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# ON ESTIMATION IN ECONOMETRIC SYSTEMS IN THE PRESENCE OF TIME-VARYING PARAMETERS

by

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AKADEMISK AVHANDLING

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Abstract

Economic systems are often subject to structural variability. For the achievement of correct structural specification in econometric modelling it is then important to allow for parameters that are time-varying, and to apply estimation techniques suitably designed for inference in such models. One realistic model assumption for such parameter variability is the Markovian model, and Kalman filtering is then assumed to be a convenient estimator. In the thesis several aspects of using Kalman filtering approaches to estimation in that framework are considered. The application of the Kalman filter to estimation in econometric models is straightforward if a set of basic assumptions are satisfied, and if necessary initial specifications can be accurately made. Typically, however, these requirements can generally not be perfectly met. It is therefore of great importance to know the consequences of deviations from the basic assumptions and correct initial specifications for inference, in particular for the small sample situations typical in econometrics. If the consequences are severe it is essential to develop techniques to cope with such aspects.

For estimation in interdependent systems a two stage Kalman filter is proposed and evaluated, theoretically, as well as by a small sample Monte Carlo study, and empirically. The estimator is approximative, but with promising small sample properties. Only if the transition matrix of the parameter model and an initial parameter vector are misspecified, the performance deteriorates. Furthermore, the approach provides useful information about structural properties, and forms a basis for good short term forecasting.

In a reduced form framework most of the basic assumptions of the traditional Kalman filter are relaxed, and the implications are studied. The case of stochastic regressors is, under reasonable additional assumptions, shown to result in an estimator structurally similar to that due to the basic assumptions. The robustness properties are such that in particular the transition matrix and the initial parameter vector should be carefully estimated. An estimator for the joint estimation of the transition matrix, the parameter vector and the model residual variance is suggested and utilized to study the consequences of a misspecified parameter model. By estimating th transitions the parameter estimates are seen to be robust in this respect.

Keywords and phrases

Econometric systems, time-varying parameters, Kalman filtering, stochastic regressors, unknown transition matrix, robustness, simulation.

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The thesis consists of this summary and the following four papers

[A] Robustness properties of a Kalman filter estimator in interdependent systems with time-varying parameters. Statistical Research Report 1979-5. To appear in Charatsis, E.G. (Ed) Selected papers on contemporary econometric problems. North-Holland. Amsterdam. (1981)

[B] On the estimation of time-varying parameters for forecasting and control. Statistical Research Report 1980-10.

[C] On Kalman filtering in econometric systems. Statistical Research Report 1978-8 (Revised August 1980).

[D] Joint estimation of parameters and their rate of change in varying parameter regression. Statistical Research Report 1980-6.

## 1. Introduction

The formation of economic policy is a complex process in which many individuals are involved, and in which their intuitive and qualitative information is mixed with results from applied quantitative analysis. At many levels of decision making in society the usefulness of econometric models is now fully recognized, and models form basic devices in policy formation processes, e.g. Ball (1978).

The primary uses of the econometric models in such a process are threefold, e.g. Zackrisson (1977) and Intriligator (1978). Firstly, in establishing knowledge of the current and historical periods structural analyses by econometric models play an important role. In this stage the acquirement of knowledge of causes and effects, adjustment processes, etc. are key factors. Secondly, in the offensive stage of preparing future economic policy, the assessment of likely future developments of factors both within and outside the control of the decision makers is important. The forecasting of outside factors largely affects the forecasting of the future activities to be planned and is hence very important. Thirdly, in the choice between alternative policies and/or in their preparation, policy evaluation techniques based on econometric models can be most rewarding.

Generally in econometric analysis the data used are time series, and the statistical techniques employed to estimate the econometric model assume that unknown parameters are constant over time. This assumption, like some other assumptions underlying econometric model buildning must, however, be regarded as approximations to reality. In fact, numerous reasons have been provided to demonstrate that the parameters of many macroeconometric models should be regarded as time-varying, e.g. Cooley (1971) and Rosenberg (1973a). The question then arises whether deviations from the traditional constant parameter assumption severely affect inferences to be drawn on the basis of the conventionally estimated model. It appears as if the consequences for inference may be severe (e.g. Cooley, 1971 and Cargill and Meyer, 1978), and this has resulted in important research into estimation techniques for time-varying parameters.

Only recently, however, the first approaches to structural form estimation of time-varying parameters have been presented, cf. Aagaard-Svendsen (1979) and the reports underlying the thesis summary.

The major purposes of this summary are to indicate why the constant parameter assumption is not always a reasonable one, and to summarize the results of the underlying reports with respect to estimation in econometric systems, when the parameters are time-varying.

The plan of the summary is as follows. Section 2 deals with time-varying parameters, giving reasons for that based on economic theory and data, as well as indicating some of the consequences for empirical use. Further, a brief survey of approaches to estimation of time-varying parameters is provided. In section 3 the reports of the thesis are summarized. The merits and drawbacks of the developed approaches are indicated. The final section points at areas where further research is needed, with special stress given to the question of empirical applicability.

## 2. Time-varying parameters

### 2.1 Sources of parameter variation

A number of reasons has been given for the parameters of econometric models to be time-varying rather than constant over time. The reasons relate to the two basic components of the econometric problem, i.e. to the theory and the data, and to their conciliation.

A complete review of the economic theoretical reasons is outside the scope of this thesis, and only two reasons for time-varying parameters will be mentioned. It has e.g. long been wellknown that the relationship between quantity and price in a free market is a nonconstant one - the relation switches between a demand and a supply regime. The further development of this theme has led into what is sometimes called disequilibrium econometrics (e.g. Bowden, 1978). In the presence of rational expectations it has been argued (Lucas, 1976) that policy making brings about an altering structure of the econometric model. The variation in parameters is then dependent on changes in the agents' behaviour. The consequences for estimation have been thoroughly studied by e.g. Wallis (1980) and Wall (1980).

As to reasons for time-varying parameters in data, distinction must be made between reasons in the process of obtaining data from the real world and the real world itself. The gathering of data is based on some conception of the phenomena of interest. As knowledge changes by a changing conception of the real world and/or as an implication of collected data, altered demands may be placed on the data (Morgenstern, 1963 ch. 5). Examples of such alterations are changes in the timing and coverage of variables. In the official production of economic statistics such changes are usually clearly stated and the time series adjusted according to the new definitions or measurement procedures. In cases where no such adjustments are performed or where the quality of adjustment is in doubt it is obvious that the relationship between variables is a changing one. In time series covering longer periods of time several redefinitions may have occurred.

In the real world such things as the composition of the population and the institutional setup brings about structural changes. Time invariant parameters, thus, can be regarded as as approximation which is better the shorter is the studied period.

In the actual model specification, i.e. when theory and data are combined, several new problems arise, which may imply time-variation in parameters. A model is a simplified representation of reality and as such all causes to a certain variable can not possibly be included. If then an excluded variable behaves in a nonstationary manner the intercept of the model will be time-varying. If further the excluded variable is correlated with an independent variable the parameter of that variable will be time-varying as well.

Other sources to time-varying parameters are the exclusion of nonlinear variables which are linearly included, or the replacement of nonlinear variables by linear ones, Cooley (1971). Further, the use of proxy variables in a constant parameter relationship implies time-varying parameters when the correlation between the proxy and the true variable is nonconstant, Cooley (1971).

Usually, however, the modeller can never a priori be completely certain about the prevalance of time-variation in the parameters, or structural changes. For this reason a number of tests for structural change has been proposed. A complete description of these is outside the scope of the thesis, and suffice it to mention only some of the more important ones.

The Chow test (Chow, 1960) assumes a split of the sample into two parts and tests for an abrupt change. A corresponding test has been suggested by Quandt (1960). To test for gradual structural change Brown et al. (1975) proposed several tests based on recursive residuals. An alternative to these tests with better power properties is suggested by Garbade (1977). Applications of some of the tests are given in Hackl and Katzenbeisser (1978).

## 2.2 Consequences of ignoring parameter variation

Once it has been established that there are reasons to suspect time-variation in parameters in many econometric models, one naturally poses the question of whether the conventional estimation techniques still can be properly applied.

It should here be pointed out that general statements of the consequences of ignoring time-varying parameters naturally are highly dependent on the type and the strength of parameter variation. In the following, the shortcomings of conventional least squares methods are briefly summarized for cases where in fact the parameters are continously time-varying.

Most of the works reported on this subject are concerned with single equations or reduced forms, and only some limited experience for structural forms is available. Cooley (1971), in a single relation framework, demonstrates that in general the ordinary least squares (OLS) estimator is biased as is the residual variance estimator; see also Rosenberg (1973a). By relating the time-varying parameters to each other through a Markovian model, Freebairn (1978) shows that a recursive OLS estimator has the ability of tracking a varying parameter path but more slowly than a time-varying parameter estimator. By utilizing the approach of Sant (1977), it is possible to show that OLS is an unbiased and consistent estimator of the mean of the parameter process, and that it is inefficient relative to an estimator taking the time-variation into account. This statement is, however, only valid for the special case when the Markovian model simplifies to a random walk model.

In considering the consequences for forecasting Cooley (1971) demonstrates by Monte Carlo studies that time-varying parameter estimators give better short term forecasts than OLS. The same conclusions for empirical models are noted by Cooley (1971, 1975), and McWhorter et al. (1977).

As to structural form estimation only some limited experience is available, and then mainly from comparing single equation time-varying parameter estimators with e.g. two stage least squares (2SLS). Brännäs and Westlund (1980b) compare a two stage time-varying parameter estimator analytically with 2SLS and versions of generalized least squares (GLS). In a small sample Monte Carlo study it is observed that OLS and 2SLS perform slightly worse when the parameters obey a random walk model, and that OLS and 2SLS are more badly off when the transitions in the parameter model are different from unity. In forecasting exercises Cooley (1975) and Baudin and Westlund (1980) note improvements in short-term forecasting accuracy over OLS and 2SLS, respectively, when using single equation time-varying parameter estimators. The same is noted in Brännäs (1980b) in comparing 2SLS with both one and two stage Kalman estimators. In forecasting several steps ahead 2SLS is generally better (Brännäs, 1980b and McWhorter et al., 1977).

## 2.3 A brief survey of approaches to estimation of time-varying parameters

Since the causes of parameter variations are of a number of different types, it is not surprising to find that many approaches to modelling and estimation have been proposed. A first and rather rough division of approaches is into methods dealing with only a limited number of changes in parameter values and methods allowing for many changes.

In the first group, which are natural correspondents to changes in variable definitions, strikes, wars, etc., one finds some of the by now most wellknown methods. An early approach is the use of dummy variables. The effect of using such variables is that the intercept is allowed to vary, and that the times of variation are assumed a priori known. This approach is now standard in most econometric textbooks, in part because OLS is still an appropriate estimator.

When the slope parameters are allowed to vary over time a small number of times, there is a choice between switching regression and its complements, segmented and spline regressions. Quandt (1958) introduced switching regressions for the case of a deterministic switching point. A more general approach, suggested by Goldfeld and Quandt (1972), is to allow the switches to depend on other variables but still in a deterministic fashion. Stochastic alternatives are discussed by e.g. Quandt (1972). Usually the estimation is based on the likelihood function (cf. Goldfeld and Quandt, 1973 and Bowden, 1978); for a Bayesian alternative, see Ferreira (1975). The actual estimation in cases with unknown switching points is computationally troublesome and to obtain well-behaved likelihood functions, further information has usually to be incorporated.

Segmented and spline regressions can be regarded as special cases of switching regression where various continuity restrictions are imposed in the switching points. The switches are then continous and smooth. The additional restrictions are often reasonable and they result in more well-behaved likelihood functions. For a survey of these approaches and some applications, see e.g. Poirier (1976).

In the second major group of time-varying parameter estimators the parameters are typically assumed to take on an infinite number of values and in a random manner. A motivation behind this group is that the modeller believes in a changing structure but does not know much about turning points, nor the size and the type of change. The estimators in this group can be distinquished by the amount of structure that is imposed on the sequence of parameter values.

Models that are not assumed to have a dependent sequence of parameter values are usually called random coefficient regressions. This is the standard setting for the regression problem as posed by the Bayesians, e.g. Zellner (1971). In fact, much of the work in the area has Bayesian origins, with the sampling theory results obtained as the special case of diffuse prior densities. In recent years great influence originates from the works of Swamy, e.g. Swamy (1971). For bibliographies, see Johnson (1977, 1980), and for a survey Johnson (1978).

Presently, it seems that the most active area of research is into models with parameters that are random and correlated over time. The present thesis is an example of this. Most of the works has either followed an orthodox approach of extending GLS (e.g. Cooley, 1971 and Rosenberg, 1973b), or a filtering approach originating in Kalmans work (Kalman, 1960) in engineering. Only recently, it was formally shown that the two approaches coincide in certain instances, Sant (1977).

Cooley (1971) introduced a model and an estimator that logically corresponds to the practice of adaptive forecasting. The parameter variation is assumed to be governed by a permanent and a transitory component, respectively, i.e.

$$\beta_t = \beta_t^P + u_t$$

$$\beta_t^P = \beta_{t-1}^P + v_t$$

where $\beta_t^P$ represents the permanent and $u_t$ the transitory component, respectively. By substitution of the parameter model into the general linear model it is found that the new residuals are heteroscedastic and dependent on unknown covariances for $u_t$ and $v_t$. These are assumed to be given by

$$\text{Cov } u_t = (1-\gamma)\ \sigma^2\ \Sigma_u$$

$$\text{Cov } v_t = \gamma\ \sigma^2\ \Sigma_v$$

By forming the likelihood function, under normality assumptions, the unknowns $\gamma$, $\sigma^2$ and $\beta_t$ are estimated, with the elements of $\Sigma_u$ and $\Sigma_v$ assumed a priori known. The approach has been analytically studied by e.g. Cooley (1971) and Cooley and Prescott (1976) and its performance in forecasting by e.g. Cooley (1971, 1975).

An alternative to the Cooley-approach is the varying parameter regression approach in which the parameters are assumed to follow the first order system

$$\beta_t = A_t \beta_{t-1} + v_t$$

Combining this with the general linear model results in a system of the same appearence as the system utilized by Kalman (1960) in deriving filtering equations for $\beta_t$. A survey of the applicability of the Kalman filter in econometrics is given in Athans (1974). The further treatise of the Kalman filter is postponed to the next section, as it forms the core of the thesis and requires special attention.

An alternative approach to the Kalman filter is the one which is based on substitution of the parameter model into the structural relation. Rosenberg (1973a,b) develops and evaluates estimators for this approach. By entering all the unknowns into the estimation problem the approach, however, runs into severe numerical problems, that by the corresponding filtering approach are significantly reduced; e.g. Brännäs (1980a).

# 3. The Kalman approach to estimation of stochastic and time-varying parameters

## 3.1 General background

The Kalman filter (Kalman, 1960) was originally introduced as a computationally attractive method for filtering problems related to stochastic difference (and differential) equations. Such equations can be written in the alternative representation of the state space form through a realization; for economic expositions, see e.g. Preston and Wall (1973), and Aoki (1976).

The Markovian parameter model

(3.1) $\beta_t = A_t \beta_{t-1} + v_t$

in combination with the linear model

(3.2) $y_t = x_t \beta_t + e_t$

takes the form of a state space representation, and hence the Kalman approach is readily applicable. In systems theory (3.1) is called the system equation and (3.2) the measurement equation. Though (3.1) here represents a model for parameter variation, the approach both can and has been successfully used for other purposes in econometrics as well (e.g. Chow, 1975 and Wall, 1980). In the case of stochastic and time-varying parameters (3.1) is e.g. easily extended to incorporate explanatory variables (e.g. Belsley, 1973 and Swamy and Tinsley, 1980).

The focus for the estimation is the parameter (state) vector $\beta_t$, which is stochastic and time-varying. In sampling theory vocubulary the problem would then be described as one of prediction. The concept filtering is here used when, in the estimation of $\beta_t$, there is information on y and x up through time t. If the available information covers a shorter period than t, it is a prediction problem, and if the amount of information covers a period longer than t, a smoothing problem.

The Kalman approach is designed to operate on a model of the form (3.1)-(3.2). If, however, the model is solved in terms of $\beta_N$ (or $\beta_0$), a model with serially correlated residuals is obtained (for details see Sant, 1977 or Brännäs, 1980a). In observation form this will be

$$(3.3) \qquad Y = X\,A\,\beta_N + G - X\,B\,V$$

Sant (1977) showed formally that the GLS estimator can be written in a form equivalent to the Kalman filter, when the transition matrix $A_t$ equals I. The GLS (or the likelihood estimator) is the special case of a Bayes estimator where the prior density is diffuse. The estimation of $\beta_0$ has been studied by Sarris (1973) in a Bayesian framework, and by Cooper (1973) using the tools of mixed estimation.

Several alternative derivations of the Kalman filter have been provided. Kalman (1960) used the theory of orthogonal projections, while other researchers have devised derivations based on likelihood, Bayesian, and least squares arguments; for a review, see e.g. Jazwinski (1970). Disregarding which approach is chosen, the density of $\beta_t$ is estimated by its first and second order conditional moments $\beta_{t|t} = E(\beta_t|y_1,\ldots,y_t)$ and $\Sigma_{t|t} = E(\beta_t-\beta_{t|t})(\beta_t-\beta_{t|t})|y_1,\ldots,y_t)$.

For the model (3.1), and under the assumptions given below, the filtering equations are given as (e.g. Jazwinski, 1970)

$$(3.4) \qquad \beta_{t|t-1} = A_t\,\beta_{t-1|t-1}$$

$$(3.5) \qquad \Sigma_{t|t-1} = A_t\Sigma_{t-1|t-1}\,A_t' + Q$$

$$(3.6) \qquad K_t = \Sigma_{t|t-1}\,x_t'\,[x_t\,\Sigma_{t|t-1}x_t' + R]^{-1}$$

$$(3.7) \qquad \Sigma_{t|t} = \Sigma_{t|t-1} - K_t\,x_t\,\Sigma_{t|t-1}$$

$$(3.8) \qquad \beta_{t|t} = \beta_{t|t-1} + K_t(y_t - x_t\beta_{t|t-1})$$

$\beta_{t|t}$ is the estimate of $\beta_t$ given information up through time t, and $\Sigma_{t|t}$ is the corresponding covariance matrix. In deriving the equations the following assumptions are essential;

(i) $x_t$ is a matrix of fixed exogenous variables

(ii) the residual vectors $e_t$ and $v_t$ satisfy

$$Ev_t = 0;\ Ee_t = 0;\ Ev_tv_s' = \delta_{ts}Q;\ Ee_te_s' = \delta_{ts}R;\ Ev_te_s' = 0$$

where $\delta_{ts}$ is the Kronecker delta. Q and R are assumed to be a priori known

(iii) the transition matrices $A_t$, t=1,2,...,N are a priori known

(iv) the initial parameter vector $\beta_{0|0}$ is a priori known, as is the associated covariance matrix $\Sigma_{0|0}$.

Clearly, these assumptions make an application of the filtering equations primarily suitable for reduced forms (fixed $x_t$), and further in particular for situations where previous experience exists (known covariances, etc.).

At a theoretical level the present thesis can be regarded as a step towards relaxing these assumptions in the case of reduced as well as structural forms. The efforts put into the work originates from the desire of making the filtering approach a realistic complement to the standard econometric techniques, designed for time-varying parameter problems.

In the case of reduced forms, assumption (i) is relaxed in Brännäs (1978), and assumption (iii) and part of (ii) in Brännäs (1980a). For interdependent models assumption (i) is clearly violated and the consequences of this is studied in Brännäs (1978), which is based on the filtering approach suggested in Brännäs and Westlund (1978). This estimator is approximative with respect to assumption (i). The estimator is evaluated in Brännäs and Westlund (1979), where the sensitivity to misspecification with respect to assumptions (ii), (iii) and (iv) are considered both for the reduced and the structural form, see also Brännäs (1980b). The possible consequences of a misspecified parameter model studied in Brännäs (1980a). The papers referred to in this paragraph form the basis for the thesis and are summarized in more detail in the next sections.

## 3.2 Reduced form estimation

The application of the Kalman filter to reduced forms is straightforward when assumptions (i)-(iv) above are satisfied. Under these and a model of the form (3.1)-(3.2) the statistical properties of the Kalman filter are good.

It is possible to show that $\beta_{t|t}$ is an unbiased and consistent estimator of the mean of $\beta_t$ (e.g. Brännäs and Westlund, 1980b). The filter is best linear, and best under normality assumptions, in the sense of minimum mean square error (e.g. Jazwinski, 1970). When, however, some or all of the assumptions are violated the performance of the estimator is deteriorated. Jazwinski (1970) gives analytical expressions for the bias as a function of misspecifications of $\beta_{0|0}$, Q and R. The expressions are, however, too complex for making inferences of joint effects.

As long as the transition matrices $A_t$ are equal to the identity matrix OLS (and GLS) give unbiased and consistent estimates of the mean $E\beta_t$. This is easily demonstrated by using the reformulation of Sant (1977) (Brännäs and Westlund, 1980b).

### 3.2.1 Stochastic regressors

In many models of economic phenomena it is not reasonable to make an explicit assumption of fixed regressors, though this for purely technical reasons may be convenient. Firstly, if lagged endogenous variables are among the regressors assumption (i) is clearly violated. This case has been studied by Åström and Wittenmark (1971) and it results in the usual Kalman filter. Secondly, by definition a variable is fixed if it can be predicted without error. Variables satisfying this are trend and intercept variables but most other economic variables must be regarded as stochastic. The argument is further amplified by the common fact that an exogenous variable of one model well can serve as a random and endogenous variable in another model.

In Brännäs (1978) assumption (i) is relaxed and it is shown that the filtering equations are given as

(3.9) $$\beta_{t|t-1} = A_t\, \beta_{t-1|t-1}$$

(3.10) $$\Sigma_{t|t-1} = A_t \Sigma_{t-1|t-1}\, A_t' + Q$$

(3.11) $$K_t^* = \Sigma_{t|t-1}\, x_{t|t-1}' [x_{t|t-1} \Sigma_{t|t-1} x_{t|t-1}' + \Sigma_{t|t-1}^{\beta\tilde{x}} + R]^{-1}$$

(3.12) $$\Sigma_{t|t} \simeq \Sigma_{t|t-1} - K_t^* x_{t|t-1} \beta_{t|t-1} + H_t$$

(3.13) $$\beta_{t|t} = \beta_{t|t-1} + K_t^*(y_t - x_{t|t-1}\beta_{t|t-1})$$

The structure of these equations is the same as that of (3.4)-(3.8). However, it is immediately clear that $x_t$ is replaced by $x_{t|t-1}$ and that a new covariance matrix $\beta_{t|t-1}^{\beta\tilde{x}}$ has entered into $K_t^*$ in (3.11). In the covariance expression $\Sigma_{t|t}$ a new covariance matrix $H_t$ has entered, and the approximation depends on setting $\Sigma_{t|t}^{\beta\tilde{x}} = \Sigma_{t|t-1}^{\beta\tilde{x}}$. These changes all originate in the form of the innovation sequence $\eta_t = (y_t - y_{t|t-1})$, which incorporates the information at time t not previously known. In the fixed regressor case this takes the form

(3.14) $$\eta_t = x_t(\beta_t - \beta_{t|t-1}) + e_t = x_t\tilde{\beta}_{t|t-1} + e_t$$

i.e. it is a function of the model residual $e_t$ and the prediction error $\tilde{\beta}_{t|t-1}$ in the parameter vector. If the regressors are stochastic the innovation sequence is given as

(3.15) $$\eta_t^* = x_{t|t-1}\tilde{\beta}_{t|t-1} + \tilde{x}_{t|t-1}\beta_t + e_t = x_t\tilde{\beta}_{t|t-1} + \tilde{x}_{t|t-1}\beta_{t|t-1} + e_t$$

i.e. $\eta_t^*$ is a function of the model residual $e_t$ and the prediction error $\tilde{\beta}_{t|t-1}$, but now also a function of the prediction $\bar{x}_{t|t-1}$ and the prediction error $\tilde{x}_{t|t-1}$.

Since the Kalman gain $K_t^*$ and the covariance $\Sigma_{t|t}$ are functions of the innovations the changes apparent in (3.9)-(3.13) follows. The effect of the incorporation of $\Sigma_{t|t-1}^{\beta\tilde{x}}$ in $K_t^*$ is that $K_t^*$ will become smaller which in turn implies a slower correction of old estimates in $\beta_{t|t}$. Thus, it confirms the intuition in the sense that if uncertainty is increased new information is to be used conservatively. From a practical point of view these filtering equations are not directly applicable. The covariance matrices need further consideration and a convenient stratey for obtaining $x_{t|t-1}$ must be determined. Only after this may it numerically be judged whether the modification with the increased computational burden is worthwhile.

### 3.2.2 Sensitivity to misspecifications

In Brännäs and Westlund (1979, 1980a) the robustness properties with respect to assumptions (ii), (iii), and (iv) are considered as a by-product to the evaluation of the two stage Kalman filter estimator suggested in Brännäs and Westlund (1978).

It is demonstrated by Monte Carlo simulations that the estimator of $\beta_t$ is remarkably robust with respect to the specification of Q and R (assumption (ii)). The sensitivity to a misspecified $\beta_{0|0}$ (assumption (iv)) is greater and in particular the impact of an incorrect $A_t$ (assumption (iii)) is serious. These results are well in line with the results of earlier corresponding studies (McWhorter et al., 1976), and hence further underline the need for research related to the estimation of the transition matrix $A_t$ and the initial parameter vector $\beta_{0|0}$.

### 3.2.3 Joint estimation of a time invariant transition matrix and the time-varying parameter vector

Motivated by the apparent sensitivity to misspecified transition matrices Brännäs (1980a) suggests and to some extent evaluates an estimator for a time invariant transition matrix A, the parameter vector $\beta_t$, and the model residual covariance matrix R.

This estimator belongs to the class of prediction error minimizing techniques and is based on the likelihood of the prediction error sequence $\eta_t = y_t - y_{t|t-1}$, where $y_{t|t-1}$ is obtained from a Kalman filter. By basing the likelihood on this sequence, the likelihood function is significantly simplified and the computational burden reduced. The estimates of A are obtained by nonlinear maximization of the concentrated likelihood function. The obtained estimates of A and R are substituted into the filtering equations to yield the desired estimates of $\beta_t$.

The identifiability questions for this estimation problem are discussed in Tse and Weinert (1975) and in Pagan (1980). Some results on the asymptotic properties are given in Ljung (1978), Ljung and Caines (1979), and Pagan (1980).

In a small sample Monte Carlo study it is seen that the estimator performs well in the sense of yielding small bias and mean square error. It is further demonstrated that the specification of the covariance matrix Q in this particular situation improves the bias properties, if increased. When the true parameter model is different from the one assumed, an increasing Q reduces the bias and improves the model fit. Generally, the bias properties of $\beta_t$ are promising and the technique computationally convenient.

It is possible to obtain estimates of $\beta_{0|0}$ and Q by the same likelihood function as long as the identifiability criteria of Pagan (1980) are satisfied.

## 3.3 Structural form estimation

Structural form estimation is in the general case different from reduced form estimation with respect to the presence of right hand endogenous variables. By this, assumption (i) of section 3.1 is violated, and further there exist nonzero correlations between these endogenous variables and the residuals. In the case of constant parameters, this implies that single equation estimators like OLS are biased even asymptotically, as would the Kalman filter be in the present framework.

To the knowledge of the author, two approaches to structural form estimation, when the parameters are time-varying have been suggested. Aagaard-Svendsen (1979) suggests that the estimation is to be performed over the linearized reduced form (with respect to structural parameters). The approach is thus an application of the extended Kalman filter (e.g. Jazwinski, 1970). Aagaard-Svendsen applies the estimator to the estimation of constant parameters in a small model of the Danish economy.

An alternative approach is suggested by Brännäs and Westlund (1978). This estimator is functionally parallel to two stage least squares, in the sense that right hand endogenous variables are replaced by instrumental variables (variables that are uncorrelated with the residuals but correlated with the variables that are replaced). In a second stage the traditional Kalman filter is applied in the same manner as OLS is applied in 2SLS. The instrumental variables are obtained by regressing each endogenous variable on the set of predetermined variables.

3.3.1 The estimator

The estimator suggested by Brännäs and Westlund (1978) (see also Brännäs and Westlund, 1979) is based on the linear structural form

$$(3.16) \qquad y_t = \tilde{y}_t C_t + x_t B_t + \varepsilon_t = z_t \beta_t + \varepsilon_t$$

where the parameters are assumed to vary according to

$$(3.17) \qquad \beta_t = A_t \beta_{t-1} + w_t .$$

In the first stage the reduced form corresponding to (3.16)-(3.17) is utilized to obtain $y_{t|t} = x_t \pi_{t|t}$, where $\pi_{t|t}$ are the reduced form parameter estimates. The estimation in stage one is suggested to be performed by a Kalman filter, where the a priori quantities are obtained from the structural form correspondents. In the second stage $\tilde{y}_{t|t}$ is used instead of the right hand endogenous $\tilde{y}_t$ vector in (3.16) and the Kalman filter applied a second time.

The estimating equations are thus given as

$$(3.18) \qquad \beta_{t|t-1} = A_t \beta_{t-1|t-1}$$

$$(3.19) \qquad \Sigma_{t|t-1} = A_t \Sigma_{t-1|t-1} A_t' + QS$$

$$(3.20) \qquad K_t = \Sigma_{t|t-1} z_{t|t}' [z_{t|t} \Sigma_{t|t-1} z_{t|t}' + RS]^{-1}$$

$$(3.21) \qquad \Sigma_{t|t} = \Sigma_{t|t-1} - K_t z_{t|t} \Sigma_{t|t-1}$$

$$(3.22) \qquad \beta_{t|t} = \beta_{t|t-1} + K_t (y_t - z_{t|t} \beta_{t|t-1})$$

where QS and RS are defined analogously to Q and R of section 3.1. The matrix $z_{t|t}$ of explanatory variables is given as $z_{t|t} = (\tilde{y}_{t|t}, x_t)$.

3.3.2 An exact estimator

If the $x_t$ matrix contains fixed exogenous variables the reduced form estimator satisfies assumption (i) of section 3.1. On the other hand, the instrumental variable part of $z_{t|t}$ is stochastic in $\pi_{t|t}$ and thus assumption (i) is violated in the second stage.

Brännäs (1978) recognizes this fact and studies the consequences of relaxing the assumption. It is shown that the impact of random regressors in the present context is of the same kind as in the reduced form case. The estimator (cf. Brännäs and Westlund, 1978) is thus an approximative one and is shown to have a larger Kalman gain. If $y_t$ in (3.16) is assumed to be generated through $y_{t|t}$ instead of $y_t$, the estimator of Brännäs and Westlund (1978) would be an exact one.

It must be recognized, however, that the estimator of Brännäs and Westlund (1978) is a computationally convenient estimator, whereas the estimator of Brännäs (1978) is numerically complex and not of a fully recursive nature.

### 3.3.3 Sensitivity to misspecifications

The structural form estimator is for its use dependent on a priori specifications on model and parameter residual covariances, the initial parameter vector and the transition matrix in a still more tricky way than in the reduced form case. A natural approach to specifying these is to initialize in terms of the structural form and then to transform into reduced form quantities (cf. Brännäs and Westlund, 1979).

In Brännäs and Westlund (1979, 1980a) the estimator is evaluated by Monte Carlo experiments, both in cases where initial specifications are correct and where misspecifications (in assumptions (ii)-(iv)) are present. With respect to bias and mean square error (MSE) the estimator does well in the correctly specified cases, but also when the covariances (QS and RS) are misspecified. The performance is deteriorated when $\beta_{0|0}$ is misspecified and more so when the transition matrix is incorrect.

The results are well in line with earlier results from studies on the reduced form properties (e.g. McWhorter et al., 1976).

The estimator (2SKF) is compared with two stage least squares (2SLS) and OLS by Brännäs and Westlund (1980b). In a Monte Carlo study corresponding to that of Brännäs and Westlund (1979) it is noticed that the bias of 2SLS is very close to that of 2SLS when the transition matrix is equal to the identity matrix. When structure two ($A_t = 1.02 \cdot I$) is utilized the bias properties of 2SLS are worsening. The MSE is throughout considerably higher in 2SLS than in 2SKF. The performance of OLS is weaker than that of both 2SLS and 2SKF.

### 3.3.4 Empirical results

In an empirical application of both 2SKF and KF on the model of Brännäs and Eklöf (1980), Brännäs (1980b) analyses the robustness both with respect to a priori specifications in the structural form and in the reduced form. It is observed that the terminal time estimates ($N = 52$) are rather insensitive to misspecifications. However, the impact on intermediate time estimates is such that inferences about structural properties are hard to make when the specification in particular of $\beta_{0|0}$ and $\Sigma_{0|0}$ is uncertain.

In a forecasting comparison KF and 2SKF are seen to outperform 2SLS, while when forecasting over several steps ahead 2SLS, is in general better (see also Cooley, 1975, McWhorter et al., 1977, and Baudin and Westlund, 1980). The KF forecasts are slightly better than those of 2SKF. In a certainty equivalence control framework it is demonstrated that the impact of changes in parameter estimates is stronger than changes in the loss function, but less important than bad exogenous forecasts. In an ex-ante comparison it is seen that more stress is to be placed on the use of certain instrument variables than when using 2SLS. This matter is explained in terms of a more rapid adaption to new parameter values of the KF and 2SKF estimators.

### 3.4 Summarizing comments

The results reported on the application of the Kalman filter in econometrics have focused at the estimation of time-varying parameters. The filter in itself is, of course, more general and can as long as the problem can be formulated in a form corresponding to (3.1)-(3.2) be applied. Examples of other applications are the incorporation of extra information in improving preliminary statistics (Conrad and Corrado, 1979), and the explicit treatment of measurement errors (e.g. Chow, 1975), and rational expectations (Wall, 1980).

In econometric applications the assumptions underlying the filtering equations are only seldom fulfilled. The research reported in Brännäs (1978, 1980a,b), and Brännäs and Westlund (1979) aims at relaxing and evaluating whether the assumptions are critical with respect to the estimates. These considerations are all important steps in gaining experience and knowledge for the further development in the area.

The evaluations are all concerned with small sample situations, as these are the important ones in most econometric applications. For such cases, the present results may be regarded as predecessors to exact theoretical results, that are the ultimate goals. The asymptotic properties of the estimators are not explicitly dealt with.

As is quite evident from the results care should in particular be exercised with respect to the initial parameter vector and its covariance as well as to the transition matrix. In a reduced form framework the technique utilized in Brännäs (1980a) can cope with both of these problems. The empirical results of Brännäs (1980b) by and large support the findings of earlier studies with respect to time-varying parameters and their robustness. Further, the performance of 2SKF is seen to be close to that of KF, both in forecasting and control.

## 4. Some notes of further research

In the reduced form framework several of the more urgent research problems have been satisfactorily solved. By the identification criteria of Pagan (1980) the elements of A, $\beta_t$, Q and R may be uniquely estimated. However, practical experience with the technique of Brännäs (1980a) and Pagan (1980) is still almost nonexistent, and the small sample properties only partly investigated (Brännäs, 1980a).

For structural form models several problems still wait for their solutions. In particular, the question of identifiability must be satisfactorily solved. Progress in this respect is believed to be achievable by the approach of Pagan (1980). As to the estimation of the necessary supplement information no solution has, to the author's knowledge, so far been offered. Quite evidently, however, the approach of extended Kalman filtering seems a promising one, and the techniques of Brännäs (1980a) and Pagan (1980) are believed to be applicable.

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