Weighted L^p –inequalities for multi–parameter Riesz type potentials and strong fractional maximal operators. *

Tord Sjödin

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Abstract

We prove weighted L^p –inequalities for multi–parameter Riesz type potentials, strong fractional maximal operators and their dyadic counterparts. Our proofs avoid the Good– λ inequalities used earlier in the R^m –case and are based on our integrated multi–parameter summation by parts lemma, that might be of independent interest.

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1 Introduction.

Estimating and comparing potential operators and maximal operators is a question of great importance in harmonic analysis. The case of Riesz potentials and the corresponding fractional maximal operators is particularly well known and often work as a model for more general situations. See [9], [10], [11], [12], [13] and [18] for the general theory and [8], Section 3 for applications to mathematical physics.

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We will study such operators in a product space setting. Let $R^d = R^m \times R^n$ and denote points in R^d by $\xi = (x, s)$ and $\eta = (y, t)$, where $x, y \in R^m$ and $s, t \in R^n$. We put $\Phi(x, s) = \varphi(x) \cdot \psi(s)$, where φ and ψ are nonnegative and radially nonincreasing functions, and define a multi-parameter potential operator

$$T_\Phi \nu(\xi) = \int \Phi(x - y, s - t) d\nu(y, t)$$

and a strong fractional maximal operator

$$M_\Phi \nu(\xi) = \sup_{\xi \in I \times J} \varphi(l(I)) \cdot \psi(l(J)) \cdot \nu(I \times J),$$

where ν is a non-negative Borel measure in R^d and I, J are cubes in R^m and R^n respectively, not necessarily of the same size. Kerman, Sawyer [8], p. 215 proved L^p -inequalities for such operators in R^m , where T_φ and M_φ are defined using φ only. In particular, they proved that

$$\|T_\varphi f\|_p \leq C \cdot \|M_\varphi f\|_p, \quad (1.1)$$

for $1 < p < \infty$ and with C only depending on m and p . The case when $\varphi(x) = |x|^{\rho-m}$, $0 < \rho < m$, is the Riesz kernel goes back to Muckenhoupt, Wheeden [9], Theorem 1, where (1.1) is proved for weighted $L^p(R^m)$ -spaces. The present author extended their result to weighted mixed norm spaces $L^{p,q}(R^d)$ in [14], Theorem 2.1.

The main new difficulties in the multi-parameter case are essentially due to the following two facts:

- (i) The potential kernel $\Phi(x, s) = \varphi(x) \cdot \psi(s)$ is singular not only at the origin, as in the R^m -case, but on the two hyperplanes $R^m \times \{0\}$ and $\{0\} \times R^n$,
- (ii) The standard covering theorems for cubes fail for rectangles [3].

We therefore avoid the Good- λ inequalities in [8], [9] and use a new technique to handle dyadic rectangles (Lemmas 4.1 and 4.2), that might be of independent interest. All our results are first proved in the dyadic case. We begin with the dyadic operators T_Φ^d and M_Φ^d (see Section 2 for the definitions) and prove that

$$\|T_\Phi^d \nu\|_{p,w} \leq C \cdot \|M_\Phi^d \nu\|_{p,w}, \quad (1.2)$$

where $1 < p < \infty$ and w is a weight satisfying a growth conditions slightly more general than $A_\infty(R^m \times R^n)$ (Theorem 3.1). The non-dyadic case

is given in Theorem 3.2, while Theorems 3.3 and 3.4 handle the multi-parameter Riesz kernels $\Phi(x, s) = R_{\rho, r}(x, s) = |x|^{\rho-m} \cdot |s|^{r-n}$. The simpler case when $0 < p \leq 1$ is treated in Theorems 3.5 and 3.6. As an application of these results we prove equivalence of the operators T_{Φ}^d , T_{Φ} , M_{Φ}^d and M_{Φ} in $L^p(w)$ (Theorem 3.7).

2 Notation and definitions.

Throughout this paper we consider the Euclidean product space $R^d = R^m \times R^n$ and denote its points by $\xi = (x, s)$ and $\eta = (y, t)$, where $x, y \in R^m$ and $s, t \in R^n$. We let I and J be cubes in R^m and R^n respectively, with their sides parallel to the axes. Denote their side length by $l(I)$ and $l(J)$ and let $R = I \times J$ be the associated rectangle in R^d . If I and J have the same side length we call $Q = I \times J$ a cube in R^d . We let $\mathcal{D}_m$ and $\mathcal{D}_n$ denote the standard dyadic nets in R^m and R^n respectively. A dyadic rectangle R in R^d is a rectangle $R = I \times J$, where $I \in \mathcal{D}_m$ and $J \in \mathcal{D}_n$. The class of dyadic rectangles in R^d is denoted by $\mathcal{D}$. If I and J have the same side length we call $Q = I \times J$ a dyadic cube in R^d . The ordinary closed ball in R^m is written $B(x, a)$. We let $B(\xi; a, b)$ denote the bidisc $B(x, a) \times B(s, b)$, while $B(\xi, r)$ is the ordinary closed Euclidean ball in R^d . The set of integers is denoted by Z .

The Lebesgue measure of a set E is written $|E|$ and integration with respect to Lebesgue measure is denoted by $d\xi$. The dimension of the Lebesgue measure will usually be clear from the context. We let $M_+(R^d)$ denote the class of nonnegative Borel measures in R^d that are finite on compact sets. Constants depending on $\alpha, \beta, \dots$ are denoted by $C(\alpha, \beta, \dots)$ or C . Constants that only depend on m and n are denoted by c , and may vary from one case to another. Two quantities A and B are equivalent if their quotient A/B is bounded from above and below by positive constants.

We consider nonnegative and radially nonincreasing functions (potential kernels) $\varphi(x) = \varphi(|x|)$ and $\psi(s) = \psi(|s|)$, in R^m and R^n respectively, which are locally integrable and satisfy $\lim_{r \rightarrow 0} \varphi(r) = \lim_{r \rightarrow 0} \psi(r) = \infty$ and $\lim_{r \rightarrow \infty} \varphi(r) = \lim_{r \rightarrow \infty} \psi(r) = 0$.

We will consider the following set of growth conditions on φ and ψ :

$$\sum_{k \leq i} \varphi(2^k)^t \cdot 2^{k\delta m} \leq B \cdot \varphi(2^i)^t \cdot 2^{i\delta m}, \quad \sum_{k \leq i} \psi(2^k)^t \cdot 2^{k\delta n} \leq B \cdot \psi(2^i)^t \cdot 2^{i\delta n}, \quad (2.1)$$

for some positive numbers t, δ and B and all $i \in \mathbb{Z}$. Note that (2.1) implies and is only slightly stronger than the conditions $\varphi(u) \leq C \cdot (v/u)^{\delta m/t} \cdot \varphi(v)$ and $\psi(u) \leq C \cdot (v/u)^{\delta n/t} \cdot \psi(v)$, $0 < u \leq v$. In particular, φ and ψ are doubling i.e. $\varphi(u) \leq C \cdot \varphi(2u)$ and $\psi(u) \leq C \cdot \psi(2u)$, for $u > 0$.

Remark. If there are positive numbers δ and B such that (2.1) holds for $t = t_1$ and $t = t_2$, then (2.1) also holds for all $t_1 < t < t_2$. This follows easily from Hölder's inequality. In the Riesz case $\phi(x) = |x|^{\rho-m}$, $\psi(s) = |s|^{r-n}$ (2.1) holds if and only if $\delta > t \cdot \max(1 - \rho/m, 1 - r/n)$.

When I is a cube we define $\varphi(I) = \varphi(l(I))$ and analogously for $\psi(J)$. We also let

$$b_{I \times J}(\xi) = b_{I \times J} \cdot \chi_{I \times J}(\xi) \quad \text{and} \quad b_{I \times J} = \varphi(I) \cdot \psi(J) \cdot \nu(I \times J), \quad (2.2)$$

for any rectangle $I \times J$. Both $b_{I \times J}(\xi)$ and $b_{I \times J}$ depend on φ , ψ and ν but this will always be clear from the context.

We now put $\Phi(x, s) = \varphi(x) \cdot \psi(s)$ and define the multi-parameter potential operator

$$T_\Phi \nu(\xi) = \int \Phi(x - y, s - t) d\nu(y, t)$$

and the strong fractional maximal operator

$$M_\Phi \nu(\xi) = \sup_{I \times J} b_{I \times J}(\xi), \quad (2.3)$$

for $\nu \in M_+(R^d)$, where supremum is over all rectangles $I \times J$ containing ξ .

We also define a dyadic fractional maximal operator $M_\Phi^d \nu(\xi)$, where the supremum in (2.3) is over all dyadic rectangles $I \times J$ containing ξ and a dyadic potential operator

$$T_\Phi^d \nu(\xi) = \sum_{I \times J \in \mathcal{D}} b_{I \times J}(\xi).$$

In the case of the multi-parameter Riesz kernels $R_{\rho,r}(x, s) = |x|^{\rho-m} \cdot |s|^{r-n}$ we denote T_Φ^d , T_Φ , M_Φ^d and M_Φ by $T_{\rho,r}^d$, $T_{\rho,r}$, $M_{\rho,r}^d$ and $M_{\rho,r}$ respectively.

A nonnegative and locally integrable function is called a weight. If w is a weight we define a measure $dw(\xi) = w(\xi)d\xi$ and the dw -measure of a set A is written $w(A)$. The Muckenhoupt classes $A_p(R^m)$ are well known, see [17], Ch. IX. The corresponding classes $A_p(R^m \times R^n)$ in the multi-parameter case were defined in [5], Section 3. See also [6], Ch. IV. We say that $w \in A_p(R^m \times R^n)$, $1 < p < \infty$, if there is $C < \infty$ such that

$$(|R|^{-1} \int_R w(\xi) d\xi) \cdot \left(|R|^{-1} \int_R w(\xi)^{-1/(p-1)} d\xi \right)^{p-1} \leq C,$$

for all rectangles $R = I \times J$. The smallest constant C is called the $A_p(R^m \times R^n)$ -norm of w . A weight w belongs to $A_\infty(R^m \times R^n)$ if there are positive numbers D and δ such that

$$w(A)/w(R) \leq D \cdot (|A|/|R|)^\delta, \quad (2.4)$$

for all rectangles R and all sets $A \subset R$.

The weights in our theorems satisfy growth conditions slightly weaker than (2.4), but with specified values on δ and D . We consider growth conditions

$$\frac{w(K \times L)}{w(I \times J)} \leq D \cdot \left(\frac{|K \times L|}{|I \times J|} \right)^\delta, \quad (2.5)$$

for all *dyadic* rectangles $K \times L \subset I \times J$, where D and δ are given positive numbers. If w is any weight we denote by $L^p(w)$ the standard Lebesgue space with norm $\|f\|_{p,w} = (\int |f(\xi)|^p dw(\xi))^{1/p}$.

3 The main results.

In this section we state our results. We start with the case when $1 < p < \infty$ and postpone the case $0 < p \leq 1$ to the end of the section. We first consider the general dyadic potentials $T_\Phi^d \nu$ and the corresponding dyadic fractional maximal functions $M_\Phi^d \nu$.

Theorem 3.1 *Let $1 < p < \infty$, $\nu \in M_+(R^d)$, let w be a weight in R^d satisfying (2.5), for some positive numbers D and δ , and let Φ be a kernel. Then*

$$\int T_\Phi^d \nu(\xi)^p dw(\xi) \leq C^p \cdot \int M_\Phi^d \nu(\xi)^p dw(\xi), \quad (3.1)$$

where C depends on p, B, D and δ , in the following two cases:

- (i) $1 < p \leq 2$ and Φ satisfies (2.1) for $t = 1$,
- (ii) $p > 2$ and Φ satisfies (2.1) for $t = 1$ and $t = p - 1$.

Next we give the non-dyadic version of Theorem 3.1.

Theorem 3.2 *Under the assumptions in Theorem 3.1 we have*

$$\int T_{\Phi} \nu(\xi)^p dw(\xi) \leq C^p \cdot \int M_{\Phi}^d \nu(\xi)^p dw(\xi),$$

where C only depends on d, p, B, D and δ .

In Theorems 3.3 and 3.4 below we let $\Phi(x, s) = |x|^{\rho-m} \cdot |s|^{r-n}$, $0 < \rho < m$, $0 < r < n$.

Theorem 3.3 *Let $1 < p < \infty$, $0 < \rho < m$, $0 < r < n$, $\nu \in M_+(R^d)$ and let w be a weight satisfying (2.5), for some positive numbers D and δ . Then*

$$\int T_{\rho,r}^d \nu(\xi)^p dw(\xi) \leq C^p \cdot \int M_{\rho,r}^d \nu(\xi)^p dw(\xi),$$

where C depends on d, p, ρ, r, D and δ , in the following two cases:

- (i) $1 < p \leq 2$ and $\delta > \max(1 - \rho/m, 1 - r/n)$,
- (ii) $p > 2$ and $\delta > (p - 1) \cdot \max(1 - \rho/m, 1 - r/n)$.

Theorem 3.4 *Under the assumptions in Theorem 3.3 we have*

$$\int T_{\rho,r} \nu(\xi)^p dw(\xi) \leq C^p \cdot \int M_{\rho,r}^d \nu(\xi)^p dw(\xi),$$

where C depends on d, p, ρ, r, D and δ .

Remark. Let $w(\xi) \equiv 1$, $\delta = 1$ and $0 < \rho < m$, $0 < r < n$. Then Theorems 3.3 and 3.4 hold for $1 < p \leq 2$ and for $2 < p < 1 + \max(2 - \rho/m, 2 - r/n)$, respectively. Putting $q = p/(p - 1)$, the dual index of p , this is equivalent to $q > \max(2 - \rho/m, 2 - r/n)$. These lower bounds are known from the R^m -theory, see [2] p. 280. Our methods will not allow us to prove Theorems 3.3 and 3.4 in the full range $1 < p < \infty$.

The next two theorems handle the case $0 < p \leq 1$, where the proofs are much simpler.

Theorem 3.5 *Let $0 < p \leq 1$, let w be a weight satisfying (2.5) and assume that Φ satisfies (2.1) with $t = p$ and δ replaced by $\delta + p - 1 > 0$. Then*

$$\int T_{\Phi}^d \nu(\xi)^p dw(\xi) \leq B^2 \cdot D \cdot \int M_{\Phi}^d \nu(\xi)^p dw(\xi),$$

for all $\nu \in M_+(R^d)$.

Theorem 3.6 *Let p, w and Φ be as in Theorem 3.5. Then*

$$\int T_{\Phi} \nu(\xi)^p dw(\xi) \leq 5^{dp} \cdot B^2 \cdot D \cdot \int M_{\Phi}^d \nu(\xi)^p dw(\xi).$$

Finally, we apply the results above to prove equivalence of norms for the four operators T_{Φ}^d , T_{Φ} , M_{Φ}^d and M_{Φ} .

Theorem 3.7 *Let $0 < p < \infty$, $\nu \in M_+(R^d)$ and let Φ be a kernel that satisfies the assumptions of Theorems 3.1, for $1 < p < \infty$, and Theorem 3.5, for $0 < p \leq 1$. Then*

$$\begin{aligned} \|M_{\Phi}^d \nu\|_{p,w} &\leq \|T_{\Phi}^d \nu\|_{p,w} \leq C_1 \cdot \|M_{\Phi} \nu\|_{p,w} \leq C(d, B) \cdot C_1 \cdot \|T_{\Phi} \nu\|_{p,w} \leq \\ &\leq C(d, B) \cdot C_1 \cdot C_2 \cdot \|M_{\Phi}^d \nu\|_{p,w}, \end{aligned}$$

where C_1 denotes the constant in Theorem 3.1 or 3.5 and C_2 denotes the constant in Theorem 3.2 or 3.6.

4 Some lemmas.

The main tool in all our proofs is an integrated multi-parameter version of a summation by parts lemma in [4], Lemma 2.6, which might be of independent interest. For ease of the exposition we split this result in two separate cases: $1 < p \leq 2$ and $2 < p < \infty$.

Lemma 4.1 *Let $1 < p \leq 2$, $\nu \in M_+(R^d)$ and assume that the kernel Φ satisfies (2.1) with $t = 1$ and the weight w satisfies (2.5), for some positive numbers B, D and δ . Then*

$$\begin{aligned} &\int \left(\sum_{I \times J \in \mathcal{D}} b_{I \times J}(\xi) \right)^p dw(\xi) \leq \\ &\leq C \cdot \int \sum_{I \times J \in \mathcal{D}} b_{I \times J}(\xi) \cdot \left(\sum_{K \times L \subset I \times J} b_{K \times L}(\xi) \right)^{p-1} dw(\xi), \end{aligned} \quad (4.1)$$

where $b_{I \times J}(\xi)$ is defined by (2.2) and $C = 2 \cdot (1 + B^2 \cdot D^2)$.

Remark. In the case when $\phi(x) = |x|^{\rho-m}$ and $\psi(s) = |s|^{r-n}$ Lemma 4.1 holds for $\delta > \max(1 - \rho/m, 1 - r/n)$. In particular, Lemma 4.1 holds for Lebesgue measure, $w = 1$ and $\delta = 1$, and all $0 < \rho < m$ and $0 < r < n$.

Proof of Lemma 4.1. We will consider sums over classes of dyadic rectangles and the idea of the proof is to estimate certain sums by a constant times the term for the largest cube, using (2.1) and (2.5). This process reduces the original sum over pairs of dyadic rectangles R_1 and R_2 to a sum where $R_1 = R_2$. All cubes in the proof are dyadic cubes. We rewrite the left hand side of (4.1) as

$$\begin{aligned} S &= \int \left(\sum_{I \times J} b_{I \times J}(\xi) \right)^p dw(\xi) = \\ &= \int \sum_{I \times J} \sum_{K \times L} b_{I \times J}(\xi) \cdot b_{K \times L}(\xi) \cdot \left(\sum_{M \times N} b_{M \times N}(\xi) \right)^{p-2} dw \end{aligned}$$

and assume without loss of generality that $K \subset I$. We consider separately the sums where $K \times L \subset I \times J$ and $K \subset I, J \subset L$ denoted by S_1 and S_2 respectively. In the first case

$$S_1 \leq \int \sum_{I \times J \in \mathcal{D}} b_{I \times J}(\xi) \cdot \left(\sum_{K \times L \subset I \times J} b_{K \times L}(\xi) \right)^{p-1} dw(\xi)$$

as desired. In the second case we fix I, J and L and consider

$$\begin{aligned} &\int b_{I \times J}(\xi) \cdot b_{K \times L}(\xi) \cdot \left(\sum_{M \times N} b_{M \times N}(\xi) \right)^{p-2} dw \leq \\ &\leq b_{I \times J} \cdot b_{K \times L} \cdot b_{I \times L}^{p-2} \cdot w(K \times J) \leq \\ &\leq D \cdot 2^{(k-i)\delta m} \cdot b_{I \times J} \cdot b_{K \times L} \cdot b_{I \times L}^{p-2} \cdot w(I \times J) \end{aligned}$$

by (2.5), where $l(K) = 2^k \leq 2^i = l(I)$. The interesting factors here are $F_K = 2^{(k-i)\delta m} \cdot \varphi(K) \cdot \nu(K \times L)$ and summing over all $K \subset I$ gives

$$\begin{aligned} \sum_{K \subset I} F_K &= \sum_{k \leq i} \sum_{l(K)=2^k} F_K \leq \nu(I \times L) \cdot \sum_{k \leq i} 2^{(k-i)\delta m} \varphi(2^k) \leq \\ &\leq B \cdot \varphi(I) \cdot \nu(I \times L) = B \cdot F_I \end{aligned}$$

by (2.1), i.e. the sum of all F_K with $K \subset I$ is at most a constant times F_I . Inserting this estimate in S_2 above gives

$$S_2 \leq B \cdot D \cdot \sum_{I, J \subset L} b_{I \times J} \cdot b_{I \times L}^{p-1} \cdot w(I \times J).$$

This is the basic step in our reduction process. Next we repeat this step for the cubes $J \subset L$. Fix I and L and let $l(J) = 2^j \leq 2^l = l(L)$, then the interesting part of the last sum is

$$\begin{aligned} \sum_{J \subset L} b_{I \times J} \cdot w(I \times J) &\leq D \cdot \sum_{J \subset L} \varphi(I) \cdot \psi(J) \cdot \nu(I \times J) \cdot 2^{(j-l)\delta n} \cdot w(I \times L) \leq \\ &\leq B \cdot D \cdot b_{I \times L} \cdot w(I \times L) \end{aligned}$$

as above and it follows that

$$\begin{aligned} S_2 &\leq B^2 \cdot D^2 \cdot \sum_{I \times L \in \mathcal{D}} b_{I \times L}^p \cdot w(I \times L) \leq \\ &\leq B^2 \cdot D^2 \cdot \int \sum_{I \times L \in \mathcal{D}} b_{I \times L}(\xi) \cdot \left(\sum_{M \times N \subset I \times L} b_{M \times N}(\xi) \right)^{p-1} dw(\xi). \end{aligned}$$

This completes the proof of Lemma 4.1 with $C = 2 \cdot (1 + B^2 \cdot D^2)$. $\triangle$

Remark. The proof of Lemma 4.1 also gives the following slightly more general result: Fix any dyadic rectangle $I_0 \times J_0$, integers k_0 and l_0 and define

$$\mathcal{A} = \{I \times J \in \mathcal{D}; I \times J \subset I_0 \times J_0, l(I) \geq 2^{k_0} \text{ and } l(J) \geq 2^{l_0}\} \quad (4.2)$$

then

$$\begin{aligned} &\int \left(\sum_{\mathcal{A}} b_{I \times J}(\xi) \right)^p dw(\xi) \leq \\ &\leq C \cdot \int \sum_{\mathcal{A}} b_{I \times J}(\xi) \cdot \left(\sum_{K \times L \subset I \times J, K \times L \in \mathcal{A}} b_{K \times L}(\xi) \right)^{p-1} dw(\xi), \quad (4.3) \end{aligned}$$

with C in (4.1). To see this we note that all the manipulations in the proof of Lemma 4.1 are done inside $I_0 \times J_0$ and increase the size of the rectangles, hence they preserve the class $\mathcal{A}$.

The analogue of Lemma 4.1 for $p > 2$ requires that Φ satisfies (2.1) also for

$t = p - 1$. The proof is more complicated but uses the same ideas. The proof in the integer case uses a slight modification of the proof of Lemma 4.1, while in the noninteger case we make use of Lemma 4.3.

Lemma 4.2 *Let $2 < p < \infty$ and assume that the kernel Φ satisfies (2.1) with $t = 1$ and $t = p - 1$ and the weight w satisfies (2.5), for some positive numbers B , D and δ . Then*

$$\begin{aligned} & \int \left(\sum_{I \times J \in \mathcal{D}} b_{I \times J}(\xi) \right)^p dw(\xi) \leq \\ & \leq C \cdot \int \sum_{I \times J \in \mathcal{D}} b_{I \times J}(\xi) \cdot \left(\sum_{K \times L \subset I \times J} b_{K \times L}(\xi) \right)^{p-1} dw(\xi), \end{aligned} \quad (4.4)$$

where C depends on d, p, B, D and δ .

Proof. All cubes in the proof are dyadic cubes. We start with the case when $p = k$ is an integer, $k \geq 3$, which is slightly easier than the general case. Then

$$\begin{aligned} S &= \int \left(\sum_{I \times J} b_{I \times J}(\xi) \right)^k dw(\xi) = \\ &= \int \sum b_{I_1 \times J_1}(\xi) \cdot b_{I_2 \times J_2}(\xi) \cdots b_{I_k \times J_k}(\xi) dw(\xi), \end{aligned}$$

where the sum is over all k -tuples of rectangles $I_\nu \times J_\nu$, $1 \leq \nu \leq k$. Further,

$$S \leq k! \cdot \sum b_{I_1 \times J_1} \cdot b_{I_2 \times J_2} \cdots b_{I_k \times J_k} \cdot w(I_1 \times T),$$

where $I_1 \subset I_2 \subset \cdots \subset I_k$ and T is the dyadic cube $\bigcap_{\nu=1}^k J_\nu$, by Lemma 4.4. Fix $I_2, I_3, \dots, I_k$ and J_ν , $1 \leq \nu \leq k$, then summing over all $I_1 \subset I_2$ gives

$$S \leq k! \cdot B \cdot D \cdot \sum_{I_1=I_2} b_{I_1 \times J_1} \cdot b_{I_2 \times J_2} \cdots b_{I_k \times J_k} \cdot w(I_2 \times T), \quad (4.5)$$

by the basic step in the proof of Lemma 4.1. Next we fix $I_3, \dots, I_k$ and J_ν , $1 \leq \nu \leq k$. The key factor in the right hand side of (4.5) is

$$F_I = \varphi(I)^2 \cdot \nu(I \times J_1) \cdot \nu(I \times J_2) \cdot w(I \times T),$$

where $I = I_1 = I_2$. Let $l(I_3) = 2^j$ and $l(I) = 2^i$ then a modification of the basic step gives

$$\begin{aligned} \sum_{I \subset I_3} F_I &\leq D \cdot \sum_{i \leq j} \sum_{l(I)=2^i} \varphi(I)^2 \cdot (|I|/|I_3|)^\delta \cdot \nu(I \times J_1) \cdot \nu(I \times J_2) \cdot w(I_3 \times T) \leq \\ &\leq D \cdot \nu(I_3 \times J_1) \cdot \nu(I_3 \times J_2) \cdot w(I_3 \times T) \cdot \sum_{i \leq j} \varphi(2^i)^2 \cdot 2^{(i-j)\delta m} \leq B \cdot D \cdot F_{I_3}, \end{aligned}$$

by (2.5) and (2.1) with $t = 2$. Continuing in this way $k - 1$ times we get

$$\begin{aligned} S &\leq k! \cdot (B \cdot D)^{k-1} \cdot \sum b_{I \times J_1} \cdot b_{I \times J_2} \dots b_{I \times J_k} \cdot w(I \times T) \leq \\ &\leq (k!)^2 \cdot (B \cdot D)^{k-1} \cdot \sum b_{I \times J_1} \cdot b_{I \times J_2} \dots b_{I \times J_k} \cdot w(I \times J_1), \end{aligned}$$

where $I = I_k$ and $J_1 \subset J_2 \subset \dots \subset J_k$. It follows that

$$S \leq (k!)^2 \cdot (B \cdot D)^{k-1} \cdot \int \sum_{I \times J \in \mathcal{D}} b_{I \times J}(\xi) \cdot \left(\sum_{K \times L \subset I \times J} b_{K \times L}(\xi) \right)^{k-1} dw(\xi).$$

This proves Lemma 4.2 when $p = k \geq 3$ is an integer.

Next let $k - 1 < p < k$, where $k \geq 3$ is an integer. Then

$$\begin{aligned} S &= \int \left(\sum_{I \times J \in \mathcal{D}} b_{I \times J}(\xi) \right)^k \cdot \left(\sum_{M \times N} b_{M \times N}(\xi) \right)^{p-k} dw(\xi) = \\ &= \int \sum b_{I_1 \times J_1}(\xi) \cdot b_{I_2 \times J_2}(\xi) \dots b_{I_k \times J_k}(\xi) \cdot \left(\sum_{M \times N} b_{M \times N}(\xi) \right)^{p-k} dw(\xi) \leq \\ &\leq k! \cdot \int \sum b_{I_1 \times J_1}(\xi) \cdot b_{I_2 \times J_2}(\xi) \dots b_{I_k \times J_k}(\xi) \cdot \left(\sum_{M \times N} b_{M \times N}(\xi) \right)^{p-k} dw(\xi), \end{aligned}$$

where $I_1 \subset I_2 \subset \dots \subset I_k$, by Lemma 4.4. Fix I_k and J_ν , $1 \leq \nu \leq k$, and let $T = \bigcap_{\nu=1}^k J_\nu$. Then

$$S \leq k! \cdot \sum b_{I_1 \times J_1} \cdot b_{I_2 \times J_2} \dots b_{I_k \times J_k} \cdot \left(\sum_{I_k \times T \subset M \times N} b_{M \times N} \right)^{p-k} \cdot w(I_1 \times T).$$

In analogy with the integer case we get

$$\begin{aligned} S &\leq k!(BD)^{k-1} \cdot \sum b_{I \times J_1} \cdot b_{I \times J_2} \dots b_{I \times J_k} \cdot \left(\sum_{I \times T \subset M \times N} b_{M \times N} \right)^{p-k} \cdot w(I \times T) \leq \\ &\leq k!^2 (BD)^{k-1} \cdot \sum b_{I \times J_1} \cdot b_{I \times J_2} \dots b_{I \times J_k} \cdot \left(\sum_{I \times T \subset M \times N} b_{M \times N} \right)^{p-k} \cdot w(I \times J_1), \end{aligned}$$

where $I = I_k$ and $J_1 \subset J_2 \subset \dots \subset J_k$. It follows that S is at most $k!^2 (BD)^{k-1}$ times the sum

$$S' = \sum b_{I \times J_1} \cdot b_{I \times J_2}^{p-k+1} \dots b_{I \times J_k} \cdot w(I \times J_1). \quad (4.6)$$

We can now apply Lemma 4.3 (k-1) times and get

$$S' \leq \sum_{I \times J} (BD)^{k-1} \cdot b_{I \times J}^p \cdot w(I \times J)$$

and (4.4) follows. The proof of Lemma 4.2 is complete. $\triangle$

Remark. Lemma 4.2 also holds for the classes $\mathcal{A}$ in (4.2) with the same constant C , by the remark following Lemma 4.1.

It is essential for the proof of Lemma 4.2 that the factor $b_{I \times J_\nu}^{p-k+1}$ in (4.6) with exponent $p-k+1 < 1$ does not have a uniquely smallest or uniquely largest cube J_ν . If so, the proof only works if we assume that ϕ and ψ satisfy (2.1) for som $t < 1$ or some $t > p-1$. This would give a weaker result.

Our third lemma is a weighted multi-parameter analogue of a result in [4], p. 355. The proof uses the basic step in the proof of Lemma 4.1.

Lemma 4.3 *Let $\nu \in M_+(R^d)$, let w be a weight satisfying (2.5) and assume that Φ satisfies (2.1) with $t = \alpha \geq 1$, for some positive numbers δ , B and D . Then*

$$b_{I \times J}^\alpha \leq w(I \times J)^{-1} \cdot \sum_{K \times L \subset I \times J} b_{K \times L}^\alpha \cdot w(K \times L) \leq B^2 \cdot D \cdot b_{I \times J}^\alpha, \quad (4.7)$$

for all dyadic rectangles $I \times J$. In the special case is when $K = I$ and ψ satisfies (2.1) for $t = \alpha \geq 1$ we get

$$b_{I \times J}^\alpha \leq w(I \times J)^{-1} \cdot \sum_{L \subset J} b_{I \times L}^\alpha \cdot w(I \times L) \leq B \cdot D \cdot b_{I \times J}^\alpha.$$

Proof. It is sufficient to prove the second inequality in (4.7). Fix cubes I, J with $l(I) = 2^i$ and $l(J) = 2^j$ and integers $k \leq i, l \leq j$. Summing over all such K and L with $l(K) = 2^k$ and $l(L) = 2^l$ gives

$$\begin{aligned} & \sum_{l(K)=2^k, l(J)=2^j} b_{K \times L}^\alpha \cdot w(K \times L) = \\ &= \sum_{l(K)=2^k, l(J)=2^j} \varphi(2^k)^\alpha \cdot \psi(2^l)^\alpha \cdot \nu(K \times L)^\alpha \cdot w(K \times L) \leq \\ &\leq D \cdot \varphi(2^k)^\alpha \cdot \psi(2^l)^\alpha \cdot (2^{(k-i)m} \cdot 2^{(l-j)n})^\delta \cdot \nu(I \times J)^\alpha \cdot w(I \times J), \end{aligned}$$

by (2.5), since $\alpha \geq 1$. Another summation over $k \leq i, l \leq j$ finishes the proof of (4.7) using (2.1). The case when $K = I$ is proved in an analogous way. $\triangle$

Remark. In the case $\varphi(x) = |x|^{\rho-m}, \psi(s) = |s|^{r-n}, 0 < \rho < m$ and $0 < r < n$, Lemma 4.3 holds if and only if $\delta > \alpha \cdot \max(1 - \rho/m, 1 - r/n)$.

We conclude this section with a simple lemma used in the proofs of Lemma 4.1 – 4.2 above.

Lemma 4.4 *Let k and N be positive integers, then*

$$\left(\sum_{i=1}^N x_i \right)^k \leq k! \cdot \sum_{i_1 \leq i_2 \leq \dots \leq i_k} x_{i_1} \cdot x_{i_2} \cdot \dots \cdot x_{i_k},$$

for all $x_i \geq 0, 1 \leq i \leq N$, where the constant $k!$ is best possible.

Proof. The lemma follows from the Multinomial Theorem

$$\left(\sum_{i=1}^N x_i \right)^k = \sum_{i_1 < i_2 < \dots < i_\nu} \sum_{\alpha_1 + \dots + \alpha_\nu = k} \frac{k!}{\alpha_1! \dots \alpha_\nu!} \cdot x_{i_1}^{\alpha_1} \cdot \dots \cdot x_{i_\nu}^{\alpha_\nu}.$$

Taking all $x_i = 1$ and letting N tend to infinity proves that the constant $k!$ is best possible. $\triangle$

5 Proofs.

This section contains the proofs of our theorems. We start with Theorems 3.1 and 3.2, which are proved in detail. The proofs combine Lemma 4.1

and 4.2 with Lemma 4.3 and Hölders inequality as in [4], Proposition 4.2. Theorems 3.3 and 3.4 are special cases and the proofs are very short. The proofs of Theorems 3.5 and 3.6 are much simpler, but use the same ideas. Theorem 3.7 follows from the results above and the doubling property of Φ .

Proof of Theorem 3.1. We first assume that $1 < p \leq 2$. Let $\mathcal{A}$ be any class of rectangles defined in (4.2) then

$$\int \left(\sum_{\mathcal{A}} b_{I \times J}(\xi) \right)^p dw(\xi) \leq C \cdot \int \sum_{\mathcal{A}} b_{I \times J}(\xi) \cdot \left(\sum_{K \times L \subset I \times J} b_{K \times L}(\xi) \right)^{p-1} dw(\xi)$$

by (4.3). Hence $\int \left(\sum_{\mathcal{A}} b_{I \times J}(\xi) \right)^p dw(\xi)$ is at most C times

$$\begin{aligned} & \sum_{\mathcal{A}} b_{I \times J} \cdot \int \left(\sum_{K \times L \subset I \times J} b_{K \times L}(\xi) \right)^{p-1} dw(\xi) \leq \\ & \leq \sum_{\mathcal{A}} b_{I \times J} \cdot w(I \times J) \cdot \left(w(I \times J)^{-1} \cdot \int \sum_{K \times L \subset I \times J} b_{K \times L}(\xi) dw(\xi) \right)^{p-1} = \\ & = \sum_{\mathcal{A}} b_{I \times J} \cdot w(I \times J) \cdot \left(w(I \times J)^{-1} \cdot \sum_{K \times L \subset I \times J} b_{K \times L} \cdot w(K \times L) \right)^{p-1} = \\ & = \int \sum_{\mathcal{A}} b_{I \times J}(\xi) \cdot \left(w(I \times J)^{-1} \cdot \sum_{K \times L \subset I \times J} b_{K \times L} \cdot w(K \times L) \right)^{p-1} dw(\xi) \leq \\ & \leq (B^2 D)^{p-1} \cdot \int \left(\sum_{I \times J \in \mathcal{A}} b_{I \times J}(\xi) \right) \cdot \left(\sup_{I \times J \subset I_0 \times J_0} b_{I \times J}(\xi) \right)^{p-1} dw(\xi) \leq \\ & \leq (B^2 D)^{p-1} \cdot \int \left(\sum_{I \times J \in \mathcal{A}} b_{I \times J}(\xi) \right) \cdot M_{\Phi} \nu(\xi)^{p-1} dw(\xi), \end{aligned}$$

by Lemma 4.3 and the definition of $M_{\Phi} \nu$, since $0 < p - 1 \leq 1$. Hölders inequality now gives

$$\int \left(\sum_{I \times J \in \mathcal{A}} b_{I \times J}(\xi) \right)^p dw(\xi) \leq C^{p/p-1} \cdot (B^2 D)^p \cdot \int_{I_0 \times J_0} M_{\Phi} \nu(\xi)^p dw(\xi), \quad (5.1)$$

since the left hand side is finite. Letting $k_0, l_0 \rightarrow -\infty$ and taking supremum over all $I_0 \times J_0$ proves Theorem 3.1 for $1 < p \leq 2$.

Now assume that (5.1) has been proved for $k-1 < p \leq k$ and all classes $\mathcal{A}$ in (4.2), where $k \geq 2$ is an integer, and assume that $k < p \leq k+1$. Let $\mathcal{A}$ be any such class then $\int (\sum_{\mathcal{A}} b_{I \times J}(\xi))^p dw(\xi)$ is at most

$$\begin{aligned} & C \cdot \int \sum_{\mathcal{A}} b_{I \times J}(\xi) \cdot \left(\sum_{K \times L \subset I \times J, K \times L \in \mathcal{A}} b_{K \times L}(\xi) \right)^{p-1} dw(\xi) = \\ & = C \cdot \sum_{\mathcal{A}} b_{I \times J} \cdot \int_{I \times J} \left(\sum_{K \times L \subset I \times J, K \times L \in \mathcal{A}} b_{K \times L}(\xi) \right)^{p-1} dw(\xi) \end{aligned}$$

by the remark following the proof of Lemma 4.2. Fix any dyadic rectangle $I \times J$. Then by our induction hypothesis, with $\mathcal{A}$ replaced by the class $\{K \times L \in \mathcal{A}; K \times L \subset I \times J\}$, the last integral is at most

$$C \cdot \int_{I \times J} M_{\Phi} \nu(\xi)^{p-1} dw(\xi).$$

Hence

$$\int \left(\sum_{\mathcal{A}} b_{I \times J}(\xi) \right)^p dw(\xi) \leq C \cdot \int \left(\sum_{\mathcal{A}} b_{I \times J}(\xi) \right) \cdot M_{\Phi} \nu(\xi)^{p-1} dw(\xi)$$

and we can finish the proof as in the case $1 < p \leq 2$ by an application of Hölders inequality. $\triangle$

Proof of Theorem 3.2. We first show that $T_{\Phi} \nu$ is dominated by a variant $\tilde{T}_{\Phi}^d \nu$ of the dyadic operator $T_{\Phi}^d \nu$ in Theorem 3.1. Let $\xi = (x, s) \in R^d$ be fixed then

$$T_{\Phi} \nu(\xi) = \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \int_{A_{k,l}} \varphi(x-y) \cdot \psi(s-t) d\nu(y, t),$$

where $A_{k,l}$ is the set of points (y, t) such that $2^k < |y-x| \leq 2^{k+1}$ and $2^l < |t-s| \leq 2^{l+1}$. Let $I_k \times J_l$ be the dyadic rectangle with $l(I_k) = 2^k$, $l(J_l) = 2^l$ containing ξ . Then $B(\xi, 2^{k+1}, 2^{l+1}) \subset 5 \cdot I_k \times J_l \equiv \widetilde{I_k \times J_l}$, where

$\widetilde{I_k \times J_l}$ is the disjoint union of 5^d dyadic rectangles congruent to $I_k \times J_l$. Define $\widetilde{b_{I \times J}}(\xi) = \varphi(I) \cdot \psi(J) \cdot \nu(\widetilde{I \times J}) \cdot \chi_{I \times J}(\xi)$ and

$$\widetilde{T_\Phi^d} \nu(\xi) = \sum_{I \times J \in \mathcal{D}} \widetilde{b_{I \times J}}(\xi),$$

then $T_\Phi \nu(\xi) \leq \widetilde{T_\Phi^d} \nu(\xi)$. The next step is to prove that Theorem 3.1 holds with $T_\Phi^d \nu$ replaced by $\widetilde{T_\Phi^d} \nu$. We first show that Lemma 4.1 and Lemma 4.2 hold with $b_{I \times J}(\xi)$ replaced by $\widetilde{b_{I \times J}}(\xi)$. Define a set function $\tilde{\nu}$ on rectangles by $\tilde{\nu}(I \times J) = \nu(\widetilde{I \times J})$, then

$$\sum_i \tilde{\nu}(R_i) \leq 5^d \cdot \tilde{\nu}(R) \quad (5.2)$$

for any disjoint subdivision $R = \bigcup_i R_i$ of a rectangle R into congruent sub-rectangles R_i . An inspection of the proofs, and taking (5.2) into account, shows that Lemma 4.1 and Lemma 4.2 hold when $b_{I \times J}(\xi)$ is replaced by $\widetilde{b_{I \times J}}(\xi)$ with an extra factor c^p in the right hand side. A similar argument also applies to Lemma 4.3, which holds for $\widetilde{b_{I \times J}}$, with an extra factor 5^{ad} in the right hand side. We can now apply the proof of Theorem 3.1 in the following way. First let $1 < p \leq 2$ and assume that ν has compact support. Let $\mathcal{A}$ be any class in (4.2) such that $I_0 \times J_0$ contains the support of ν , then

$$\begin{aligned} I &= \int \left(\sum_{\mathcal{A}} \widetilde{b_{I \times J}}(\xi) \right)^p dw(\xi) \leq \\ &\leq C \cdot \int \sum_{\mathcal{A}} \widetilde{b_{I \times J}}(\xi) \left(\sum_{K \times L \subset I \times J} \widetilde{b_{K \times L}}(\xi) \right)^{p-1} dw(\xi) \leq \\ &= C \cdot \sum_{\mathcal{A}} \widetilde{b_{I \times J}} \cdot \int_{I \times J} \left(\sum_{K \times L \subset I \times J} \widetilde{b_{K \times L}}(\xi) \right)^{p-1} dw(\xi) = \\ &\quad \sum_{I \times J \subset I_0 \times J_0} \widetilde{b_{I \times J}}^p \cdot w(I \times J) \leq \\ &\leq \widetilde{b_{I_0 \times J_0}}^p \cdot w(I_0 \times J_0) = b_{I_0 \times J_0}^p \cdot w(I_0 \times J_0) \leq C \cdot \int_{I_0 \times J_0} M_\Phi^d \nu(\xi)^p dw(\xi), \end{aligned}$$

since $\widetilde{b_{I_0 \times J_0}} = b_{I_0 \times J_0}$. Letting $k_0, l_0 \rightarrow -\infty$ and $I_0 \times J_0 \rightarrow R^d$ proves Theorem 3.2 for $1 < p \leq 2$.

Now assume that $k < p \leq k+1$, for some integer $k \geq 2$. Let $\mathcal{A}_1$ be any class in (4.2) then

$$\begin{aligned} I &= \int \left(\sum_{\mathcal{A}_1} \widetilde{b_{I \times J}}(\xi) \right)^p dw(\xi) \leq \\ &\leq C \cdot \sum_{\mathcal{A}_\infty} \widetilde{b_{I_1 \times J_1}} \cdot \int_{I_1 \times J_1} \left(\sum_{I_2 \times J_2 \subset I_1 \times J_1} \widetilde{b_{I_2 \times J_2}}(\xi) \right)^{p-1} dw(\xi). \end{aligned}$$

Fix $I_1 \times J_1$ and define $\mathcal{A}_2 = \{I_2 \times J_2 \in \mathcal{A}_1; I_2 \times J_2 \subset I_1 \times J_1\}$ then the last integral is at most a constant C times

$$\sum_{\mathcal{A}_2} \widetilde{b_{I_2 \times J_2}} \cdot \int_{I_2 \times J_2} \left(\sum_{I_3 \times J_3 \subset I_2 \times J_2} \widetilde{b_{I_3 \times J_3}}(\xi) \right)^{p-2} dw(\xi).$$

Continuing in this way k times gives

$$\begin{aligned} I &\leq C \cdot \sum \widetilde{b_{I_1 \times J_1}} \cdots \widetilde{b_{I_k \times J_k}} \cdot \int_{I_k \times J_k} \left(\sum_{K \times L \subset I_k \times J_k} \widetilde{b_{K \times L}} \right)^{p-k} dw(\xi) \leq \\ &\leq C \cdot \widetilde{b_{I_1 \times J_1}} \cdot \widetilde{b_{I_2 \times J_2}} \cdots \widetilde{b_{I_k \times J_k}}^{p-k+1} \cdot w(I_k \times J_k), \end{aligned}$$

by Lemma 4.3 since $0 < p - k \leq 1$. Repeated use of Lemma 4.3 now gives

$$\begin{aligned} I &\leq C \cdot \sum_{I \times J \subset I_0 \times J_0} \widetilde{b_{I \times J}}^{p-1} \cdot w(I \times J) \leq C \cdot \widetilde{b_{I_0 \times J_0}}^p \cdot w(I_0 \times J_0) \leq \\ &\leq C \cdot \int_{I_0 \times J_0} M_\Phi^d \nu(\xi)^p dw(\xi). \end{aligned}$$

Now letting $k_0, l_0 \rightarrow -\infty$ and $I_0 \times J_0 \rightarrow R^d$ completes the proof of Theorem 3.2. $\triangle$

Theorems 3.3 and 3.4 are special cases of theorems 3.1 and 3.2 and we only have to check condition (2.1) and Lemma 4.3 for the Riesz kernels $R_{\rho, r}$.

Proof of Theorems 3.3 and 3.4. As remarked above, (2.1) holds for $\delta > t \cdot \max(1-\rho/m, 1-r/n)$, while Lemma 4.3 is true for $\delta > \max(1-\rho/m, 1-r/n)$. Theorems 3.3 and 3.4 now follow from Theorems 3.1 and 3.2. $\triangle$

Proof of Theorem 3.5. Fix any dyadic rectangle $I \times J$ then, since $0 < p \leq 1$,

$$\begin{aligned} \int \left(\sum_{K \times L \subset I \times J} b_{K \times L}(\xi) \right)^p dw(\xi) &\leq \int \sum_{K \times L \subset I \times J} b_{K \times L}(\xi)^p dw(\xi) = \\ &= \sum_{K \times L \subset I \times J} b_{K \times L}^p \cdot w(K \times L) \leq B^2 \cdot D \cdot b_{I \times J}^p \cdot w(I \times J), \end{aligned}$$

by a variant of the argument in the proof of Lemma 4.3. Hence

$$\int \left(\sum_{K \times L \subset I \times J} b_{K \times L}(\xi) \right)^p dw(\xi) \leq B^2 \cdot D \cdot \int_{I \times J} M_{\Phi}^d \nu(\xi)^p dw(\xi)$$

by the definition of $M_{\Phi}^d \nu$ and Theorem 3.5 follows by letting $I \times J$ tend to R^d . $\triangle$

Proof of Theorem 3.6. We first recall that $T_{\Phi} \nu(\xi) \leq \tilde{T}_{\Phi} \nu(\xi)$, where $\tilde{T}_{\Phi} \nu$ is defined as in the proof of Theorem 3.2. Assume without loss of generality that ν has compact support and let $I \times J$ be a dyadic rectangle containing the support of ν . Then

$$\begin{aligned} \int \left(\sum_{K \times L \subset I \times J} \tilde{b}_{K \times L}(\xi) \right)^p dw(\xi) &\leq \sum_{K \times L \subset I \times J} \tilde{b}_{K \times L}^p \cdot w(K \times L) \leq \\ &\leq 5^{dp} \cdot B^2 \cdot D \cdot \tilde{b}_{I \times J}^p \cdot w(I \times J) \end{aligned}$$

by the variant of the argument of Lemma 4.3 used in the proof of Theorem 3.2. Now $\tilde{b}_{I \times J} = b_{I \times J}$ and hence

$$\int \left(\sum_{K \times L \subset I \times J} \tilde{b}_{K \times L}(\xi) \right)^p dw(\xi) \leq 5^{dp} \cdot B^2 \cdot D \cdot \int_{I \times J} M_{\Phi}^d \nu(\xi)^p dw(\xi).$$

Theorem 3.6 follows by letting $I \times J$ tend to R^d . $\triangle$

Proof of Theorem 3.7. The first inequality follows easily from the definitions, while the second inequality follows from Theorems 3.1 and 3.5 and the trivial observation $M_{\Phi}^d \nu(\xi) \leq M_{\Phi} \nu(\xi)$. The third inequality is a consequence of the doubling property of Φ and the last inequality follows from Theorems 3.2 and 3.6. $\triangle$

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Address: Tord Sjödin, Department of Mathematics and Mathematical Statistics, University of Umeå, S-901 87 Umeå, Sweden.
 E-mail: tord.sjodin@math.umu.se