Dimensions and Projections

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Abstract

This thesis concerns dimensions and projections of sets that could be described as *fractals*. The background is applied problems regarding analysis of human tissue. One way to characterize such complicated structures is to estimate the dimension. The existence of different types of dimensions makes it important to know about their properties and relations to each other. Furthermore, since medical images often are constructed by x-ray, it is natural to study projections.

This thesis consists of an introduction and a summary, followed by three papers.

Paper I, Anders Nilsson, *Dimensions and Projections: An Overview and Relevant Examples*, 2006. Manuscript.

Paper II, Anders Nilsson and Peter Wingren, *Homogeneity and Non-coincidence of Hausdorff- and Box Dimensions for Subsets of* $\mathbb{R}^n$, 2006. Submitted.

Paper III, Anders Nilsson and Fredrik Georgsson, *Projective Properties of Fractal Sets*, 2006. Submitted.

The first paper is an overview of dimensions and projections, together with illustrative examples constructed by the author. Some of the most frequently used types of dimensions are defined, i.e. Hausdorff dimension, lower and upper box dimension, and packing dimension. Some of their properties are shown, and how they are related to each other. Furthermore, theoretical results concerning projections are presented, as well as a computer experiment involving projections and estimations of box dimension.

The second paper concerns sets for which different types of dimensions give different values. Given three arbitrary and different numbers in $(0, n)$, a compact set in $\mathbb{R}^n$ is constructed with these numbers as its Hausdorff dimension, lower box dimension and upper box dimension. Most important in this construction, is that the resulted set is homogeneous in the sense that these dimension properties also hold for every non-empty and relatively open subset.

The third paper is about sets in space and their projections onto planes. Connections between the dimensions of the orthogonal projections and the dimension of the original set are discussed, as well as the connection between orthogonal projection and the type of projection corresponding to realistic x-ray. It is shown that the estimated box dimension of the orthogonal projected set and the realistic projected set can, for all practical purposes, be considered equal.

Key words: Hausdorff dimension, box dimension, packing dimension, projections, fractals.

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Introduction

Many objects in nature have an irregular non-smooth structure which may be hard to describe with traditional geometry using straight lines or smooth curves. Moreover, sometimes this structure looks similar at different scales, a property called self-similarity. Objects with this type of complicated geometry are frequently called *fractals*. The word fractal comes from the latin word fractus, which means broken, and was made popular by the mathematician Benoit Mandelbrot during the 1970's (see e.g. [Man75], [Man77]).

There is no formal definition of fractals. Suggestions of possible definitions have been presented (see for example [Man82], [Tay86]), but they always seem to exclude some interesting cases. This leaves the concept of fractals a little bit fuzzy. Thus, people may have different opinions regarding the fractality of a given set. However, there are some properties that are typically associated with fractals. A set F may be considered fractal if it has some of the following properties (see for example [Fal90]).

- F has a fine, detailed structure in all scales.
- F has a complicated structure, which makes it hard to describe with traditional geometry (with lines, circles and so on).
- F has some kind of self-similarity.
- The fractal dimension of F (defined in some reasonable way) is greater than its topological dimension.
- F can be described as a limit set of a sequence of geometrically simple sets, maybe in some recursive way.

Regardless of the fractality or non-fractality of a given set, it is possible to use tools originating from the area of fractal geometry to examine that set. However, these tools may of course be more useful for some sets than others. For example, there is no need for advanced fractal geometry when handling simple shapes such as lines, circles, squares, cubes, etc.

It may be useful in some situations to distinguish different sets in terms of their geometric properties. One way of doing this is to determine their dimensions, using some reasonable definition of dimension. There are several different types of dimensions, some of them with multiple names. A few of the most frequently used types of dimensions are Hausdorff dimension, upper and lower box-dimension, and packing dimension.

The dimension of a set is a non-negative number, reflecting in some sense how much space the set fills. For example, a line is one-dimensional, a square is two-dimensional, and a cube is three-dimensional. Other sets may have dimensions that are not integers. It is a central part of fractal geometry to investigate different types of dimensions, and how to determine or estimate dimensions for families of sets in order to be able to characterize them.

The existence of different types of dimensions, makes it natural to ask questions about the possible non-coincidence of the corresponding dimension values. For simple shapes, such as a line, a square, or a cube, any reasonable definition of dimension should result in the values one, two and three. However, it is actually possible for a set to have different dimensions depending on what definition is used (see for example [Fal90, p.43], [LG92], [McM84], [PW96, pp.129-132],[Spe99], [NW06]). This may seem to be a serious drawback. However, this is a consequence of the fact that some types of dimensions are designed to be easy to use in applications, while others are designed to have appealing mathematical properties.

Numerous investigations have shown that many objects and processes in nature have properties typically associated with fractals. Some examples are branching of trees, blood-vessels, rivers, clouds, coastlines, internet traffic, etc. Thus, it should be possible to find useful practical applications for fractal techniques. One interesting area is analysis of medical images. For example, x-ray images of human tissue have been shown to exhibit typical fractal properties (see e.g. [Kal98]).

When studying medical images, it is in many cases essential to be able to generate information showing differences between healthy and non-healthy tissue. Some researchers claim that cancer tumors have another fractal dimension than normal tissue (see e.g. [BJ00] and [CSH+90]), but some skepticism have been put forward with the motivation that other research groups have not been able to repeat the results (see e.g. [HLD94], [JF98], [VGvdM+96]).

The analysis of mammograms is one interesting example of a medical area where fractal techniques may be useful. A mammogram image is produced by exposing a breast to x-rays. As a result, some of the x-rays are absorbed, while others pass through the breast to expose a film. The image produced on the film in this way shows the internal structures of the breast. In a mathematical setting, this process corresponds to a plane projection of a mass distribution in space, i.e. a measure. The resulted projection is also a mass distribution.

Fractal analysis of a mass distribution is called multifractal analysis, and that is an area lying outside the scope of this thesis. However, with the application of mammograms in mind, it is still natural to study projections of sets. For simple shapes, it is easy to imagine the projections. For example, the orthogonal projection of a sphere in space onto a plane is always a disc, and the projection of a line is in general a line. For more complicated sets, it may be difficult to make precise statements about the projections.

Intuitively, it is natural to expect that a set with high dimension has projections with high dimensions, and that a set with low dimension has projections with low dimensions. This is generally true, but one has to be careful when dealing with dimensions and projections, since different types of dimensions have different projection properties. Relationships between the dimension of the projections and the dimension of the original set, are formulated and investigated e.g. in [Mar54], [Mat75], and [FH96].

Summary of papers

This thesis consists of three papers, and they are summarized below.

Paper I, Anders Nilsson, *Dimensions and Projections: An Overview and Relevant Examples*, 2006. Manuscript.

This paper concerns dimensions and projections of sets in Euclidean spaces. The area is considered from a mathematical point of view, but with applied problems in mind, regarding analysis of human tissue. An overview is given, followed by a number of illustrative examples constructed by the author.

The text begins with the concept of dimension, which is a central part of fractal geometry. There are many different types of dimensions, and some of the most frequently used types of dimensions are defined here, i.e. Hausdorff dimension, lower and upper box dimension, and packing dimension.

Some properties of the dimensions are shown, such as how the dimension of a set is related to subsets, and how different types of dimensions are related to each other. Iterated function systems are discussed briefly, giving situations in which the dimension can easily be determined. Moreover, certain mappings are described, that map sets without affecting the dimension.

Projections of sets in $\mathbb{R}^n$ onto lower-dimensional subspaces are discussed, with a focus on connections between the dimension of a set and the dimensions of its projections. It is shown that in general, a stronger statement can be made for Hausdorff dimension than for other types of dimensions. The connection between central projection and orthogonal projection is also discussed.

The paper ends with a number of examples, some of them presented for the first time, that illustrate some of the results given earlier in the paper. In these examples, differences between theory and applications are discussed. The first two examples concern a certain statement about almost all projections of a set. The first example shows that the exceptional angles of projections may be a dense set. In the second example, a computer approximation of a subset of $\mathbb{R}^3$ is generated, and the box dimensions of projections onto planes are estimated. The next example shows a problem with the frequently used practical method of estimating the box dimension of a set. The last example is a construction of a set with different Hausdorff dimension, lower box dimension and upper box dimension.

Paper II, Anders Nilsson and Peter Wingren, *Homogeneity and Non-coincidence of Hausdorff- and Box Dimensions for Subsets of* $\mathbb{R}^n$, 2006. Submitted.

This paper concerns sets for which different types of dimensions give different values. A general Cantor set construction is presented, that provides such sets. Furthermore, the resulting sets have a certain homogeneity property with respect to the types of dimensions involved. The result is formulated in the following theorem.

Theorem. *Given the Euclidean space* $\mathbb{R}^n$ *and* $r, s, t \in (0, n)$, $r < s < t$, *there is a compact set* $K \subset \mathbb{R}^n$, *such that for each non-empty set* U, *relatively open in* K,

$$\overline{\dim}_B(U) = t,$$

$$\underline{\dim}_B(U) = s,$$

$$\dim_H(U) = r.$$

One of the most important properties of K in the theorem above, is that every non-empty and relatively open set in K has the same dimensions as K. It is easy to associate this type of homogeneity with the typical fractal property of self-similarity, which also is a connection between local and global characteristics. One consequence of this homogeneity is that the packing dimension coincides with the upper box dimension.

Paper III, Anders Nilsson and Fredrik Georgsson, *Projective Properties of Fractal Sets*, 2006. Submitted.

This paper is about sets in space and their projections onto planes. The situation is studied with applied problems in mind, regarding analysis of x-ray images of human tissue. In this application, it is desirable to be able

to draw conclusions about a structure in space based on knowledge about its projections.

A mathematical background is presented, where Hausdorff dimension and box dimension are defined. Some of their properties are shown, and how they are related to each other. Moreover, connections between the dimension of a set and the dimensions of its orthogonal projections are presented. These results show that a bound on the dimension of the original set can be established by estimating the dimension of its projections.

Two types of projections are considered, central projection and orthogonal projection. A connection between these two types of projections are presented, as well as the relation to the type of projection corresponding to a realistic x-ray. A computer approximation of a Sierpinski tetrahedron is generated, in order to be able to compare orthogonal projection and realistic x-ray projection. It is shown that the estimated box dimension of the orthogonally projected set and the realistically projected set can, for all practical purposes, be considered equal.

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