 3 Conclusions

The present report provides a preliminary mathematical analysis on a novel metric, that is also introduced in the report, for the evaluation of the contextual similarity between documents. The proposed metric is computationally superior than the Euclidean distance which is oftenly employed in similar tasks. Further investigation will be performed towards the direction of the biasness of the introduced metric (investigate whether the proposed metric is biased or not).

#### A Is it a metric?

In order to prove that the proposed statistic,  $D_{ij}$ , is also a metric distance the following has to be proven:

**Positiveness:** Since  $f_i(m)$  and  $f_j(m)$  for  $m = 1, 2, \dots, N_w$  contains the total probability mass of the  $i$ th and  $j$ th

RV the following stems out:

$$\begin{aligned}
 & \left. \begin{aligned} 0 \leq x_{im} < 1 \text{ and } \sum_{m=1}^{N_w} x_{im} = 1 \\ 0 \leq x_{jm} < 1 \text{ and } \sum_{m=1}^{N_w} x_{jm} = 1 \end{aligned} \right\} \Rightarrow \\
 & 0 \leq x_{im}x_{jm} \leq 1 \Rightarrow \\
 & 0 \leq \sqrt{x_{im}x_{jm}} \leq 1 \Rightarrow \\
 & 0 \leq \sum_{m=1}^{N_w} \sqrt{x_{im}x_{jm}} \leq 1 \Rightarrow \\
 & 0 \leq 2L_{ij} \leq 2 \Rightarrow \\
 & 2 \leq 2 + 2L_{ij} \leq 4 \Rightarrow \\
 & 2 \leq D_{ij} \leq 4
 \end{aligned} \tag{25}$$

In case where  $i = j$  then  $L_{ii} = \sum_{m=1}^{N_w} \sqrt{x_{im}x_{jm}} = \sum_{m=1}^{N_w} x_{im} = 1 \Rightarrow D_{ii} = 2 + 2L_{ii} = 4$ .

**Symmetry:** Since  $x_{im}x_{jm} = x_{jm}x_{im} \Rightarrow D_{ij} = D_{ji}$ .

**Triangular inequality:** In order to prove the triangular inequality it can be proven that:

$$D_{ij} + D_{jm} \geq D_{im} \Rightarrow 2 + 2L_{ij} + 2 + 2L_{jm} \geq 2 + 2L_{im} \Rightarrow 1 + L_{ij} + L_{jm} \geq L_{im} \tag{26}$$

which is obvious since  $L_{ij}, L_{jm} \geq 0$  and  $1 \geq L_{im}$ .

## B Critical values

The critical values for the hypothesis test associated with the RV  $D_{ij}$  are computed using the following:

$$\begin{aligned}
 \frac{1}{2\sqrt{2}\sigma} (2(1+\mu) - z_\alpha) \int_0^{e^{-t^2}} dt &= \frac{\alpha\sqrt{\pi}}{2} \operatorname{erf}\left(\frac{(\mu-1)}{\sqrt{2}\sigma}\right) + \frac{(1-\alpha)\sqrt{\pi}}{2} \operatorname{erf}\left(\frac{\mu}{\sqrt{2}\sigma}\right) \Rightarrow \\
 z_\alpha &= 2(1+\mu) - 2\sqrt{2}\sigma \operatorname{erf}^{-1}\left(\alpha \operatorname{erf}\left(\frac{(\mu-1)}{\sqrt{2}\sigma}\right) + (1-\alpha) \operatorname{erf}\left(\frac{\mu}{\sqrt{2}\sigma}\right)\right),
 \end{aligned} \tag{27}$$

where  $\operatorname{erf}^{-1}$  is the inverse of the error function [2]. Using Eq. (27) and pre-defined significance levels two tables of critical values for hypothesis test were computed. Table I<sup>2</sup> corresponds to a significance level of  $\alpha = 0.90$  when the dimensionality of the feature vectors is  $N_w = 2000$ , whereas, table II<sup>3</sup> corresponds to a significance level of  $\alpha = 0.95$  under the same feature vector dimensionality.

<sup>2</sup>The values of the table a scaled version of the original values. Original values = 3.8+scaled value\*10<sup>-9</sup>.

<sup>3</sup>The values of the table a scaled version of the original values. Original values = 3.8+scaled value\*10<sup>-9</sup>.

## References

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Table I: Distribution tables for the RV  $D_{ij}$  and for  $N_w = 2000$  at  $\alpha = 0.90$  (10% Confidence Level).

1.05	1.10	1.15	1.20	1.25	1.30	1.35	1.40	1.45	1.50	1.55	1.60	1.65	1.70
1.05	1.2081	1.2117	1.2146	1.2183	1.2234	1.2256	1.2277	1.2314	1.2314	1.2328	1.2365	1.2372	1.2365
1.10		1.2161	1.2197	1.2263	1.2299	1.2336	1.2372	1.2408	1.243	1.2452	1.2459	1.2481	1.2488
1.15			1.2263	1.2328	1.2379	1.2438	1.2481	1.251	1.2547	1.2576	1.259	1.2612	1.2627
1.20			1.2394	1.2467	1.2539	1.2583	1.2634	1.2634	1.2667	1.271	1.2736	1.2761	1.2776
1.25			1.2561		1.2619	1.2685	1.2754	1.2812	1.2852	1.2889	1.2889	1.2929	1.2958
1.30					1.2725	1.2816	1.2885	1.2954	1.3012	1.3056	1.3056	1.3092	1.3132
1.35						1.2929	1.3023	1.31	1.3161	1.3231	1.3282	1.3322	1.3322
1.40							1.3151	1.3238	1.3329	1.3405	1.3471	1.3529	1.3529
1.45								1.3383	1.3489	1.358	1.366	1.3733	1.3733
1.50									1.3642	1.3751	1.3853	1.3938	1.3938
1.55										1.3915	1.4027	1.4135	1.4135
1.60											1.4195	1.4313	1.4313
1.65												1.4484	1.4484
	1.75	1.80	1.85	1.90	1.95	2.00							
1.05	1.2372	1.2379	1.2379	1.2394	1.2387	1.2394							
1.10	1.2488	1.2503	1.2503	1.251	1.251	1.2503							
1.15	1.2641	1.2649	1.2656	1.2659	1.2663	1.2659							
1.20	1.279	1.2805	1.2805	1.2823	1.2823	1.282							
1.25	1.2972	1.2983	1.2998	1.3001	1.3016	1.3012							
1.30	1.3151	1.3172	1.3187	1.3201	1.3205	1.3216							
1.35	1.3362	1.338	1.3409	1.3427	1.3442	1.3445							
1.40	1.3569	1.3609	1.3634	1.3656	1.3674	1.3682							
1.45	1.3787	1.3831	1.3871	1.3898	1.3918	1.3935							
1.50	1.4005	1.406	1.4113	1.4144	1.4173	1.4196							
1.55	1.4216	1.4287	1.4344	1.4389	1.4424	1.4451							
1.60	1.4418	1.4502	1.4575	1.4628	1.4673	1.4706							
1.65	1.4604	1.4704	1.4784	1.4851	1.49	1.4942							
1.70	1.4769	1.4882	1.4975	1.5059	1.5119	1.517							
1.75		1.5046	1.5153	1.5241	1.5315	1.537							
1.80			1.5301	1.5402	1.5483	1.5544							
1.85				1.5539	1.5626	1.5697							
1.90					1.5752	1.5824							
1.95						1.593							
2.00													

Table II: Distribution tables for the RV  $D_{ij}$  and for  $N_w = 2000$  at  $\alpha = 0.95$  (5% Confidence Level).

	1.05	1.10	1.15	1.20	1.25	1.30	1.35	1.40	1.45	1.50	1.55	1.60	1.65	1.70
1.05														
1.10		6.3825	6.397	6.4116	6.4552	6.448	6.4625	6.4771	6.4916	6.5062	6.4989	6.5207	6.528	6.5207
1.15			6.4188	6.4334	6.4698	6.4843	6.5134	6.5207	6.5498	6.5571	6.5644	6.5716	6.5935	6.6007
1.20				6.4625	6.5134	6.528	6.5571	6.5935	6.608	6.6226	6.6299	6.6517	6.6517	6.6662
1.25					6.5425	6.5789	6.6153	6.6444	6.6808	6.6844	6.7099	6.7244	6.739	6.7463
1.30						6.6226	6.659	6.699	6.7317	6.7608	6.7826	6.8045	6.8227	6.8372
1.35							6.7172	6.7608	6.7936	6.8409	6.8736	6.8918	6.91	6.9354
1.40								6.8227	6.8736	6.9063	6.9464	6.99	7.0046	7.0373
1.45									6.9464	6.9864	7.0373	7.081	7.1101	7.1392
1.50										7.0591	7.121	7.1646	7.2083	7.2483
1.55											7.201	7.2556	7.3102	7.3538
1.60												7.3393	7.4029	7.4593
1.65													7.4902	7.5557
														7.643

	1.75	1.80	1.85	1.90	1.95	2.00
1.05	6.5207	6.5425	6.5425	6.528	6.5498	6.5425
1.10	6.6007	6.608	6.5935	6.608	6.6007	6.6007
1.15	6.659	6.6808	6.6808	6.6808	6.6844	6.6808
1.20	6.7499	6.7608	6.7608	6.7645	6.7645	6.7608
1.25	6.8445	6.8518	6.8554	6.859	6.8663	6.8736
1.30	6.9391	6.9573	6.9609	6.9645	6.9718	6.9827
1.35	7.0482	7.0628	7.0773	7.0846	7.0919	7.0992
1.40	7.161	7.1828	7.1937	7.2083	7.2119	7.2228
1.45	7.2738	7.3029	7.3211	7.3338	7.3484	7.352
1.50	7.3938	7.4193	7.4448	7.4666	7.4811	7.4921
1.55	7.4993	7.5393	7.5703	7.5957	7.6121	7.6285
1.60	7.6066	7.6539	7.6921	7.7231	7.7413	7.7613
1.65	7.7049	7.7594	7.8067	7.8377	7.8631	7.8868
1.70	7.7922	7.8559	7.905	7.9468	7.9814	8.005
1.75		7.9413	7.9959	8.0432	8.0814	8.1123
1.80			8.0778	8.1287	8.1705	8.2051
1.85				8.2015	8.2506	8.2833
1.90					8.3142	8.3506
1.95						8.407
2.00						