To Explore and Verify in Mathematics

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Akademisk avhandling som med tillstånd av rektorsämbetet vid Umeå universitet för avläggande av filosofie doktorsexamen framlägges till offentlig granskning torsdagen den 1 november 2001 klockan 10.15 i Hörsal Ma 121, Matematik och informationsteknologihuset, Umeå universitet.
To my family
Abstract.

This dissertation consists of four articles and a summary. The main focus of the studies is students’ explorations in upper secondary school mathematics.

In the first study the central research question was to find out if the students could learn something difficult by using the graphing calculator. The students were working with questions connected to factorisation of quadratic polynomials, and the factor theorem. The results indicate that the students got a better understanding for the factor theorem, and for the connection between graphical and algebraical representations.

The second study focused on the last part of an investigation, the verification of an idea or a conjecture. Students were given three conjectures and asked to decide if they were true or false, and also to explain why the conjectures were true or false. In this study I found that the students wanted to use rather abstract mathematics in order to verify the conjectures.

Since the results from the second study disagreed with other research in similar situations, I wanted to see what Swedish teachers had to say of the students’ ways to verify the conjectures. The third study is an interview study where some teachers were asked what expectations they had on students who were supposed to verify the three conjectures from the second study. The teachers were also confronted with examples from my second
study, and asked to comment on how the students performed. The results indicate that teachers tend to underestimate students' mathematical reasoning.

A central focus to all my three studies is explorations in mathematics. My fourth study, a revised version of a pilot study performed 1998, concerns exactly that: how students in upper secondary school explore a mathematical concept. The results indicate that the students are able to perform explorations in mathematics, and that the graphing calculator has a potential as a pedagogical aid, it can be a support for the students' mathematical reasoning.

**Keywords:** Explorations, mathematical reasoning, empirical investigations, graphing calculators, conjectures.
This dissertation consists of five parts, four articles and a summary. In the summary a description of the theoretical framework used in the articles is presented, together with an introduction to the research area and short presentations of the four articles. The summary ends with a discussion of the results of the studies and of possible implications for the Swedish school. The four articles are:

I Gymnasieelever undersöker ett matematiskt begrepp med grafräknare (Bergqvist, 1999a).
II How students verify conjectures (Bergqvist, 2000).
III How students verify conjectures: Teachers expectations (Bergqvist, 2001a).
IV Secondary school students using graphing calculators. Revised version (Bergqvist, 2001b).

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1. Introduction

In spring 1998 I visited a class at an upper secondary school. The students were working with tasks from the textbook. Since I knew the teacher, I was walking around in the class, prepared to answer questions from the students. One girl raised her hand and asked me for help. She showed me a question of this kind:

Find the values of $a$ and $b$ in the function 
$$f(x) = x^3 + 2ax + b$$ if $f(1) = 5$ and $f'(0) = 2$.

When I started asking her what kind of function she had and what $f(1) = 5$ could mean, she said “Stop! Just tell me what to do. Exactly how to get the right answer. Nothing else!” She made it very clear to me that she wanted to know exactly what method to use in order to solve this specific question, nothing else. I told her that I would not give her the solution, she would have to think for herself for a while first. She then told me to leave, so I walked away. About one minute later she called upon the attention of her regular teacher, and asked him the same question. He told her what method to use to get correct answer, she did what she was told to do, and she got the right answer. Everyone was happy. Except me, maybe.

This little story convinced me that there must be possibilities to change school mathematics, in a way that makes students more interested in mathematical reasoning and mathematical ideas, and teachers less interested in just provide correct methods to the students.

The student in the example was working in a very practical and also (possibly) a very successful way. She got instructions how to solve the problems and she memorised the methods. The teacher who showed her the methods also constructed the exams, so she did probably very well at the exams too. However, from my point of view she wasn’t doing much mathematics. It is probably possible to get high grades in mathematics in upper secondary school, without doing much mathematics at all. In the literature one can find many indications that what students remember from their calculus is that you move the exponent down in front of the $x$ and then you write a new exponent which is one less than the original. I believe that the method of differentiation of polynomials could be learned in primary school as well.
But are students in upper secondary school really able to do something else? To learn more about this question I designed a small pilot project: Secondary school students using graphing calculators. (Bergqvist, 1998).

Three pairs of students were given the following open ended question:

| Describe how the expression $x^2 + bx + c$ changes when the values of $b$ and $c$ varies. Summarise your results in a short report. |

Figure 1. Question to the students.

The students used the graphing calculator to investigate quadratic polynomials of the given kind. They seemed to use the graphs in order to enhance the discussions, and to examine ideas of what would happen when the values of $b$ and $c$ varied. The students looked at a few graphs at a time to see the difference between two graphs. They also came up with guesses of different effects the changes of $b$ and $c$ could have. One example of this is when one pair looked at the graphs of the two functions $y = x^2 + 2x + 1$ and $y = x^2 + 4x + 1$ in the same window. They could see that one of the functions was 'lower down' and they said “a lower value of $b$ gives a lower graph”. This is not true, something the students found out when they examined the situation more closely. The conflict that appeared when the students found that their idea did not fit what they saw on the calculator was very clear.

By presenting a short (but maybe not so simple) question to the students, I initiated a lot of student activity which, according to the students, was very unusual for them. This was very interesting and also very intriguing. Could this be a way to get students to increase the amount of discussions around mathematical concepts and ideas?

When I now look back at my five years as a research student, I think that the pilot study presented above has played a very important role in the development of my research focus. That is also one reason for the revision of the report from the pilot study. The revised report can now be found as article no. IV in this dissertation.
2. Theoretical framework

In this section I will present the framework that has guided me through my research. In order to better understand the environment in which the research has taken place, I start with a short description of mathematics in upper secondary school in Sweden.

This dissertation deals with mathematics at one of the many different educational programs in Swedish upper secondary school, the Natural science program (in Swedish: Naturvetenskapsprogrammet). Approximately 15% of the time is supposed to be mathematics. Besides the core subjects (Swedish, English, Mathematics, Civics etc), the focus of the education is on advanced mathematics, physics, chemistry and biology. The education is for three years, and the students get access to most technical and scientific educations at the university. The mathematics in the program is mainly meant to support future studies at the university and it is supposed to cover the following mathematical themes:

- Arithmetics
- Algebra
- Geometry
- Probability
- Statistics
- Precalculus (functions, trigonometry etc.)
- Calculus (with differential equations)

The mathematical focus of this dissertation is precalculus and calculus, mainly in the second year of the three-year programme (the students’ eleventh school year).

2.1. Investigations in mathematics education. What can students really do when they get the opportunity to investigate and use their imagination in mathematics? How can this be described? This was some of the central questions I had when I started the study Gymnasieelever undersöker ett matematiskt begrepp med grafräknare (Bergqvist, 1999b, article no. I). Are students able to state conjectures when they work with unfamiliar concepts in mathematics? If they can, what happens next? Can they test it also?
In the work with these questions, it was very natural to look into the work of Schoenfeld (1985). He describes the process of problem solving as a set of chunks called episodes. There are six different episodes, and Schoenfeld describes them like this (pp. 297-300, words in italics are quotations):

**Reading:** The reading episode begins when a subject starts to read the problem statement aloud. It includes the time spent ingesting the problem conditions and may continue through any silence that may follow the reading - silence that may indicate contemplation of the problem statement, the (non-vocal) rereading of the problem, or blank thoughts. It continues as well through vocal rereadings and verbalisations of parts of the problem statement.

**Analysis:** In analysis an attempt is made to fully understand a problem, to select an appropriate perspective and reformulate the problem in those terms, and to introduce for consideration whatever principles or mechanisms might be appropriate. The problem might be simplified or reformulated.

**Exploration:** Exploration, on the other hand, [compared to analysis] is less well structured and is further removed from the original problem. It is a broad tour through the problem space, a search for relevant information that can be incorporated into the analysis-plan-implementation sequence.

**Planning:** To devise a plan for the solution of the problem.

**Implementation:** Solution of the problem according to the plan.

**Verification:** The nature of the episode is obvious.

One interesting result from Schoenfeld’s research is the apparent difference in performance in problem solving between students and expert mathematicians, even when the mathematical focus is far away from the experts normal body of knowledge. The students used more than half the time to the implementation of some methods, while the expert was mixing analysis, planning, implementation and verification in a very different way. Schoenfeld shows clearly that strategies and working methods are of vital importance in problem solving.
In article no. I (Bergqvist, 1999b) I let students investigate a mathematical concept, the factor theorem. There are several differences between mathematical investigations and problem solving, but some parallels can be seen. The investigations I have observed in my research can be divided into three main parts:

- Visualisation
- Conjecture
- Verification

Using visualisations is to let the objects be observed, using concrete representations, in order to identify different possible connections between the objects. One example can be a function, which can be described by an algebraic expression, a table, or a graph. The graph is of course the most obvious example of a visualisation, and here is the graphing calculator a natural tool. The analysis and exploration episodes in Schoenfeld’s description are carried out through visualisations.

Stating a conjecture. A conjecture is an assumption or an idea which could become a result or part of a result. It will mostly derive from visualisations, together with other observations and previous knowledge. To state a conjecture could be a result of Schoenfeld’s exploration, and a part of the planning episode.

Verification is an investigation of a conjecture. This can be done in many different ways, and also based on more or less stable reasons. To test an example, to see if the conjecture works for other types of objects, to try to find counterexamples or to prove a conjecture using deductive methods, are all examples of verification activities. Verification would of course answer to Schoenfeld’s verification episode.

### 2.2. Mathematical reasoning and proof.

**Mathematical Reasoning in School Mathematics.**

What is mathematical reasoning really? The expression can be found in many different articles and documents (for example Skolverket, 2001; Ernest, 1989; Stiegler and Hiebert, 1999). What is meant when the syllabus for mathematics in upper secondary school in Sweden (2001) talks about mathematical reasoning? It says:
The school shall strive for the students to develop their ability to follow and carry out mathematical reasoning, and to show their thoughts verbally and in writing. (p. 1)

Reasoning can be interpreted as several different activities:

- Logical reasoning in a traditional sense, following practices from research mathematics meaning deductive proof.
- Discussions about mathematical concepts or ideas, communication aiming towards the understanding of a situation.
- Finding the appropriate solution strategy in a specific situation.
- Problem solving, to use mathematics in a new situation.

The Swedish syllabus states the goals for the educations, the orientation of the work in the school, but not how these goals are supposed to be reached. The way to reach the goals are left to the expertise of the teachers. This is a positive situation because the teachers must reflect over their own teaching. However, it can also be problematic since the goals not are stated clearly.

*NCTM Principles and Standards for School Mathematics* (NCTM, 2000) have a section called ‘Reasoning and Proof Standard for Grades 9 - 12’:

Instructional programs from prekindergarten through grade 12 should enable all students to -

- recognise reasoning and proof as fundamental aspects of mathematics;
- make and investigate mathematical conjectures;
- develop and evaluate mathematical arguments and proofs;
- select and use various types of reasoning and methods of proof.

These four points are followed by eight pages of discussions and examples to illustrate what proof and reasoning should look like in upper secondary school. Several examples are presented, and different aspects of proof and reasoning are considered. One example is the following discussion concerning indirect proofs:
Because conjectures in some situations are not conducive to direct means of verification, students should also have some experience with indirect proofs. And since iterative and recursive methods are increasingly common, (...) students should learn that certain types of results are proved using the technique of mathematical induction. (p. 344)

This is written in a very different, almost opposite, tradition compared to the Swedish syllabus. The Standards say clearly what is meant by ‘Reasoning and Proof’. It is also very clear that reasoning and proof should be a normal part of the mathematics education. “Mathematics should make sense to students” is a central idea in the document. One problem with the Standards is that the teacher might feel restricted only to use methods and ideas mentioned in the document.

Mathematical Reasoning and Proof in my Research.
In two of my studies I have used a hierarchy of proof levels constructed by Balacheff (1988) (original in French: Balacheff, 1987) in order to classify and analyse student activities.

The levels, listed below, are not proofs in a strict mathematical meaning, but rather something that is recognised as such by the student. In the presentation I have used the concept of a linear function which in school mathematics often is defined a bit looser than in undergraduate mathematics. A linear function is here a function that can be written as \(y = kx + m\). The levels are:

**Level 1:** Naive empiricism.
To be convinced that a conjecture is true, and to argue that it is true after verifying some cases. *Example:*
*Conjecture:* All linear functions intersect the \(y\)-axis.
*I tested four different linear functions and it was true every time.*

**Level 2:** The crucial experiment.
Two different versions of this level was discussed by Balacheff. The first version is an experiment designed to make a choice between two possibilities. The result should clearly show
which one of the two conjectures that must be rejected. Note that this does not say that the other conjecture is true.

The second version means testing the conjecture in a special case and drawing the conclusion that “if it works even for this it will always work”. The difference from Level 1 is mainly that the students are aware of the problem of generality. Only the second version was identified in my research.

Example (second version):

Conjecture: All linear functions intersect the y-axis.
I tested the function $y = 1000x - 1000$. Even that one intersect the y-axis, in spite of the fact that it is vertical in the calculator window. Then I’m sure that all linear functions intersect!

Level 3: The generic example.
To show the truth by manipulating an object which is used as a representative of all similar objects. The proof is indicated by the effect of the operations. Example:

Conjecture: All linear functions intersect the y-axis.
$y = 2x + 3$ is a linear function. To intersect the y-axis means that $x$ is 0. In this functions we get $y = 3$ when $x = 0$. This can be done for all linear functions.

Level 4: The thought experiment.
An abstract member of a class is discussed. The proof is indicated by looking at the properties of the objects, not on the effects of operations on the object. Example:

Conjecture: All linear functions intersect the y-axis.
A linear function is described by a polynomial of degree 1. Such a polynomial is defined for all $x$, including $x = 0$. Therefore all linear functions intersect the y-axis.

Balacheff argues that Level 1 and 2 are pragmatic proofs, based on statements of facts, and that Level 3 and 4 are conceptual proofs, based on reasons. In his study he found evidence that there exists a break between these two main types of proof.

It is important to remember that these levels deal with the working method that is being used, not with how correct or successful the
students are. A student activity can be classified to be at Level 4, without being fully correct or complete.

When he had constructed his four levels, Balacheff performed a study on some students (13-14 years of age). The students were asked to answer a question in geometry. They got the following instruction:

You are to write a message which will be given to other pupils of your own age which is to:

provide a means of calculating the number of diagonals of a polygon when you know the number of vertices it has.

(p. 220)

The predominant answer to the question was \( \frac{n}{2} \), where \( n \) is the number of vertices in the polygon, which is correct only for a quadrilateral. (Since a polygon can have an odd number of diagonals, the solution is incorrect.) One pair out of thirteen managed to find the correct formula, \( \frac{n(n - 3)}{2} \).

In his study, Balacheff found examples of all four levels. Naive empiricism (Level 1) was found in many situations. The student Christophe supported the conjecture \( f(n) = 2n \), since it was true for \( n = 7 \) (the only case where the conjecture is true). The crucial experiment (Level 2) was observed in both versions. The first version was observed when two students had different conceptions, “as a weapon in discussions of validity between the pupils from the same pair” (p. 224). The second version was found when students where checking an idea against a specific polygon, with the specific goal of testing it. The generic example (Level 3) was in three cases found in order to test the conjecture that the number of diagonals could be calculated using the formula \( f_2(n) = (n - 3) + (n - 3) + (n - 4) + \ldots + 2 + 1 \). In all these three cases the generic example convinces a doubting partner. The thought experiment (Level 4) was found when two students tried to formulate an argument in support of the conjecture that from each vertex there will be \( n - 3 \) diagonals. They discussed in a general way that you must subtract 3 from the number of vertices since there will not be any diagonals to the two adjacent vertices and not to the vertex itself.
To characterise mathematical reasoning the way I have been doing in my research, to use Balacheff’s levels, covers only one aspect of the concept. One other way to look at mathematical reasoning is presented in Section 3.3.

2.3. The didactical contract. The concept, coined by Brousseau 1984 (Brousseau, 1997), has turned out to be a very powerful lens when it comes to look at various kinds of educational or didactical situations. The idea of an invisible agreement in a classroom makes it possible to look at situations in the classroom in new ways.

A normal teaching situation would be that a teacher presents a good problem. The student engages in the problem, and if the student can do something with the problem, learning occurs. When there is a problem, if the student avoids the problem, or doesn’t solve it, the teacher has an obligation to help. The relationship that now forms, the rules for what the two involved persons are supposed to do, is a type of a contract. The part that deals with the mathematical content is called the didactical contract.

Brousseau argues that it is impossible to give explicit details of the contract, since it deals with outcomes of education, and there are no known ways to guarantee that students will learn a specific concept. The following general consequences are offered:

- The teacher is supposed to create sufficient conditions for the appropriation of knowledge and must “recognise” this appropriation when it occurs.
- The student is supposed to be able to satisfy these conditions.
- The didactical relationship must “continue” at all costs.
- The teacher therefore assumes that earlier learning and the new conditions provide the student with the possibility of new learning. (p. 32).

Brousseau also says that the most important part is not the contract in itself, but the “hypothetical process of finding a contract” (p. 32).

The didactical contract is also always invisible. Balacheff (1999) writes:
Everything happens as if there was a contract, but that contract can never be agreed upon and whenever it is efficiently agreed upon, it cannot be enforced. (p. 27)

Balacheff argues that the concept of the didactical contract is almost like a paradox, there must always exist a contract, but if it is spelled out, it must fail.

Blomhøj (1994) describes the very clear game between the teacher and the student that takes place in an individual supervising situation. He mentions several explicit characteristics of the didactic contract in mathematics:

- The teacher is supposed to explain the methods and algorithms that are presented in the book.
- The teacher is supposed only to pose questions which the students are able to solve using those methods and algorithms.
- A task is solved, when the easiest question in it is answered.
- Answers to the questions should be short, i.e. a number, a graph or, if absolutely necessary, a sentence.
- The students will demand marking of solved exercises.
- The assessment is based only on how successful the students are in the solving of the given exercises.
- The students will do their best to try to solve the exercises.

When the teacher has explained the methods and algorithms of the day, she/he will help students individually. In the communication, the student is busy trying to catch signals from the teacher, to read the hints that the teacher is giving, to find out “if the number 17 in the text should be above or under the fraction bar”¹ (p. 38). At the same time, the teacher is trying to interpret signals from the student, signals which can be read as signs of understanding. “That kind of signs is the teacher’s alibi to believe that the student now has sufficient prerequisites to engage in the solution of the exercise, and the teacher can proceed to the next student with a clean conscious”² (p. 38). The risk is here that the teacher believes that the student’s

¹Author’s translation
²Author’s translation
correct actions are founded in an understanding of the mathematics, when the student only is reacting on signals from the teacher. Altogether, this would mean that the only way for learning to take place is if the contract is broken. The student must engage in the problems, and seize control of the activity.

3. Relevant related research

3.1. Calculators in upper secondary school mathematics. Technology (mainly calculators and computers) as a tool in mathematics is seldom in dispute. When it comes to the use of the same technology in education, however, the case is not as clear. Discussion concerning the risk of causing the students to loose all mental calculations skills and becoming ‘button-pressers’, who can’t solve anything by using paper and pencil, arises in the press at regular intervals. Interesting is here the following quote:

*If men learn writing, it will implant forgetfulness in their souls; they will cease to exercise memory because they will rely only on that which is written, calling to remembrance no longer from within themselves, but by means of external marks.* (Plato, 380 BC)

In research, nowadays, the discussion seldom concerns the existence of calculators and computers, but rather what role they should play in mathematics education. Grugnetti and Jaquet (1996) say:

> It is no longer a question of accepting or rejecting the new technology, but rather of determining its place in the teaching of mathematics and of trying to evaluate its effects on teaching practices and on student learning. There are however questions which concerns the necessary conditions for this process of integration and the obstacles which confront it. (p 629)

The change in the use of calculators and the role of calculators have also been discussed by several researchers. Dick (1996) says:

> Graphs generated by technology can be used to effectively communicate and discuss the meaning of the derivative rather than using the derivative as a tool for
Pomerantz (1997) has a similar opinion:

In the past, students studied advanced mathematics (calculus) to learn how to draw graphs accurately. Now computer-generated graphs can be used to study important mathematical concepts” (p. 16).

Pomerantz also discusses and tries to counter some of the ideas about calculators in mathematics that many people seem to hold. She calls that part of her article “dispelling the myth” (p. 6), where the myths are

1. Calculators are a crutch: They are used because students are too lazy to compute the answers on their own; they do the work for the students.
2. Because calculators do all of the work for the student, he/she will not be stimulated or challenged enough.
3. ‘If I didn’t need to use technology to learn math, then neither does my child. After all, I turned out just fine.’
4. The use of calculators prevents students from effectively learning the basic mathematics they will need when they enter the workforce.
5. People will become so dependent on calculators that they will be rendered helpless without one. (e.g.: What if the battery dies or some student has to perform a computation when no calculator is available?)

One answer to the myths that Pomerantz offers is that the calculator can only do routine skills, no thinking without the user. Another answer is that the calculator, “when properly used”, can enhance the learning of mathematics. I find this list of myths rather interesting, I recognise many of the arguments from discussions with parents and mathematics teachers, and her answers are to a rather large extent the same as I tend to use. Even if there are findings in research that support the answers, more research is needed to stabilise the argumentation.
So, what has happened in research concerning technological aids in mathematics the last twenty years? In Section 3.1.1 I will look at two reviews of calculator research. In the earlier years (1981 - 1995), one part of the research was comparative studies (see for example Quesada and Maxwell, 1994; Ruthven, 1990; Mesa, 1997), aiming at determining the effects of the use of calculators or computers in teaching (see Penglase and Arnold, 1996, for further references). I will here (in Section 3.1.2) give two examples of this kind of study. Later research has more and more turned towards qualitative studies, aimed at describing the possibilities and problems in the use of educational technology (see for example Healy and Hoyles, 1999; Doerr and Zangor, 2000). Section 3.1.3 contains two examples of this kind of studies.

The last few years, the interest for CAS (Computer Algebra Systems) and SC (Symbolic Calculators) has increased, and today many of the studies that are performed deal with this issue (Heid et al., 2000; Kutzler, 2000). One might expect the third wave of the same kind: first to see what students can do, the comparacies with students without CAS and finally qualitative studies to examine possibilities and different strategies in the use of the technology.

3.1.1. Reviews on calculator research. Ruthven (1996) offers a clear review over research on calculators in mathematics education. He focuses on calculators, since computers are too expensive. He talks about “computational technologies which are already widely available to students as an individual resource” (p. 441). However, he argues that in time the costs for portable computers will fall, and “it will become increasingly realistic to envisage the portable computer as a personal resource” (p. 441). Today, five years later, computers for all students is still not an option in most schools. In Sweden the development has gone in the other direction: five years ago all students on the natural science program in Umeå was given a graphing calculator as a free educational material, but today the students are supposed to buy their own calculator.
In his survey, Ruthven points at several researchers who have tried to analyse what impact the calculators have on the teaching of mathematics. Up to 1990, the general conclusions was that the use of the arithmetic calculator in primary and lower secondary school was “little more than marginal” (p. 443). Ruthven also mentions the strong reservations from different directions (parents, employers and politicians among others). The fear that the calculator will cause the students to become lazy, that they will lack number sense and that they will be unused to writing mathematics when it comes to public examinations.

After 1990 the amount of calculators in school had gone up dramatically (from 10 % to 75 % in 10 years), but still the use of the calculators in the teaching was limited. The reason for this was, according to Ruthven, the teachers lack of confidence in calculator use. Ruthven mentions several calculator projects where the teachers have been guided towards a more developed model of teaching, and the general experience from these studies is that it takes a long time for teachers to develop confidence in using calculators in their teaching, and even longer time to use them outside specific, well prepared, teaching activities.

When the graphing calculator became available in school (around 1990), the research started all over again. What is the impact? Is it good or bad? What do the students do with the graphing calculators? Ruthven performed a study (Ruthven, 1996) in an attempt to answer the last of these questions. He found that

the calculators were primarily used in three distinct ways: to support routine calculations and graphing; to mediate a numeric and/or graphic treatment of a new idea prior to a symbolic one, or parallel to it; and to stimulate open-ended activities intended to promote mathematical exploration and reflection. (p. 448)

But how effective is the use of graphing calculators in mathematics education? Is that question possible to answer in a clear way? Dunham and Dick (1994) looked at seven studies, and found three where the use of graphing calculators had a positive impact, three
studies where no differences was found, and one study where there was a negative impact.

Dunham and Dick also looked at what kind of changes in student performance the researchers found. The report students who use graphing technology (compared to students who do not), in general

- have a better ability to interpret graphs.
- can connect the graphs to other representations to a larger extent
- are better problem solvers

Another effect of the use of graphing calculators in the classroom is that the demands on the teacher changes. The loss of teacher control that some studies report as an effect of an increased use of investigations and explorations in the classroom, is an indication that teachers will need to be more flexible in their teaching. “They can no longer only follow a script of the day’s lesson” (p. 443). The conclusions Dunham and Dick draw from their investigation are that the graphing calculator is a powerful tool in mathematics education, a tool which has a positive effect on the learning of mathematics. They also say that it is too early to draw any conclusions concerning the teaching of mathematics.

Another very clear and thorough review of graphing calculator research is offered by Penglase and Arnold (1996). They found that research results have indicated positive effects in several areas:

- understanding of function and graphing concepts
- reading and interpreting graphs
- relationship between functions and graphs
- spatial visualisation skills
- understanding of mathematical modelling

However, they also found that the research could not tell whether the graphing calculator was effective for developing understanding of transformations of functions or for combining algebraical and graphical knowledge of functions and graphs.
Penglase and Arnold are very critical to many comparative studies, meaning that often “the effects of the calculators cannot be distinguished from that of the instructional program” (p. 63). They finish their review by pointing out that the potential of the graphing calculator in education is clear, but “it is one thing to use a tool, but quite another to use it effectively” (p. 85).

3.1.2. **Comparative studies.** The first example is a quantitative study by Quesada and Maxwell (1994), where 710 students participated. The study took place during a three-semester period, which means that smaller parts of the participants were active in each semester. The method used was an experimental group versus a control group. The control group was taught in the traditional way, while the experimental group was using a graphing calculator (Texas TI-81 and Casio G-7000), and a new textbook designed for teaching with graphing utilities. All students took the same tests (at least 90% correspondence). In addition to the written tests, all students in the experimental group answered a survey about their thoughts on the use of the calculator.

The results of the study show significantly better results in the experimental group than in the control group. The average mean was 14.21 points higher (maximum 105 points) in the experimental group. In the article the authors discuss possible underlying factors:

- The students in the experimental group were aware of their participation in the experiment.
- The exams may have been biassed towards the experimental group, since it was prepared by one of its teachers.
- The groups were using different textbooks. The one used in the experimental group contained many new ideas.
- The fact that different teachers taught the various groups raises the possibility that the effects were due to the instructors rather than the treatment. (pp. 212 - 213)

The open-ended questions to the experimental group indicated three main positive effects: “(i) facilitates understanding, (ii) provides
the ability to check answers, and (iii) saves time on tedious calculations” (p.212). The most common negative aspect was that the students might become too dependant on the graphing calculator.

A number of students were also interviewed, and among them, some expressed that the calculator was a mean not only to check answers, but also to look at problems graphically, before using algebra to solve them. Students also expressed the concern that they might get problems in the next course (calculus).

The authors end their article by marking that “it is not clear what causes the improvement in scores when the graphing calculator is used” (p. 214). This statement can in a way summarise the problems with this kind of research studies. There are so many possible factors that can affect the students and the results of comparative tests, so it is very hard to isolate a single one. The fact that the use of the graphing calculator is supposed to effect the methods and maybe also the mathematical content, is one reason that this kind of studies are hard to find in more recent research.

The ADM-project was a Swedish analogy to the study by Quesada and Maxwell. In the study, Björk and Brolin (1995) looked at the effects of decreasing the amount of time used for graphing by hand, maximisation problems using differentiation, solving equations and calculation of determined integrals by hand. The time was instead put on the use of a graphing calculator or a corresponding computer program called Matematikverkstad. The students were supposed to use a specially designed booklet, covering the following topics:

- Graphing functions and extreme-value problems.
- Numerical solutions of equations.
- Setting up and solving integrals.
- Problem solving.
- Test.

In the end of the study the students were given a test. The same test was also given in 70 randomly selected control classes (each consisting

\footnote{Swedish for Mathematics workshop}
of 20 - 30 students). The results show a significant better performance by the experimental classes on problem solving and questions designed to test understanding. In the case of routine tasks there were no statistical difference between the experimental group and the control group.

The researchers then draw the conclusion that the use of graphing technology can enhance the quality of mathematics education. Again there is a problem of isolating a single factor in a whole new design, but this problem is not addressed by the authors.

3.1.3. Qualitative studies. Lauten et al. (1994) looked at five students during one semester 1991, with a mathematical focus on functions and limits. According to the authors this was one of the first studies with students who had the calculator as a normal part of their learning situation. They used the outcomes of clinical interviews as a basis for their analysis. After the initial interviews, the research was focused on one student, Amy. The authors claim that the study indicates that Amy had a dynamic notion of functions, and that she even might see a curve as animated in some way. Amy clearly used an image of $x$ and $y$ as particles, moving along the curve, where the particles never reached the limit point. In the end they suggest some further research areas, especially concerning changes over time, in the ways students use the calculator.

Dahland and Lingefjärd (1996) looked at how students, in a test situation, tell a reader how the calculator was used, and how the results was interpreted. A test with six questions was given in four classes (approximately 600 analysed student solutions). In the answer to the first question in the test, the students normally informed the reader that the calculator had been used to draw one or two graphs, or that the Newton-Raphson method had been used, without further description of the work. The second question was to draw the graph and describe the function $y = \frac{x^3}{x^2-1}$. This was in most cases answered only by a graph and a short sentence similar to “The graphing calculator has been used” (p. 45). However, many errors in the copying
of the graph were made. The authors come to the conclusion that students (and teachers), need to develop a double competence: to use the calculator and be aware of benefits and drawbacks, and to have a basic mathematical knowledge to be able to interpret results from the graphing calculator. As a conclusion, the authors argue that the traditional methods will change but the mathematical core must remain unchanged.

3.2. Proof in mathematics education.

3.2.1. What is proof? Hersh (1993) addresses this question, and in his discussion of the concept he distinguishes between “proof among professional mathematicians” (p. 389) and “proof in our classrooms” (p. 396). He says that

The role of proof in the classroom is different from its role in research. In research its role is to convince. In the classroom, convincing is no problem. Students are all too easily convinced. Two special cases will do it. [...] What a proof should do for the student is provide insight into why the theorem is true. (p.396)

He argues that the purpose of proof in education is understanding. Although we cannot really understand what it is ‘to understand’ something, we can recognise understanding when we see it. Therefore we can also teach in order to foster ‘understanding’. Hersh finishes his article with the following statement:

Mathematical proof can convince, and it can explain. In mathematical research, its primary role is convincing. At the high-school or undergraduate level, its primary role is explaining. (p. 398)

3.2.2. Proof in school. Many researchers agree that proof must be a part of mathematics education (Coe and Ruthven, 1994; Hanna and Jahnke, 1999). The main reason to use proof in mathematical education can not be to show that a theorem is true. Proving that Pythagoras’ theorem is true has been done thousands of times by the best thinkers in mathematics during the last few millennia. It would
be rather ridiculous to doubt that theorem. So why do we prove things in school?

Hanna and Jahnke (1996) says:

*Proof is an essential characteristic of mathematics and as such should be a key component in mathematics education.* (p. 877)

And:

*A proof that we propose to use in the classroom must be well structured, and almost any proof could presumably be restructured to make it more teachable. Yet proofs do differ greatly in their inherent explanatory power, and it is useful to make a distinction between proofs that prove and proofs that explain.* (p. 903)

Proofs that explain. To use proofs in this way would be a good way both to make use of proofs in school mathematics and to show students how to benefit from discussions around proofs. In undergraduate education a simplification of what a lesson consists of would be *definitions, theorems and proof*. I think that education in upper secondary school can learn from undergraduate mathematics and vice versa, not to replace anything, but to supplement each other.

Several attempts to classify students’ different approaches when they are supposed to prove some mathematical statement has been done. The classifications are of different kinds, and Hoyles (1997) describes the different dimensions that have been used. “From pragmatic involving recourse to actions to conceptual arguing and relationships, (van Dormolen, 1977 and Balacheff, 1988); from weak to strong deduction (for example Bell, 1976; Coe & Ruthven, 1994); according to different modes - enactive, visual and manipulative (Tall, 1995), or proof schemes (Harel & Sowder, in press).” (p. 7, see Hoyles (1997) for references.) Hoyles finds in her survey that students in general have rather weak understanding of what proof really is and what it is used for. She points at three different student misconceptions or difficulties:

(1) Empirical arguments are preferred over deductive reasoning.
A deductive proof is still only evidence.

It is hard to understand under what premises a proof will hold, and to follow a logical argumentation throughout a proof.

Hoyle also gives references in support for this list (p. 7). However, to a very large extent, these studies are focused on students in lower secondary school (12 - 15 years of age). There are examples of studies on older and more advanced students, but there are few studies and the results are still much in line with the list above.

In the light of the argumentation and research results in this section, the results in my article *How Students Verify Conjectures* (2000) appear almost strange. I found that most students used conceptual argumentation and obviously were aware that a few examples is not a proof. See article number II and III for further discussions.

### 3.2.3. Students conceptions of proof

Above are examples of how, at least some, researchers in mathematics education look at the concept of proof. Equally important are students’ beliefs or conceptions of proof.

Chazan (1993) studied if an extensive use of hands-on measuring in geometry class would hinder students’ ideas about proof. He argues that this is not the case, even if he found that many students (in his study) had beliefs that were not compatible with the common idea of what a proof is. He observed two sets of beliefs, which have been found in earlier research:

1. *Evidence is proof.* That an empirical argument is sufficient as ‘proof’ is often found in research ((Balacheff, 1988; Hoyle, 1997)). An example from geometry is when students believe that if you take a triangle, cut of the vertices and show that they together form a straight line, you have a proof that the sum of the angles in a triangle is $180^\circ$.

2. *Deductive proof is just evidence.* Students don’t accept the generality of a deductive proof, “deductive proof provides no safety from counterexamples” (Chazan, 1993, p. 372).
To change the situation he proposes that teachers, by clearly showing different ways of knowing in mathematics, can help the students to better understand why mathematics claims to be a “unique and important human endeavour, different from other human activities (e.g. empirical science).” (p. 385). That way we can also help students to understand why deductive proof is so highly valued by the mathematical community.

Healy and Hoyles (2000) looked at how students chose among a collection of written proofs, when they were asked to

i choose the one they would use themselves to answer the question and
ii choose the one they believed their teacher would give the best mark.

Interesting is that the choice that were “the most popular for the students’ own approaches turned out to be the least popular when it came to choosing for best mark, and vice versa” (p. 407). They also found indications that “most students were aware that empirical arguments has limitations” (p. 410). A connection to Hersh’s descriptions of the dual function of proof can be made: Healy and Hoyles argues that the reason that students avoid algebraical approaches is their limited possibility to explain using algebra. Few students in the study chose algebraic arguments, and even fewer could construct them. They conclude with the following important question:

If students do not see algebra as a language with which they can explain phenomena in mathematics classrooms in which explanations are highly valued, what motivation can there be for those who can successfully construct informal arguments to learn how to reexpress them algebraically? (p. 425)

3.2.4. Classification of proof and proof-like activities. Dreyfus is giving a review over different approaches to study students’ conceptions of proof as one part of his article “Why Johnny can’t prove” (1999). He says:
Mathematics educators have attempted to classify students’ developing notions of proof. Balacheff (1988), for example, distinguishes pragmatic proofs and intellectual proofs, subdividing each into several subclasses; and Harel and Sowder (1998) propose a large set of schemes intended to make a classification of collage students’ proof-like productions possible. (p. 94)

The expression *proof-like productions* fits very well to the kind of student activities I have been examining in my studies. I have used Balacheff’s classification in my research (for a description of his hierarchy of proof levels, see Section 2.2). Here I will compare the levels presented by Balacheff with the proof schemes constructed by Harel and Sowder.

Harel and Sowder (1998) have created a rather complex tree-structure to illustrate their three categories (with several subcategories). The three main schemes are:

- **External conviction proof schemes**: The ground for a proof is something outside the mathematical content.
- **Empirical proof schemes**: The conclusion is founded on facts and examples.
- **Analytical proof schemes**: Logical deduction of some kind.

Each of the three categories has several subcategories. For the first category, external conviction proof schemes, three subcategories are proposed. The doubts of an argument are removed by following a *ritual* (‘It looks like a proof...’), obeying an *authority* (‘the teacher said so’ or ‘it says in the book’) or the *symbolic* form (the proof is written without understanding the meaning of it).

The second category, empirical proof schemes, is divided in two parts: the *inductive proof scheme*, which is when students argue that a conjecture is true on the basis on quantitative evaluation in one or some cases, and the *perceptual proof scheme*, when students use mental images (based on observations) but fail to see what happens when the images change (transforms).
The third (and final) category, analytical proof schemes, has two main subcategories: transformational and axiomatic proof schemes. Transformational means that the argumentation “involves operations on objects and anticipations of the operations’ results” (p. 258). A person possesses the axiomatic proof scheme if she or he is aware that axioms and other undefined terms are in the bottom of a mathematical justification.

The system of proof schemes proposed by Harel and Sowder contains seventeen subcategories on different levels. The large number of possibilities create a problem when it comes to understand their ideas, and for some of the subcategories I find it hard to see why the examples justify that specific subcategory.

Balacheff has four different levels, which all in some way have to do with level of abstraction and awareness of abstraction in the reasoning. The empirical proof schemes by Harel and Sowder roughly match Balacheff’s pragmatic proofs (Level 1 and 2), while the analytical proof schemes in many ways match Balacheff’s conceptual proofs (Level 3 and 4). Of course there is not an exact match, examples which contradict this can certainly be found. To me the closest connection is between Harel and Sowder’s transformational proof scheme and Balacheff’s the generic example, since both base the assertion of a statement on operations and anticipation of results of operations.

As a final comment Harel and Sowder claims that “key to the analytical proof schemes is the transformational proof scheme” (p. 276), and that education of students towards transformational proof schemes must begin in early age.

3.3. Research on mathematical reasoning. It is difficult to find definitions of what should be included in the concept mathematical reasoning. One way to avoid the problem is to say that all mathematical activities are mathematical reasoning, and that the reasoning can be of different types and also of different quality. In Section 2.2 I discussed the way I have looked at quality in mathematical reasoning. Lithner (2000) studies another aspect of quality in mathematical
reasoning, when he discusses whether the base for the reasoning lies in the mathematics involved or in (perhaps mathematically superficial) experiences from the learning environment. He bases his clarification of the concept on Pólya’s idea of plausible reasoning. Lithner defines two kinds of reasoning where one kind rests on the mathematical properties involved, and the other on experiences from the learning environment. The first, Plausible Reasoning (abbreviated PR), is characterised as reasoning which

(i) is founded on mathematical properties of the components involved in the reasoning, and

(ii) is meant to guide towards what probably is the truth, without necessarily having to be complete or correct.

The second kind, reasoning based on Established Experiences (abbreviated EE), is reasoning which

(i) is founded on notions and procedures established on the basis of the individual’s previous experiences from the learning environment, and

(ii) is meant to guide towards what probably is the truth, without necessarily having to be complete or correct.

In his research Lithner shows that the students’ global strategy choices are guided to a large extent by EE: “to apply methods they know from similar tasks” (p. 187). Only a few situations are found where students use PR, and those situations are very local and “are all too limited to address or resolve the student’s main difficulties” (p. 187).

Other research on mathematical reasoning can be found in related areas, for example research on proof (Balacheff, 1988; Lakatos, 1976; Chazan, 1993) and research on investigations (Schoenfeld, 1996; Krainer, 1993).

3.4. The teaching gap. One part of the TIMSS study, Third International Mathematics and Science Study, was a video study which compared the mathematics teaching of eight-graders in USA, Germany and Japan. The results from the study are described in the book “The Teaching Gap” by Stiegler and Hiebert (1999). The results are in a way quite alarming and show great differences between
the three countries. One invited researcher summarised the findings like this:

“In Japanese lessons, there is the mathematics on the one hand, and the students on the other. The students engage with the mathematics, and the teacher mediates the relationship between the two. In Germany, there is the mathematics as well, but the teacher owns the mathematics and parcel it out to students as he sees fit, giving facts and explanations at just the right time. In U.S. lessons, there are the students and there is the teacher. I have trouble finding the mathematics; I just see interactions between students and teachers.”

pp. 25-26

In each country, many mathematics lessons in grade eight were videotaped. The final sample included 100 lessons in Germany, 50 lessons in Japan and 81 lessons in USA, all randomly selected. Several interesting differences were found. One was in the presentation of how much time the students spent on different kinds of tasks in their seatwork. Seatwork is here what the students are doing on their own in the classroom.

![Figure 2: Average percentage of seatwork time spent in three kinds of tasks (p. 71).](image)

30
The activities were divided into three categories: “Practising routine procedures, apply concepts or procedures in new situations, and invent something new or analyse situations in new ways” (p. 70). In Figure 2 the difference between Japan and the two other countries is very striking. To have 95.8% of the time spent on practising routine procedures as in USA, is probably not a good situation.

Another focus in the study was on the quality of the mathematical content. A group of four experienced university mathematics teachers analysed the lessons, making a subjective judgement of what they thought was relevant for student learning. The group did not know which lessons came from which country. What they found can be seen in Figure 3.

![Figure 3](image)

**Figure 3.** Percentage of lessons rated as having low, medium, and high quality of mathematical content (p. 65).

The experienced mathematicians and mathematics teachers in the group judged that “American students were at a clear disadvantage in their opportunities to learn, at least as indicated by the content of their lessons” (p. 65).

The results found in the study indicate that American students have a clear disadvantage when it comes to learn mathematics. The results of the TIMSS study of eight graders (Skolverket, 1996) show that Japan was one of the top countries while USA was below average,
something that supports the view that the teaching in USA is in great need of improvement.

The authors do not propose a new way of teaching mathematics in the United States. They claim that “improving teaching, as we have said, is a cultural change and thus must happen in small steps” (p. 167). They argue that the change will take a long time, and that all teachers must work together to improve the mathematics teaching.

“The star teachers of the twenty-first century will be teachers who work every day to improve teaching - not only their own but that of the whole profession” (p. 179).

4. **Short abstracts of my four studies**

I started my PhD-studies in 1995. I was full of expectation and after 7 years of teaching at upper secondary school I wanted to change the teaching of mathematics completely. I believed that the graphing calculator was the key to better teaching and learning. The calculator would make it possible to visualise mathematics in new ways, and the connections between different representations were the most important part of upper secondary school mathematics.

I soon found out that it wasn’t that easy.

Between 1995 and 1997 I planned and carried out several pilot studies aimed at showing the positive effects of the calculator. The results, when I got any, did not say much. At this point I asked myself “What are the students really able to do in mathematics?” To answer this question I planned a new pilot study, the study mentioned in Section 1. The results from the study indicated that the students could use the calculator to learn new mathematics and to examine their own ideas and conjectures.

Now I wanted to look more into these results. Are students really able to learn something difficult by using the graphing calculator? To learn about this I designed Study no. I, “Gymnasieelever undersöker
ett matematiskt begrepp med miniräknare” which translates to “Secondary school students examining a mathematical concept using a graphing calculator”.

The other important indication from the pilot study was that the students could use the calculator to examine ideas and conjectures. How well can they do that? This led me to Study no. II, “How students verify conjectures”.

The results from study no. II, that most students tried to verify the conjectures at a relatively high level of abstraction, surprised me a little. Would other teachers also find it surprising? What are the teachers’ view of how students work with mathematics? So, I designed an interview study, Study no. III, “How students verify conjectures: Teachers expectations”.

Now it was time to look back at my research. All three studies followed on the pilot study. When I examined the pilot study I was not so pleased with the report. However, after a few looks at the video tapes, I decided that I wanted to revise the analysis and write a new report, since I found the data very good. The result of the revision can be found as Study no. IV, “Secondary School Students Using Graphing Calculators. Revised version”.

4.1. Gymnasieelever undersöker ett matematiskt begrepp med grafräknare. In this study I used a series of questions concerning the factor theorem (sometimes referred to as a direct deduction from the remainder theorem). The theorem says that for all polynomials, $p(x)$, it holds that:

$$p(x) = (x - a) \cdot q(x) \iff p(a) = 0$$

where $q(x)$ is another polynomial. From the left to the right this means that if a polynomial $p(x)$ has a factor, $(x - a)$, then $x = a$ is a zero to the polynomial, i.e. $p(a) = 0$. From the right to the left the theorem means that if a polynomial $p(x)$ has a zero at $x = a$, then the polynomial has a factor, $(x - a)$, meaning that $p(x) = (x - a) \cdot q(x)$ The meaning of the theorem from the left to the right is

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4A shorter version of this article is published in NOMAD, (Bergqvist, 1999b)
easier to understand. The study was aiming for the students to gain understanding of the theorem from the right to the left.

In each session, two students were working together with a teacher (myself). The students were presented a series of questions about factorisation of quadratic polynomials, where the difficulty of the questions increased all the time. The difference in difficulty was both numerical and concerning level of generality. The first question was to find polynomials \(f(x)\) and \(g(x)\) if \(f(x) \cdot g(x) = x^2 + x - 6\), and the last question was to write an instruction on how you are to factorise a general quadratic polynomial. The students were told to draw all graphs on the calculator, the quadratic polynomial and the two functions \(f(x)\) and \(g(x)\) (can be seen as straight lines) in the same window. Many students reacted directly when they saw that the straight lines intersected the \(x\)-axis at the same places as the quadratic polynomial. They said things like “Look! The zeroes!” or similar statements.

The research question in the study was:

> Are students able to learn something difficult, which normally is treated in a purely algebraic way, by
> - working visually, i.e. using graphical representations of functions
> - using a graphing calculator
> - investigating the mathematical concept together with the teacher?

All students in the study could do this, in more or less advanced ways. Interesting observations in the study are that students with high grades\(^5\) in mathematics were most able to state conjectures and to use deductive reasoning. The students with medium grades\(^6\) appeared to benefit the most from the use of the calculator while the students with the lowest grades\(^7\) in mathematics only could solve the exercises, and not explain anything about the mathematics involved.

I believe that I have found some support in the study, that investigations using graphing calculators can give students a better

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\(^5\)MVG, translates to very well approved
\(^6\)VG, translates to well approved
\(^7\)G, translates to approved
understanding of factorisation and the factor theorem. Students can also get a better understanding for the connection between graphical and algebraical representations of functions.

4.2. How students verify conjectures. When I saw how the students were working with investigations, I became fascinated by how the students tried to verify their conjectures. Most of the time, they only looked at one extra example and then they were satisfied. Was this all they could do, or are students able to do more than just look at examples? I decided to take a closer look at verification, the third step in the description of investigations in mathematics (see Section 2.1). At the same time, I got in contact with an article by a French researcher, Balacheff (1988). He had constructed a hierarchy of proof levels (see Section 2.2) that might be possible to use in order to analyse how students verify conjectures. I also found it possible to use the concept of the didactical contract (see Section 2.3) in the analysis.

Five pairs of students participated. They got three conjectures to verify, and were told to say if the conjectures were true or false, and write an explanation of why they were true or false. There were two research questions in the study: How do students verify conjectures? and Can the didactical contract explain the students’ strategy approach?

The results indicated that the students tried to verify the conjectures at a high level according to the proof levels by Balacheff. Out of the fourteen classified situations, eight were at the highest level. In several situations the students showed very clearly that they wanted to use abstract or advanced mathematics. However, they seemed to need more practice.

4.3. How students verify conjectures: Teachers expectations. The results in the previous study, that the students tried to verify conjectures at such a high level, surprised me as a teacher. I wanted to know if the results would be as surprising for other teachers in Sweden. In order to answer that question, I planned an interview
study where I would ask some teachers questions connected to the results of study no. II.

Eight teachers were interviewed, six male and two female teachers. Each session was divided into three parts, the first part was to ask the teachers what they thought students might do. In the second part I presented the results from the previous study, and invited the teachers to comment the students’ performance. The third part was a more general discussion around the type of student activities that could be found in study no. II.

The research question in the study was *How do teachers expect students to work when trying to verify conjectures?* The findings were that the teachers in the study tended to underestimate the students’ reasoning levels, and that the teachers seemed to think that only a few students in a class can use high level reasoning.


**Revised version.** When I in spring 2001 looked back at the pilot study mentioned in Section 1, I found the data very intriguing, but the presentation of the study (Bergqvist, 1998) not so good. I decided to make a new analysis of the video recordings, using some of the knowledge and experiences I now had. The new article, no. IV in this dissertation, is based on the theoretical framework presented in article no. I (and in Section 2.1) concerning investigations in mathematics.

The article describes how the students worked in an investigation of a quadratic expression, $x^2 + bx + c$, mainly using the graphing calculator. The results from the study show that the students in the study used the calculator to visualise the expression as graphs and that they were able to state conjectures concerning the behaviour of the graphs. The students also used the calculator to verify their conjectures, at least for themselves. The verifications were always very superficial, in the sense that the students only checked one or two extra examples.

The implications of these results are that the graphing calculator could be an important part in the educational system. The calculator could be used to more than just calculations, it might be used as a
means to understand new concepts and to develop the teachers’ and students’ creativity.

5. Discussion

In my four studies I have found indications of some problems that seem to be related to mathematical reasoning in the mathematics education at upper secondary school in Sweden. Teachers seem to believe that many students are unable or at least very limited when it comes to mathematical reasoning. At the same time, students are interested in using fairly advanced mathematical reasoning. This discrepancy might also be self-supporting in the sense that teachers who believe that the students are unable to carry out mathematical reasoning might not want to use mathematical reasoning in exercises and discussions. I have also found that some teachers think that only a small group of students in a class can use mathematical reasoning of a more advanced type, the rest are unable to use it. This division of a class in a small group who can and a large group who cannot use mathematical reasoning of a higher quality could also add to the resistance. Teachers in Sweden are encouraged to spend a lot of time on students who are on the limit of passing the course, because it is very important for a school to have a very small percentage of student failure. If a teacher believes that the class is divided into two groups and they focus a lot on the weak students, they will probably not include high quality mathematical reasoning in their teaching.

The result that many students in the study wanted to use rather advanced or abstract mathematics differs from what most other researchers have found (Balacheff, 1988; Chazan, 1993, and others). Hoyles (1997) found that results from a number of studies indicate that most students on the one hand prefer empirical argument over deductive reasoning, and on the other hand think that proof is just evidence. However, she also discusses the problem of the great diversity of how and when proof is introduced in different countries. Other differences can also be found, like the mathematical focus in the studies. Some of the studies Hoyles refers to, look at students working with proof in geometry, where examples might be a more
common way to describe concepts than what it is in calculus. Another difference between my studies and most other studies is the age of the students. Balacheff (1988) worked with 13-year-old students and Hoyles (1997) presents a study with 15-year-old students. In my studies the students were older, 17 years of age.

I have also found indications in support of the notion that the graphing calculator has a potential as a pedagogical aid (Dick, 1996; Pomerantz, 1997). Students seem to benefit from the use of a graphing calculator as a support for their reasoning. The calculator works in two ways:

- as a tool; it is used to carry out calculations or look at graphs in ways that otherwise would become a serious obstacle for the student, and
- as a catalyst; it helps the student to analyse the problem and to spark ideas concerning possible solutions.

I found that students often use the calculator to visualise mathematics, for example looking at graphs of functions, and also to examine special features such as stationary points and zeroes. Most students in my studies seemed confident in the handling of the calculator, and some of them also expressed that they saw the calculator as a normal and important part of their mathematical environment. These findings are close to what other researchers have found (Selinger and Pratt, 1997; Movshovitz-Hadar et al., 1994; Doerr and Zangor, 2000). Smart (1995) let some students explore parabolas and reports that “The reports they [the students] handed in showed that not only had they seen patterns on their calculator screens, they were able to write coherently about the mathematics they had developed” (pp. 205-206).

Letting students in upper secondary school explore mathematical concepts or objects, using the graphing calculator in relevant situations, can be a way to increase the amount of high quality mathematical reasoning. This way it might be possible to reduce the apparent unbalance in the Swedish school, where the teaching and learning of routine skills appears to dominate the students every-day work to a
very large extent. I must point out that practising routine skills must take a lot of time in school, but it must not be the only important activity for the students.

It is not an easy task to change a long tradition of mathematics teaching, but I believe that it can be done. It is possible that a small change can give visible effects. A transfer of 10% of the time from practice of routine skills to explorative activities might be a way to start the change.
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Tomas Bergqvist
Gymnasieelever undersöker ett matematiskt begrepp med miniräknare

Tomas Bergqvist

SAMMANFATTNING. Artikeln beskriver ett försök där gymnasieelever får undersöka faktorisering av andragradspolynom med hjälp av grafiska representationer av funktioner. Eleverna leds in i ett för dem nytt arbetssätt, där de tillsammans med en lärare får arbeta med ett antal uppgifter med hjälp av en grafräknare. Resultaten visar att eleverna kommer med egna hypoteser och använder grafräknaren på egen initiativ i vissa situationer. Resultaten visar också att eleverna i försöket i viss mån kunde använda grafräkna-ren i ett undersökande arbetssätt.

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1. Inledning

Är grafritande miniräknare en del av matematikundervisningen i gymnasieskolan för sin egen skull eller finns det områden i matematiken som grafräknaren kan ge eleverna ökad möjlighet att arbeta med? Finns det arbetsmetoder som grafräknaren möjliggör? Förändrar grafräknaren elevens inlärningsmiljö? Kan det vara så att grafräknaren finns i skolan för att det ligger i tiden att utnyttja tekniska hjälpmedel? Har de ‘teknikfrälsta’ tagit kommandot utan att bli ifrågasatta?

I Sverige har de flesta eleverna inom de matematikintensiva programmen tillgång till en grafritande miniräknare; en grafräknare. Detta kraftfulla verktyg har införts som hjälpmedel i matematikundervisningen eftersom många matematiklärare och matematikdidaktiker tror att det kan underlätta elevers matematikinlärning i stor utsträckning. Den tekniska utvecklingen och ekonomiska intressen har också bidragit till att grafräknarna nu är en självklar del av gymnasieskolans matematikundervisning.

Det är mycket svårt att avgöra om elever presterar bättre om de får använda en grafräknare än om de arbetar utan detta hjälpmedel. Svårigheterna grundar sig framför allt på att det finns så många andra faktorer som också påverkas om grafräknare införs i undervisningen. Denna studie kommer därför inte att jämföra elever som arbetar med respektive utan grafräknare utan istället beskriva vad som händelserna när grafräknaren används i gymnasieämnets innehåll.

För att det matematiska innehållet ska vara relativt nytt för eleverna och lämpa sig för en liten undersökning, har en liten del av gymnasieämnets innehåll, nämligen faktorsatsen, valts ut. Anledningen till detta är att faktorsatsen kan beskrivas i flera olika representationsformer där grafiska och algebraiska representationer är de viktigaste men där även numeriska representationer kan användas. En intressant fråga är också om elever måste ha ett fungerande funktionsbegrepp för att kunna tillgodogöra sig faktorsatsen. I en lärobok för gymnasiet (Björk och Brolin, 1996a) formuleras Faktorsatsen så här:

\[ Ett \text{ polynom } f(x) \text{ har faktorn } x - a \text{ om och endast om } f(a) = 0. \]

De flesta elever brukar inse att vänstra ledet medför det högra ledet, men ha betydligt svårare med omvändningen. Detta projekt går ut på att ta fram en undervisningssituation där elever får arbeta med den svårare delen av faktorsatsen. Undervisningsmodellen bygger på att studenterna använder ett undersökningsarbetsätt med visualiseringar av funktioner med hjälp av grafräknare. Ett mål är att eleverna ska kunna formulera faktorsatsen för andragradspolynom. Det kan tänkas att vissa elever också kan generalisera resonemanget till den allmänna faktorsatsen. Eleverna skall alltså genom att arbeta självständigt under viss handledning komma fram till något som liknar detta:

\[ Ett \text{ andragradspolynom kan skrivas som produkten av två förstegrads-}\]
\[ \text{polynom om andragradspolynomet har två nollställen.} \]
\[ \text{Om andragradspolynomet } p(x) \text{ har två nollställen, a och b, så kan } p(x) \text{ skrivas så här: } p(x) = (x-a)(x-b). \]

Detta är ingen fullständig beskrivning av faktorsatsen eller ens av vad faktorsatsen innebär för andragradspolynom, men om eleverna har kommit fram till detta till stor del på egen hand kommer den delaktigheten i arbetet att ge en större förståelse än om faktorsatsen presenteras på det sätt som beskrivs i gymnasiet läroböcker i matematik.
Ett mål är också att eleverna ska utveckla en viss förtrogenhet med ett undersökande arbetsätt. Se kapitel 2.2 för en noggrannare diskussion om begreppet undersökande arbetsätt.

2. Teori


One might expect that, after so much enthusiastic rhetoric and so many studies specifically intended to explore the effectiveness and limitations of the graphics calculators as a tool for teaching and learning of mathematics, we might enter the second decade of their use well-prepared.

...Sadly, the answers offered by research to these questions at the end of this first decade remain elusive and conflicting. (p.59)


Denna artikel syftar till just detta; att belysa teknikens möjligheter och svårigheter, att ge exempel på en annorlunda undervisningssituation och att ta fram en systematisk analys av elevernas och lärarens arbete.

2.1. Frågeställningar. Den övergripande didaktiska frågeställningen i detta projekt är

Kan elever lära sig någonting svårt som normalt behandlas rent algebraiskt genom att arbeta
• visuellt, dvs. använda grafiska representationer av funktioner
• med en grafräknare, dvs. utnyttja modern teknik
• med ett undersökande arbetsätt, dvs. tillsammans med läraren leta sig fram till den kunskap och förståelse som krävs

Projektets inriktning ger också upphov till ett antal mer specialiserade frågor. Dessa behandlar bland annat elevers inlärning, elevers arbete med grafräknaren och det arbetsätt som eleverna tillsammans med läraren använder:

I vilka situationer använder eleverna grafräknaren, och på vad grundar de beslutet att använda den?
Hur använder eleverna grafräknaren?
Till vilka arbetsuppgifter?
Kan det arbetssätt som beskrivits ovan ge eleverna en bild av vad faktorsatsen innebär och på så sätt underlätta förståelsen av den algebraiska formuleringen?
Vilka svårigheter stöter eleverna på när de arbetar på detta sätt vad gäller arbetssätt respektive matematiskt innehåll?
Vilka nya svårigheter (som inte finns i traditionell undervisning) ställs läraren/handledaren inför?
Förekommer deduktiva resonemang i elevernas arbete?
I vilken utsträckning kan eleverna utföra det undersökande arbetssättet?

Givetvis kommer inte alla dessa frågor att besvaras i denna artikel, men förhoppningsvis kommer flera av frågorna att belysas i någon utsträckning. Beskrivningen av hur eleverna använder grafräknaren och vad som ligger bakom deras sätt att använda den är en central del i analysen av försöket. Att försöka förstå om och i så fall hur grafräknaren hjälper eleverna i arbetet är också mycket viktigt.
Lärarens/handledarens svårigheter och vad som måste beaktas i försöks situationen beskrivs noggrannare i kapitel 3.3.


Han delar in processen i sex steg eller episoder, kort sammanfattat ser beskrivningen ur så här:

- **Reading:** Innefattar att läsa uppgiften högt, begrunda tyst, upprepa vissa delar, läsa tyst med mera.
- **Analysis:** Består av analys av problemet, att försöka förstå problemet fullständigt, att formulera om problemet och att introducera principer och metoder som kan passa.
- **Exploration:** Ett relativt ostrukturerat utforskande av området omkring problemet, ofta en bit ifrån själva frågeställningen.
- **Planning:** Planering av lösningsarbetet. Kan ofta vara svårt att skilja från själva lösningsarbetet.
- **Implementation:** Utförandet av lösningen.
- **Verification:** Verifikation och kontroll av lösningen, kan också innehålla en diskussion om lösningen är realistisk.

Problemlösning skiljer sig på flera sätt från ett undersökande arbetssätt, men man kan ändå se flera paralleller. Den typ av undersökande arbetssätt som behandlas i denna artikel kan delas upp i tre huvuddelar:

**Visualisering:** Att låta de ingående objektten beskrivas i konkret form när man vill observera olika möjliga samband. En funktion kan beskrivas med ett algebraiskt uttryck, en tabell eller en graf. Visualiseringen är här det tredje alternativet, en graf. I detta försök sker visualiseringarna oftast med hjälp av grafräk-naren, genom att eleven ritar en eller flera funktioner. Analys- och utforskande episoderna i Schoenfeldts beskrivning sker här via visualisering. Visualiseringar an-vänds för att ge en representation av objektet, en


2.3. Faktorsatsen.

2.3.1. Matematisk beskrivning av faktorsatsen. För alla polynom $p(x)$ gäller att:

\[ p(x) = (x - a) \cdot q(x) \iff p(a) = 0 \]
Denna ekvivalens kan läsas från båda håll. Från vänster till höger betyder det att om \( p(x) \) har en faktor \( x - a \), så är \( p(a) = 0 \). Från höger till vänster innebär satsen att om \( a \) är ett nollställe till polynomet \( p(x) \) så kan \( p(x) \) skrivas som produkten av förstgradspolynomet \( x - a \) och ett annat polynom \( q(x) \). \( q(x) \) har då ett gradtal som är 1 mindre än gradtalet för \( p(x) \).

**Figure 1.** \( y = 4x^3 + 7x^2 - 4x - 4 \)

I figur 1 ser vi ett tredjegradspolynom, \( 4x^3 + 7x^2 - 4x - 4 \) som har ett nollställe \(-2\). Då kan alltså \( p(x) \) skrivas som \((x - (-2)) \cdot q(x)\). I detta fall blir \( q(x) = 4x^2 - x - 2 \). I figur 2 ser vi då \( q(x) \) och \( x + 2 \) i samma bild. Dessa två funktioner har tillsammans samma nollställen som det ursprungliga tredjegradspolynomet.

**Figure 2.** \( y = x + 2 \) och \( y = 4x^2 - x - 2 \)

Elever i gymnasiet verkar ha relativt lätt att acceptera att om \( p(x) \) kan faktoriseras i \((x + 2) \cdot q(x)\) så är \(-2\) ett nollställe till \( p(x) \), det vill säga faktorsatsen från vänster till höger. Många elever verkar däremot ha svårt med omvändningen, att om \(-2\) är ett nollställe till \( p(x) \) så måste \( x + 2 \) vara en faktor i \( p(x) \). Detta försök går ut på att låta elever arbeta med denna den svårare delen av faktorsatsen, dock på en betydligt elementärrare nivå. Tanken är att eleverna ska skriva ett andragradspolynom på faktorerad form genom att använda de nollställen som andragradspolynomet har.

### Faktorisering av typen \( x^2 - 5x + 6 = (x - 2)(x - 3) \)
Vi löser först ekvationen \( x^2 + 2x - 15 = 0 \)
\[ x = -1 \pm \sqrt{1 + 15} = -1 \pm 4 \]
Rötterna är \( x = -5 \) och \( x = 3 \).
Ekvationen \( (x + 5)(x - 3) = 0 \) har samma rötter som ekvationen \( x^2 + 2x - 15 = 0 \). Är de båda ekvationerna identiska?
Vi multiplicerar samman \( (x + 5)(x - 3) \).
\[ (x + 5)(x - 3) = x^2 - 3x + 5x - 15 = x^2 + 2x - 15 \]
Produkten är lika med vänstra ledet i första ekvationen. Ekvationerna är tydligt identiska.
Vi har fått en metod att faktorisera ett andragradspolynom.

Efter denna inledning följer en blåfärgad metadruta:

<table>
<thead>
<tr>
<th>Faktorisering</th>
</tr>
</thead>
<tbody>
<tr>
<td>Polynomet ( p(x) = x^2 + 2x - 15 ) ska delas upp i faktorer.</td>
</tr>
<tr>
<td>1. Lös ekvationen ( x^2 + 2x - 15 = 0 ). Rötterna är (-5) och (3).</td>
</tr>
<tr>
<td>2. ( p(x) = x^2 + 2x - 15 = (x + 5)(x - 3) ). [Observera: ombytt tecken!]</td>
</tr>
</tbody>
</table>


I läroboken för E-kursen (Björk och Brolin 1996b) tas faktorsatsen upp på ett mer formellt sätt tillsammans med restsatsen. Man jämför polynomdivision med heltalsdivision och visar på det sättet att om en division av ett andragradspolynom med ett förstagradspolynom inte ger någon rest så måste förstagradspolynomet vara en faktor i andragradspolynomet. Faktorsatsen formuleras så här:

Ett polynom \( f(x) \) har faktorn \( x - a \) om och endast om \( f(a) = 0 \).

Erfarenheten säger att många elever verkar ha svårt att förstå och tillämpa regeln från höger till vänster, det vill säga att om ett polynom har ett nollställe \( x = a \) så kan det faktoriseras med en faktor \( x - a \).

I en annan läroboksseries E-kursbok (Björup et al., 1996) formuleras och bevisas faktorsatsen på en halv sida, i samband med arbetsområdet komplexa tal. Ytterligare en annan lärobok (Jacobsson et al., 1996) tar inte upp faktorsatsen i E-kursboken annat än att en tillämpning av den visas i samband med lösning av tredjegradsekvationer.
Inte i någon av de analyserade läroböckerna används någon grafisk representation av polynom i samband med diskussioner runt faktorsatsen. Tolkningen av detta är att de flesta elever endast kommer i kontakt med faktorsatsen i dess algebraiska form.

Eftersom faktorsatsen inte är central i gymnasiets matematikkurser så finns det flera läroböcker som inte berör den alls.

3. Metod

Eleverna arbetade i försöket två och två. På så sätt blev diskussionen runt matematiken i uppgifterna tydligare eftersom eleverna skulle hjälpas åt. De uppmanades också att diskutera och förklara för varandra sina tankar omkring uppgifterna.

Eleverna fick tillsammans med läraren lösa ett antal uppgifter som gradvis blev svårare och svårare för att avslutats med generaliseringar omkring faktorsatsen. En mer detaljerad beskrivning av uppgifterna och tanken med elevernas arbete med dessa kommer i avsnitt 3.2. I försöket fungerade jag själv som lärare.

Uppgiften som lärare var att komma med lämpliga ledtrådar för att arbetet inte skulle stanna upp eller gå för långsamt. Det är en svår uppgift att avgöra vad och hur mycket som ska sägas till eleverna. Därfor sattes en lista på användbara ledtrådar upp i förväg, ledtrådar som gradvis ger mer och mer hjälp till eleverna (se avsnitt 3.3).


Försöket kompletterades med en elevintervju i direkt anslutning till varje försök. Inför dessa intervjuer förbereddes ett antal intervjupunkter av vilka vissa var direkt kopplade till de överordnade frågor som jag hade inför detta delprojekt (se vidare avsnitt 3.4).

3.1. Försöket. Eleven och läraren skall tillsammans utföra det undersökande arbetsättet. Läraren ska leda eleven på rätt spår genom att ta fram lämpliga visualiseringar (om inte eleven själv föreslår sådana), se kapitel 3.2 som beskriver de uppgifter som eleverna skall arbeta med. Eleven ska förhoppningsvis vara den som ställer hypoteserna och utför kontrollen. Detta kan vara svårt i vissa situationer och för att kunna kontrollera hur mycket läraren leder eleverna finns i kapitel 3.3 en i förväg definierad beskrivning av hur läraren skall handleda eleverna.
Eleverna har inte full frihet att själva utföra det undersökande arbetssättet. De får till exempel inte själva välja vilka andragradspolynom de skall arbeta med. Detta är nödvändigt av följande orsaker:

- delar av det matematiska innehållet riskerar att gå förlorat, till exempel kan fallet med dubbelrot bli helt förbigående.
- om eleverna själva skall ange andragradspolynomen är det sannolikt så att de väljer andragradspolynom med förhållandevis komplicerade nollställen i stället för heltalsnollställen.
- en stegvis ökad svårighetsgrad underlättar betydligt för eleverna eftersom de då inte behöver formulera hypoteser som innehåller flera komponenter, utan kan utöka hypoteserna lite i taget.


### 3.2. Uppgiften


1. Om $f(x)$ och $g(x)$ är förstagradspolynom, hur kan då $f(x)$ och $g(x)$ se ut om $f(x) \cdot g(x) = x^2 + 3x - 4$?
2. Kan ni lösa uppgiften om $f(x) \cdot g(x) = x^2 - 3x - 4$ i stället?
3. Vad händer om $f(x) \cdot g(x) = x^2 - 4x + 4$?
4. Vad händer om $f(x) \cdot g(x) = x^2 - 2x + 3$?
5. Vad händer om $f(x) \cdot g(x) = x^2 - 4x + 1$?
6. Gå tillbaks till uppgift 1. Kan $g(x)$ och $f(x)$ vara något annat än det ni fick fram i ert svar?
7. Skriv en instruktion som förklarar hur man ska göra för att faktorisera ett allmänt andragradspolynom i två förstagradspolynom.

Tanken med att dela upp uppgifterna på detta sätt är att eleverna ska arbeta med uppgifter med stigande svårighetsgrad, för att sen kunna generalisera arbetsuppgifterna steg för steg. I uppgift 1 är uppgiften relativt rakt på sak, även om den kanske innehåller en av de största svårigheterna i och med att eleverna måste ta första steget till att använda faktorsatsen. För att underlätta denna initiala svårighet fick eleverna börja med att utföra multiplikationen $(x + 3)(x - 4)$.

Normalt brukar eleverna lösa uppgift 1 genom att prova sig fram och genom att använda uteslutningsmetoden. De får också veta att de kan koncentrera sig på förstagradspolynom med $x$-koefficient 1. Detta innebär att resonemanget förenklas utan att det matematiska

I uppgift 6 ber jag eleverna gå tillbaka till uppgift 1 för att diskutera lösningens entydighet. Denna uppgift är svår och kräver någon form av deduktivt resonemang för att kunna lösa på ett relativt allmängiltigt sätt. I uppgift 7 slutligen ska eleverna lösa uppgiften utan faktiska funktioner. Om eleverna har sett sambanden mellan parabliernas och linjernas nollställen bör de ha en god möjlighet att kunna lösa även denna uppgift.

3.3. Handledning. I handledningssituationen uppstår ett klassiskt problem eftersom försöksledaren är både lärare och forskare. Problematiken behöver inte innebära en alltför stor svårighet eftersom målet med studien är att undersöka och beskriva en undervisningssituation, inte att mäta elevernas prestationer. Handleningen av eleverna kan ge upphov till svårigheter eftersom försöksledaren (F) måste tänka på ett flertal aspekter under hela försöket:

- Balansen mellan att göra för lite och att göra för mycket är svår. Om F är för aktiv blir det han som utför arbetet och är F för passiv kan eleverna fastna i ett tidigt skede av försöket.
- F ska i stor utsträckning ifrågasätta elevernas påståenden, både när de är felaktiga och när de är korrekta. På så sätt kan diskussionen mellan eleverna bli tydligare och underlätta beskrivningen av situationen.
- F måste använda ett korrekt matematiskt språk i samtala med eleverna samtidigt som han måste avdramatisera situationen med lämpligt ”småprat”.

Viss del av handledningen kan bestå av ledträd som har formulerats i förväg. Några av dessa är allmänna idéer medan andra är kopplade till speciella deluppgifter. Här följer en lista på tänkbara handledning:

- Gissa och prova er fram.
- Rita upp funktionerna.
- Vad skiljer från föregående uppgift?
- Rita \( f(x), g(x) \) och andragradspolynomet i samma fönster.
- Det kan vara praktiskt att använda grafräknaren.
- \( f(x) = x - 2 \) (uppgift 1)
- Kan \( f(x) = x + 5 \) användas? (uppg. 2)
• Kan ni säga något om funktionernas nollställen?
• Hur menar du då?
• Kan du beskriva tydligare hur du menar?
• Kan du visa?

3.4. Intervjuer. I direkt anslutning till försöket intervjuades eleverna om försöket. Intervjun omfattade frågor inom följande områden:

- Situationen, med frågor om
  - stress.
  - det var en ovan situation.
  - den här typen av arbete är roligt.
  - grafräknaren (handhavande, attityd, nytta).
- Uppgiften, med frågor om
  - det var en ny typ av uppgifter.
  - det var svår.
  - svårighetsgraden var stegrad.
- Matematiken, det vill säga faktorsatsen, med frågor om
  - vad de lärde sig.
  - svaren de kom fram till.
  - de tror att svaren gäller för andra funktioner än andragradspolynom.

Intervjun skedde förstas i form av ett samtal vilket ledde till att även andra frågor än de som tagits upp här diskuterades.

4. Empirisk undersökning


4.1. Klara och Karro. Klara och Karro har båda två goda studieresultat i matematik, betyg omkring VG. När försöket börjar har de svårt att förstå vad det handlar om, vilket leder till att försöksledaren (F) ger dem den inledande uppgiften, att utföra multiplikationen \((x + 3)(x - 4)\). När de har gjort det går det lättare att förstå vad uppgift 1 innebär, och de löser den snabbt. De ritar graferna utan annan kommentar än att linjerna är parallella. Uppgift 2 går också bra och de ritar upp graferna på min anmodan. Följande samtal utspelar sig:

F: År det någon skillnad från första uppgiften?
Karro: Dom är ju förskjutna åt höger, men är fortfarande parallella och skär på samma ställe på x-axeln. Det gjorde dom förra också.
F: Skär på samma ställe på x-axeln, varför gör dom det?
tyst
Klara: Ingen aning faktiskt.
tyst
Karro: Samma nollställen. Om man sätter en andragradsekvation till noll så får man två nollställen. Om man sätter förstgradspolynomen till noll så blir det samma. Det är det enda sättet jag kan se. Stämmer det?
F: Ja det stämmer. Andragradspolynomet \(x^2 - 3x - 4\) och faktoriseringen \((x - 4)(x + 1)\) är ju identiska funktioner som måste ha samma nollställen.
Karro: Då behöver man ju inte gå den här vägen för att hitta förstgradspolynomen utan man kan utgå från andragradsekvationen.

Under fortsättningen av försöket använder de räknaren för att ta fram nollställen till andragradspolynomen och sedan använda dessa för att ta fram faktoriseringen.

I uppgift 3 (fallet med dubbelrot) blir eleverna lite tveksamma, men inser snart att det blir ett kvaderat förstgradspolynom. Uppgift 4 ställer inte heller till några stora problem i och med att Karro ser att ”det blir negativt under rotecknet” om man försöker lösa andragradsekvationen. Hon säger också att ”det finns inga reella rötter” och ”då kan vi inte lösa uppgiften”.

När de i uppgift 5 får fram att ena nollstället är \( x = 0,2679... \) försöker de först att med hjälp av räknaren omvandla det till ett bråk. När detta misslyckas vill de använda närmevärdelen. F påpekar att då får de inte rätt svar. F säger också att det uppenbarligen finns exakta nollställen. De kommer då fram till att de måste lösa andragradsekvationen och ”använda roten ur”. De skriver ner en korrekt faktorisering av polynomet efter viss diskussion om tecknen. Den riktiga formuleringen motiverar de med att produkten av konstanterna ska vara positiv.

På uppgift 6 svarar de snabbt att ”det går inte om man ser på nollställena”. De menar att man måste använda de nollställen som finns. F påpekar då att de har antagit att koefficienten framför \( x \) är 1 men att det går att välja andra koefficienter och då svarar Karro ”så kan man göra men det blir mycket krångligare”.

Som svar på uppgift 7 tar de fram en tydlig beskrivning. Elevernas problem var behandlingen av tecknen. De ville inte riktigt använda uttrycket ”byt tecken på nollställena” och fick då istället hjälp av mig med att formulera den instruktion som följer här:

1. Ta reda på nollställena (Med hjälp av miniräknare el genom att lösa andragradsekvationen)
2. Fall 1: Två nollställen \( a \) och \( b \)
   Då kan \( p(x) \) skrivas \( (x - a)(x - b) \)
   Fall 2: Ett nollställe (dubbelrot) \( a \)
   Då kan \( p(x) \) skrivas \( (x - a)^2 \)
3. Inga nollställen.
   \( p(x) \) kan inte faktoriseras med reella tal

4.2. Kristina och Eldina. Kristina och Eldina är två elever med mycket goda studieresultat, betyg MVG. De har inga som helst svårigheter att lista ut hur faktoriseringarna av andragradspolynomen ska se ut. De reagerar inte nämnvärt när F ber dem rita graferna till funktionerna. F fortsätter då och ger dem andragradspolynomin ur extrauppgifterna. De har inga problem med dessa förrän F ger dem uppgift A9. De ser ingen enkel lösning och F upptäcker då plötsligt att detta polynom inte har de relativt enkla rötter som hade räknats ut i förväg. F går då direkt vidare till A10, \( f(x) \cdot g(x) = x^2 - \frac{1}{3}x - \frac{2}{3} \). De inser att produkten av konstanterna i \( f(x) \) och \( g(x) \) ska vara \(-\frac{2}{3}\) och provar med \(-2\) och \(\frac{1}{3}\) utan framgång. De testar även att förlänga hela uttrycket och sedan bryta ut en faktor utan att komma närmare en korrekt faktorisering. När eleverna har varit tysta en längre stund (ca 30 sekunder) kommer följande samtal:

   F: Blev det besvärligt?
   Kristina: Ja, faktiskt.
   F: Mmm. Vad kan man göra då?
Eldina: Om vi ... man kollar på miniräknaren.
Kristina: Ja, gör det. Sätt in den där [pekar på $x^2 - \frac{1}{3}x - \frac{2}{3}$].
Eldina ritar funktionen på grafräknaren.
Kristina: Men vänta, där den skär $x$-axeln, där $y$ är noll ...
Eldina: Ja, annars kan vi lösa ut ...andragradsekvationen.
Kristina: Ja, då får vi ju samma. Vi får ju svaren direkt om vi gör med räknaren.
F: Hur menar du då?
Kristina: Om vi kollar den skär $x$-axeln så får vi fram dom möjliga $x$-en vi kan ha.
F: Varför då?
Kristina: Det känns rätt.
Nu löser Eldina andragradsekvationen och får fram rötterna 1 och $-\frac{4}{6}$. Efter en kort tvekan sätter de in rötterna i $(x - 1)(x + \frac{4}{6})$ och kontrollerar genom att utföra multiplikationen. Efter detta börjar F fråga ut eleverna om varför denna metod fungerar. De har mycket svårt att svara på det. De verkar ha svårt att överhuvudtaget förstå vad F frågar efter, de har ju visat att metoden fungerar. Till slut kommer följande samtal:
F: Varför är det just rötterna till andragradsekvationen man ska sätta in?
Eldina: För då får man fram funktionerna.
Kristina: Men om vi kollar på räknaren, hur gick dom hän linjerna?
Kristina ritar funktionerna på grafräknaren.
Kristina: Då gick dom ju... precis där den skär $x$-axeln skär ju också linjerna. När det här uttrycket blir noll [andragradspolynomet] ... När $y$ blir noll här... Ja, jag vet inte ...
Mer diskussion
Kristina: När den [andragradskurvan] skär $x$-axeln har vi sagt att den ena delen av den här[faktoriseringen] har blivit noll. Antingen $f(x)$ eller $g(x)$ har blivit noll. Nu har eleverna kommit fram till en relativt korrekt förklaring till varför metoden fungerar. De får nu använda denna metod till att lösa uppgift A8. De ritar funktionen $[x^2 + 2x - 15]$, hittar nollställena och faktoriserar polynomet på kort tid. För att se om eleverna har kontroll över hur förstagradspolynomen ska tecknas ger F dem ytterligare en uppgift, $f(x) \cdot g(x) = x^2 + 5x + 6$. Med hjälp av räknaren hittar de nollställena $-2$ och $-3$ och skriver direkt ner den felaktiga faktoriseringen $(x - 2)(x - 3)$. Då säger Eldina:
Just det. Och liksom, på kurvan, om det är $+3$ så måste det vara $x - 3$ för att $x$ ska vara $3$ så att det ska bli $0$. Om vi får på kurvan att $x$ ska vara $3$,
då måste det bli $-3$ här för att $x$ ska vara $3$ och det här ska bli $0$.
Kristina accepterar förklaringen. Hon ändrar till $(x + 2)(x + 3)$ och de två kan nu gå vidare till uppgift 3, 4 och 5.
Uppgift 6 sätter igång en ganska livlig verksamhet där eleverna försöker hitta alternativa lösningar till uppgiften. När de misslyckas kan de inte förklara varför, men de kommer
gång på gång tillbaka till att det bara finns två nollställen till andragradsfunktionen och därmed bara två möjliga $x$-värden.

Sista uppgiften går ganska bra förutom att de återigen får problem med hur de ska behandla tecknen. Efter en ganska lång diskussion kommer följande samtal:

**F:** Om nollställena är $x = a$ och $x = b$, då är $f(x) =

**Kristina:** $f(x)$ . . . typ $(x + a)$ . . .

**Eldina:** $(x - a)$. Motsatt tecken.

**Kristina:** Joo, jaa, så kan man säga. Och . . . [skriver $g(x) = x - b$]

**Eldina:** Motsatt tecken, eftersom vi vill att funktionen ska bli noll.

Arbetet resulterar i följande instruktion:

$$p(x) = g(x) \cdot h(x)$$

Hur kan vi skriva andragradspolynomet $p(x)$ i två förstgradspolynom $g(x)$ och $f(x)$?

På miniräknaren:

1. Rita kurvan
2. Ta reda på nollställena (dvs $y = 0$)
3. Om nollställena är $x_1 = a$ och $x_2 = b$
   - då är $f(x) = x - a$
   - och $g(x) = x - b$

Om inte kurvan skär $x$-axeln så finns det inga nollställen och alltså inga lösningar.

4.3. Jenny och Nazanin. Jenny och Nazanin har båda två goda studieresultat i matematik, betyg omkring VG. Första uppgiften innebär en del besvär för eleverna. De kan dock resonera sig fram till att det måste vara $x$ i båda parenteserna. De diskuterar några olika varianter och kommer så fram till en korrekt faktorisering. **F** ber dem rita upp graferna och när graferna kommer fram säger de:


**Janny:** Jag vet inte . . .

**Nazanin:** Jag tycker att det var nåt sånt.

**Jenny:** Det är det säkert.

De fortsätter raskt till uppgift 2 och hittar snart en korrekt faktorisering. När Nazanin ritar graferna säger hon ”hoppas att jag ritade rätt”, och när graferna syns säger hon ”Ja!”

Nästa uppgift (A1) har lösningen $(x+1)(x-6)$. När Nazanin ska rita de tre funktionerna skriver hon fel och ritar $y = x^2 - 7x - 7$ i stället för $y = x^2 - 6x - 7$. När graferna kommer fram utspelar sig följande samtal:

**Nazanin:** Nej, nu har jag skrivit fel för annars skulle den där inte vara där. Den ska väl skära i $x$-axeln. [ändrar] Så!

**F:** Hur visste du att du hade gjort fel?

**Nazanin:** Den skärde inte i nollpunkterna. Strecket gick inte på . . .

**F:** Varför ska den göra det?
Nazanin: För att det är väl... När man faktoriserar, då får man väl ut vilka punkter den skär x-axeln, då y är 0.
F: Varför får man det?
F: Men varför får du det i dom där förstawgradspolynomen?
Torr jag. Att det är -1 och 7 [pekar på \((x - 7)(x + 1)\)]
F: Jaha, just det... Varför då?
Nazanin: [skratt] Jenny, hjälp mej!
Nazanin: Det lärde vi oss när vi gick B-kursen.


När F påpekar att betydelsen av att man använder nollställena är att man använder de punkter där andragradspolynomet är 0 säger de:

Nazanin: Andragradspolynomet är noll. Då är \((x + 2)(x - 3) = 0\). Blev vi smartare av det?
Jenny: Då är dom här också lika med noll

\[
0 = (x + 2) \mid (x - 3) = 0
\]
Nazanin: Nej, så kan man nog inte göra.
Jenny: Det är ju så vi gör med andragradarna, vi faktoriserar.
Nazanin: Ja! Nu förstår jag varför det blir ombytta tecken! För att 3 - 3 är 0 [pekar på \((x - 3)\)] och...
Jenny: -2 + 2 är noll [pekar på \((x + 2)\)]

Nu är det dags för uppgifterna 3,4 och 5. Uppgift 3 tar lite tid innan de inser dels att parabeln verkligen har ett nollställe och dels att det blir två likadana parenteser. I uppgift 4, där andragradsfunktionen saknar nollställen, är Nazanin mycket säker på att uppgiften inte går att lösa men har svårt att formulera en motivering. De verkade dock förstå precis vad F menade och förklaringen att ett förstgradspolynom utan nollställe inte existerar. Uppgift 5 som har irrationella nollställen gör att eleverna blir lite osäkra. De kommer fram till att de måste lösa andragradsekvationen men har svårt att sätta upp förstgradsekvationerna. De gör ett teckenfel och föreslår lösningen \((x - (2 - \sqrt{3}))(x + (2 + \sqrt{3}))\). När F frågar vad som händer med att den senare parentesen när man sätter in nollstället ändrar de genast tecknet.

Som svar på uppgift 6 föreslår de bland annat "omvända tecken", vilket kan tolkas som en annan lösning beroende på vad man menar. Faktoriseringen \((-x + 2)(-x - 3)\) är korrekt och förstawgradspolynomen skiljer sig från den lösning de fick när de löste uppgift 1, faktoriseringen \((x - 2)(x + 3)\). Dock kan uttrycket "omvända tecken" syfta till den felaktiga faktoriseringen \((x + 2)(x - 3)\). Ett annat förslag som kommer fram är "ha flera..."
x som sedan försvinner" vilket också kan vara korrekt om det utvecklas. Faktoriseringen 
$(3x-6)(\frac{1}{2}x+1)$ är också korrekt, men det är tveksamt om det var en sådan som eleverna 
hade i tankarna när de kom med förslaget.

Uppgift 7 klarar eleverna av mycket snabbt utan att fråga om någonting och under 
endast sporadisk diskussion. Här följer deras instruktion:

### Instruktionen

- Rita upp 2gradspolynomen på miniräknaren och finn nollpunkterna för x. Dessa är (med omvända tecken) konstanterna i paranteserna.
- Undantag Om talet ej är exakt, lösa som andragradsekv.
- Om den ej skär (rör) x-axeln finns ej lösning.
- Dubbelrot= en lösning.

Instruktionen innehåller också en skiss som förklarar fallet med en dubbelrot. Skissen visar grafen till $x^2 - 6x + 9 = (x - 3)^2$ som tangerar x-axeln.

### 4.4. Haris och Jens.

Haris och Jens har båda två goda studieresultat i matematik, betyg omkring VG. Den första uppgiften (efter uppmjukningsuppgiften) löser Haris direkt. När jag ber dem rita upp graferna på räknaren ritar Jens $Y_1 = x + 3$, $Y_2 = x - 2$ och $Y_3 = Y_1 \cdot Y_2$. De kör sen fast lite på uppgift 2 eftersom de inte ser att -4 kan faktoriseras i $(-4) \cdot 1$.

De försätter på samma sätt och löser snabbt uppgifterna A1, A2 och A7 utan problem. De ritade också graferna utan kommentarer. Uppgift A10 tar lite längre tid eftersom det inte är så lätt att se faktoriseringarna när polynomet har rationella koefficienter. När eleverna får uppgift A11 tar det stopp. Följande samtal äger rum:

**F:** Tog det stopp?
**Jens:** Mmm. Stopp och full back.
**F:** Vad kan man göra då?
**Jens:** Ja... [tyst]
**F:** Skulle man kunna använda räknaren? Ha nyttå av den på nåt sätt?
**Jens:** Inte vad jag vet.
[tyst]
**Jens:** Men vännta... Jag ska se... [Ritar andragradspolynomet på räknaren] Sen kollar man på vilka ställen... där den skär x-axeln...

De avläser skärningspunkterna på räknaren, $x = 3$ och $x = 1,5$, och skriver upp en faktorisering med: $(x + \frac{1}{2})(x - 3)$. Denna faktorisering innehåller dels fel tecken och dels $\frac{1}{2}$ i stället för $\frac{3}{2}$. Haris upptäcker att Jens har skrivit fel tal och de inser tillsammant att de måste byta tecken i båda parenteserna. När jag ber om en förklaring på vad de har gjort säger de:

**Jens:** Hur vi använde miniräknaren? Kollade var den skärde x-punktarna för då får vi reda på vilka två...
**F:** Varför får man det?
**Haris:** Ja... Varför får man det...
**Jens:** Jamen... Då får man ju reda på vilka x-värden som gäller för y.
**F:** För vaddå? För y?
Jens: För den här ekvationen, \( [x^2 + \frac{3}{2}x - \frac{9}{2}] \) vilka x-värden då \( y \) är 0.

F: Jaa... Varför ska dom x-värdena stå här? [Pekar på parenteserna]

Haris: Men...

Jens: Därför att...

Haris: Man tar den typ så här [skriver \( y = 0 \) under \((x + 3)\)] och sen om man ska skriva 0 = x+ den som du vet inte, och sen \( x = \) siffran du har här.

F: Varför ska den vara 0 då?

Jens: Man sätter den här \([(x - \frac{3}{2})]\) som 0 sen också. Då får du ju två ställen, det här är ju liksom en faktor, det är ju som \( x \cdot x \), om du sätter den här [pekar på vänstra x-et] till 0 så får du ut den andra, värdet här [pekar på högra parentesen i faktoriseringen], och sen sätter man den andra till 0. Det hända ju på två ställen, när den skär... y-axeln...


Nu är det dags att eleverna ska få faktorisera med hjälp av den metod som de har kommit fram till. I uppgift A5 hittar de två nollställen, −3 och 2. De skriver ner en faktorisering, \((x - 3)(x + 2)\) men upptäcker att den är felaktig när de utför multiplikationen. De byter tecken och får en riktig faktorisering. Jag frågar ännu en gång om varför man ska byta tecken och nu säger Haris:

Detta är kanske dumt men... om vi har \( f(x) = x - 2 \) och skriver istället \( f(x) = 0 \), det ska bli \( 0 = x - 2 \) och sen om vi flyttar \( x \) till vänstra sidan får vi \( -x = -2 \). för att vi ska få bara \( x \) vi måste typ gångra med −1 så att vi får \( x = 2 \) och den är positiv.


Nu ger jag eleverna uppgift 6. De kommer först med förslaget att lösa andragradsekvationen. De löser den, får lösningarna \( x = 2 \) och \( x = -3 \). Då säger Jens "Nej, det gick inte." När jag tar upp träden om andragradspolynoms nollställen ska återfinnas i förstagradspolynomen föreslår Jens "−x", men det är ju lite dumt. Dom går ju uppifrån och ner. De kommer inte så mycket längre men jag visar på möjligheten att faktorisera polynomet i \((−x + 2)(−x − 3)\).
I sista uppgiften ska eleverna skriva en instruktion. De behöver lite hjälp för att komma igång, de har svårt att lägga instruktionen på en lämplig nivå. Till exempel vill de skriva ner formeln för hur man löser en andragradsekvation, men jag påpekar då att det räcker med att skriva att man ska ta fram nollställena till andragradspolynomet. På min uppmaning lägger de till ett exempel och tar upp fallet när inga nollställen finns. Så här ser deras instruktion ut:

- Använd miniräknaren, vid problem: utnyttja andragradsekvationsformeln.
  (för att få ut eventuella nollställen)
- Byt tecken på värdena för nollställena
- He in värdena i 2 st. förstagradspolynom
  ex: \( x_1 = 2 \) \((-2)\)
  \( x_2 = -3 \) \((3)\)


Nu ber jag eleverna rita upp de tre funktionerna på grafväxaren. De gör det utan kommentarer och jag ger dem uppgift 2. Johan skriver genast upp \((x+ )\)(\(x-\)) vilket följs av denna diskussion:

**F:** Varför satte du minus i den?

**Johan:** Varför jag satte minus?

**F:** Ja, du satte plus i den ena och minus i den andra?

**Johan:** Jaha, det är minus i båda där! \([x^2 - 3x - 4]\)

**F:** Nej, jag sa inte att det var fel, jag frågade bara varför du gjorde så.

**Johan:** För att jag skulle få bort ett tal, så att jag får ner det till ... eller så att jag får ... eller ... så att jag ska få tre i stället för fyra olika ... så att jag skulle få bort dom där två [pekar till höger om parenteserna]

De provar med \((x + 2)(x - 2)\) men ser att de inte kan få \(3x\) och kommer efter en stunds funderande fram till rätt faktorisering. På min uppmaning ritar de funktionerna och när jag frågar om det är någon skillnad säger de att parabeln är förskjuten till höger.

Uppgift A1 löser de genom att se likheten med uppgift 2. De säger:

**Fredrik:** Förra var ungefär likadan, fast då var det 3 och 4.

**Johan:** Det blir 7 och 1 i stället.

*de löser uppgiften klart*
Fredrik: Blir det fler likadana uppgifter?
F: Jaa.
Johan: Inga logiska uppgifter i stället för ekvationer?
F: Är ni less på sana hår uppgifter? Är det träkigt?
F: Tror ni att man skulle kunna göra på något annat sätt för att lösa dom här uppgifterna? I stället för att sitta och gissa?
Johan: Nää. [skratt]
F: Är det bara att gissa som gäller?
Fredrik: Nej.
F: Ni tror inte att man skulle kunna använda räknaren för att faktorisera?
Fredrik: Vi har inte lärt oss det. Jag vet inte om det går.

Nu försöker jag leda in eleverna på att använda räknaren och titta på nollställen. Jag ber dem beskriva graferna och de säger saker som att ”linjerna är parallella”, ”dom går diagonal” och ”dom korsar vid x-axeln”. Genom att fråga var andragradskurvan skär x-axeln visar jag eleverna att den skär på samma ställen som linjerna. Då säger Johan ”Man skulle kunna använda det till att lösa uppgiften.” Eleverna får prova detta på uppgift A4. De finner att nollställena är -1 och 5 och Johan skriver ner faktoriseringen \((x−5)(x+1)\). De utför multiplieringen och ser att det stämmer. När jag frågar varför metoden fungerar kan de inte säga någonting.

Grafen till polynomet i uppgift 3 kommenteras av Fredrik som ”en kluring”. De ser att kurvan bara har ett nollställe och försöker med faktoriseringen \((x−2)(x+2)\). De ändrar till \((x+2)(x+2)\) när Fredrik pekar att ”Det står att x är plus 2.” När detta inte stämmer provar de att utföra multiplieringen på det första förslaget också. Efter några korta frågor från mig inser de att förstagradspolynomen måste vara identiska och Johan provar \((x−2)(x−2)\), vilket stämmer. Samtidigt som Johan kontrollerar förslaget algebraiskt gör Fredrik samma sak genom att rita både \(x+2\) och \(x−2\) på räknaren. De kommer fram till att faktoriseringen är korrekt ungefär samtidigt. Nu frågar jag:

F: Vad kan man dra för slutsats här, hur linjen ska se ut utifrån vilket nollställe man har?
Fredrik: Om båda går genom samma punkt så borde dom vara lika.
F: Jaa. Det är den ena saken. Sen säg ni att när ni hade nollstället 2 så såg linjen ut så här: \((x−2)\). Kan ni säga nåt om det?
Johan: Tja...
F: En sak som ni har haft problem med är ju vilket tecken ni ska ha på dom här [pekar på faktoriseringen]
Johan: Jaa... Dom blir negativa negativa när den där är positiv.
F: Blev det tvärt om också?
Fredrik: Ja.

Längre än så kommer inte eleverna när det gäller att förklara varför metoden fungerar.

I uppgift 4 får eleverna arbeta med ett andragradspolynom utan nollställen. De försöker då istället använda koordinaterna för kurvans minimipunkt, \(x = 1\) och \(y = 2\). När de ser att detta inte fungerar tar det helt stopp. Jag försöker hjälpa dem genom att prata om att andragradsfunktionen aldrig blir 0, men de kan inte dra några slutsatser. Till slut förklarar
jag för dem att uppgiften inte kan lösas eftersom det inte finns några förstagradspolyom som inte har något nollställe. De verkar inte helt övertygade. Fredrik avslutar arbetet med uppgiften med att säga ”En kluring!”

De har också svårt att lösa uppgift 5. Med hjälp kommer de fram till att man måste lösa andragradsekvationen för att få fram exakta värden, och för att komma fram till lösningen på uppgiften behöver eleverna en tydlig handledning från min sida. I detta läge hoppar jag över uppgift 6 och går direkt till uppgift 7. Johan säger:

Om man ritar upp det [andragradspolynomet] på miniräknaren så... kan man ju rita upp funktionen först, sen kolla vart... vart nollställena är. Då får du två stycken rötter som du sätter in i en formel, eller i en...

Utifrån detta formulerar eleverna följande instruktion:

- Rita upp Funktionen i en graf.
- kolla nollställen.
- Sätt in dem i funktionen:
  \((x - \text{nollställe}1)(x - \text{nollställe}2)\)
  det är alltså det inverterade värdet av nollstället som sätts in i formeln.
  \(x - \text{nollställen ska bli } 0\)
  Om bara ett nollställe erhålls, så ska polynomen vara likadana.
  Om inget nollställe erhålls, så är det omöjligt att faktorisera.


5. Analys och resultat

Grunden för analysen består i de frågeställningar som formulerades i avsnitt 2.1:

* Kan elever lara sig någonting svårt som normalt behandlas rent algebraiskt genom att arbeta
  - visuellt, dvs använda grafiska representationer av funktioner
  - med en grafräknare, dvs utnyttja modern teknik
  - med ett undersökande arbetssätt, dvs tillsammans med läraren leta sig fram till den kunskap och förståelse som krävs?

5.1. Kan eleverna arbeta visuellt? Det vill säga kan eleverna använda grafiska representationer av funktioner? Hur använder de grafräknaren i dessa situationer? Varför väljer de att använda grafräknaren?

De olika sätt som grafräknaren används på kan delas in i några olika varianter:
Rita en eller flera grafer.
Att rita grafer medför inte några som helst problem för någon elev. De visar alla god fördrogenhet med detta.

Undersök nollställen.
Alla eleverna börjar med att använda ”zoom”-funktionen. På så sätt kan de alla gissa vilken rot de undersöker så länge som de arbetar med andragradspolynom med heltalsnollställen. När detta inte fungerar visar jag eleverna ”zero”-funktionen. Denna funktion tar fram ett mycket exakt värde på ett nollställe. Flera elever kan direkt använda funktionen, andra elever behöver en kort förklaring för att inse vad ”Left Bound?” (vänster gräns för ett intervall som innehåller nollstället), ”Right Bound?” (höger gräns) och ”Guess?” (gissning av nollstället) innebär. Alla elever använder funktionen utan problem vid slutet av försöket.
Några elever insåg att produkten av nollställena ska vara lika med konstantermen i andragradspolynomet och att det därför räcker att ta fram ena nollstället med räknaren.

Numeriska beräkningar.
De numeriska beräkningar som förekommer är av tre typer:
Undersök om ett decimaltal (från ”zero”-funktionen) är ett bråk.
Undersök om $(2 + \sqrt{3}) \cdot (2 - \sqrt{3}) = 1$ för att kontrollera faktoriseringen i uppgift 5.
Annat, till exempel för att beräkna $(-2)^2$.

När det gäller de numeriska beräkningarna sker de flesta på elevernas egna initiativ. När eleverna i uppgift 5 får fram ett decimaltal försöker Klara och Karro omvandla detta till ett brak med ”Frac”-funktionen. Fredrik och Johan är också inne på att det kan vara ett brak och gör samma sak sedan jag har påpekat att en sådan funktion finns. Jenny och Nazanin tycker att det verkar konstigt att $(2 + \sqrt{3}) \cdot (2 - \sqrt{3}) = 1$ och kontrollerar detta med grafrekaren.
Att eleverna har nytta av de grafiska representationerna syns tydligt flera gånger under försöket. I samtliga fall utom ett användes miniräknaren för att ta fram de grafiska representationerna. Undantaget var Jennys och Nazanins skiss i samband med uppgift 7. Här följer några exempel på situationer när eleverna utnyttjade visualiseringar:

**Kristina och Eldina:** (sidan 13) ritar en funktion och utbrister ”Men vänta, där den skär $x$-axeln...”
Genom att titta på grafen kan eleverna formulera sin hypotes om hur förstagradspolynomen ska se ut.
Jenny och Nazanin: (sidan 14) När grafen kommer fram säger Nazanin ”Nej, nu har jag skrivit fel... ” Nazanin ser att hon har gjort fel eftersom linjerna inte skär x-axeln i parabelns nollställen.

Jens och Haris: (sidan 16) Jens ritar andragradspolynomet på räknaren och säger ”Sen kollar man på vilka ställen... där den skär x-axeln... ” Jens faktorerar polynomet genom att avläsa parabelns nollställen och sätta in dessa i förstgradspolynomen.

Både Kristina och Jens använder andragradspolynomets nollställen för att faktorisera. Resonemanget bakom detta verkar mycket osäkert. Det skulle kunna vara så att de har en svag minnesbild av något som de har sett i lärbooken eller som deras lärare har visat på tavlan och när de ser nollställena så kopplas detta mer eller mindre undermedvetet till minnesbilden och de ”känner” att de är på rätt spår. Detta sätt att arbeta kan vara framgångsrikt i gymnasie-matematiken. År detta något som kännetecknar elever som lyckas bra i skolan?

Å andra sida förekommer det också att eleverna inte använder de grafiska representationerna i situationer då detta hade varit både enkelt och effektivt. Detta gäller ofta då eleverna kommer i en situation de inte riktigt kan hantera, till exempel i uppgift 6, när de söker andra lösningar till uppgift 1. När de arbetar med uppgift 6 har de flesta eleverna någorlunda klart för sig att nollställena måste vara de samma i andragradspolynomet som i faktoriseringen, men de verkar ha mycket svårt att använda den kunskapen. De återgår då till den första metoden, gissa och prova, trots att flera av dem har formulerat och använt en metod för att faktorisera. En kort diskussion utifrån grafen till polynomet i uppgift 1 skulle direkt kunna visa eleverna att enbart två nollställen är möjliga. Några elever för också ett resonemang i denna riktning, men de har mycket svårt att utnyttja sin kunskap.

Det förekommer också att eleverna inte kan dra några egna slutsatser genom att titta på graferna utan att slutsatserna grundas på en diskussion med mig och på den hjälp de får. Johans hypotes i slutet av följande citat är ett exempel på detta:

Nu försöker jag leda in eleverna på att använda räknaren och titta på nollställen. Jag ber dem beskriba graferna och de sager saker som att ”linjerna är parallella”, ”dom går diagonal” och ”dom korsar vid x-axeln”.

Genom att fråga var andragradskurvan skär x-axeln visar jag eleverna att den skär på samma ställen som linjerna. Då säger Johan ”Man skulle kunna använda det till att lösa uppgiften.”

Johans hypotes kommer först efter det att jag har gjort honom uppmärksam på att andragradskurvans nollställen utgörs av de samma ställen som linjernas nollställen. Hypotesen grundas troligen till en mycket liten del på den grafiska representationen och i betydligt större utsträckning på samtalen med mig.

Att använda grafräknaren för att ta fram visualiseringar av funktioner verkar vara mycket naturligt för eleverna. Grafräknaren bereder dem inga egentliga problem och hanteras på ett ganska självlivligt sätt. När det gäller att utnyttja visualiseringarna är skillnaderna mellan eleverna betydligt större. Vissa elever verkar ”triggas” av graferna så att de snabbt kan komma med hypoteser medan andra inte verkar ha någon nytta alls av visualiseringsarna.

5.2. Undersökande arbetssätt. I avsnitt 2.2 beskrivs ett undersökande arbetssätt som bestående av tre huvuddelar: visualisering, hypotes och kontroll. Begreppet visualisering

Ett exempel på hur eleverna formulerar hypoteser är när Kristina och Eldina kör fast på en extrauppgift. Efter en stunds tystnad säger de:

**Eldina:** Om vi ... man kollar på miniräknaren.
**Kristina:** Ja, gör det. Sätt in den där [pekar på $x^2 - \frac{1}{3}x - \frac{2}{3}$]
**Eldina:** ritar funktionen på graf räknaren.
**Kristina:** Men vän ta, där den skär $x$-axeln, där $y$ är noll ...
**Eldina:** Ja, annars kan vi lösa ut ... andragradekvationen.
**Kristina:** Ja, då får vi ju samma. Vi får ju svaren direkt om vi gör med räknaren.
**F:** Hur menar du då?
**Kristina:** Om vi kollar var den skär $x$-axeln så får vi fram dom möjliga $x$-en vi kan ha.


Alla elever i försöket kunde formulerar någon typ av hypotes, men dessa var underbyggda på olika sätt. Förmågan att formulerar relevanta hypoteser verkar överensstämma med den matematiska förmåga som elevernas lärare anser att de besitter. De elever som enligt läraren hade betyget MVG kunde i betydligt större utsträckning formulerar egna hypoteser och klarade av ett deduktivt resonemang bättre än de som hade betyget VG eller G. MVG-eleverna var också de elever som bäst klarade av att förklara varför metoden med nollställen fungerar.


Eleverna med betyg MVG och VG verkar ha förstått varför satsen fungerar och G-eleverna kan använda satsen och graf räknaren för att utföra faktoriseringar på nytt sätt.
Grunden för dessa påståenden ligger i de beskrivningar som finns i kapitel 4. Jag anser att Kristina och Eldina, eleverna med betyget MVG, har nått en viss förståelse om varför faktorsatsen fungerar. Detta grundar jag på bland annat Kristinas uttalande på sidan 13:

När den andragradskurvan skär x-axeln har vi sagt att den ena delen av den här faktoriseringen har blivit noll. Antingen \( f(x) \) eller \( g(x) \) har blivit noll.

Kristina menar alltså att om polynomet \( p(x) \) har ett nollställe \( a \), det vill säga \( p(a) = 0 \), så är \( f(a) = 0 \) eller \( g(a) = 0 \). De har tidigare faktorerat ett polynom genom att hitta polynomens nollställen och skriva \( f(x) = (x - a) \) och \( g(x) = (x - b) \), där \( a \) och \( b \) är polynoms nollställen. De har då i stort sett visat att om \( p(a) = 0 \) så är \( (x - a) \) en faktor i polynomet \( p(x) \). Det är just faktorsatsen från höger till vänster (se sidan 5) som Kristina beskriver här, även om det är ganska osäkert och inte helt korrekt formulerat.

I slutet av försöket kan också Eldina motivera förstagradspolynoms utseende när hon säger ”Motsatt tecken, eftersom vi vill att funktionen ska bli noll.” Dessa situationer tyder på att Kristina och Eldina har nått en viss förståelse om sambandet mellan faktorisering av andragradspolynom och nollställen hos andragradspolynom.

När det gäller eleverna med betyget VG verkar Jenny och Nazanin ha insett varför faktoriseringen utnyttjar nollställena. Följande diskussion äger rum sedan eleverna har insett att andragradspolynomet och förstagradspolynomen har samma nollställen, \(-2 \) och \(3\):

**Nazanin:** Andragradspolynomet är noll. Då är \((x + 2)(x - 3) = 0\). Blev vi klokare av det?

**Jenny:** Då är dom här också lika med noll

[krigger \(0 = (x + 2)(x - 3) = 0\)]

**Nazanin:** Nej, så kan man nog inte göra.

**Jenny:** Det är ju så vi gör med andragradarna, vi faktoriserar.

**Nazanin:** Ja! Nu förstår jag varför det blir ombytta tecken! För att \(3 - 3\) är \(0\) [pekar på \((x-3)\)] och ...

**Jenny:** \(-2 + 2\) är noll [pekar på \((x + 2)\)]

Jenny och Nazanin verkar känna igen resonemanget från situationer där de har arbetat med lösning av andragradsekvationer med hjälp av faktorisering. En möjlighet är att de kopplar denna uppgift till hur de brukar lösa andragradsekvationer av typen \(x^2 - 3x = 0\) genom att faktorisera: \(x^2 - 3x = x(x - 3)\).

Lite tveksamare är det hurruvida Haris och Jens är helt på det klara med varför man utnyttjar nollställena, men Jens sista uttalande i den andra refererade diskussionen i avsnitt 4.4 kan tolkas så. Han säger:

Man sätter den här \([(x - 3/2)]\) som \(0\) sen också. Då får du ju två ställen, det här är ju likom en faktor, det är ju som \(x \cdot x\), om du sätter den här [pekar på vänstra x-et] till noll så får du ut den andra, värdet här [pekar på högra parentesen i faktoriseringen], och sen sätter man den andra till \(0\). Det hända ju på två ställen, när den skär... y-axeln...

Uttalandet kan dock tolkas på flera sätt. En möjlighet är att Jens faktiskt vet hur faktoriseringar används vid ekvationslösning, det vill säga att \(x \cdot y = 0 \iff x = 0\) eller \(y = 0\). Eftersom Jens pekar på det ena x-et och (felaktigt) säger att man då kan få fram det andra, kan det kan också tyda på en stor osäkerhet om vad som gäller när en produkt av två faktorer ska bli noll.

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Det är uppenbarligen mycket svårt att dra några säkra slutsatser huruvida eleverna verklig har lärt sig någon matematik under försöket. I denna undersökning kommer jag inte att gå längre än så här i analysen av vad eleverna eventuellt har lärt sig. Däremot vill jag påpeka att flera elever enligt analysen i avsnitt 5.1 och 2.2 har kunnat använda ett undersökande arbetssätt. Eleverna har med hjälp av visualiseringar av funktioner på en grafiräknare kunnat formulera hypoteser omkring sambandet mellan funktioners nollställen och möjligheten att faktorisera funktioner. De har också kunnat kontrollera dessa hypoteser. Detta är, anser jag, en god indikation på att eleverna faktiskt har lärt sig någonting, eller åtminstone arbetat på ett sådant sätt att de har haft goda möjligheter att lära sig någonting.

6. Diskussion

6.1. Grafiräknaren i ett undersökande arbetssätt. Vilka slutsatser kan då dras utifrån det försökte som har genomförts? Några iakttagelser är följande:

- Alla elever har lärt sig någonting under försöket. Det kan då tyda på att metoden med ett undersökande arbetssätt är positiv för elevers lärning, men det är kanske lika troligt att en lärare ensam med två elever i en timme är en viktigare orsak till framstegen.
- Elever med betyget VG verkar vara de som kan dra störst nytta av det arbetssätt som användes i försöket. Arbetssättet för svårt för elever med betyget G, eller är det uppgiften som är lämpligast för elever med relativt goda prestationer i matematik? Skulle MVG-eleverna ha kunnat dra större nytta av arbetssättet om uppgiften hade varit svårare?
- Eleverna tyckte att uppgifterna och arbetssättet var intressant och roligt, men ingen hade gjort något liknande förut. I kursplanen för matematik (Skolverket, 1995) står det bland annat att "undervisningen skall sträva efter att eleverna skall få uppleva tillfredsstillande av den typ som användes i försöket.

6.2. Elevernas uppfattning om matematik. "Jag älskar min grafiräknare" säger en av eleverna, "jag använder den tämligen, utan den känner jag mig så osäker." Det finns flera situationer där elever använder grafiräknaren utan att egentligen veta varför. Motiven kan vara till exempel att man inte kommer på något annat att göra, man brukar använda den i liknande situationer eller att man tror att läraren vill att man ska använda den. Den sista anledningen kan vara mer frekvent än undersökningen visar då det didaktiska kontraktet, den uttalade överenskommelsen om hur arbetet i klassrummet ska fungera, kan vara mycket starkt, särskilt hos lägrepresterande elever. Ett exempel på hur det didaktiska kontraktet kan styra elever är följande situation:

När Johan och Fredrik ska lösa uppgift 2 börjar Johan helt korrekt med att skriva upp två parenteser på detta sätt: \((x + 1)(x - 1)\). När jag frågar varför han har skrivit olika tecken säger Johan "Jaha!" och börjar ändra det han har skrivit.
Johans reaktion kan vara mycket naturlig utifrån hur hans matematikundervisning har sett ut under de elva år som han har vistats i skolan. När en lärare frågar "Varför har du gjort så?" är det ofta något som är fel. Vad är det som saknas när denna fråga får en sådan reaktion? Är diskussionen om matematik så ovanlig att den bara förekommer då det är något fel?

Uttalandet om kärleken till grafväxten gäller endast de tre elevparen med betyget VG. Eleverna med betyget G respektive MVG talar inte om samma beroende. Om detta är en generaliserbar egenskap hos elever i gymnasieskolan, det vill säga om de elever med betyget VG är de som använder grafväxten mest och de som då troligen drar mest nytta av tekniken, eventuellt på bekostnad av G- och MVG-elevernas inlärning, så kan det vara av stor vikt att undersöka effekterna av detta.

Eleverna i denna undersökning var mycket ovana att arbeta självständigt på det sätt som frågorna inbjöd till. Flera kommentarer under försöket eller under intervjun stöder detta och visar även på elevernas föreställningar om vad matematik är. Några exempel:

**Haris:** Det är så i matten, när man räknar, det är huvudsaken att man får ett resultat, inte hur man tänker. Man frågar inte hur man tänker. Vi som går natur vi gör bara så där och så där, vi vet inte varför.

**Jenny:** (Svar på frågan ”var är det egentligen som händer?”) Det är såna frågor man aldrig hinner fråga sig på en mattelekction, för att då måste man räkna ikapp och få rätt på uppgifterna.

Om åsikterna bakom dessa uttalanden är representativa för elever i det svenska gymnasiet finns det anledning att diskutera uppläggning och innehåll i matematikundervisningen. Flera initiativ har tagit i den riktningen, till exempel har Bedömningsgruppen för studenternas förkunskaper i matematik, tillsatt av högskoleverket, kommit med sin slutrapport "Räcker kunskaperna i matematik?" (1999) där flera frågor om matematikens roll och gymnasieelevers kunskaper diskuteras.


Det är svårt att avgöra vilken effekt elevernas arbete har haft på deras inlärning i denna studie, men jag anser mig ändå ha fått ett visst stöd för att ett undersökande arbetsätt med hjälp av grafritande miniräknare ger möjlighet för elever att få en bättre förståelse för begreppet faktorisering och för faktorsatsen samt en större insikt i kopplingen mellan grafiska och algebraiska representationer av funktioner och ekvationer (se avsnitt 5.3). Hur denna typ av arbetsätt kan användas i det dagliga arbetet i matematikklassrummet och inom vilka matematiska områden det passar in är viktiga frågor som bör bli föremål för framtida forskning.

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Appendix A. Extra uppgifter

Här följer ett antal extra uppgifter som kan användas efter uppgift 2 om eleverna inte börjar diskutera alternativa lösningsmetoder. De första åtta andragradspolynomen har heltalsnollställen, de sista fyra har rationella nollställen.

\[ f(x) \cdot g(x) = x^2 - 6x - 7 \quad \text{rötter: } x = -1 \text{ och } x = 7 \quad (A.1) \]
\[ f(x) \cdot g(x) = x^2 - x - 6 \quad \text{rötter: } x = -2 \text{ och } x = 3 \quad (A.2) \]
\[ f(x) \cdot g(x) = x^2 - 6x + 8 \quad \text{rötter: } x = 2 \text{ och } x = 4 \quad (A.3) \]
\[ f(x) \cdot g(x) = x^2 - 4x - 5 \quad \text{rötter: } x = -1 \text{ och } x = 5 \quad (A.4) \]
\[ f(x) \cdot g(x) = x^2 + 5x + 6 \quad \text{rötter: } x = -3 \text{ och } x = -2 \quad (A.5) \]
\[ f(x) \cdot g(x) = x^2 - 2x - 48 \quad \text{rötter: } x = -6 \text{ och } x = 8 \quad (A.6) \]
\[ f(x) \cdot g(x) = x^2 - 8x + 12 \quad \text{rötter: } x = 2 \text{ och } x = 6 \quad (A.7) \]
\[ f(x) \cdot g(x) = x^2 + 2x - 15 \quad \text{rötter: } x = -5 \text{ och } x = 3 \quad (A.8) \]
\[ f(x) \cdot g(x) = x^2 - \frac{1}{2}x - 1 \quad \text{rötter: } x = 2 \text{ och } x = -\frac{1}{2} \quad (A.9) \]
\[ f(x) \cdot g(x) = x^2 - \frac{1}{3}x - \frac{2}{3} \quad \text{rötter: } x = -\frac{2}{3} \text{ och } x = 1 \quad (A.10) \]
\[ f(x) \cdot g(x) = x^2 + \frac{3}{2}x - \frac{9}{2} \quad \text{rötter: } x = -3 \text{ och } x = \frac{3}{2} \quad (A.11) \]
\[ f(x) \cdot g(x) = x^2 + x - \frac{15}{4} \quad \text{rötter: } x = \frac{3}{2} \text{ och } x = -\frac{5}{2} \quad (A.12) \]

Felet i uppgift A9 upptäcktes under försöket. Korrekta rötter skall vara \( x = \frac{1}{4} \pm \sqrt{\frac{17}{4}} \).

References


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How Students Verify Conjectures.

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Abstract. This article considers the issue of verification of conjectures. Students are asked to verify or reject conjectures given by the instructor and also conjectures made by themselves. The analysis indicates that the students seem inclined to perform verifications at a rather abstract level, that they succeed to a certain degree, and that there are differences between the way students write down verifications and the way they discuss the conjectures.

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1. Introduction

This article will discuss how upper secondary students verify conjectures. Students (age 16-17) will be presented some conjectures and asked to check if they are correct. They will also be asked to make a written explanation of why they are true or false. The article consists of a theory section where the theoretical background and the framework for the analysis is presented, a description of the experiment and a discussion of the analysis and results.

2. Theory

2.1. Background and earlier research. Deductive proofs are the core of the presentation of research mathematics. A conjecture may be formulated as a result of inductive reasoning, but it must be clearly examined and proved to be true by deductive reasoning before it is accepted. School mathematics differs from research mathematics on this point. In school mathematics there are many more accepted ways to verify conjectures, and, according to Hanna and Jahnke (1996), one important goal should be to explain the conjectures, not to prove them.

The ability to decide whether a conjecture is true or false is essential to a mathematician. To do this she or he uses strict mathematical reasoning to make a deductive proof. For students it can also be an important part of their work to verify conjectures, and opportunities for students to discuss and argue about conjectures are, according to Dreyfus (1999), easy to find. Dreyfus mentions some examples: “A student may want to convince a classmate of a guess or conjecture during a collaborative phase; another student may have asked for help; or the teacher may try to obtain clarification about student’s thinking[... ]” (p. 85). Discussions with some teachers on this issue indicates different opinions whether this is true in the Swedish school. One opinion is that there might be few opportunities for collaborative activities in the Swedish upper secondary school.

However, students in such situations often use a different kind of reasoning. Lithner (1999) discusses these different kinds of reasoning and introduces the concept of plausible reasoning. Plausible reasoning, an expression first mentioned by Pólya (1945), differs from strict mathematical reasoning in such a way that the grade of certainty requested is lower.

According to Lithner, plausible reasoning

(i) is founded on mathematical properties of the components involved in the reasoning and
(ii) is meant to guide towards what is probably true without necessarily having to be complete or correct. (p. 5)

This less strict form of reasoning is also encouraged by school practices. Students sometimes get credit for verifications which can be very superficial and for answers which can be partly or even completely wrong if the student can show some kind of (plausible) reasoning.

Several other researchers have also discussed the very complex question concerning how students deal with mathematical proof. Chazan (1993) found that some students accepted empirical evidence as proof while some students had a clear picture of the difference between empirical evidence and deductive proof. He also found some students that claimed that “a deductive proof provides no safety from counterexamples” (p. 372).

Balacheff (1988) looked at students working in geometry in order to find a hierarchy of types of proof. What he calls “proof” is something that is recognized as such by the
producer. In this way it is possible to discuss how well students can handle mathematical proof or how close to an (by mathematicians) acceptable mathematical proof students can get.

The principal aim of this study is to use the hierarchy constructed by Balacheff, together with the theory of the didactical contract, in order to describe and analyze how students deal with verification of conjectures.

2.2. Verification of conjectures. To describe the work performed by the students part of a description of exploration learning will be used. The description was presented in the study Gymnasieelever undersöker ett matematiskt begrepp med grafräknare (Bergqvist, 1999), which translates into Upper secondary school students examine a mathematical concept using a graphing calculator. There, the work by the students is divided into three parts, visualization, hypothesis and verification. Verification is the last part and the part that will be examined in this article. The verification part can be described like this:

**Verification:** The meaning of verification is to examine the conjecture. More exactly, the aim is to check if the idea is correct in some meaning, by testing or mathematical reasoning. This can be done in different ways and on different grounds. One way can be to check an example, another can be trying to find a counter-example. Another yet can be to try to prove the conjecture in a deductive way. To discuss if the conjecture is reasonable can also be part of the verification process.

The expression **conjecture** will be used for all statements that can be rewritten in *if-then* form and contains some degree of mathematical uncertainty. An example:

“It’s when you get a negative number under the square root sign that the quadratic curve is above the x-axis all the time, right?”.

This can be written in the following way:

*If* you use the formula to solve a quadratic equation and get a negative number under the square root sign *then* the graph of the corresponding quadratic function will everywhere be above the x-axis.

The expression **local conjecture** will be used when students pose their own conjectures, or when the students discuss a small part of a conjecture.

2.3. The four levels. In order to categorize the students’ work on verification the four main types of proof or verification listed by Balacheff in his article “Aspects of proof in pupils’ practice of school mathematics” (1988) will be used. Balacheff claims that the four different types form a hierarchy so that a higher level of generality and understanding is needed for a higher level of proof. This means that moving from one type to another involves some kind of change in understanding or practice.

The descriptions in italics below are quoted from Balacheff’s article. My interpretation of each level is presented together with examples after each quotation.

**Level 1:** Naive empiricism.

*Naive empiricism consists of asserting the truth of a result after verifying several cases. This very rudimentary (and as we know, insufficient) means of proving is one of the first forms in the process of generalization (Piaget, 1978). But arising from a collection of problems posed to fifteen year olds, Bell (1979) found that 25 per cent of them based their answers only on the verification of a few cases. We can therefore expect naive empiricism to constitute a form resistant to generalization.*

**Interpretation of level 1:** To be convinced that a conjecture is true, and to argue that it is true after verifying some cases.
Example:
Conjecture: “You can always use the zeroes to factorize!”
\[x^2 - x - 6\] has zeroes \(x = 3\) and \(x = -2\), so \(x^2 - x - 6 = (x - 3)(x + 2)\).
Multiplying gives \(x^2 - 3x + 2x - 6 = x^2 - x - 6\). So, it is true!

**Level 2:** The crucial experiment.
The expression ‘crucial experiment’, coined by Francis Bacon (Novum Organum, 1620), refers to an experiment whose outcome allows a choice to be made between two hypotheses, it having been designed so that the outcome should be clearly different according to whether one or other hypothesis is the case. (Whether this experiment allows the rejection of one hypothesis or not, it does not allow us to assert that the other is true.)

We use the same expression for a slightly different process, one of verifying a proposition on an instance which ‘does not come for free’, asserting that ‘if it works here, it will always work’. Here is an example from Bell (1976) (cp. 10, p.12): 'Jayne shows a complicated polygon, she can definitely say that the statement is true.' This type of validation is distinguishable from naive empiricism in that the pupil poses explicitly the problem of generality and resolves it by staking all on the outcome of a particular case that she recognizes to be not too special.

**Interpretation of level 2:** Version 1:
An experiment designed to make a choice between two possibilities. The result should clearly show which one of the two conjectures that must be rejected.

Example:
“Say that you have a polynomial, and that you can use the zeroes to factorize. How are you supposed to deal with the signs? If the zeroes are \(x = 3\) and \(x = -2\) it must be \((x + 3)(x - 2)\) or \((x - 3)(x + 2)\). Lets check!”

Version 2:
Testing a conjecture in a rather special case (but not extreme) and draw the conclusion “if it works even for this…” The difference from naive empiricism is that the conjecture should be formulated in a general way and tested in a not too special case.

Example:
“The graph of \(y = x^3 + 43\) cuts the \(x\)-axis. Then it’s probably true that all third degree polynomials cuts the \(x\)-axis.”

**Level 3:** The generic example.
The generic example involves making explicit the reasons for the truth of an assertion by means of operations or transformations on an object that is not there in its own right, but as a characteristic representative of its class. The account involves the characteristic properties and structures of a class, while doing so in terms of the names and illustrations of one of its representatives. Below is an example taken from Bezout (Notes on Arithmetic, 1832, p. 23):

The reminder on dividing a number by \(2 \times 2\) or \(5 \times 5\) is the same as the reminder on dividing the number formed by the rightmost two digits by \(2 \times 2\) or \(5 \times 5\)...
To fix these ideas, consider the number 43728 and the divisor \(5 \times 5\). The number 43728 is equal to 43700 + 28. However, 43700 is divisible by \(5 \times 5\), because is the product of 437 and 100, and as \(100 = 10 \times 10\), or \(5 \times 2 \times 5 \times 2\), the factor 100 is divisible by \(5 \times 5\). The reminder on dividing 43728 by \(5 \times 5\) or 25 is therefore the same as dividing 28 by 25.
Interpretation of level 3: To show the truth by manipulating an object which is used as a representative of all objects in the class. The proof is indicated by the effect the operations have.

Example: “How can a polynomial with certain zeroes be factorized? Let's say that we have an arbitrary polynomial, for example \( p(x) = x^2 - x - 6 \), with zeroes \( x = 3 \) and \( x = -2 \). How can it be factorized? If it can be written \( p(x) = (x - a)(x - b) \), \( x = 3 \) must be a zero to that expression to! One possibility could be \( a = 3 \).”

Level 4: The thought experiment.

The thought experiment invokes action by internalizing it and detaching itself from a particular representation. It is still coloured by an anecdotal temporal development, but the operations and foundational relations of the proof are indicated in some other way than by results of their use, something which is the case for the generic example. (For more on the thought experiment, see Lakatos, 1976.) Here, for example, is the proof that Cauchy gave for the intermediate value theorem in his Cours d’Analyse of 1821:

It suffices to show that the curve with equation \( y = f(x) \) will cross the line \( y = b \) at least once in the interval which includes the ordinates which correspond to the absissae \( x_0 \) and \( X \); however, it is clear that this will happen under the given hypotheses. Indeed, as the function \( f(x) \) is continuous between \( x = x_0 \) and \( x = X \), the curve with equation \( y = f(x) \) and which passes firstly through the point with coordinates \( x_0, f(x_0) \), and secondly through the point \( X, f(X) \), will be continuous between these two points; and as the constant ordinate \( b \) of the line whose equation is \( y = b \) lies between the ordinates, \( f(x_0) \) and \( f(X) \), of the two points under consideration, the line necessarily passes between these two points, something it cannot do without meeting the above-mentioned curve in the interval.

Interpretation of level 4: An abstract member of a class is discussed and the proof is indicated by looking at the properties of the objects, not on the effects of operations on the object.

Example:

“A third degree polynomial is continuous and approaches \( -\infty \) and \( +\infty \) when \( x \) goes towards \( -\infty \) and \( +\infty \) or vice versa, because for large \( x \)-values the \( x^3 \)-term will dominate. Therefore it must always cut the \( x \)-axis”

2.4. The didactical contract. Blomhøj (1994) describes how traditional mathematics education can be characterized using the concept “didactical contract”. The didactical contract, which was first introduced by Brousseau around 1980 (Brousseau, 1997), can be described as a mutual, unseen understanding between teacher and students. Some of the implications of this contract are, according to Blomhøj:

(1) The teacher is supposed to explain the methods and algorithms that are presented in the textbook.
(2) The teacher is supposed only to pose questions which the students are able to solve using those methods and algorithms.
(3) A task is solved, when the easiest question in it is answered.
(4) Answers to the questions should be short, i.e. a number, a graph or, if absolutely necessary, a sentence.
Blomhøj also talks about the way students try to figure out what the teacher want them to do. He argues that the most important thing for the students will be to keep their part of the didactical contract, to comply with the teachers wish. In the analysis the students’ work will be examined also to see if the didactical contract can explain why students choose a specific strategy. Eventual effects on the level of the reasoning or presented written explanations will also be examined.

2.5. Research questions. The central questions in this study are:

- How do students verify conjectures?
- Can the didactical contract explain students’ strategy approach?

Using the description of verification, the hierarchy of proof levels and the description of the didactical contract I will try to cast some light on these questions, questions concerning how students test conjectures and how they argue when they want to convince themselves and others.

3. Method

3.1. The students. In this study, students chosen from an eleventh grade were working in pairs with verification or rejection of three given conjectures. The students were all in a natural science program, and the same year had had their first meeting with calculus. Five pairs of students were chosen and then asked if they were willing to participate. The selection was made in cooperation with their teacher so that both high and low performing students were included. The teacher and researcher also paired the students so that each pair would have students of the same gender and approximately the same achievement level. There were six boys and four girls in the study.

3.2. The conjectures. The following three conjectures were presented to the students:

1. A linear function and a quadratic function always intersect in two points.
2. If the graph to a quadratic function cuts the x-axis in two points there is a point between the intersections where the tangent to the graph is horizontal.
3. The graph of a third degree polynomial always cuts the x-axis.

Conjecture 1 is incorrect, conjectures 2 and 3 are correct. For a mathematical discussion of the conjectures see appendix A. The first conjecture was included partly as an easy introduction, a ‘warm-up’, and partly to show the students that the conjectures might be false. It is not really possible to apply Balacheff’s levels to a rejection that is made using a counterexample.

The students were asked to decide if the conjectures were correct and then make a written explanation why they were correct or not. They were asked to write an explanation aimed at upper secondary school students (their classmates). The students were also encouraged to be as careful and accurate as possible “so that you convince even those who are in doubt.”

When the students had given their answers they were asked the following questions:

Q1: Are you convinced that your answer is correct? Why or Why not?
Q2: Are you satisfied with your explanation? Why or Why not?
Q3: Do you think your explanation will convince your classmates? Why or Why not?
Q4: Do you want to make any changes?
The students were given a conjecture which they were asked to examine. In their work they also stated local conjectures of different complexity, so they worked both with conjectures that were given to them by the researcher and with conjectures created by themselves. During the sessions the researcher took notes concerning interesting situations and after each session the students were interviewed on those issues.

3.3. Documentation. The sessions took place in the students' normal classroom, but only the two students and the researcher were present in each session. The sessions were videotaped using a video camera placed directly above the table and focused on a paper in front of the students. The recordings showed the paperwork the students performed and also how they used the graphing calculator. The discussions around the questions Q1 - Q4 and the interviews were also recorded on video. The videotapes have been transcribed and analyzed. There was no time limit to the groups, however no group worked more than one hour with the three conjectures.

4. Empirical data

In this section, a short presentation of the students work on the three conjectures will be given. The idea is to give an overall picture of the students' reasoning and performance.


A linear function and a quadratic function always intersect in two points.

All students immediately knew that conjecture 1 was false. Some of the students had barely time to read it before they said “No, not true” or something similar. They also almost directly showed each other a counterexample. However, three of the five groups were not satisfied with that. They said things like “It can’t be that simple”, or “How do we prove this in maths?” One proof was made by two students who said “\(y = x^2\) won’t intersect \(y = x - 10\) since the the equation \(x^2 = x - 10\) has no roots.”

Two groups started working with, what in their eyes might have been, more advanced mathematics. One of them made an explanation they didn’t think would work. On a question if they thought their explanation would convince their classmates they answered “No. They would be more convinced just by looking at it, seeing that it’s not the case.”

4.2. Conjecture 2.

If the graph to a quadratic function cuts the x-axis in two points there is a point between the intersections where the tangent to the graph is horizontal.

When conjecture 2 was presented to the students three of the groups were immediately certain that it was true. They all looked at sketchy drawings and concluded that the interesting point will be at the minimum (or maximum) of the function. Two groups discussed the sign of the derivative: negative derivative in one zero and positive in the other must mean that the derivative is zero somewhere in between. One of these groups stated that this must be true because the function is continuous. [This is not correct, it is the derivative that must be continuous.] Two groups made almost identical proofs of conjecture 2, they found that \(x\)-value of the zero of the derivative is the same as the \(x\)-value of the symmetryline (see Example 5 on page 12).
4.3. Conjecture 3.

The graph of a third degree polynomial always cuts the x-axis.

Conjecture no 3 was clearly the most difficult for the students to discuss. It was a difficult problem and the diversity of the ideas the students followed was rather large. Three main ideas could be identified:

(1) *Ideas about sign and the meaning of odd or even exponents.*

Three pairs worked rather much with this approach. The major problem was to show that it assumes both positive and negative values. The interpretation of this was totally left out. No group discussed anything about continuity or anything related to the intermediate value theorem. In the interviews after the sessions they all said that it was “clearly understood” that if they could show that the functions assumed both positive and negative values, the proof was done.

(2) *Ideas about the general shape of the graph, derivatives and slopes.*

One group referred directly to the shape and said “If it looks like this it will always cut the x-axis.” Another group said that the graph must continue in the same general direction and because of that it will cut the x-axis.

(3) *Ideas about how to solve the corresponding equation.*

Three groups mentioned this idea but two of them immediately realized that they couldn’t solve it. One group worked a lot with this idea because they found that they could solve the cubic equation if the constant term was zero.

No group could fully explain the behavior of a third degree polynomial, but all the groups could say something. Some examples are:

“When \( x = 100, x^3 \) is much bigger than \( x^2 \).”

“Even \( x^3 + 10000x^2 + 3x + 100 \) is negative for some \( x \)-values.”

“Since the exponent is odd the \( y \)-value will have the same sign as the \( x \)-value.”

“\( x^3 \) will in the end always be larger than \( x^2 \).”

5. Interpretation of the empirical data

In the following, excerpts from the students’ work will be presented. Examples and quotations will be used to describe the different kinds of reasoning and the written explanations the students produced. In the interpretation I will use the four levels and the theory of the didactical contract in order to analyze the data.

The five pairs will be called pair A, B, C, D and E, and the students in each pair will have assumed names. There was one low performing pair (pair A), three average performing pairs (pairs B, C, and D) and one high performing pair (pair E).

5.1. Level 1. Naive empiricism. Not so many situations where naive empiricism was used could be found in the study, the only three clear examples will be presented here.

**Example 1.** Conjecture 2.

Pair A became a little confused over the second conjecture. They said things like “between these intersections, that should be here (points at the x-axis), or?” and also “between, there is no tangent at all...” At this point the researcher intervened by explaining the meaning of the word between in this context, and the students said:

**Anna:** OK, then it’s true.

**Agnes:** Yes, and this we can explain by...

**Anna:** [Traces a line in the air with her hand, indicating a tangent to an imagined curve]
**Agnes:** It’s in the minimum point that it’s horizontal.

**Anna:** Yes, if you draw tangents, they will be more and more sloping and finally its like this [traces a horizontal tangent in the air]

The word “sloping” is used in a non-mathematical way, a flagpole which starts to lean more and more in one direction can be looked at as more and more sloping. They came back to this idea a little later:

![Graph](image)

**Figure 1. Discussion of conjecture 2**

**Agnes:** You can see here [points at A, see figure 1] that it should be like this [the tangent], and then more and more [sloping] . . . and finally [horizontal] . . .

**Anna:** But that is not a good explanation!

**Agnes:** You can calculate it!

**Anna:** Umm, you do that.

**Agnes:** No, I don’t remember how it’s done. We have done it. You use the tangents . . . kind of . . . the slope is zero in the tangent.

**Anna:** You differentiate and put it as zero.

They have the same discussion a little later, a struggle between the idea about drawing several tangents and the idea of differentiating a specific function:

**Agnes:** It’s like . . . the slope will became smaller and smaller. In the end it will be zero.

**Anna:** Yes, but it’s still not a good explanation.

**Agnes:** You can draw several tangents to show it.

**Anna:** Yes . . .

[silence]

**Agnes:** If you differentiate you will get the slope, and then put the slope equal to zero.

After the last discussion they decided to verify the conjecture by solving one example, so they differentiated $y = x^2 + 3x - 5$ and found the $x$-value of the minimum point, see figure 2. Note that the graph is not the graph of the function they differentiated.

The text in the figure translates to:

*differentiated the function and put $y'$ equal to zero to find out where the slope is horizontal. We got an $x$-value which was compared with the graph on the calculator. Saw that it was the lowest $y$-value.*
The students' final written explanation was a level 1 verification of conjecture 2, they showed that the conjecture was true by solving one example. Agnes had several times tried to explain the situation by the way the slope of the tangents changed, but Anna said every time “that’s not a good explanation” or something similar. What grounds Anna had for this is difficult to say, but one possibility is that the didactical contract played an important part. Anna may have thought that in order to explain something you must be very formal, or perhaps she realized that the discussion about tangents was incomplete.

The students seemed convinced that the conjecture was true long before they came up with the written example, they convinced themselves using a few examples, naive empiricism.

An interesting remark is here that the discussion was at a higher conceptual level than the written work. The discussion about how tangents to a quadratic curve behave is an incomplete level 4 verification, a thought experiment on the properties of a quadratic function and its tangents, while the written explanation was a single example, a verification at level 1, naive empiricism.

**Example 2.** Local conjecture: *Is it a fourth degree polynomial?*

When the students in pair A were working with conjecture 3 they discussed the general shape of the graph of a third degree polynomial. They said things like “what does it look like?”, “Is it, like, bent or?” and “Can it have two maximum points?” They also graphed four different third degree polynomials on the graphing calculator to find out more. At one point they discussed if the graph could “go down again” [similar to the quadratic function $y = -x^2 - 1$] instead of cutting the $x$-axis.

In the following discussion they worked with a local conjecture: “if the graph goes down again then it is a fourth degree polynomial.”

- **Agnes:** Have we mixed it up with a fourth degree?
- **Anna:** Have we been working with those?
- **Agnes:** No.
- **Anna:** Let’s try a fourth degree polynomial to see if starts going down. Or a fifth degree...
They looked at one example of a graph of a fourth degree polynomial (which did cut the $x$-axis) and then something in the discussion or in the appearance of the graph to the fourth degree polynomial made the students abandoned the conjecture immediately. It is difficult to find out why the students behaved like this. Some possibilities could be the following:

- The students didn’t really believe that the fourth degree polynomial would be present in the work since they hadn’t “been working with those” (see Section 5.6), an example of the first implication of the didactical contract mentioned in Section 2.4. The fourth degree polynomial is also very rare in their textbook (Björk and Brolin, 1996), only three exercises where the students were supposed to draw the graph to a fourth degree (or higher) were found. The didactical contract may have affected the students so that only one example was needed to convince the students, an example of naive empiricism. It is even possible that the students had decided to leave the fourth degree polynomial before they saw the graph.

- Since the conjecture was about a function that would “go down again”, the students might have recognized the graph as a counterexample to the conjecture. The fourth degree polynomial was not the case since the example they looked at did not “go down again”, it cut the $x$–axis in two points. This could be characterized as a level 4 verification, a thought example, since the properties of the fourth degree polynomial was used when the conjecture was rejected.

These two possibilities are almost opposites in relation to the four levels. My personal opinion is that the first possibility is more probable than the second, an opinion based on the students’ performance in the work with the other conjectures.

**Example 3. Conjecture 3.**

Pair D used the graphing calculator rather much. Their work on conjecture 3 mainly concerned three issues: trying to remember what the graph of a third degree polynomial looked like, discussing the signs of $x$ and $x^3$ and regarding the graph as two halves of quadratic functions, symmetrical with one “going down” and one “going up”. In the attempts to remember what a third degree polynomial looked like they used guesses, memory images, examples on the graphing calculator and attempts to find counterexamples. In this work the following discussion took place:

**Doris:** A quadratic function doesn’t have to cut, it can go like this. But this one [a third degree polynomial] must, right?

**Dagmar:** Yes, I think so. It must always go up.

**Doris:** If we do like this [graphs $y = -x^3$ on the graphing calculator]. No it will only go in the other direction [compared to $y = x^3$]. But... [graphs $y = x^3 + 2x$] only that one [points at the curve around the inflection point] gets bigger. It’s still hard to prove.

**Dagmar:** Yeah, I think so too.

**Doris:** [Graphs $y = x^3 - 5$]

**Dagmar:** The terrace, or that one [points again at the inflection point], will be at minus five.

**Doris:** Yes. It still seems as if it always must cut the $x$-axis.

After several examples on the calculator they appear to be convinced that the conjecture is true, but they also seemed to be aware that the examples and graphs they had been using were not enough for a proof. The students here used verification at level 1 to convince
themselves, several examples of third degree polynomials were used. It must be pointed out that a higher level verification was later used in their written work (see Example 7). They seem to use the examples in order to understand the problem, the first step in Polya’s strategy for problem solving (Pólya, 1945).

Two of the three situations at level 1 in this study were observed in the work of pair A, the only low performing pair in the study.

5.2. Level 2. The crucial experiment. Only one example of level 2 verification occurred in this study:

**Example 4.** Local conjecture: *Can you always get a negative value?*

Pair E had some problems with conjecture 3. They discussed two different ideas. One was that you will always get both positive and negative function values, and the other was that you can solve the corresponding equation if the constant term is zero. The following discussion illustrates how the two different ideas were active all the time. Erik argues about factorizing the polynomial and Egon tries to verify the local conjecture “if you have a third degree polynomial then you can always get both positive and negative function values”:

- **Egon:** Wait, what you said about this \([x^3 + 10000x^2 + 3x]\]
- **Erik:** We can throw that away.
- **Egon:** But if you have...
- **Erik:** You can just factorize \(x\).
- **Egon:** If it is plus, say, 10, 100, \([writes x^3 + 10000x^2 + 3x + 100]\) can you always get a negative value? You should since it’s to the third.
- **Erik:** You must still be able to get...
- **Egon:** minus... ten millions... to the third... It must be smaller, plus this, ten millions to the second, times ten thousand, should be a negative value.
- **Erik:** Yeah... but a proof... It’s only this stupid constant term.

The last comment by Erik shows that the students are discussing different things. One interpretation is that Erik agrees that the local conjecture is true, but continues to argue that in order to verify conjecture 3 they need something more, preferably a way to deal with the constant term.

Egon found that *even this function will get negative eventually.* This is an example of a level 2 verification, version 2, since the conjecture is tested in a rather special case. The conjecture, “if you have a third degree polynomial then you can always get a negative value”, is also formulated in a general way.

5.3. Level 3. The generic example. Two groups, pair B and E, performed very similar verifications of conjecture 2. These two were the only level 3 verifications found in the study. The work of pair B will be described here.

**Example 5.** Conjecture 2.

Pair B almost immediately found a way to deal with conjecture 2:

- **Bengt:** Since it’s symmetrical, the quadratic function, it looks the same on both sides. Then you should be able to say, if you solve it you will get where it cuts, and then the derivative, if you put that equal to zero, it should be exactly midway between the intersections.
- **Bosse:** Yes, it can’t possibly be any other way.
Bengt: Yes, the maximum or minimum value, if you solve a quadratic equation you will get, like, the symmetry line, what is in front of the square root in the solution is the middle, and that’s where the derivative should be zero.

After this discussion it didn’t take long for them to complete an explanation. They worked with the function $y = kx^2 + px + m$. They showed that the derivative of the function is $y' = 2kx + p$ and that the derivative is zero at $x = -\frac{p}{2k}$. They then found the solution of the equation $y = 0$, or $kx^2 + px + m = 0$, which is $x = -\frac{p}{2k} \pm \sqrt{\left(\frac{p}{2k}\right)^2 - \frac{m}{k}}$. The students now saw that the point $x = -\frac{p}{2k}$ was present in both expressions. From this they drew the conclusion that $x = -\frac{p}{2k}$ was between the solutions of the quadratic equation. They also commented that the point always will exist since $k \neq 0$. They made the following written explanation including some arrows to the solutions mentioned above:

The point $x = -\frac{p}{2k}$ is between the solutions. The value of $x = -\frac{p}{2k}$ can always be solved [meaning found] since $k \neq 0$. If $k = 0$ the function would be a first degree.

This verification of conjecture 2 is an example of a level 3 verification, the generic example. The truth of the conjecture is shown by the effects of operations on a representative of all quadratic functions. This is a good and accurate deductive proof which would be accepted by mathematicians.

5.4. Level 4. The thought experiment. This was the most frequent level in the study, many discussions were at a relatively abstract level. The students often had problems trying to formulate a written explanation which would correspond to and summarize their discussion (see Example 1). Three examples of student discussion at level 4 will be presented here:


When the students in pair C started working with conjecture 2 they knew directly that it was true. They also said that it would be in the minimum or maximum point of the function. They said:

Curt: We can look at $x^2 - 5$.
Carl: Then we have to differentiate.
Curt: But it must be always. You can say like this, if it’s supposed to cut in two points it must look... either $x^2 \pm ax - b$ or $-x^2 \pm ax + b$ [probably meaning $(x \pm a)^2 - b$ and $-(x \pm a)^2 + b$], because if it’s a plus in front it must be placed below and vice versa. If that helped. Yes, you can say... the derivative will be negative on one side and positive on the other side. Then it must be zero somewhere in between.

The two first comments would lead to a level 1 verification, but in the third comment Curt pointed out that in order to prove the conjecture they must show that it is always true. The last part of the third comment contains the core of their written explanation. They later finish the discussion:

Carl: Then we just have to prove this in some way.
Curt: It should really be enough for a proof, since the slope where it cuts is not zero. [...] If it cuts two times it must cut the other direction the second time.

After this they wrote down their explanation:

If the curve cuts the x-axis in two points the derivative in the intersections must be $> 0$ in one point and $< 0$ in the other. As a quadratic function
The students drew conclusions from the properties of the objects. The derivative of a quadratic function must have different signs in the two zeroes and it must become zero somewhere in between. However, they claimed that this was true since the function is continuous. This is wrong, it is the derivative that must be continuous. The discussion was not entirely correct, but it can still be considered a level 4 verification.

Example 7. Conjecture 3.
The work by pair D on conjecture 3 was mainly about three things: trying to remember what a third degree polynomial looked like, discussing the signs of $x$ and $x^3$ and regarding the graph as two halves of quadratic functions, symmetrical with one “going down” and one “going up”. The first part has been described in Example 3. The second part came mostly from Doris. The following discussion describes her ideas:

Doris: If the exponent is odd, like in a third degree polynomial, it will always be the same, the $x$ and the $y$-value will always have the same value [this is incorrect, even if the student means the same sign it is only true for sufficiently large values of $x$]. But then, if it’s a minus in front it will be different.

Dagmar: Will this explain the situation?
Doris: I think it does, because...
Dagmar: That it will cut the $x$-axis?
Doris: It must do that since the values below zero will have $y$-values below zero and the values above zero will have $y$-values above zero.

In the third part the students discussed the possibility to construct the third degree polynomial using two halves of two quadratic functions [this is not correct, a third degree polynomial is not a combination of two quadratic functions]:

Dagmar: It’s like a quadratic function, but it goes up instead.
Doris: One half of a quadratic function.
Dagmar: Yes...
Doris: Or, you can... if you have the two base-quadratics [students own terminology], those are $x^2$ and $-x^2$, the only ones. One goes like this [sketches a graph on the paper, see figure 3] and the other one like this [sketches more]. And the third degree polynomial is half of each [sketches the combination]...

When they finally decided to use the second part to explain why the conjecture was true they wrote:

Since the exponent is an odd number in a third degree polynomial the $y$-value will always have the same sign as the $x$-value. This shows that the curve must be both positive and negative in $y$ and therefore it must cross the $x$-axis. This is true.

They believed that they would be able to convince their classmates if they together with their explanation “could talk and show graphs”. The last sentence in the written explanation, “this is true”, was added when they remembered that they were supposed to determine if the conjecture was true or false.

This is a written verification at level 4, the thought experiment. The students discuss the situation by looking at the properties of the object involved, a third degree polynomial. It is not complete and not correct, they have not mentioned that the cubic term will dominate only for sufficiently large values of $x$ or that the function is continuous.
Example 8. Conjecture 3.
When the students in pair B started working on conjecture 3 their first idea gave them a good start:

Bengt: Will it always cut? I don’t know…
Bosse: Yes it does, since a negative $x$-value always will become negative and a positive $x$-value always will become positive. It’s only in the middle it’s messing around.

The statement by Bosse is only correct for some special third degree polynomials, but the last part indicates that the student have some knowledge about the general behavior of the polynomial. This idea could probably lead to a rather accurate verification of the conjecture, and they continued to discuss the idea for a while.

Bosse: $x^3$ is always more than $x^2 + x$.
Bengt: Is it? Not for $x=1$! You said always! [laugh]
Bosse: Yeah, but, except for 1. When $x$ is more than one… But on the other hand, the equation may look like this: $ax^3 + bx^2 + cx + m$. $c$ can be really big, and… $b$ can also be very big.
Bengt: $a$ also.
Bosse: Yeah, but…
Bengt: That won’t help us much.
Bosse: $x^3$ will in the end always be more than both $b$ and $c$. According to the holy rule. When you get to 100, $x$ to the third will be much more than $x$ squared.

At this point they suddenly changed their minds about how to deal with the conjecture. They started looking at the slope: "If we differentiate that one we will get a zero-point. And on both sides… On one side it will go up and on the other down. And it will be steeper and steeper.” Now the students started writing something about the function and the rate of change of the function and the derivative of the function. They quickly got out in very deep water, they really didn’t know what they were doing. The students noticed this too, they used the expressions “totally insane” and “we are loosing ourselves.” They completed the following written explanation:

$x$ much bigger than 0 makes positive and negative rates of changes respectively, for $f(x)$.
much smaller than 0 makes negative and positive rates of changes respectively, for \( f(x) \).

When the rate of change is \( \pm \) on one side and \( \mp \) on the other side the function will cut the \( x \)-axis in between.

What this explanation really means or what it is about is very hard to understand, and the students probably did not understand it either. When asked why they were convinced that the conjecture was true they said:

**Bosse:** Because you know that if \( x \) is negative \( x^3 \) will be negative too. For very big \( x \)-values \( x^3 \) will be more than \( x^2 \) and \( x \), and the same for positive values. So the graph will always increase and decrease. Therefore it will always cut the \( x \)-axis.

**Tomas:** And that is what you have written here?

**Bengt:** [laugh]

**Bosse:** Yes, but it feels like it’s a bad explanation.

On a question if their classmates would accept their explanation they answered “very doubtful”, probably because they didn’t really understand it themselves.

The students’ written explanation is incorrect. It is very weakly connected to the conjecture. However, the students’ spoken answer to the question why they were convinced is an example of a level 4 verification, it is close to a correct discussion of the conjecture (see Appendix A). It is a level 4 verification since the truth of the conjecture is indicated using the properties of the third degree polynomial, not operations on the polynomial.

5.5. Not categorizable. There were also some situations that could not be categorized. A few local conjectures, most of them quickly abandoned, were not possible to categorize either. This is not really a problem since in those situations the discussions were either mostly wrong or mostly irrelevant. One example of a not categorizable situation is the written explanation made by pair B on conjecture 3, see Example 8. This explanation is mostly irrelevant and also very hard to understand and therefore it is not categorizable. Another example is the work by pair D on conjecture 2:

**Example 9.** Conjecture 2.

The students in pair D read the conjecture, sketched a quadratic function and said:

**Dagmar:** The tangent, yes, like that [draws a tangent], at the bottom, where the slope is zero.

**Doris:** [reads the conjecture again]

**Dagmar:** It always turns and the slope is zero.

**Doris:** It should be like that. Ok... [starts writing]

They started to write down an explanation almost at once after they had read the conjecture. There was no hesitation and no discussion if the conjecture was true or false, they only discussed how to formulate the written explanation. They wrote:

\[
\text{In a quadratic function the derivative of the maximum or minimum value is always zero, and then the tangent is horizontal.}
\]

After this they were satisfied, about 6 minutes after they first saw the conjecture. The researcher then asked if they thought it was a sufficient answer to the question but did not get any response. He said:

**Tomas:** You have said that there is a point where the tangent is horizontal. You have not discussed where that point is, if it is between the intersections.

**Doris:** It will be like... This is a symmetry line [points at the \( y \)-axis], it’s the same on both sides and where they meet it’s horizontal.
The students’ written work on this conjecture is not categorizable using the four levels, since they have only stated some facts. However, it is possible that what the students really meant was relevant, that many of the students’ thoughts about the conjecture remained unspoken.

Another possibility is that they believed that they had answered the question. That would be an example of the third implication of the didactical contract listed in Section 2.4, “a task is solved when the easiest question in it is answered”. The students’ written explanation is not an answer to the question “why is the conjecture true?” It is an answer to the question “where will a quadratic function have a horizontal tangent?” which can be considered one of the easier parts of the verification of the conjecture. From the students’ point of view it might be an attempt to clarify the situation, or a part of an explanation. However, it is not a full description. The argument by Doris concerning the symmetry of the curve should clearly have been a part of the explanation to make it more stable. The didactical contract might have affected the students to answer below their abilities, they had knowledge that should have been included.

In Example 2 it is difficult to categorize the work by the students. Two almost opposite possibilities are mentioned. The problem, however, is not in the use of Balacheff’s levels, but rather in the interpretation of the students work.

5.6. The didactical contract. A general observation in the study was the students’ wish to use abstract or advanced mathematics. The following excerpt from the work of pair B on conjecture 1 shows this wish:

The students directly stated that the conjecture was false. They seemed a bit surprised that it was so easy to see:

Bengt: That is not correct [laugh] because... it can just go above.
Bosse: Or below.
Bengt: It’s really enough with an example where it doesn’t work and the conjecture fails.
Bosse: Yes, but we can also do like this, a bit more fancy...

It seems like the students tried to find something which in their eyes was more mathematical than just a counterexample. After some work they found that the difference between the two functions was a new quadratic function, and for certain values of the coefficients this new function would have no zeroes. They made a few errors in this procedure, but the main idea was correct as well as the final written answer. Here the students discussed the new function \( y = kx^2 + px + m \):

If the \( m \)-value is larger than the total of the \( x \)-values \( (kx^2 + px) \) at the maximum/minimum of the function, the function never cuts the \( x \)-axis. It then has no solution. If the function has no solution, the two separate functions doesn’t cross each other.

When they were finished with their answer the following discussion between the researcher and the students took place:

Tomas: Why are you convinced that you have a correct answer?
Bengt: We saw it on the calculator.

[...] Tomas: Do you think your classmates will be convinced by your explanation?
The students seemed inclined to use abstract mathematics to verify the conjecture. However, they believed that their classmates would be more convinced by an example. The origin of the wish to use abstract mathematics could be how the students believe they are supposed to deal with problems of this type. In their textbook (Björk and Brolin, 1996) almost every new concept is introduced using abstract reasoning, but very few exercises where abstract reasoning is needed can be found. One might also expect that most teachers follow the textbook rather much. The students might recognize the conjectures, and the task to verify them, as discussions around concepts. The normal way to deal with new concepts is to discuss them at a high level.

There was another clear situation where the students wanted to do more than “just” a counterexample when they were working on conjecture 1. Pair E drew a sketchy graph of two functions that did not intersect and then said “can it be that easy?” This resulted in a discussion about two inequalities, $f(x) \leq g(x)$ and $f(x) \geq g(x)$ where $f(x)$ and $g(x)$ denoted a linear and a quadratic function respectively.

When the students were asked about counterexamples directly after the session one answer was “we have been told many times that one example is never sufficient as a proof”.

In these examples, and also the situations discussed in examples 2 and 9, the students’ choice of strategy can be explained to some extent using the theory of the didactical contract. The students may attempt to verify the conjectures at a higher level than what they themselves think is necessary, because they want the work to match the ‘normal’ way, the way it is done by the teacher and in the textbook. The last answer by Bengt in the discussion above is one example where the students thought that their classmates would be more convinced by some examples than by the written explanation.

6. Discussion

The overall picture of the students’ work with the conjectures is that most students discuss and try to verify the conjectures at a high level according to Balacheff’s hierarchy. In most cases the students fail to accomplish a correct verification at level 4. One reason for this might be that they are unused to this kind of activity.

The students’ discussions of the conjectures are sometimes at a higher level than their written explanations, and several spoken discussions are more correct than the produced written work. The students have problems when it comes to formulating something in writing, but even so, only one written explanation at level 1 and no written explanation at level 2 were found. The written explanation at level 1 was preceded by a discussion concerning the properties of the objects involved (see Example 1). All examples of verifications at level 1 and 2 were followed by higher level discussions and written explanations. The difference between spoken and written answer is also discussed by Biggs and Collis (1982):

There is no doubt that individual testing - the students answering orally to the teacher’s questions - would result in higher level responding. (p. 179)

The reason for this is, according to Biggs and Collis, that “there has been little or no opportunity to question ambiguities in the students’ response, to clarify his meaning, or to allow him to elaborate”. The setting in this study is somewhat different from the situation discussed in the book by Biggs and Collis, but the arguments are still valid.
Examples of students activities at all four levels have been found in the study, but no clear connections between student achievement and level of reasoning have been found. The only indication that high performing students use higher level reasoning than low performing students, is that two of the three level 1 activities described in the study were performed by the only low performing pair. On the other hand, the only level 2 verification was made by the only high performing pair. To say anything about connections in this area is not possible from the data in this study.

The students’ difficulty when it comes to formulate something in writing combined with their wish to use abstract reasoning, could imply that students in upper secondary school in Sweden are inclined to use abstract reasoning to a rather large extent, but that they suffer from lack of practice. This result, that the students wish to use abstract reasoning, differs from many other studies where students often have been found to use naive empiricism to convince themselves and others. Hoyles (1997) says:

Many students have a limited awareness of what proof is about. On the one hand, they show a preference for empirical argument over any sort of deductive reasoning and seem to fail to appreciate the crucial distinction between them: for example, many students judge that after giving some examples which verify a conjecture they have proved it. Yet, on the other hand, students tend to assume that deductive proof provides no more than evidence, with the scope of the proof’s validity being merely the diagrams or examples in the text. (p. 7)

The students in this study show a certain will to use deductive, or at least abstract, reasoning. They all try to verify the conjectures in what they think is a mathematical way. One reason for the difference between this study and previous studies might be that the students in this study are a little older than in most other studies.

The students higher age could also be one important reason why Balacheff’s levels sometimes were inappropriate in this study, the students mainly presented verifications at level 4. The students in this study were older than the students in Balacheff’s study, 16 - 17 years compared to 13 - 14 years. Older students should in general have a greater mathematical experience, since they have been exposed to more mathematics, especially abstract mathematics. It is possible that Balacheff’s levels fail to discriminate between the students’ different ways to verify conjectures since the students mainly use abstract reasoning. Another possible reason is the difference in mathematical focus, calculus compared to geometry. It might be easier or more natural to use examples in geometry than in calculus. The fact that Balacheff performed his study in France and this study took place in Sweden may also have caused some differences.

The way the students interpreted what they were supposed to do could to some extent be explained using the the didactical contract. It was found to be a possible explanation of why the working methods, both form and content, were chosen in the way they were. The students often seemed inclined to produce something ‘mathematical’, something that would be normal with this type of questions. They seemed to believe that in order to verify a conjecture you must discuss it in a general way, a few examples are not enough. Several examples where students have ideas about mathematical proofs and about the types of mathematics they were supposed to use have been found in the study, see Section 5.6.

Future research in this area will be necessary in order to analyze the teachers’ practice when dealing with verification of conjectures in upper secondary school. It would be interesting and important to see how the teaching and the textbooks correspond to the
students’ possible wish to use abstract reasoning. If there is a difference in how teachers view students, and how the students really work, this difference must be enlightened in order to help teachers to adjust their teaching to an appropriate level.
Appendix A. The mathematics in the conjectures

Conjecture 1: “A linear function and a quadratic function always intersect in two points.”
A correct mathematical language would be “The graph of a linear function and the graph of a quadratic function always intersect in two points”. However, in everyday speech it is rather common to say “intersection of functions” when what you really mean is “intersection of the graphs of functions”. A counterexample to this conjecture is the following: the graph of the linear function \( f(x) = x - 5 \) never intersect the graph of the quadratic function \( g(x) = x^2 \) since the equation \( x^2 = x - 5 \) has no real roots.

Conjecture 2: “If the graph of a quadratic function cuts the x-axis in two points there is a point between the intersections where the tangent to the graph is horizontal.”
The derivative will have different signs in the two zeroes. Since the derivative of a second degree polynomial is a continuous function, the value of the derivative must be zero somewhere between the zeroes which means that the tangent will be horizontal.

Conjecture 3: “The graph of a third degree polynomial always cuts the x-axis.”
For sufficiently large (or large negative) x-values the cubic term of a third degree polynomial will dominate; a general third degree polynomial, \( ax^3 + bx^2 + cx + d \) where \( a > 0 \) can be factorized into \( x^3(a + \frac{b}{x} + \frac{c}{x^2} + \frac{d}{x^3}) \), and for sufficiently large values of \( x \) the sum of the three terms \( \frac{b}{x} + \frac{c}{x^2} + \frac{d}{x^3} \) will be smaller than \( a \) and therefore the expression in the parenthesis will be positive and so the polynomial will have the same sign as \( x^3 \) and also the same sign as \( x \). Because of this the function will assume both negative and positive values and since the function is continuous it will assume all intermediate values, including zero.

Appendix B. Acknowledgement

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How Students Verify Conjectures: Teachers’ Expectations

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Abstract. Eight teachers were interviewed about student activities on verification of conjectures. The study is a sequel to a previous study, “How Students Verify Conjectures” (Bergqvist, 2000). Teachers’ expectations of students’ reasoning and how students perform are examined, and also how they wish students would work. The results indicate that the teachers tend to underestimate the students’ reasoning levels and that some teachers believe that only a small group of students in each class can use higher level reasoning in mathematics.

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1. Introduction

When students in the Swedish upper secondary school were given some mathematical conjectures and asked to decide if they were true or false, the students seemed inclined to work
in a relatively abstract way (Bergqvist, 2000). Most discussions of the conjectures were
about the properties of the involved mathematical objects, or operations on objects. Very
few students seemed to believe that empirical evidence could be used to verify the conjectures. Section 2 is a short presentation of the study “How Students Verify Conjectures”
(Bergqvist, 2000), below referred to as ‘the previous study’.

This study will compare the results of the previous study with the outcomes of interviews
with upper secondary school teachers in Sweden. The teachers were asked how they
thought students would perform when asked the questions in the previous study. The
teachers were also asked questions about how they would like students to deal with the
conjectures.

2. How Students Verify Conjectures

The study examined how five pairs of students worked when they were told to verify or
reject three conjectures. In order to analyse the different approaches, the hierarchy of
proof levels constructed by Balacheff (1988) was used.

2.1. The four levels. The types listed below are not proofs in a strict mathematical
meaning, but rather something that is recognised as such by the student. The levels are:

Level 1: Naive empiricism.
To be convinced that a conjecture is true, and to argue that it is true after verifying
some cases. Example:
All linear functions intersect the y-axis. I tested four different linear functions and
it was true every time.

Level 2: The crucial experiment.
Two different versions of this was discussed by Balacheff, but only the following
was appropriate in the study:
Testing the conjecture in a special case and draw the conclusion that “if it works
even for this it will always work”. The difference from Level 1 is mainly that the
students are aware of the problem of generality. Example:
All linear functions intersect the y-axis. I tested the function y = 1000x – 1000.
Even that one intersect the y-axis. Then I’m sure that all linear functions intersect!

Level 3: The generic example.
To show the truth by manipulating an object which is used as a representative of all
similar objects. The proof is indicated by the effect of the operations. Example:
All linear functions intersect the y-axis. y = 2x + 3 is a linear function. To
intersect the y-axis means that x is 0. In this functions we get y = 3 when x = 0.
This can be done for all linear functions.

Level 4: The thought experiment.
An abstract member of a class is discussed. The proof is indicated by looking
at the properties of the objects, not on the effects of operations on the object.
Example:
All linear functions intersect the y-axis. A linear function is described by a polyno-
mial of degree 1. Such a polynomial is defined for all x, including x = 0. Therefore
all linear functions intersect the y-axis.

It is important to remember that these levels deal with the working method that is being
used, not with how correct or successful the students are. A student activity can be
classified to be at Level 4, without being fully correct or complete.
2.2. The students. In the study, students were chosen from a class in their second year of a three-year natural science program (eleventh school year, 16-17 years of age). The same year they had had their first meeting with calculus. Five pairs of students were chosen and then asked if they were willing to participate. The selection was made in cooperation with their teacher so that both high and low performing students were included. The teacher and researcher also paired the students so that each pair would have students of the same gender and approximately the same achievement level. There were six boys and four girls in the study.

2.3. The conjectures. The students were asked to decide if the conjectures were true or false, and to make a written explanation why. The written explanation should be aimed at students in upper secondary school (their classmates). The following three conjectures were presented to the students:

- **Conjecture 1**: The graphs of a linear function and a quadratic function always intersect in two points.
- **Conjecture 2**: If the graph of a quadratic function cuts the $x$-axis in two points there is a point between the intersections where the tangent to the graph is horizontal.
- **Conjecture 3**: The graph of a third degree polynomial always cuts the $x$-axis.

Conjecture 1 is false, Conjectures 2 and 3 are true.

2.4. Results. Examples of activities at all four levels could be found in the study. Fourteen situations were classified, three at Level 1, one at Level 2, two (identical) situations at Level 3 and the remaining eight at Level 4. This differs from previous research where researchers have found that students prefer Level 1, naive empiricism, when they are asked to verify or prove something (Hoyles, 1997; Chazan, 1993, see).

The inclination to use abstract reasoning and also some other examples of student behaviour could be interpreted using the concept of the didactical contract (Brousseau, 1997; Blomhøj, 1994, see). In several situations the students’ wishes to use abstract or advanced mathematics were very clear. This could be explained by looking at their textbook (Björk and Brolin, 1996), where the sections in most cases start by looking at a new concept in an abstract way but very few student tasks that demand abstract reasoning can be found. Maybe the students recognise the conjectures as discussions about new concepts which should be discussed in an abstract way because “that is how it’s done in the book”.

The results of the study could be summarised like this: Students in upper secondary school in Sweden are inclined to use abstract reasoning in mathematics, but they suffer from lack of practice.

3. Reasoning in school mathematics

The word ‘reasoning’ is often used in various school documents. One example of this is the Swedish syllabus for upper secondary school mathematics (Skolverket, 2001), according to which the school shall strive for the students to “...develop their ability to follow and carry out mathematical reasoning...”. Most of the time, mathematical reasoning is referred to as something positive, something more than only saying ‘multiplication gives this result’, without any closer definition.

Discussions around and, in some cases, definitions of mathematical reasoning, or some aspect of mathematical reasoning, have been made by some researchers (see for example
Schoenfeld, 1985; Lithner, 2000b). A lot more can be found connected to the concept of proof (Chazan, 1993; Dreyfus, 1999; Hanna and Jahnke, 1996; Hoyles, 1997, and others).

Mathematical reasoning can be of different types and also of different quality. Lithner (2000a) studies one aspect of quality in mathematical reasoning, when he discusses whether the base for the reasoning lies in the mathematics involved or in (perhaps mathematically superficial) experiences from the learning environment. Balacheff (1988) focuses on another quality in the reasoning, a quality which has to do with the level of abstraction. The expression ‘higher quality reasoning’ will here be used in a broad sense to indicate some kind of more developed mathematical reasoning.

In this article I will use the same focus as Balacheff, the aspect of mathematical reasoning that can be described using his four levels. The students’ level of reasoning will be related to the hierarchy of proof levels, in such a way that work at Level 1 and Level 2 will be called lower level reasoning and work at Level 3 and Level 4 will be called higher level reasoning. It would not be correct to say that higher quality reasoning is the same as higher level reasoning. It is certainly possible to find examples where students use high quality reasoning around an example, something that possibly could be defined as an activity at Level 1, lower level reasoning.

In the study, interviewees sometimes use expressions like ‘analytical reasoning’, ‘advanced mathematical reasoning’ and ‘abstract reasoning’. These will all be treated as higher quality reasoning, and, when appropriate, higher level reasoning.

4. Research questions

The main research question in this study is

- How do teachers expect students to work when they are trying to verify conjectures?

This question can be divided into smaller parts:

(1) What level of abstraction are the teachers expecting from students?
(2) Will the teachers describe their expectations differently when they can choose among a list of invented examples (see Appendix A) of student activities?
(3) What kind of differences between teacher expectations and student performance can be found?
(4) What are the teacher comments on the (possible) differences?

5. Method

To use qualitative interviews in order to get information about teachers’ expectations was a somewhat obvious choice of method, even if a questionnaire or an analysis of teachers’ grading of student exercises was considered. Few other methods would give the kind of information wanted. As Kvale (1997) says, it is possible to perform rather similar interviews if you have well structured and tested interview guide. Therefore, the interview layout and discussion topics was tested in two interviews with upper secondary school teachers. The pilot interviews were performed in advance, and the interview questions and interview setting were revised after each pilot interview.

In the study eight teachers were interviewed about their expectations on students working with the three conjectures presented in Section 2. The interviews were audiotaped and transcribed by the researcher.
The analysis was made using the computer software QSR NUD*IST Vivo\textsuperscript{1}, which is a program for handling qualitative data analysis.

5.1. The teachers. Eight teachers were interviewed in the study, six men from one school and two women from another school. All teachers accepted to participate when they were asked by a contact person at their school. The age of the teachers ranged from 29 up to 62, rather evenly distributed. The teachers all had the mathematical education required to teach at upper secondary school in Sweden. All teachers had received the three conjectures in advance and been told that the interview would be about how students would deal with the conjectures. All teachers except one had been teaching students on the natural science programme, and all teachers had experience from teaching the mathematical content in the previous study (Bergqvist, 2000)).

5.2. The interviews. The interviews were divided into three parts. In the first part the teachers were asked how they thought students would deal with the three conjectures. They were also shown four invented examples of student activities, one at each level (see below), and asked to try to decide which description would match their own expectations best.

In the second part of the interview the results from the previous study were presented. The teachers were invited to comment on the students’ performances and reasoning levels during the presentation.

The third part was a more general discussion concerning the type of activity used in the study, the possible effects in class and what possibilities the teachers have to use different methods and new ideas in class.

The following topics were discussed in all interviews:

- How do you think the students dealt with the conjectures?
- Here are examples\textsuperscript{2} of student activities at Conjecture 2 and 3. Which examples describe best how you think students would work?
- What would you like them to do?
- What is the right way?
- Here are the results of the previous study. Comments?
- Can students benefit from discussions about conjectures?
- Do you use this kind of exercise in class?
- Is there any time to do such things?
- Have you studied any more mathematics than what is necessary for your profession?

The interviews in this study were transcribed by the researcher. The reason for this was an attempt to minimise the transformation of the material which follows every transcription. Each interview took approximately one hour.

6. Analysis

The central findings in this study are the following:

- Teachers tend to underestimate students’ levels of reasoning, but the estimation is more accurate when the teachers can choose among a list of invented examples of student activities.

\textsuperscript{1}More information can be found here: http://www.scolari.co.uk/qsr

\textsuperscript{2}The examples can be found in appendix A
• Some teachers are inclined to think that an algebraic solution or the use of mathematical expressions indicates high performing students.
• Teachers seem to think that there are only a few students in each class who can use high level reasoning.

6.1. Underestimation of level of reasoning. The teachers mainly believed that the students would work at a relatively low level of reasoning. The major part of a class would certainly work with examples, at Level 1 and 2 according to the hierarchy used in the previous study.

Example 1. Excerpt from interview with teacher Martin (invented name) concerning the students work on Conjecture 2: *If the graph of a quadratic function cuts the x-axis in two points there is a point between the intersections where the tangent to the graph is horizontal.*

**Martin:** Let’s see... I would say that it’s true. And I have taught the students, or told the students this, earlier we did all the graphing by hand, and we made models of parabolas, what the quadratic function looked like. How you could use the coefficients to make it broader, reflect it, move it sideways and up and down. They need to understand the concept, the parabola, how it is constructed. What kind of... \( y = ax^2 + bx + c \), you can have a coefficient in front of the \( x \) also, all this decides the width, the height and sideways. The student got to cut shapes out of paper, I remember. Partly standard parabolas. A lot of drawing. Then they will catch the concept, what the parabola looks like. It’s like... between the intersections you have the mirror image, and... you have the turning point there also. I think they realise that.

**Researcher:** That the conjecture is true?
**Martin:** Yes.
**Researcher:** Because they recognise the shape?
**Martin:** Right, it’s symmetrical.
**Researcher:** Okey. They realise that it’s true. How do you think they try to explain why?

*silence*

**Martin:** I think they would show a lot of examples of quadratic equations, what they can look like. Not to prove it generally, but showing a lot of special cases. That all parabolas have something in common.

**Researcher:** Some kind of generalisation?
**Martin:** Yes, and that between the intersections there is a turning point, a top or a bottom. The students would... I don’t think they would start discussing the derivative, to find that the slope is zero between the points. That is also a way, but I don’t think they would do it like that.

The excerpt shows that Martin expects the student to work at Level 1, *Naive empiricism*. The students would “...show a lot of examples ...” He explicitly states that the students would not use the properties of the derivative to verify the conjecture. Martin shows his opinion again when discussing the third conjecture, *the graph of a third degree polynomial always cuts the x-axis*. He says:

I think they could verify this conjecture, by drawing a number of third degree polynomials and check. Students at this level are often satisfied with that, just convinced, not shown generally.
Other teachers have similar expectations on students’ reasoning on Conjecture 2. The teacher Jimmy said:

To solve this more generally, you should start with $ax^2 + bx + c$ and then show that...it should work...that the derivative is zero. If there is a solution to that equation you can show that the zeros of the derivative is in between. You must be able to see that if you use the coefficients in some way. [...] But I think few would be able to sort that out. Very few.

In the previous study all students used high level reasoning, discussing the properties of the functions and derivatives, when dealing with the conjecture, so what Martin and Jimmy express are underestimations of level of reasoning. Even if the situation for the students in the study was unusual, and might have inspired the students to work differently, the difference between the teachers expectations and the students work was considerable.

The discussion about student performance was often accompanied by other statements, some of them very strong, concerning student behaviour. The invented example of student activity at Level 3 for Conjecture 2 (see Appendix A) was commented by several teachers. The teacher John said:

You know that a student in the second year of upper secondary school doesn’t like to work with this kind of expression.

Another teacher, Lisa, made a similar comment about the same example:

I think that in school, at upper secondary, they very much want something concrete, they prefer numbers in stead of... just the $h$ in the derivative is difficult. ‘$h$? Its enough with $x$ and $y$!’ they say.

In the previous study almost all students worked with abstract expressions, so compared to that study, the comments above show underestimations of level of reasoning.

The difference between expectations and student performances was reduced when the teachers could choose among examples of student activities. It is possible that one reason for the teachers’ difficulty to describe their expectations of student performance is that the questions were of an unusual type. The teachers were simply not used to seeing students work with this kind of questions, but when they could see some examples of possible student activity it was easier to anticipate what they would do. That could indicate that what the teachers said in the choice among examples should be more reliable statements. Another reason for the apparent difference between teacher expectations and student performance could be that the teachers tried to estimate how successful the students would be, while the study examined what reasoning level the students used when working with the conjectures. There is a big difference between these two points of view. A student could work at a high level but not be able to conclude anything from the work. In that case, the teacher expectations of student performance would be correct if the teachers were looking at how successful the students were. At the same time the student’s reasoning could be at Level 4, the highest level, according to Balacheff. But even if this is the case there was a difference between what teachers expected and what really occurred. This can be seen in many comments from teachers, like “that is very good!” and “I’m impressed by their work”. The difference could also be found when looking at what the teachers said in the choice among examples. Even if some teachers now had different opinion and said “possibly more at Level 4 than at Level 1 and 2” and also “I think they start at Level 1 but continue up to Level 3”, most teachers said things like “80 - 90 % at Level 1, a few at Level 2 and 3, and hardly any at Level 4”.

7
6.2. Signs of mathematical understanding. Some teachers in the study expressed that students had a good mathematical ability if the teacher could see a structured algebraic way of working (see Examples 2 and 3). There was also an indication that the use of advanced mathematical expressions was seen as a sign of high performing students (see Example 4).

Example 2. Excerpt from interview with John (invented name) during the presentation of the work by a group of students on Conjecture 1 and 2. The students first said that it would be enough with one example to reject the first conjecture. They changed their minds and started to work algebraically with the expressions \( y_1 = ax + n \) and \( y_2 = kx^2 + px + m \). The work ended up in a discussion that a function defined as the difference between the two functions \( y_1 \) and \( y_2 \), sometimes could have no zeroes, which is almost the same thing as a simple rejection of the original conjecture.

John: They were reasoning about the mathematics all the time... they could have said... gone further so to say.
Researcher: When they first saw the conjecture one of them said that it would be sufficient with one example. The other student answered “yes, but we can also do like this, a bit more flashy”.
John: They realised the solution, but wanted to spice it up a little.

John: Did you say that these students were average performing? I really think they approach the problems at a higher level than an average student. Their method of working can be seen in both conjectures. This is almost the same method as we would use.

The students’ algebraic work on Conjecture 1 was unnecessary and the statement “almost the same methods as we would use” is here not really correct, since mathematically you don’t have to do anything more than to show one counterexample. In spite of this, John seemed to like what they did, and also indicated that the algebraic approach would be a sign that the students are high performing. After John had seen the work of all groups he said

There was one pair early in the presentation [he refers to the same group as in Example 2] that had a very good algebraic reasoning, even if they came out wrong on the third conjecture.

The fact that he again wanted to comment on the students who used an algebraic approach indicates his preference for algebraic methods.

Example 3. The preference for algebraic methods can also be found in this excerpt from the interview with David. The discussion here concerned the invented examples of student solutions for Conjecture 2 (see Appendix A).

David: [...] I think they have a good picture of the shape of a quadratic function, and they can arrive at conclusions about how it is. If you are of the opinion that this is Level 4 [points at the example], to reason around a graph like that, I think most students would do so. But that might just be what I’m hoping for.

Researcher: How would you do this (Conjecture 2) yourself?
David: I would do like that (Level 4). If someone asked me in class I would draw graphs to reason around.
Researcher: And maybe discuss this, about increasing and decreasing...
David: Yes, and this could be a nice... That would be the first step and to show it then... but what did you say these levels were? I would think that this (Level 3) is higher than this (Level 4).

A proof at Level 3 is based on operation while a proof at Level 4 is based on the properties of the concepts involved. Both are acceptable as solid proof by mathematicians, but a proof based on properties would in most cases need a higher level of understanding. This is also what Balacheff says when he argues that the four proof levels form a hierarchy. David’s opinion that Level 3 is ‘higher’ than Level 4 and his statement “If you are of the opinion that this is Level 4...” indicates that in his opinion, the form in which the mathematics is presented is equally important or even more important than the content of the mathematics. It is possible that John and David view mathematics mainly as a formal and descriptive science, something Ernest (1989) has labelled as “the instrumentalist view of mathematics”. Ernest argues that a teacher will teach in a way closely related to her/his view about mathematics. He says:

The instrumentalist view is likely to be associated with the instructor model of teaching. (p. 3)

In the instructor model the teacher’s role is to show mastery of skills and correct performance, so that the students can imitate and duplicate her/his actions. This could explain the teachers’ comments in the above interview excerpts. It would mean that if John and David hold the instrumentalist view of mathematics, they would prefer a teaching model close to the instructor model. Then they would clearly be of the opinion that an algebraic approach is a sign of mathematical understanding and that Level 3 is higher than Level 4.

Example 4. Another sign of mathematical ability seemed to be use of advanced mathematical expressions. This is an excerpt from the interview with teacher Lisa (invented name) concerning the work by one group on Conjecture 2. The group presented the following written explanation why Conjecture 2 was true:

If the curve cuts the $x$-axis in two points the derivative in the intersections must be $> 0$ in one point and $< 0$ in the other. As a quadratic function is continuous the derivative must assume the value $0$ somewhere between the points.

Lisa: They managed to include continuous...

Researcher: Yes, but it still came out a little wrong, it’s the derivative that must be continuous.

Lisa: Yes, but to include continuous at all!

Researcher: This function (shows $y = |x|$) fulfils the reasoning, but it has no horizontal tangent. But they still have reached far.

Lisa: They use continuous. I must nag more on my boys.

The word ‘continuous’ was not used in a correct way, but Lisa seemed very impressed by the students’ writing, she also emphasised the word ‘continuous’ in a way that supports this interpretation. That students use advanced terminology could be explained by using the concept of the didactical contract. The students try to do what they believe the teachers want, one of the most central parts of the theory of the didactical contract (Brousseau, 1997; Blomhøj, 1994). If the students think that the teacher, or researcher, will be impressed if they use a mathematical expression, it is not surprising that they use the word continuous in their work, even if they don’t really know what it means.

Of course, one possibility is that the students really were aware that the derivative must be continuous, but since the concept of continuity of derivatives is not a part of the
Swedish upper secondary mathematics syllabus, they used the expression they were more familiar with, the continuity of the function. There is another possible reason why the students used the continuity of the function instead of the derivative, and that is that almost all continuous functions they have met have continuous derivatives. So, maybe the teacher was right in being impressed by the students.

Whether the students were using the word continuous to impress the researcher, or if they understood the concept to some degree is really not possible to say from their written explanation. Therefore, Lisa’s view of the students must be based on something else than the students’ mathematical performance. The use of an advanced expression was to Lisa a sign of high performing students.

### 6.3. Only a few students can use high level reasoning.

Almost all teachers expressed the thought that in each class there is only a small group of students who can use high level reasoning.

**Example 5.** Excerpt from interview with David (invented name) concerning the students’ work on Conjecture 2.

David: I think that many would take a function, invent a quadratic function, and calculate where it cuts the x-axis. Then they would... they have been working with differentiation?

Researcher: Yes, this was in the spring term, year 11.

David: Then I think they would find the derivative, some of them. Find a point, calculate where the derivative is zero, and see that the zero is between the intersections. I have done like this:

David shows something similar to the invented Level 3 example.

Researcher: That’s a good way to deal with the conjecture, right?

David: Among my former students, the high performing [could do like that]. I don’t think the average performing could... maybe a few of the better ones could have done this.

Silence

Researcher: Is this what you would like all to be able to do?

David: Solve it in this general way?

Researcher: Yes.

David: That would be nice.

Researcher: It does show some understanding.

David: Then there are some smart students, but... It will be interesting to see what they did, because you have, there are in classes you have had, smart students. Things you never figure out yourself. I often say that the better you are, the less you have to think and be smart. If you can’t do the formal mathematics, you have to be smart. It has been easy for me, I have always been able to use algorithms, no need to be smart.

David expected only the high performing students, and maybe a few more, “I don’t think the average performing could... maybe a few of the better ones could have made this”, to be able to use differentiation in order to verify the conjecture. This could imply that he expects only a small group to be able to use high level reasoning in order to verify the conjecture.

This idea can also be found in statements from other teachers. In a discussion around the Mathematics textbook he is using, Martin says
There are some exercises, it’s possible that this one [Conjecture 1] can be found in our book (Björk and Brolin, 1996). But there are only a few students who are, well, [good enough ³] so that I can ask them to think about this.

Martin seems to be of the opinion that only a few students in his class can be asked to think and reason around a conjecture of the type used in this study. In a discussion about what students would do when working with Conjecture 3, Harold, another teacher in the study, says

I guess that they would show a lot of examples and check. They will find that for every function they try, it will be true. So, they will claim that it is true, but they would not, I believe, very few would really prove it. They would make the conjecture so likely so that they could decide that it really is the case. One or two really good mathematicians in the class might start to speculate about \( x \) to the third, how fast it increases and decreases, depending on how you change the value of the variable.

According to Harold, the students would try to work at Level 1 and Level 2 when verifying the conjecture. A small group might work at Level 3 or Level 4, “one or two really good mathematicians...”.

All three teachers, David, Martin and Harold, appear to believe that only a part of the students in a class can work at Level 3 or 4. Compared to the students in the previous study, this is not correct, since almost all students in that study worked at Level 4.

One interpretation of the three teachers’ opinion is that the rest of the class can’t use reasoning at Level 3 and 4 in mathematics. If only a few students in each class are supposed to be capable of mathematical reasoning at a high level according to Balacheff’s hierarchy, it would be interesting to know if this is considered to be a permanent situation, or if the ability can be learned in school, if the rest of the class also can learn this kind of reasoning.

David uses the word “smart” in his last statement in Example 5, seemingly to describe students who have shortcomings in their mathematical basic knowledge, but who compensate this by being smart, by being able to reason in unorthodox ways. The view of mathematics that David gives here, that you don’t need to think if you know the formal mathematics, is not so easy to interpret. One possibility is that mathematics is all about methods and algorithms (i.e. the instrumentalist view of mathematics, see Section 6.2), and the most important part of mathematics is to find the right method. Another possible interpretation is that David means that if you understand the mathematics, you don’t have to guess and try to figure out solutions in strange ways. You just know how to do it, because everything is easy when you understand it.

7. Discussion

7.1. Higher quality reasoning in school mathematics. Higher quality mathematical reasoning is considered an important part of mathematics in upper secondary school. In the syllabus for mathematics at upper secondary school in Sweden (Skolverket, 2001) it says “The school shall strive for the students to develop their ability to follow and carry out mathematical reasoning, and to show their thoughts verbally and in writing.” The syllabus is written in a very general way, without explicitly specifying what is meant,
so this is only one possible interpretation. What is called *mathematical reasoning* in the syllabus could also be more related to problem-solving or the ability to implement a mathematical method. Here I make the interpretation that the meaning of the expression “mathematical reasoning” in the syllabus is the same, or at least contains, what I call higher quality reasoning.

If school mathematics in Sweden is supposed to help students to develop their ability to use higher quality reasoning, it should contain activities that require this. One aspect of higher quality reasoning is the ability to discuss mathematical concepts in an abstract way. This is one of the main features of higher level reasoning according to my description. Therefore, one way to “help students to follow and carry out mathematical reasoning” is to use exercises that require higher level reasoning.

7.2. Results of the study. There are three main results of this study:

7.2.1. *The teachers in the study underestimate the students level of reasoning.* In the study I have pointed out several situations where teachers underestimate the reasoning level students use. The teachers mainly believe that the students to a large extent would use examples to convince themselves and as arguments that a conjecture is true. All teachers in this study appeared impressed by what the student in the previous study did.

This result indicates that few teachers in the study expect students to be able to use higher level reasoning. It is not possible to draw well founded conclusions whether the teachers in the study believe that all kinds of higher quality reasoning are hard to reach for students. However the fact that all teachers in the study were impressed with the students’ work might indicate that this is the case.

7.2.2. *Algebraic approaches and use of advanced mathematical expressions are, by some teachers in the study, considered a sign of high performing students.* Some teachers indicated their preference for algebraic approaches. They preferred this before discussions about mathematical properties.

One teacher was impressed by students who used an advanced mathematical concept in an erroneous way.

7.2.3. *Several teachers in the study believe that only a small group in each class can use higher level reasoning.* According to these teachers there is a difference between a group of students who can use higher level reasoning, and the rest of the students. One teacher said that he could only ask a few students to think about the questions used in the study.

7.3. Reflections. This section consists of informal discussions around possible effects of the results. I will present a few ideas and speculations of my own, mainly supported by my experiences as a researcher and a teacher.

7.3.1. *The results.* The first result, that teachers underestimate how students work, would probably be reflected in the kind of classroom activities the teachers use. One example from my own early teaching:

In the textbook used in my classes, there were two kinds of questions the students did not like. One was of the kind “Prove the following”, and the other was “Show the following”. My advice to the students, since I ‘knew’ that they could not use higher level reasoning, was to change *prove* into *show* in the first one and *show* into *understand* in the second.
My students were of the opinion that there is a significant difference between prove and show, where showing something is supposed to be less rigorous and, more important, less difficult. Also, the students often said that they understood something if they could believe that it was correct. The result of this change was probably that the students never got a real chance to practice their high-level reasoning skills. If the teacher (i.e., myself) shows that this kind of questions are not so interesting, and avoid them in the teaching, surely the students will not work very much with them.

The second indication of the study is that some teachers seem to think that algebraic approaches are signs which indicate high performing students. This is not always a correct indication. It is easy to find examples where this is not true. One example from the previous study was when a pair of students started working with the first conjecture, using a lot of algebra instead of applying the proper (and easier) method of showing a counterexample. One important factor when a teacher is supposed to analyse how students work, is the teacher’s own mathematical ability. If the teacher does not recognize the counterexample as the way to reject the conjecture, that is, if the mathematics is very difficult for the teacher, it will be very hard for her or him to understand what the students are doing. It is also possible that the teacher will try to avoid such situations.

The third result indicates that several teachers in the study believe that only a few students in each class can perform higher level reasoning. The differentiation in a class between the small group who can use higher level reasoning and the rest of the class could be a serious problem, especially if the teacher expects this to be a permanent situation. The students that are considered unable to use higher level reasoning risk being kept out of mathematical discussions that involve higher level reasoning. Teachers who believe that not many students can understand high level reasoning, would probably not use this in the classroom, not at all in discussions with the class and only occasionally in exercises. The teacher can argue that since only a few students can do this, it will not be of any use for the rest of the class.

According to the teachers in this study, it is hard to find questions of the type used in the previous study in the most common Swedish textbook (Björk and Brolin, 1996). In the beginning of the interview with Lisa, she said

Lisa: [...] It’s difficult, I’m not used working with this kind of questions.
Researcher: Why is that?
Lisa: Well, you know, the books don’t have this type of questions, and we are, or I am, not ingenious enough [to invent questions]. Not the book I’m working with.

Other teachers also supported this opinion. Some teachers indicated that there are almost no exercises at all that require higher level reasoning. Are exercises that require higher quality reasoning of some other kind also hard to find in the textbooks? In a study by Lithner (2000a) approximately 90% of the exercises in an undergraduate calculus book could be solved by looking at, and maybe slightly modify, an example, theorem or a situation in the text. Solutions of this kind require no higher quality reasoning. The remaining questions were also more difficult because of what Lithner calls “double difficulties” (p. 23), more difficult mathematics and more demanding reasoning. An ongoing study (Johansson, 2001) indicates that the situation is the same in the most frequently used Swedish upper secondary school textbooks. Since exercises that require higher quality reasoning seem to be few, it is possible that teachers say “We don’t engage in that kind of activities much, since it’s almost absent in the textbook.”
Mathematics teaching in Sweden. I believe that many teachers in Sweden follow a very strict routine when they plan and carry through a mathematics lesson. A typical lesson could look like this:

- 5 minutes: an exercise from last lesson is shown on the blackboard (by the teacher).
- 15 minutes: a new concept is described, one or two examples showing methods to treat questions in the new area are presented.
- Remaining time: students practice the methods on their own, while the teacher walks around and answers questions about the methods.

This setting can be used with success. The students learn to solve questions of certain types, and since the teacher will construct the test, the students will probably do rather well. It is possible to do this without high quality mathematical reasoning at all. The few questions in the book that require higher quality reasoning of some kind, are marked with stars or crosses or something (Lithner, 2000a), so the students can easily avoid them.

A report that supports the view that the teaching of mathematics follows a strict routine is the book “The Teaching Gap” (1999) by Stigler and Hiebert. They present the outcomes of a video study, one part of TIMSS. Their results show an image of the American mathematics education where a lesson can be described as consisting of four parts (pp. 80-81):

- Reviewing previous material.
- Demonstrating how to solve problems for the day.
- Practising.
- Correcting seatwork and assigning homework.

They also found that the teachers state concepts in stead of letting the students develop them to a very large extent (pp. 59-61), that the mathematical content is rated to be of a low quality in the majority of the lessons observed (p. 65), and that the students use over 95% of their seatwork practising routine procedures (pp. 70-71). It seems likely that the situation in the Swedish school is rather similar to the situation in American schools. The picture of American lessons is not very different from the description of a typical Swedish lesson I proposed in the beginning of this section.

When Cooney (1999) had been observing mathematics classes in elementary school in the USA, he reported:

There were no instances of students discussing mathematics with each other. In almost every case either the lesson was drill and practice (cleverly done, nonetheless) or the mathematics was reduced to bits and pieces in which memory could suffice for delivering the correct answer. (p. 183)

Cooney argues that this will give rise to a moral dilemma. What will happen when the students first meet a real mathematical problem? He says “I suspect they will conclude something is wrong - with the teacher, with the curriculum or with themselves” (p.183). I believe that this description of school mathematics is rather accurate in upper secondary school in Sweden as well. Then it is important to ask the question in Sweden too: is it morally defendable to teach students in the way we do? Cooney continues:

Do I congratulate them on having children excited about ‘doing’ mathematics? Or should I struggle to weave into my report what I saw as a potential problem for students’ later learning of mathematics? (p. 183)

Shall we congratulate teachers in Sweden for showing students how to perform complicated solution methods, or shall we raise a warning that the students might get into trouble in
their future learning of mathematics because of the absence of high quality mathematical reasoning?

7.3.3. How can we change the mathematics teaching? If the description in the preceding section is accurate, something must be done to increase the amount of high quality reasoning that the school wants to fulfill the goals in the syllabus. One way to do this could be to let students work with questions that require higher level reasoning. The student exercises used in this study, verification of conjectures, is one example of exercises that require higher level reasoning. The students in the previous study and the teachers in this study, both confirm that it is unusual to work that way in Swedish classrooms.

Increasing the use of questions that require higher level mathematical reasoning is not so easy. The first problem is to find appropriate questions. It may not be necessary to have very well developed questions. It could be enough to put up a short sentence on the blackboard and ask the students to discuss it. However, one must be prepared to deal with a lot of different questions and supervising situations. This leads to the second problem. Teachers will not have the kind of control over the classroom situation that they are used to, and they must be able to analyze quickly what kind of help a certain student needs in a specific (and often new) situation. This kind of teacher activity is also more demanding for the teacher, when it comes to her/his subject knowledge in mathematics, since student questions may concern other areas, not only the obvious centre of the question. One example could be that if you ask the students to discuss a quadratic equation, one conceptually difficult question from the students could very well be how to use the factor theorem to solve a cubic equation. It would be even more difficult to supervise this kind of activity without knowledge about the students you are supposed to help. If you don’t know how the students think and react, how can you help them in an appropriate way?

How, then, can we change the situation in the Swedish upper secondary school? If we want to increase the use of questions that require higher level reasoning, we must make the teachers change their practice. I believe there are two ways that are the most effective if we want to influence teachers:

- More questions that require higher level reasoning in the textbooks. The questions should also be of an easier type than the (few) questions that already are present in the books.
- More questions that require higher level reasoning in the Swedish national tests.

The first point is not so easy to achieve if the questions are supposed to be easier than the present questions. The second point is already in progress and will, hopefully, also effect the first point, since teachers probably request textbooks that reflect the demands of the national tests.

7.4. Ideas for the future. The results in this study give rise to several new questions. It will be necessary to perform additional research in order to gain more insight in the classroom situation. A study designed to investigate teachers’ approaches when it comes to higher level reasoning in mathematics, and how teachers grade different kinds of student work could maybe answer some of these questions:

- What kind of teaching is normal when it comes to higher quality reasoning? What does it look like? To what extent is it present in the Swedish school?
How do teachers reward different kinds of reasoning? Is the second result of the study, that some teachers have a preference for algebraic approaches, visible in how teachers grade student work?

Are there any differences between the way ‘the group of students that can use higher level reasoning’ is treated in the classroom and the rest of the class? Is there really such a division?

Are teachers of the opinion that higher level reasoning can be taught and learned, or that this competence is something that a few gifted students are born with?

Appendix A. Examples

The invented examples shown to the teachers looked like this:

Conjecture 2: If the graph of a quadratic function cuts the x-axis in two points there is a point between the intersections where the tangent to the graph is horizontal.

Level 1: I draw the graph of $y = x^2 - 3$. I see directly that I can draw a horizontal tangent to the curve. I can also do this to the graph of $y = -x^2 + 2$.

Level 2: If you look at the graph of $y = x^2 + 0.0001x$ you can see that the graph cuts the x-axis in two points which are very close to each other, so close that the tangent is almost horizontal in both intersections. I’ll check. No, if I zoom in on the intersections I can see that the conjecture is true even for this function. Then it’s probably true for all quadratic functions.

Level 3: The quadratic function $y = ax^2 + bx + c$ has a horizontal tangent for $x = -\frac{b}{2a}$. The function cuts the x-axis on both sides of this point since the equation $y = 0$ has the solution $x = -\frac{b}{2a} \pm \sqrt{\left(\frac{b}{2a}\right)^2 - \frac{c}{a}}$. Hence the conjecture is true.

Level 4: A quadratic function which cuts the x-axis has a positive derivative in one intersection and a negative derivative in the other. Since the derivative is continuous it will assume the value 0 somewhere between the intersections.

Conjecture 3: The graph of a third degree function always cuts the x-axis.

Level 1: I’ll check $y = x^3 - 2x^2 + 3x - 4$. It cuts the x-axis. $y = 5x^3 + 4x$ also cuts the x-axis. So, a third degree function always cuts the x-axis.

Level 2: It could be true for all third degree functions. I look at $y = 2x^3 + 5x + 10000$. The last term, +10000, moves the curve very high up. It still cuts the x-axis. Then certainly all curves cut the x-axis.

Level 3: When you have a third degree function you can always find both positive and negative function values. The function $y = 5x^3 + 23x^2 - 14x + 30$ is positive for almost every $x$, but if you take a sufficiently small $x$, like $x = -100$, it will become negative. You can do like this for all third degree functions. Both positive and negative function values means that it will cut the x-axis.

Level 4: A third degree function will always assume both positive and negative function values because the cubic term will dominate for large x-values. Since the function is continuous the graph will cut the x-axis.

Appendix B. Acknowledgements

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Secondary School Students Using
Graphing Calculators.
Revised Version.

Tomas Bergqvist

Abstract. Upper secondary school students were confronted with an open-ended question in the form of an investigation about translations of quadratic functions. The students were encouraged to use a graphing calculator (a Texas TI-83). An analysis of the students use of the calculator in the investigation was made. The results indicate that the students in the study used the calculator as a means to visualise graphs, and as a tool to verify conjectures.

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1. Introduction

Students in the natural science programme in Sweden are well-equipped when it comes to graphing calculators, normally all students have a device of their own. Many researchers have found and described a lot of different possibilities and positive effects for the students when they use graphing calculators in their work. According to Dunham and Dick
(1994) there are three main positive effects of graphing calculator use: the students in
general become better at interpreting graphs, more able to use multiple representations
(i.e. connecting graphs to tables, equations etc.) and they become better problem solvers.
Pomerantz (1997) says “Using graphing calculators provides a new teaching and learning
paradigm: Graphs can now be used to study math. In the past students studied calculus
to learn how to draw graphs accurately” (p. 14). The following is a similar statement
from Dick (1996):

“Graphs generated by technology can be used to effectively communicate
and discuss the meaning of the derivative rather than using the derivative
as a tool for graphing. Graphing has become a means rather than an ends
in calculus!” (p. 38)

So what can students really do with the graphing calculator? In this study I wanted
to see if the students were able to use the graphing calculator in an investigation. Would
they be able to use it to understand more about the mathematical expression and not
only to draw a few graphs? In this study they were confronted with an unusual type of
exercise, unusual in the sense that there are a lot of ways to answer, depending on the
interpretation of the question (an open-ended question).

This article is a revised version of a pilot study (Bergqvist, 1998) performed in 1998.
The purpose of the revision was to make the article more connected to my other research,
and to make a better analysis of the data.

2. Theory

2.1. Investigations in mathematics. In a previous article (Bergqvist, 1999) I pre-
sented a way to analyse investigations in mathematics as a built on Schoenfeld’s (1985)
episodes in problem solving. Investigations are in my description divided into the following
three main parts:

- Visualisation
- Conjecture
- Verification

Using visualisations is to let the objects be observed, using concrete representations, in
order to identify different possible connections between the objects. If, for example, the
object is a function, it can be described by an algebraic expression, a table, or a graph.
The graph is of course the most obvious example of a visualisation, and here the graphing
calculator is the natural tool.

Stating a conjecture. A conjecture is an assumption or an idea which could become a
result or part of a result. It is mostly derived from visualisations, together with other
observations and previous knowledge.

Verification is a test of a conjecture. This can be done in many different ways, and also
be based on more or less stable reasoning. To test an example, to see if the conjecture
works for other types of objects; to try to find counterexamples or to prove a conjecture
using deductive methods are all examples of verification activities.

2.2. The mathematics in the question. The question given to the students in the
study was to examine the following expression:

\[ x^2 + bx + c \]

They were asked to describe how the expression changes for different values of b and c.
What kind of answers could be expected from the students? What would be a good way to deal with this problem? A start for the students would of course be to look at some examples on the graphing calculator. It is not so difficult to realize that changes of the value of $c$ will move the graph vertically, and changes of the value of $b$ will move the graph mainly diagonally. I will here try to explain some characteristics of the connection between the values of the constants $b$ and $c$ and the position of the graph.

The function $f(x) = x^2 + bx + c$ can be viewed as an addition of three different functions: $f(x) = x^2$, $f(x) = bx$ and $f(x) = c$. First we will look at the case where $b = 0$, the function $f(x) = x^2 + c$. Figure 1 shows the graphs of two different functions, $f(x) = x^2$ and $f(x) = 3$.

![Figure 1. $y = x^2$ and $y = 3$](image1)

Figure 2 shows the sum of the two functions. The function value will for each $x$ be the sum of the function values of the two added functions ($x^2$ and 3). For example: in Figure 1 the function values at $x = 2$ are 4 and 3 respectively, so in Figure 2 the function value for $x = 2$ should be 7.

![Figure 2. $y = x^2 + 3$](image2)

Now we will look at the case where $c = 0$, the function $f(x) = x^2 + bx$. The graph of the function $f(x) = x^2 + bx$ will always have the same shape as $f(x) = x^2$. This can be seen if you complete the square: $f(x) = x^2 + bx = (x + \frac{b}{2})^2 - \frac{b^2}{4}$. The function $f(x) = (x + \frac{b}{2})^2 - \frac{b^2}{4}$ will differ from $f(x) = x^2$ in two ways: by a constant, $-\frac{b^2}{4}$, which is the same as a change in $c$ above, and by a translation in $x$. A translation works like this: $f(x) = x^2$ is 9 for
$x = 3$, while $f(x) = (x - 2)^2$ is $9$ for $x = 5$. The same function value can be found two steps to the right. The curve has the same shape, but it is moved two units to the right.

As shown above, the position of the graph of the function $f(x) = x^2 + bx + c$ is depending on the values of $b$ and $c$. A change of $c$ will move the graph vertically, and a change in $b$ will move the curve both horizontally and vertically. If we compare the function $f(x) = x^2 + bx + c = (x + \frac{b}{2})^2 - \frac{b^2}{4} + c$ with the function $f(x) = x^2$, the curve is moved vertically by $-\frac{b^2}{4} + c$ and horizontally by $-\frac{b}{2}$.

This way to look at the question is rather difficult. Students will probably not be able to discuss it like this, but there are other, maybe not so exact, ways to deal with the question. Several examples of how students work can be found in Section 4.

2.3. Research questions. There are different issues in this study: the use of a graphing calculator and students’ investigations of an unfamiliar type of question. The primary focus of the analysis is on the use of the graphing calculator, with the following research questions:

1. When the students use the calculator, in what way are they using it?
   a. What graphs are drawn?
   b. How much time is the calculator used?
   c. Do they use built-in functions?
   d. What kind of reasons do they give for their choice of graphs to look at?

Questions a - c concern mainly the technical use of the calculator while question d is about a more conceptual aspect of the use of the calculator. The secondary focus of the analysis is on how students investigate an unusual type of question. The central question here is:

2. To what extent, and in what form, do the students use visualisations in the investigation of the question?
   a. Are visualisations used as a base when conjectures are formulated?
   b. How are the conjectures verified?

Another question that also may arise as a part of both Question 1 and Question 2 is:

3. In what situations do the students get stuck? How do they resolve such situations?
To get stuck is here referred to situations in which the students just stop doing something, or when they continue to work without making any progress.

3. Method

In the study I videotaped students, using a document camera, a small video camera with high resolution. The camera was placed directly above the paper where the students were supposed to work. The video shows the students’ work on the paper and with the graphing calculator (Texas TI-83), and it is also possible to hear what they were saying. After a first analysis of the tape I met with the students again to discuss whether my interpretation of their work was acceptable. This second meeting, the interview, took place no more than three days after the first session. The purpose of the second meeting was to make my interpretation of the events in the test situation more accurate.

The students were not told what the study was about. This caused some minor problems since the students kept trying to figure out what I really wanted them to do. The reason
for not telling the students was that I wanted to see if using the graphing calculator was among their first reactions.

Three pairs of students were videotaped in the study. They were all students from one class in the natural science program in the upper secondary school in Sweden. The session took place in October in their second year on the program (school year 11, 17 years of age). They were just about to get their first introduction to calculus. The students were chosen only because they were willing to participate.

They were given the question in the following form:

```
Describe how the expression
x^2 + bx + c
changes when the values of b and c varies.
Summarise your results
in a short report.
```

**Figure 3.** Question to the students.

The question was deliberately formulated to be vague. The reason was that I wanted to see how the students would react to an unfamiliar question. Would they directly connect the expression to graphs or would equations or something else be their first reaction? The idea was to see how they started out and then, if necessary guide them (or tell them) into looking at graphs using the graphing calculator.

In the study I was acting both as a teacher and as the leader of the experiment, something that can be problematic. To be an observer and researcher at the same time as you try to guide the students into a specific working area is not always a good situation, there is a risk that you influence the students to behave in a certain way. Bell (1993) says “The interviewers are people and not machines, and might therefore influence the interviewees unconsciously” (p. 92, authors translation). This problem has been reduced in two ways in this study. The first is that everything was videotaped, including the interference from the teacher, which made it possible to take such influences into account in the analysis. The second is that the session was divided into two parts, first to see how they would start (without interference from me), and then, after my possible intervention, how they would deal with the question using a graphing calculator (with a minimum of interference from me).

4. **Empirical data**

This section is a brief overview of the work of the three groups, aiming to give an introduction and an idea of how the groups performed. More specified excerpts will be presented in the analysis section, in the discussion of the research questions. Photocopies of the students’ answers (in Swedish) can be found in Appendix A.

4.1. **The first pair.** The first pair consisted of two boys, Frank and George. Seven seconds after the question was presented to them they said “OK”, and started working. Nothing in the question said anything about the value for x, so they stated that $x = 2$. They quickly found that increasing c by 1 would increase the value of the expression by 1, and increasing b by 1 would increase the value of the expression by 2. At this point I
started to guide them gently into looking at graphs: “Why should \( x \) be 2?” I asked them. They said that it was just an assumption, and started to do the same calculations for \( x = 3 \). My next suggestion was to write “\( y = \)” in front of the expression, and ask if they could see the expression in any other way, but they only appeared confused. I had to say explicitly that I wanted them to look at graphs before they started doing this. I did not mention the calculator, but they started using the graphing calculator immediately.

Now they quickly found how the graphs changed according to the values of \( b \) and \( c \). It only took a few examples for them to see that \( c \) corresponded to the intersection with the \( y \)-axis and that \( b \) decides where the symmetry line will be. Their summary:

- \( x^2 \) tells us only that this is an \( x^2 \)-curve
- \( b \) gives the symmetry line ( \( \frac{b}{2} \)=symm.line)
- \( c \) gives where the curve cuts the \( y \)-axis.
- When \( b \) is negative, the curve will be moved to the positive \( x \)-axis
- At a negative \( c \)-value the curve will cut the \( y \)-axis at the neg. part.

The students quickly realised the sign error in the second point when it was pointed out. The two last points were added after I mentioned that they had only looked at positive values for \( b \) and \( c \).

4.2. The second pair. The second pair, Hans and Irina (a boy and a girl), appeared very uncertain in the beginning, something they also confirmed in the interview. They had never been confronted with a question of that type before. They discussed the question for some time, using words like equations, graphs, increase, decrease, halving etc. They also mentioned “curve” and “symmetry line”, but the concept symmetry line was not used for about twenty minutes. After a while I started to guide them into looking at graphs. This was easy to do since they were very good at listening to what I said. (This may also be a well-developed technique to find out what to do.) So they started looking at graphs using the calculator. They made a misinterpretation about the value of \( b \). They said “a smaller \( b \) gives a smaller expression which moves the curve lower down”. This became a conflict when they revisited the first two graphs. They solved this problem by looking at several graphs, two graphs at a time, and found that they had made an error. Their summary:

- If \( c \) is changed, the curve is moved up or down
- If \( b \) increases the symmetry line will move to the left, if \( b \) decreases it will move to the right.
- If the absolute value of \( b \) increases the curve will be moved down and the zeros will get further apart.

4.3. The third pair. The third pair was a girl and a boy, Katarina and Johan. They started using the calculator after only one minute and Katarina (who seemed to be “in charge”) had very well developed methods for the calculator. One example is that they looked at the graphs \( x^2 + x + 1 \), \( x^2 + 2x + 2 \), \( x^2 + 3x + 3 \) and \( x^2 + 4x + 4 \) to verify their theories about both \( b \) and \( c \) at the same time. Johan was the most experienced user of the calculator in the hole study when it came to using all its possibilities. He was the only
student to use built-in functions on the calculator. He used the minimum-function to see if the minimum value increases with 1 when the value of \( c \) increases with 1. He also used the zoom-function and the zero-function.

The students got a major problem in their work when they focused too much on the word changes in the question. As a consequence they didn’t realise that the symmetry line was at \(-\frac{b}{2}\). Instead, they tried to find a connection between the change of \( b \) and the change of the symmetry line. This relationship is rather complex, so they couldn’t make it out. This complexity, together with shortage of time, made it necessary for me to clarify the situation in order to make it possible for them to proceed. Their summary (after my clarification):

- \( c \) is the place on the \( y \)-axis where the graph will intersect
- the \( x \)-value for the minimum point is half the value of \( b \)
- if \( c \) increases with 1 the minimum value of the graph also increases by 1

5. Analysis

In the analysis I will try to answer the research questions (see Section 2.3):

5.1. Question 1. When the students use the calculator, in what way are they using it?
   a. What graphs are drawn?
   b. How much time is the calculator used?
   c. Do they use built-in functions?
   d. Do they give reasons for their choice of graphs to look at?

1a. What graphs are drawn?
When the students used the graphing calculator to draw graphs of quadratic functions on the form \( y = x^2 + bx + c \) all three pairs used similar strategies. The most common form was to draw two, three or four graphs in the same window (standard size) and look at similarities and differences. The values of \( b \) and \( c \) were in almost all cases integers between 1 and 5. Two of the pairs drew only four graphs each (always two and two), but in several different combinations. The third pair (Johan and Katarina) was in a way more structured than the other pairs, since they drew three or four graphs at a time. They were also holding first \( c \) and then \( b \) constant. They drew a lot more graphs than the others, sixteen different graphs in different combinations. They were the only group to set one of the constants equal to zero.

1b. How much time is the calculator used?
Even though the number of graphs drawn was small for two of the pairs and larger for one pair, the amount of time the calculator was used was rather similar for all groups. The calculator was placed on the table in front of the students for more than two thirds of the time for all pairs. After the point where they were told (or guided) into looking at graphs, the calculator was the centre of their focus almost the whole time. They pointed at graphs, discussed intersections and other visible features of the graphs. It seems as if they used it as something to communicate around. Maybe it is not so surprising that the
students used the calculator a lot, since they were more or less told to use it.

1c. Do they use built-in functions?
Only one pair (and mainly one of the students in that pair) used built-in functions, like zoom or trace. They, on the other hand, used it a lot. Especially the minimum-function was used, in order to find the coordinates of the minimum points of the quadratic functions. The other two pairs could probably have benefited from the built-in functions. They complained that they couldn’t separate two functions from each other (could have been managed using the zoom-function), and that they couldn’t see the x-value of the minimum point (the minimum-function). One reason why they did not use such functions could be problems in the handling of the calculator. It is possible that they didn’t know how to use the functions, or maybe they were unaware of the existence of such functions.

1d. Do they give reasons for their choice of graphs to look at?
In many situations it was very difficult to see why the students chose specific graphs to look at. The first impression was that they just picked two values, not to small and not to big, as constants. It is possible that they were influenced by the way similar graphs are used in the classroom or in the textbook, where integers (often positive) are predominating. Another possibility is that the students were aware that they faced a situation where they looked at something new, something unknown, and that in such situations it is normally a good idea to start with small positive integers. Here follows short descriptions of how the groups chose their first graphs.

Frank and George started their work by stating that x should be 2, and then they found that the value of the expression increases by 2 when b increases by 1 and that the expression increases by 1 when c increases by 1. In their two last examples, before I told them to start working with graphs, they had b = 4 and c = 5 together with b = 20 and c = 21. At this point I intervened and told them to work with graphs. The first two graphs on the calculator was $y = x^2 + 4x + 5$ and $y = x^2 + 20x + 21$. It seemed very natural for them to take the values they had been using in the end of the first part of their work.

Hans and Irina started with $x^2 + 2x + 1$ and $x^2 + 4x + 1$ without any observable reasons:

**Hans:** We can just assume some values of b and c. Lets see...

**Hans types in** $y = x^2 +$

**Irina:** Draw two curves so we can see the difference.

**Hans:** Yeah. Plus two x... and plus one...

**Hans finishes the functions:** $x^2 + 2x + 1$ and $x^2 + 4x + 1$.

It took some time for Hans to come up with the two functions, but it was not possible to decide what he was thinking about. Maybe he wanted to find two functions appropriate for the question, or maybe he was just trying to figure out any two functions.

Katarina told Johan to start without c-value ($c = 0$):

**Katarina:** Type in x-squared, eh, plus, what shall we take...one, and two.

**Johan:** b equals one and c equals something, then the answer will be... [types $x^2 + x+$] plus c, what shall c be?

**Katarina:** We’ll take no c for the first. You don’t need to have c, right?

*The graph of y = x^2 + x appears.*
Katarina: OK. Type in one more. Let's type in different ones and look at the difference.

Katarina changed her mind, in her first sentence she proposed \( b = 1 \) and \( c = 2 \), but when Johan asked her about \( c \) she suddenly said that they should start without \( c \). This might come from insight in the question, that it will be easier to answer the question if you take one constant at a time. Another possibility is that she remembered something from her lessons, that their teacher or their textbook sometimes uses the technique to let one value be zero while changing another.

In some situations it was possible to identify the students reason for a certain choice, like when Hans and Irina looked at \( x^2 + 2x + 2 \) and \( x^2 + 2x + 1 \) to verify that “the first one will get one step higher”. Situations of this type will be further analysed in Section 5.2.

5.2. Question 2. To what extent, and in what form, do the students use visualisations in the investigation of the question?

a. Are visualisations used as a base when conjectures are formulated?

b. How are the conjectures verified?

These two questions are very much related to each other, so they will be treated together.

Visualisations using the graphing calculator seemed to be a very important part of the students’ work. They formulated conjectures based on what they saw to a very large extent. Their previous knowledge about quadratic functions was of course also an important factor. I will here give some examples of how the students in the study used visualisations when they formulated and verified conjectures.

Example 1. In this sequence we can see how Katarina and Johan dealt with the value of the constant \( c \).

Katarina: Let’s try to just change \( c \) now.

*She graphs \( x^2 + x + 1, x^2 + x + 2 \).*

Katarina: When \( c \) increases, the graph moves up.

Johan: Will it go up one unit when \( c \) increases by one?

*Johan uses the built-in minimum function, He finds \( y = 0.75 \) and \( y = 1.75 \).*

Johan: Yes, it goes up by one all the time.

Katarina: OK, check a third one too.

*Johan adds \( x^2 + x + 3 \) to the other two graphs and examine the minimum value, he finds \( y = 2.75 \).*

Katarina: Well, then!

They saw on the calculator that the graph moved up when \( c \) increased. It seems likely that the base for Johan when he proposed his conjecture was the visual appearance of the graphs, possibly connected to some earlier experience of functions, presumably linear functions. To refer to linear functions when discussing the meaning of the constant \( c \) is not so far fetched. Another student (George, see Example 4) said in a similar situation “…So \( c \) is the same as \( m \) in this?”. All students were definitely more familiar with the linear function than with the quadratic function. The linear function is normally described by the expression \( y = kx + m \). Johan and Katarina verified the conjecture by testing two specific cases. The second test was clearly made to be even more certain that the conjecture was true.
Example 2. At the end of their session, Katarina and Johan made a rather special test to check their conjectures about both $b$ and $c$ at the same time. They looked at the following four graphs:

\[ y = x^2 + x + 1, \quad y = x^2 + 2x + 2, \quad y = x^2 + 3x + 3 \text{ and } y = x^2 + 4x + 4 \]

They found that the graphs intersected the $y$-axis at $y = 1$, $y = 2$, $y = 3$ and $y = 4$, all in support of their conjecture about $c$. They also examined the minimum points of the four graphs (using the built-in function), and could verify that the $x$-value of the minimum point was $\frac{b}{2}$. It is interesting to notice that they never discussed that fact that the calculator gave a negative $x$-value, it was referred to as a positive value all the time. This sign error was also made by Frank and George.

Example 3. Hans and Irina were looking at two graphs on the calculator, see figure 4.

![Figure 4](image-url)

\[ f(x) = x^2 + 2x + 1 \text{ and } f(x) = x^2 + 4x + 1 \]

When they saw the graphs Hans said “a smaller $b$ gives a smaller $y$-value” and a little later he said “$b$ makes it go down”. They used the visualisation of the two graphs to state the conjecture: *a smaller $b$ makes the graph go lower down.* Although this conjecture is false, they used visualisations on the graphing calculator as a base for their conjecture.

Hans and Irina had stated a false conjecture. Four minutes later they were trying again to check the conjecture. The following discussion took place:

**Hans:** It [the graph] gets more to the left for bigger [he points at $b$], and it gets higher up for larger . . . [he points at $c$].

**Irina:** Higher and more to the left?

*They look at the same two graphs as in Figure 4 on the calculator.*

**Hans:** But it gets lower down . . . [Confused, since this contradicts their conjecture]

*They check again which graph is which by redrawing them.*

**Hans:** It gets lower! [Very sure]

In this discussion they found out that their first conjecture was false, and a new conjecture was formulated: *when $b$ increases the curve will go lower down.*

The students used visualisations on the graphing calculator as a base, first to state a conjecture, then to check if it was correct. When they experienced that the conjecture was false, they used visualisations again, when they stated a new conjecture.

Example 4. Here is another example of how the students tried to verify their conjectures. Frank and George stated that $c$ gives where the $y$-axis is intersected, and $b$ gives where the symmetry line is. They verified the statement by predicting results in two steps:
George: We can type in a curve, where we have the same \( b \)-value, we only change \( c \).
Frank: Mhm, let’s do that.
They look at \( y = x^2 + 4x + 5 \) and \( y = x^2 + 5x + 5 \).
George: They intersect at the same place on the \( y \)-axis.
Frank: Yeah, but that’s because they have the same \( c \)-value.
George: So \( c \) is the same as \( m \) in this?
Frank: Well, not really, this isn’t \( y = kx + m \), but... OK, you can say that.
George: Change and do the opposite now, then.
Frank: Wait, the symmetry line is at 2.5 and at 2, and that’s because we have 4 and 5... [indicating the value of \( b \)]
George: OK. Let’s take the same \( b \) and different \( c \) now.
They look at \( y = x^2 + 4x + 5 \) and \( y = x^2 + 4x + 3 \).
George: They intersect at 3 and 5 at the \( y \)-axis.
Frank: Yeah, and the symmetry line is still at 2.5. [Note: the symmetry line is at \( x = 2 \), Frank’s error.]

First they checked two graphs using the same \( c \) and different \( b \), and then they looked at the other possibility, the same \( b \) and different \( c \). In both cases their conjecture was verified. Even if they only did this check for one specific case for each constant, they were very convinced that they were right.

The answer to the research question 2 would be that the students in this study used the calculator almost all the time in their investigations. They looked at some graphs at a time, and they tried to draw conclusions from what they saw. In general, the students in this study were very quick to draw conclusions and to state conjectures. In the cases where the students tried to correct an incorrect idea it took considerably longer time (like when Hans and Irina were trying to understand what was wrong in their conjecture, see Example 3).

5.3. Question 3. In what situations do the students get stuck? How do they resolve such situations?
The students kept on working according to their initial idea even when they met serious problems. This behaviour has been documented by other researchers, for example Schoenfeld (1985). When they believed that they had found a correct method of working it took considerable effort from my side to make them change. The most obvious example of this behaviour was when I tried to guide Frank and George into working with graphs. I had to tell them very clearly that I wanted them to look at graphs. Another example of this behaviour was when Johan and Katarina tried to relate the change of \( b \) to the change of the symmetry line. They could not figure out this relationship (it is rather complicated), but they kept trying (see Section 4.3). None of these two groups made any progress until I intervened. One reason for this might be that they were not really aware that they were stuck, it is possible that the students were certain that they were using a correct strategy.

A few times the students were confronted with conflicting ideas. One clear example of this is when Hans and Irina made their incorrect conjecture about the meaning of the value of \( b \) (see Example 3). It took a long time for the students to sort it out, and several times they appeared very confused. They resolved the situation by testing their conjecture by visualising the situation, and then they modified the conjecture. Finally they could verify their new conjecture.
6. Discussion

6.1. Results from the study. Are secondary school students able to use a graphing calculator in a creative way when examining a mathematical expression? In the analysis I have pointed out several situations where the students in this study use the calculator in a way that I would call creative. Some examples:

- Frank and George looked at some graphs, claimed that \( c \) gave the intersection with the \( y \)-axis and \( b \) gave the symmetry line. First they checked two graphs using the same \( c \) and different \( b \), and then they looked at the other possibility, the same \( b \) and different \( c \). When the check verified their conjectures, they were satisfied (see Example 4).
- Hans and Irina made the misinterpretation that ”a smaller \( b \) gives a lower curve”. This created a conflict when they looked at some other graphs. After some time they made a special test for exactly this: they drew \( x^2 + 4x + 1 \) and \( x^2 + 2x + 1 \), realised their mistake and corrected it (see Example 3).
- Johan and Katarina checked their theories for \( b \) and \( c \) at the same time by drawing \( x^2 + x + 1 \), \( x^2 + 2x + 2 \), \( x^2 + 3x + 3 \) and \( x^2 + 4x + 4 \) simultaneously (see Example 2).

All groups used visualisations on the calculator as a basis for conjectures about the graphs and how they depended on the values of \( b \) and \( c \). They also used the calculator to verify different conjectures. The students would hardly have been able to work like this without the graphing calculator (or some other graphing device). Drawing graphs is a very time consuming activity, and graphs that students draw by hand are often not very accurate. Therefore it would have been necessary for the students to used some other strategy in the work with the question.

The students in the study used the calculator, directly or indirectly, almost all the time. It seemed natural for all three groups to use the calculator. Apart from the problem mentioned in the discussion about Question 1d, none of the groups had any real problems in the handling. The calculator was used in two somewhat different ways:

- Directly, as a tool to look at specific situations, certain graphs, or test a certain conjecture.
- Indirectly, as a catalyst to spark ideas and to bring the discussion forward.

These different ways were sometimes active at the same time. When the students looked at a graph for a specific reason, something else came to their mind. One example of this is the last example in the list above. When Johan and Katarina looked at the four graphs, they were intrigued by the fact that all four graphs passed through the point \((-1,-1)\). They spent some time to analyse the situation, but then went back to the more important issue of verifying their conjecture about \( b \) and \( c \).

The students also showed some limitations in their work. All students verified their conjectures using only one or two examples. One reason for this might be that the verifications always were made by the students in order to convince themselves, and not to prove the conjectures. In one of my other studies, ‘How students verify conjectures’ (Bergqvist, 2000), I found that several students distinguished between convincing themselves and proving that something was true. Two students in that study discussed a conjecture, and when they were convinced that the conjecture was true one of them said “Then we just have to prove this in some way” (p. 17). It appeared as if those students were aware of the difference between accepting that something is true and to prove that it is true.
Another limitation was that all pairs only used integers between 0 and 5 as examples of values for $b$ and $c$. To do like this could be a problem if there are differences in the behaviour of the mathematical concept when you deal with fractions, large numbers or negative numbers.

### 6.2. Other observations

It was very interesting to see the students’ reactions when the question was presented. There were clear differences between the reactions of the three pairs:

- Frank and George looked at the question for seven seconds, and then started to work with calculations, probably a rather normal reaction among upper secondary school students (Palm, 2001).
- Hans, in the second pair, appeared rather confused about the question. First asked me if the other groups got the same question, and then he asked Irina “Do you know the answer?”
- If Johan in the third pair tried to lower my expectations on his performance, or if there was something else behind his reaction is hard to tell. When he saw the question, he first laughed, and then he said “Well, math’s not my strong side...”.

Some incorrect ideas appeared of which at least one may be caused by the calculator. The calculator related misunderstanding was that Irina said “when the curve gets lower down, it gets broader”. She repeated this opinion at least three times during the session. The interpretation, that the shape of a curve is the shape you can see in a specific window, reveals a rather serious lack of understanding in the area of functions and translations of functions.

Another rather interesting situation was when Johan and Katarina used the minimum-function on the graph of $y = x^2 + 4x$. The calculator gave the result $x = -2.000001$ and $y = -4$. They directly (and correctly) interpreted the $x$-value as exactly $-2$. It is not possible to answer the question if the students were aware of the calculator’s limitations, or if they made the interpretation for some other reason.

### 6.3. Reflections about implications for the Swedish school

The results of this study indicate that exercises of the type that was used here can lead the students into using the calculator in a creative way. I argue that the calculator could be an important part of the educational setting, partly as a powerful tool and partly as a focal point in discussions around mathematical concepts. The teachers have an opportunity to let students use the calculator in their work for other things than just calculations, it may help the students to develop their creativity. I believe that in upper secondary school in Sweden today, the calculator is used almost exclusively as a tool. It is possible that the Swedish textbooks are too focused on methods and algorithms, or that the teachers use the books in a way which encourages memorisation of methods and algorithms. It is easy to find exercises to in the books where the students will not need to use anything more than to remember a method in order to solve the problem. Lithner (2000) found in a study of a calculus textbook (Adams, 1995) that around 70% of the exercises could be solved by identifying the related solved example and copy the procedure. An ongoing study (Johansson, 2001) indicates the same proportions in the most common Swedish textbook (Björk and Brolin, 1996). In such situations, the calculator is nothing more than a tool. I also believe that the teachers in Sweden teach in a way that is very closely connected to the presentation in the textbooks, which could lead to student activities and working

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1The calculator gives different results depending on how you put the left and right bound when you use the *minimum* function.
methods that are very focused on methods and on memorising solved examples from the textbook. In a large video study, presented by Stiegler and Hiebert (1999), students in America were found to use 95% of their seatwork\textsuperscript{2} time practising routine procedures. I recognise several characteristics in the description of the American school from my own experiences in the Swedish school. The planning of a normal lesson and the way teachers ask questions to the students are just two examples. If the Swedish students also spend the major part of their time practising routine procedures, the calculator is probably be used almost exclusively as a tool. In that case the calculator’s pedagogical potential is poorly used.

However, these are my personal beliefs and speculations of the situation in upper secondary school, based on the results from this study and my own experiences as a teacher.

6.4. Conclusion. The students in this study used the calculator in a creative way in several situations. They used the calculator to visualise the interesting mathematical concept in the question in order to be able to state conjectures concerning the solution. They also used the calculator in attempts to verify their conjectures.

\textsuperscript{2}individual activities in school
Appendix A. The students' answers in Swedish

Here are photocopies of the students' written answers:

\[
\begin{align*}
* x^2 & \text{ bestämmer enhet av det är en } x^2\text{-kurva} \\
* b & \text{ bestämmer symmetriliyen } (b = \text{symtérilinje}) \\
* C & \text{ bestämmer var kurvan skär } y\text{-axeln.} \\
* \text{När } b \text{ är negativ försjuts kurvan till pos. } x\text{-axeln} \\
* \text{Vid neg. } C \text{ skär kurvan } y\text{-axeln på den neg. delen.} \\
* \text{Är } x^2 \text{ pos. har kurvan en minimipunkt } \left( \begin{array}{c}
\end{array} \right) \\
* \text{Är } x^2 \text{ neg. har kurvan en maxipunkt } \left( \begin{array}{c}
\end{array} \right)
\end{align*}
\]

FIGURE 5. Answer produced by Frank and George.

\[
\begin{align*}
* \text{Om } C \text{ ändras så flyttas kurvan uppåt eller nedåt} \\
* \text{Om } B \text{ ökar flyttas symmetrilijen åt vänster, minskar } B \text{ flyttas den åt höger} \\
* \text{Om beloppet av } b \text{ ökar så flyttas kurvan nedåt och skärningspunktarna kommer längre ifrån varandra.}
\end{align*}
\]

FIGURE 6. Answer produced by Hans and Irina.

\[
\begin{align*}
C & \text{ är stället på } y\text{-axeln där grafen kommer att skriva} \\
\text{x-värdet för minimipunkten} & \text{ är hälften av } b
\end{align*}
\]

FIGURE 7. Answer produced by Johan and Katarina.


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