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Long paths and connectivity in 1-independent random graphs

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Abstract

A probability measure μ on the subsets of the edge set of a graph G is a 1-independent probability measure (1-ipm) on G if events determined by edge sets that are at graph distance at least 1 apart in G are independent. Given a 1-ipm μ , denote by \mathbf{G}_μ the associated random graph model. Let $\mathcal{M}_{1,\geq p}(G)$ denote the collection of 1-ipms μ on G for which each edge is included in \mathbf{G}_μ with probability at least p . For $G = \mathbb{Z}^2$, Balister and Bollobás asked for the value of the least p_\star such that for all $p > p_\star$ and all $\mu \in \mathcal{M}_{1,\geq p}(G)$, $(\mathbf{G}_\mu)_\mu$ almost surely contains an infinite component. In this paper, we significantly improve previous lower bounds on p_\star . We also determine the 1-independent critical probability for the emergence of long paths on the line and ladder lattices. Finally, for finite graphs G we study $f_{1,G}(p)$, the infimum over all $\mu \in \mathcal{M}_{1,\geq p}(G)$ of the probability that \mathbf{G}_μ is connected. We determine $f_{1,G}(p)$ exactly when G is a path, a complete graph and a cycle of length at most 5.

KEYWORDS

extremal graph theory, local lemma, percolation, random graphs

1 | INTRODUCTION

1.1 | Bond percolation models, 1-independence and edge-probability

Let G be a (possibly infinite) connected graph. Spanning subgraphs of G are called *configurations*. In a configuration H , an edge is said to be *open* if it belongs to H , and *closed* otherwise. A *bond percolation*

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model on the host graph G is a probability measure μ on the spanning subgraphs of G , that is, on the space of configurations. Given such a measure, we denote the corresponding random graph model by \mathbf{G}_μ , and refer to it as the μ -random graph or μ -random configuration.

In this paper, we study bond percolation models μ where the states (open or closed) of edges in subsets F_1, F_2 of E in a μ -random configuration are independent provided that the edges in F_1 and F_2 are “sufficiently far apart.” To make this more precise, we make use of the following definition.

Definition 1. Two edge sets $F_1, F_2 \subseteq E$ are k -distant if $F_1 \cap F_2 = \emptyset$ and the shortest path of G from an edge in F_1 to an edge in F_2 contains at least k edges. A bond percolation model μ on G is k -independent if for any pair (F_1, F_2) of k -distant edge sets, the intersections $\mathbf{G}_\mu \cap F_1$ and $\mathbf{G}_\mu \cap F_2$ are independent random variables.

So for example μ is 0-independent if each edge of \mathbf{G}_μ is open at random independently of all the others, that is, μ can be viewed as a product of Bernoulli measures on the edges of G . A well-studied 0-independent model is the Erdős-Rényi random graph $\mathbf{G}_{n,p}$, where the host graph is $G = K_n$, the complete graph on n vertices, and where μ , known as the p -random measure, sets each edge to be open with probability p , independently of all the others.

In this paper, we focus on the next strongest notion of independence, namely 1-independence. Measures that are 1-independent have the property that events determined by vertex-disjoint edge sets are independent. For many 1-independent models, the randomness can be thought to “reside in the vertices.” An important example of a 1-independent model is that of *site percolation* on the square integer lattice. In this case the host graph is the square integer lattice \mathbb{Z}^2 (where two vertices are joined by an edge if they lie at distance 1 apart), and the measure $\mu = \mu_{\text{site}}(\theta)$ is obtained by switching each vertex of \mathbb{Z}^2 on at random with probability θ , independently of all the others, and by setting an edge to be open if and only if both of its endpoints are switched on. Site percolation measures may be defined more generally on any host graph in the natural way.

Site percolation is an example of a broader class of 1-independent measures where we independently associate to each vertex $v \in V(G)$ a state S_v at random, and set an edge uv to be open if and only if $f(S_u, S_v) = 1$, for some deterministic function f (which may depend on u and v). We refer to such measures as *vertex-based measures* (see Section 11 for a formal definition). Vertex-based measures on \mathbb{Z} are a generalization of the well-studied notion of *two-block factors*, which are vertex-based measures on \mathbb{Z} in which the vertex states S_u are i.i.d. random variables and the function f is independent of u and v , see [24] for further details.

An important point to note is that while all 0-independent bond percolation models are a product of Bernoulli measures on the edges of G (with varying parameters), it is well-known that a graph G may support many 1-independent measures which cannot be realized as vertex-based measures or as general “block factors,” see for instance [1, 13, 21]. In particular for most graphs G , it is not feasible to generate or simulate the collection of 1-independent measures of G .

Definition 2. Given a bond percolation model μ on a host graph G , the (lower)-edge-probability of μ is

$$d(\mu) := \inf_{e \in E(G)} \mu\{e \text{ is open}\}.$$

So for instance a p -random measure has edge-probability p , while a site percolation measure with parameter θ has edge-probability θ^2 . The collection of k -independent bond percolation models μ on a graph G with edge-probability $d(\mu) \geq p$ is denoted by $\mathcal{M}_{k, \geq p}(G)$.

Remark 3. Given a measure $\mu \in \mathcal{M}_{k,\geq p}(G)$, we may readily produce a measure $\tilde{\mu} \in \mathcal{M}_{k,\geq p}(G)$ such that $\tilde{\mu}(\{e \text{ is open}\}) = p$ for all $e \in E(G)$ via *random sparsification*: independently delete each edge e of \mathbf{G}_μ with probability $p/\mu(\{e \text{ is open}\}) \in [0, 1]$. The resulting bond percolation model on G is clearly k -independent and has the property that each edge is open with probability exactly p ; the corresponding bond percolation measure $\tilde{\mu}$ thus has the required properties.

1.2 | Critical probabilities for percolation and motivation for this paper

Percolation theory is the study of random subgraphs of infinite graphs. Since its inception in Oxford in the 1950s, it has blossomed into a rich theory and has been the subject of several monographs [11, 17, 27]. The central problem in percolation theory is to determine the relationship between edge-probabilities and the existence of infinite connected components in bond percolation models.

In the most fundamental instance of this problem, consider an infinite, locally finite connected graph G , and let μ be a 0-independent bond percolation model on G . We say that *percolation* occurs in a configuration H on G if H contains an infinite connected component of open edges. By Kolmogorov's zero-one law, for G and μ as above, percolation is a tail event whose μ -probability is either zero or one. This allows one to thus define the *Harris critical probability* $p_{0,c}(G)$ for 0-independent percolation:

$$p_{0,c}(G) := \inf \left\{ p \in [0, 1] : \forall \mu \in \mathcal{M}_{0,\geq p}(G), \mu(\{\text{percolation}\}) = 1 \right\}.$$

Problem 4. Given an infinite, locally finite connected graph G , determine $p_{0,c}(G)$.

One of the cornerstones of percolation theory—and indeed one of the triumphs of twentieth century probability theory—is the Harris-Kesten theorem, which established the value of $p_{0,c}(\mathbb{Z}^2)$ to be $1/2$.

Theorem (Harris-Kesten Theorem [20, 23]). Let μ be the p -random measure on \mathbb{Z}^2 . Then

$$\mu(\{\text{percolation}\}) = \begin{cases} 0 & \text{if } p \leq \frac{1}{2} \\ 1 & \text{if } p > \frac{1}{2}. \end{cases}$$

In this paper, we focus on the question of what happens to the Harris critical probability in \mathbb{Z}^2 if the assumption of 0-independence is weakened to k -independence. In particular, how much can *local* dependencies between the edges postpone the *global* phenomenon of percolation?

Definition 5. Let G be an infinite, locally finite connected graph and let $k \in \mathbb{N}_0$. The Harris critical probability for k -independent percolation¹ in G is defined to be:

$$p_{k,c}(G) := \inf \left\{ p \in [0, 1] : \forall \mu \in \mathcal{M}_{k,\geq p}(G), \mu(\{\text{percolation}\}) = 1 \right\}.$$

¹As observed in [16], a simple k -independent variant of Kolmogorov's zero-one law shows that percolation remains a tail event when we consider k -independent models.

Problem 6. Determine $p_{1,c}(\mathbb{Z}^2)$.

Problem 6 was proposed by Balister and Bollobás [4] in a 2012 paper in which they began a systematic investigation of 1-independent percolation models. Study of 1-independent percolation far predates their work (see e.g., [1, 6, 7, 13, 22, 24]), however, due to important applications of 1-independent percolation models.

A standard technique in percolation is *renormalisation*, which entails reducing a 0-independent model to a 1-independent one (possibly on a different host graph), trading in some dependency for a boost in edge-probabilities. Renormalisation arguments feature in many proofs in percolation theory; a powerful and particularly effective version of such arguments was developed by Balister, Bollobás, and Walters [7].

Their method, which relies on comparisons with 1-independent models on \mathbb{Z}^2 (in almost all cases) and Monte-Carlo simulations to estimate the probabilities of bounded events, has been applied to give rigorous confidence intervals for critical probabilities/intensities in a wide variety of settings: various models of continuum percolation [3, 7, 8], hexagonal circle packings [10], coverage problems [5, 19], stable Poisson matchings [14, 15], the Divide-and-Color model [2], site and bond percolation on the eleven Archimedean lattices [29] and for site and bond percolation in the cubic lattice \mathbb{Z}^3 [9]. The usefulness of comparison with 1-independent models and the plethora of applications give strong theoretical motivation for the study of 1-independent percolation.

From a more practical standpoint, many of the real-world structures motivating the study of percolation theory exhibit short-range interactions and local dependencies. For example a subunit within a polymer will interact and affect the state of nearby subunits, but perhaps not of distant ones. Similarly, the position or state of an atom within a crystalline network may have a significant influence on nearby atoms, while long-range interactions may be weaker. Within a social network, we would again expect individuals to exert some influence in esthetic tastes or political opinions, say, on their circle of acquaintance, and also expect that influence to fade once we move outside that circle. This suggests that k -independent bond percolation models for $k \geq 1$ are as natural an object of study as the more widely studied 0-independent ones.

Despite the motivation outlined above, 1-independent models remain poorly understood. To quote Balister and Bollobás from their 2012 paper: “1-independent percolation models have become a key tool in establishing bounds on critical probabilities [...]. Given this, it is perhaps surprising that some of the most basic questions about 1-independent models are open.” There are in fact some natural explanations for this state of affairs. As remarked on in the previous subsection, there are *many* very different 1-independent models with edge-probability p , and they tend to be harder to study than 0-independent ones due to the extra dependencies between edges. In particular simulations are often of no avail to formulate conjectures or to get an intuition for 1-independent models in general. Moreover, while the theoretical motivation outlined above is probabilistic in nature, the problem of determining a critical constant like $p_{1,c}(\mathbb{Z}^2)$ is extremal in nature—one has to determine what the worst possible 1-independent model is with respect to percolation—and calls for tools from the separate area of extremal combinatorics.

In this paper, we continue Balister and Bollobás’s investigation into the many open problems and questions about and on these measures. Before we present our contributions to the topic, we first recall below previous work on 1-independent percolation.

1.3 | Previous work on 1-independent models

Some general bounds for stochastic domination of k -independent models by 0-independent ones were given by Liggett, Schonmann and Stacey [24]. Among other things, their results implied $p_{1,c}(\mathbb{Z}^2) < 1$.

Balister, Bollobás and Walters [7] improved this upper bound via an elegant renormalisation argument and some computations. They showed that in any 1-independent bond percolation model on \mathbb{Z}^2 with edge-probability at least 0.8639, the origin has a strictly positive chance of belonging to an infinite open component. This remains to this day the best upper bound on $p_{1,c}(\mathbb{Z}^2)$. In a different direction, Balister and Bollobás [4] observed that trivially $p_{1,c}(G) \geq \frac{1}{2}$ for any infinite, locally finite connected graph G . In the special case of the square integer lattice \mathbb{Z}^2 , they recalled a simple construction due to Newman which gives

$$p_{1,c}(\mathbb{Z}^2) \geq (\theta_{\text{site}})^2 + (1 - \theta_{\text{site}})^2, \quad (1.1)$$

where θ_{site} is the critical value of the θ -parameter for site percolation, that is, the infimum of $\theta \in [0, 1]$ such that switching vertices of \mathbb{Z}^2 on independently at random with probability θ almost surely yields an infinite connected component of on vertices. Plugging in the known rigorous bounds for $0.556 \leq \theta_{\text{site}} \leq 0.679492$ [33, 34] yields $p_{1,c}(\mathbb{Z}^2) \geq 0.5062$, while using the nonrigorous estimate $\theta_{\text{site}} \approx 0.592746$ (see e.g., [32]) yields the nonrigorous lower-bound $p_{1,c}(\mathbb{Z}^2) \geq 0.5172$.

With regards to other lattices, Balister and Bollobás completed a rigorous study of 1-independent percolation models on infinite trees [4], giving 1-independent analogs of classical results of Lyons [25] for the 0-independent case. Balister and Bollobás's results were later generalized to the k -independent setting by Mathieu and Temmel [26], who also showed interesting links between this problem and theoretical questions concerning the Lovász local lemma, in particular the work of Scott and Sokal [30, 31] on hard-core lattice gases, independence polynomials and the local lemma.

1.4 | Our contributions

In this paper, we make a three-fold contribution to the study of Problem 6. First of all, we improve previous lower bounds on $p_{1,c}(\mathbb{Z}^2)$ with the following theorems.

Theorem 7. *For all $d \in \mathbb{N}_{\geq 2}$, we have that*

$$p_{1,c}(\mathbb{Z}^d) \geq 4 - 2\sqrt{3} \approx 0.535898 \dots$$

Theorem 7 strictly improves on the previous best lower bound for $d = 2$ given in (1.1) above; moreover, it is based on a very different idea, which first appeared in the second author's PhD thesis [16]. In addition we give a separate improvement of (1.1): let θ_{site} again denote the critical threshold for site percolation. Then the following holds.

Theorem 8.

$$p_{1,c}(\mathbb{Z}^2) \geq (\theta_{\text{site}}(\mathbb{Z}^2))^2 + \frac{1}{2} (1 - \theta_{\text{site}}(\mathbb{Z}^2)).$$

Substituting the rigorous bound $\theta_{\text{site}} \geq 0.556$ into Theorem 8 yields the lower bound $p_{1,c}(\mathbb{Z}^2) \geq 0.531136$, which does slightly worse than Theorem 7. However substituting in the widely believed but nonrigorous estimate $\theta_{\text{site}} \approx 0.592746$ yields a significantly stronger lower bound of $p_{1,c}(\mathbb{Z}^2) \geq 0.554974$.

Second, motivated by efforts to improve the upper bounds on $p_{1,c}(\mathbb{Z}^2)$, and in particular to establish some 1-independent analogs of the Russo-Seymour-Welsh (RSW) lemmas on the probability of crossing rectangles, we investigate the following problems. Let P_n denote the graph on the vertex set $\{1, 2, \dots, n\}$ with edges $\{12, 23, \dots, (n-1)n\}$, that is, a path on n vertices. Given a connected graph

G , denote by $P_n \times G$ the Cartesian product of P_n with G . A *left-right crossing* of $P_n \times G$ is a path from a vertex in $\{1\} \times V(G)$ to a vertex in $\{n\} \times V(G)$. We define the *crossing critical probability* for 1-independent percolation on $P_n \times G$ to be

$$p_{1,\times}(P_n \times G) := \inf \left\{ p \in [0, 1] : \forall \mu \in \mathcal{M}_{1,\geq p}(P_n \times G), \mu(\exists \text{ open left-right crossing}) > 0 \right\},$$

that is, the least edge-probability guaranteeing that in any 1-independent model on $P_n \times G$, there is a strictly positive probability of being able to cross $P_n \times G$ from left to right.

Problem 9. Given $n \in \mathbb{N}$ and a finite, connected graph G , determine $p_{1,\times}(P_n \times G)$.

Problem 9 can be thought of as a first step toward the development of 1-independent analogs of the RSW lemmas; these lemmas play a key role in modern proofs of the Harris-Kesten theorem, and one would expect appropriate 1-independent analogs to constitute a similarly important ingredient in a solution to Problem 6. By taking the limit as $n \rightarrow \infty$ in Problem 9, one is led to consider another 1-independent critical probability. Let G be an *infinite*, locally finite connected graph. The *long paths critical probability* for 1-independent percolation on G is

$$p_{1,\ell p}(G) := \inf \left\{ p \in [0, 1] : \forall \mu \in \mathcal{M}_{1,\geq p}(G), \forall n \in \mathbb{N} \mu(\exists \text{ open path of length } n) > 0 \right\},$$

that is, the least edge-probability at which arbitrarily long open paths will appear in all 1-independent models in G .

Problem 10. Given an infinite, locally finite, connected graph G , determine $p_{1,\ell p}(G)$.

In this paper, we resolve Problem 9 in a strong form when G consists of a vertex or an edge (see Theorems 15 and 30). This allows us to solve Problem 10 when G is the integer line lattice \mathbb{Z} and the integer ladder lattice $\mathbb{Z} \times P_2$.

Theorem 11. We have that

- (i) $p_{1,\ell p}(\mathbb{Z}) = \frac{3}{4}$, and
- (ii) $p_{1,\ell p}(\mathbb{Z} \times P_2) = \frac{2}{3}$.

Note that part (i) of Theorem 11 above can be read out of earlier work of Liggett, Schonman, and Stacey [24] and Balister and Bollobás [4]. We prove further bounds on both $p_{1,\times}(P_n \times G)$ and $p_{1,\ell p}(\mathbb{Z} \times G)$ for a variety of graphs G . We summarize the latter, less technical, set of results below. Let C_n and K_n denote the cycle and the complete graph on n vertices respectively.

Theorem 12. We have that

- (i) $0.5359 \dots = 4 - 2\sqrt{3} \leq p_{1,\ell p}(\mathbb{Z} \times C_n) \leq p_{1,\ell p}(\mathbb{Z} \times P_n) \leq \frac{2}{3}$ for all $n \geq 3$;
- (ii) $p_{1,\ell p}(\mathbb{Z} \times K_3) \leq \frac{1}{16} \left(13 - \frac{55}{\sqrt[3]{128\sqrt{14}-251}} + \sqrt[3]{128\sqrt{14}-251} \right) = 0.63154 \dots$;
- (iii) $p_{1,\ell p}(\mathbb{Z} \times C_4) \leq (3 - \sqrt{3})/2 = 0.63397 \dots$;
- (iv) $p_{1,\ell p}(\mathbb{Z} \times C_5) \leq 0.63895 \dots$;
- (v) $0.5359 \dots = 4 - 2\sqrt{3} \leq \lim_{n \rightarrow \infty} p_{1,\ell p}(\mathbb{Z} \times K_n) \leq \frac{5}{9} = 0.5555 \dots$

A key ingredient in the proof of Theorems 11 and 12 is a local lemma-type result, Theorem 26, relating the probability in a 1-independent model of finding an open left-right crossing of $P_n \times G$ to the probability of a given copy of G being connected in that model. This motivated our third contribution to the study of 1-independent models in this paper, namely an investigation into the connectivity of 1-independent random graphs.

Definition 13. Let G be a finite connected graph. For any $p \in [0, 1]$, we define the k -independent connectivity function of G to be

$$f_{k,G}(p) := \inf \left\{ \mu(\exists \text{ open spanning tree}) : \mu \in \mathcal{M}_{k,\geq p}(G) \right\}.$$

Problem 14. Given a finite connected graph G , determine $f_{1,G}(p)$.

We resolve Problem 14 exactly when G is a path, a complete graph or a cycle on at most 5 vertices.

Theorem 15. Given $n \in \mathbb{N}_{\geq 2}$ and $p \in [0, 1]$, let $\theta = \theta(p) := \frac{1+\sqrt{4p-3}}{2}$ and $p_n := \frac{1}{4} \left(3 - \tan^2 \left(\frac{\pi}{n+1} \right) \right)$. We have that

$$f_{1,p_n}(p) = \begin{cases} \sum_{j=0}^n \theta^j (1-\theta)^{n-j} & \text{for } p \in [p_n, 1], \\ 0 & \text{for } p \in [0, p_n]. \end{cases}$$

Theorem 16. Given $n \in \mathbb{N}_{\geq 2}$ and $p \in [0, 1]$, let $\theta = \theta(p) := \frac{1+\sqrt{2p-1}}{2}$ and $p_n := \frac{1}{2} \left(1 - \tan^2 \left(\frac{\pi}{2n} \right) \right)$. We have that

$$f_{1,p_n}(p) = \begin{cases} \theta^n + (1-\theta)^n & \text{for } p \in [p_n, 1], \\ 0 & \text{for } p \in [0, p_n]. \end{cases}$$

In particular,

$$f_{1,p_3}(p) = f_{1,p_3}(p) = \begin{cases} \frac{3p-1}{2} & \text{for } p \in \left[\frac{1}{3}, 1 \right], \\ 0 & \text{for } p \in \left[0, \frac{1}{3} \right]. \end{cases}$$

Theorem 17. For $p \in [0, 1]$ we have that

$$f_{1,p_4}(p) = \begin{cases} 2p-1 & \text{for } p \in \left[\frac{1}{2}, 1 \right], \\ 0 & \text{for } p \in \left[0, \frac{1}{2} \right]. \end{cases}$$

Theorem 18. For $p \in [0, 1]$ we have that

$$f_{1,p_5}(p) = \begin{cases} \frac{p(3p^2-1)}{3p-1} & \text{for } p \in \left[\frac{\sqrt{3}}{3}, 1 \right], \\ 0 & \text{for } p \in \left[0, \frac{\sqrt{3}}{3} \right]. \end{cases}$$

We also consider the opposite problem to Problem 14, namely maximizing connectivity in 1-independent random graph models. Let $\mathcal{M}_{k,\leq p}(G)$ denote the collection of 1-independent measures μ on G such that $\sup_{e \in E(G)} \mu\{e \text{ is open}\} \leq p$. Set

$$F_{k,G}(p) := \sup \{ \mu(\exists \text{ open spanning tree}) : \mu \in \mathcal{M}_{k,\leq p}(G) \}.$$

Problem 19. Given a finite connected graph G , determine $F_{1,G}(p)$.

We resolve Problem 19 exactly when G is a path, a complete graph or a cycle on at most 5 vertices.

Theorem 20. For all $n \in \mathbb{N}$ with $n \geq 2$, $F_{1,P_n}(p) = p^{\lfloor \frac{n}{2} \rfloor}$.

Theorem 21. For all $n \in \mathbb{N}$ with $n \geq 2$, $F_{1,K_n}(p) = 1 - f_{1,K_n}(1 - p)$.

Theorem 22.

$$F_{1,C_4}(p) = \begin{cases} 2p - p^2 & \text{if } p \in [\frac{2}{3}, 1], \\ 2p^2 & \text{if } p \in [0, \frac{1}{3}]. \end{cases}$$

Theorem 23.

$$F_{1,C_5}(p) = \begin{cases} \frac{p(2-5p(1-p))}{5-3p} & \text{if } p \in [\frac{3}{5}, 1], \\ \frac{5p^2}{3} & \text{if } p \in [\frac{1}{2}, \frac{3}{5}], \\ \frac{5p^2(p+1)}{p+4} & \text{if } p \in [0, \frac{1}{2}]. \end{cases}$$

Together, Theorems 15–18 and 21–23 determine the complete connectivity “profile” for 1-independent measures μ on K_n , P_n , C_4 and C_5 —i.e., the range of values $\mu(\{\text{connected}\})$ can take if every edge is open with probability p . In Figure 1, we illustrate these for four of these graphs G with plots of $f_{1,G}(p)$, $F_{1,G}(p)$ and $f_{0,G}(p) := \mu(\mathbf{G}_p \text{ is connected})$, where \mathbf{G}_p is the 0-independent model on G obtained by setting each edge of G to be open with probability exactly p , independently at random.

1.5 | Organization of the paper

Our first set of results, Theorems 7 and 8 are proved in Section 2.

In Section 3, we use arguments reminiscent of those used in inductive proofs of the Lovász local lemma to obtain Theorem 26, which gives a general upper bound for crossing and long paths critical probabilities in 1-independent percolation models on Cartesian products $\mathbb{Z} \times G$. This result is used in Sections 5 and 6 to prove Theorem 11 on the long paths critical probability for the line and ladder lattices.

In Sections 5, 7 and 8 and 9, we prove our results on $f_{1,G}(p)$ and $F_{1,G}(p)$ when G is a path, a complete graph or a short cycle. We apply these results in Section 10 to prove Theorem 12. Finally we end the paper in Section 11 with a discussion of the many open problems arising from our work.

1.6 | Notation

We write \mathbb{N} for the set of natural numbers $\{1, 2, \dots\}$, \mathbb{N}_0 for the set $\mathbb{N} \cup \{0\}$, and $\mathbb{N}_{\geq k}$ for the set of natural numbers greater than or equal to k .

We set $[n] := \{1, 2, \dots, n\}$. Given a set A , we write $A^{(r)}$ for the collection of all subsets of A of size r , hereafter referred to as r -sets from A . We use standard graph theoretic notation. A graph is a pair $G = (V, E)$ where $V = V(G)$ and $E = E(G) \subseteq V(G)^{(2)}$ denote the vertex set and edge set of G .

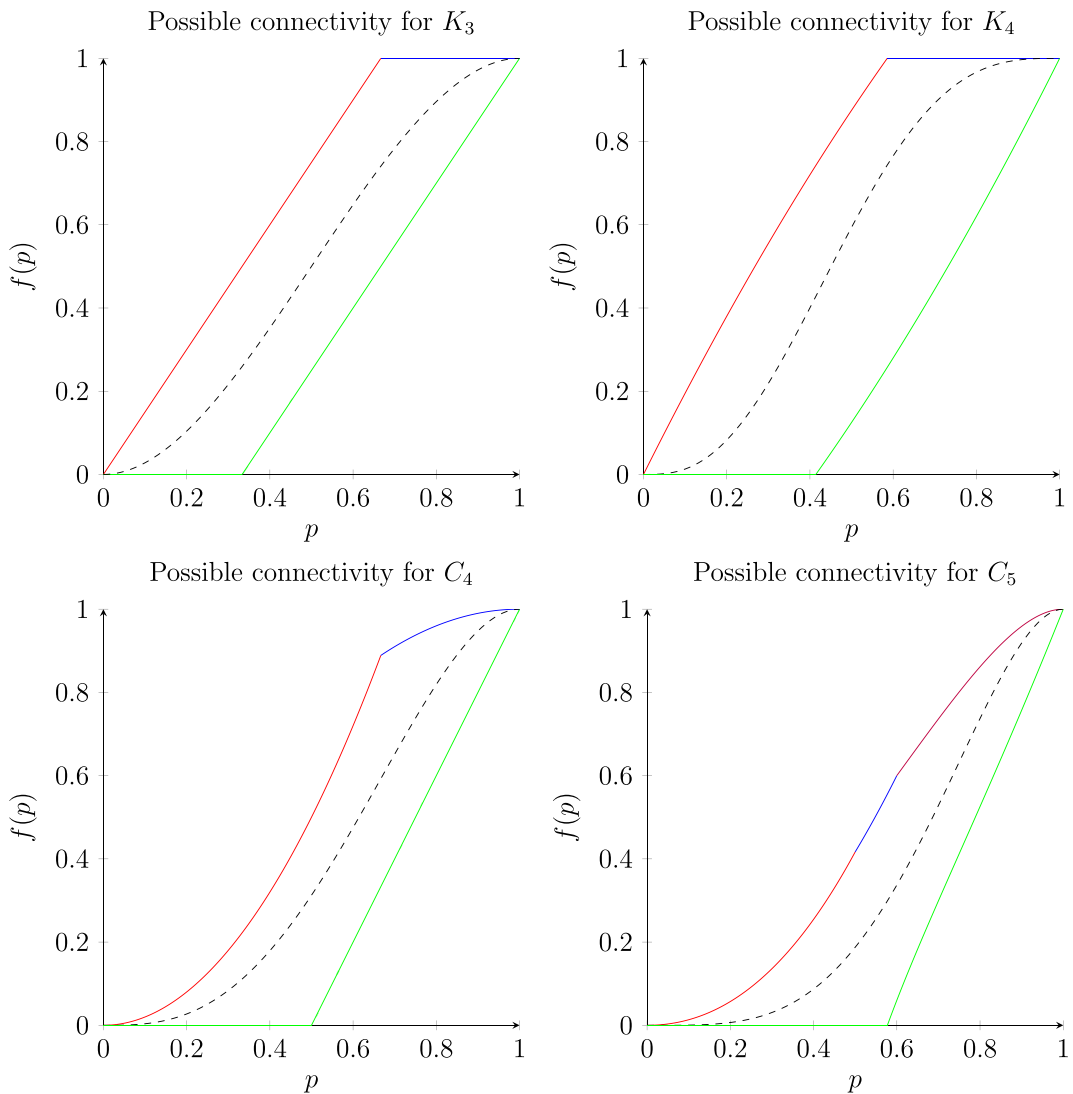


FIGURE 1 The 1-independent connectivity profile of G for $G = K_3, K_4, C_4$ and C_5 . The green curve represents $f_{1,G}(p)$, the dashed black curve $f_{0,G}(p)$, and the union of the red, blue and purple segments represent the piecewise smooth function $F_{1,G}(p)$ [Colour figure can be viewed at wileyonlinelibrary.com]

respectively. Given a subset $A \subseteq G$, we denote by $G[A]$ the subgraph of G induced by A . We also write $N(A)$ for the set of vertices in G adjacent to at least one vertex in A .

Given two graphs G and H , we write $G \times H$ for the Cartesian product of G with H , which is the graph on the vertex set $V(G) \times V(H)$ having an edge between (x, u) and (y, v) if and only if either $u = v$ and x is adjacent to y in G , or $x = y$ and u is adjacent to v in H .

Throughout this paper, we shall use k -ipm as a shorthand for “ k -independent percolation model/measure.” In a slight abuse of language, we say that a bond percolation model μ on an infinite connected graph G percolates if $\mu(\{\text{percolation}\}) = 1$. We refer to a random configuration \mathbf{G}_μ as a μ -random subgraph of G . Finally we write \mathbb{E}_μ for the expectation taken with respect to the probability measure μ . For any event X , we write X^c for the complement event.

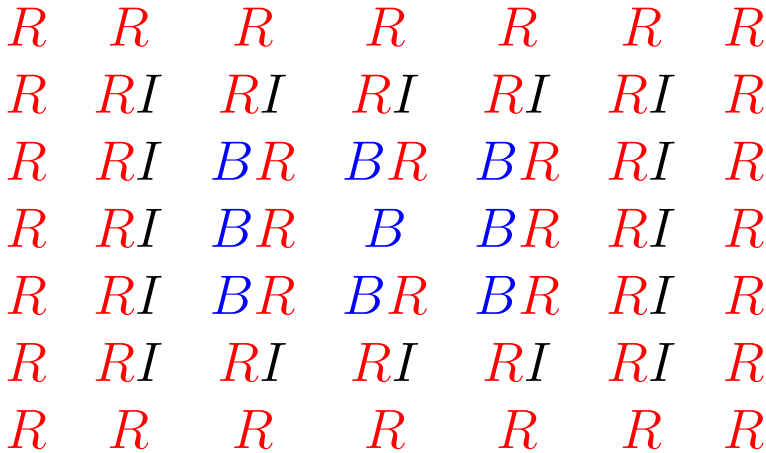


FIGURE 2 The possible states of the vertices in T_0, T_1, T_2 and T_3 when $d = 2$. The letter B stands for Blue, the letter R stands for Red, and the letter I stands for the Inwards state [Colour figure can be viewed at wileyonlinelibrary.com]

2 | LOWER BOUNDS ON $P_{1,C}(\mathbb{Z}^D)$

Proof of Theorem 7. Let $d \in \mathbb{N}_{\geq 2}$. For $k \in \mathbb{N}_0$, let $T_k := \{(x, y) \in \mathbb{Z}^d : \max(|x|, |y|) = k\}$. Let $q := \sqrt{3} - 1$. For each vertex in \mathbb{Z}^d , we color it either Blue or Red, or set it to state I , which stands for *Inwards*. The probability that a given vertex will be in each of these states will depend on which of the T_k the vertex is in, and we assign these states to each vertex independently of all other vertices.

- If v is a vertex in T_k , where $k \equiv 0 \pmod{6}$, then we color v Blue.
- If v is a vertex in T_k , where $k \equiv 1 \pmod{6}$, then we color v Red with probability $q/2$ and color it Blue otherwise.
- If v is a vertex in T_k , where $k \equiv 2 \pmod{6}$, then we color v Red with probability q and put it in the Inwards state I otherwise.
- If v is a vertex in T_k , where $k \equiv 3 \pmod{6}$, then we color v Red.
- If v is a vertex in T_k , where $k \equiv 4 \pmod{6}$, then we color v Blue with probability $q/2$ and color it Red otherwise.
- If v is a vertex in T_k , where $k \equiv 5 \pmod{6}$, then we color v Blue with probability q and put it in the Inwards state I otherwise.

Note that the rules for $T_{k+3}, T_{k+4}, T_{k+5}$ are the same as those for T_k, T_{k+1}, T_{k+2} respectively, except with red and blue interchanged. See Figure 2 for the possible states of the vertices in T_0, T_1, T_2 and T_3 when $d = 2$. Now suppose that $e = \{v_1, v_2\}$ is an edge in \mathbb{Z}^d . First we say that the edge e is open if either both v_1 and v_2 are Blue or both v_1 and v_2 are Red. We also say the edge e is open if, for some k , we have that $v_1 \in T_k, v_2 \in T_{k+1}$, and v_2 is in state I . In all other cases we say that the edge e is closed. It is clear that this gives a 1-independent measure on \mathbb{Z}^d as it is vertex-based, and it is also easy to check that every edge is present with probability at least $4 - 2\sqrt{3}$.

Call this measure μ , and let $G := \mathbb{Z}^d$. We claim that in \mathbf{G}_μ , for all $k \equiv 0 \pmod{3}$, there is no path of open edges from T_k to T_{k+3} . Suppose this is not the case, and P is some path of open edges from a vertex in T_k to T_{k+3} . We first note that P cannot include a vertex in state I , as such a vertex would be in T_{k+2} and would only be adjacent to a single edge. Thus every vertex of P is either Blue or Red. However, as one end vertex of P is Blue and the other end vertex is Red, and there are no open edges with different

colored end vertices, we have that such a path P cannot exist. As a result, every component of \mathbf{G}_μ is sandwiched between some T_{k-3} and T_{k+3} , where $k \equiv 0 \pmod 3$, and so is of finite size. Thus we have that $p_{1,c}(\mathbb{Z}^d) \geq 4 - 2\sqrt{3}$. ■

The construction in Theorem 7 can in fact be generalized to certain other graphs and lattices. Given an infinite, connected, locally finite graph G , and a vertex set $A \subseteq V(G)$, let \bar{A} be the closure of A under 2-neighbor bootstrap percolation on G . That is, let $\bar{A} := \bigcup_{i \geq 0} A_i$, where $A_0 := A$ and for $i \geq 1$

$$A_i := A_{i-1} \cup \{v \in V(G) : v \text{ has 2 or more neighbors in } A_{i-1}\}.$$

We say that G has the *finite 2-percolation property* if, for every finite set $A \subseteq V(G)$, we have that \bar{A} is finite.

Corollary 24. *If G has the finite 2-percolation property, then $p_{1,c}(G) \geq 4 - 2\sqrt{3}$.*

Proof. Partition $V(G)$ in the following way: pick any vertex v and set $T_0 := \{v\}$. For $k \geq 1$ let

$$T_k := \overline{N(T_{k-1})} \setminus \bigcup_{j=0}^{k-1} T_j.$$

We have that if $w \in T_k$, then w is only adjacent to vertices in T_{k-1} , T_k and T_{k+1} . Moreover, w is adjacent to at most one vertex in T_{k-1} —this is the crucial property needed for our construction. Since G has the finite 2-percolation property, each T_k is finite. Thus we can use the T_k to construct a nonpercolating 1-ipm on G in the exact same fashion as done for \mathbb{Z}^d in Theorem 7 (the key being that vertices in state I are still dead ends, being incident to a unique edge), which in turn shows that $p_{1,c}(G) \geq 4 - 2\sqrt{3}$. ■

An example of a lattice with the finite 2-percolation property is the lattice $(3, 4, 6, 4)$, where here we are using the lattice notation of Grünbaum and Shephard [18]. Riordan and Walters [29] showed that the site percolation threshold of this lattice is very likely to lie in the interval $[0.6216, 0.6221]$. Thus this estimate, together with Newman's construction (see equation (1.1)), shows (nonrigorously) that $p_{1,c}((3, 4, 6, 4)) \geq 0.52981682$. As this is less than $4 - 2\sqrt{3}$, we have that our construction gives the (rigorous) improvement of $p_{1,c}((3, 4, 6, 4)) \geq 4 - 2\sqrt{3}$.

Proof of Theorem 8. Fix $\varepsilon > 0$ sufficiently small so that $q := \theta_{\text{site}}(\mathbb{Z}^2) - \varepsilon$ is strictly larger than $1/4$. For each vertex $v \in \mathbb{Z}^2$, we assign to it one of three states: On, L or D , and we do this independently for every vertex. We assign v to the On state with probability q , we assign it to the L state with probability $\frac{1}{2}(1 - q)$, and else we assign it to the D state with probability $\frac{1}{2}(1 - q)$.

We now describe which edges are open based on the states of the vertices. We first say that the edge e is open if both of its vertices are in the On state. If a vertex is in state L , then the edge adjacent and to the left of it is open. Similarly, if a vertex is in state D , then the edge adjacent and down from it is open. All other edges are closed. See Figure 3 for an example of this construction.

It is easy to see that this is a 1-independent measure on \mathbb{Z}^2 as it is vertex-based, and every edge is present with probability $q^2 + \frac{1}{2}(1 - q)$. Call this measure μ and let $G := \mathbb{Z}^2$. We will show that every component of \mathbf{G}_μ has finite size. We begin by first proving an auxiliary lemma. Let $t \in [0, \frac{1}{2}]$, and let us define another 1-independent measure on \mathbb{Z}^2 , which we call the *left-down* measure with parameter t . In the left-down measure, each vertex of \mathbb{Z}^2 is assigned to one of three states: Off, L or D , and we do this independently for every vertex. For each vertex $v \in \mathbb{Z}^2$, we assign it to state L with probability t , we assign it to state D with probability t , and we assign it to state Off with probability $1 - 2t$. As above,

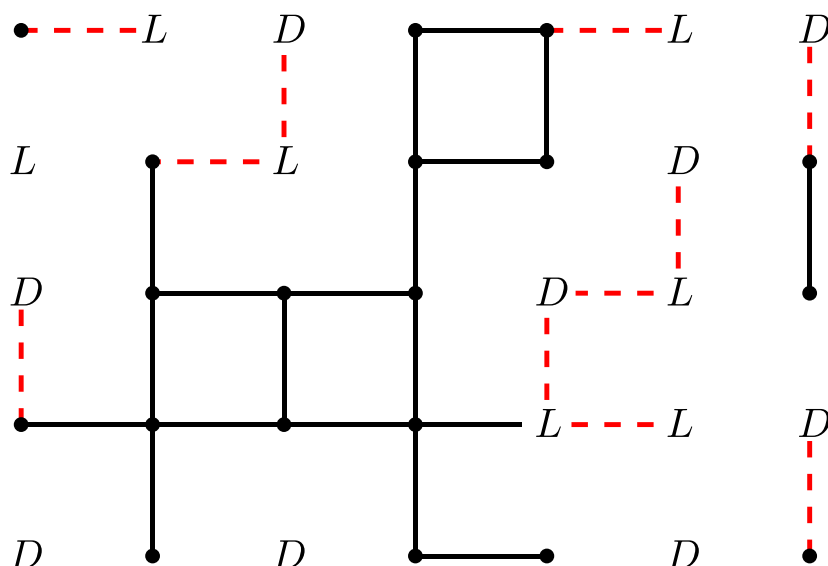


FIGURE 3 This figure shows the open edges of the construction on a small subset of \mathbb{Z}^2 . The unlabeled vertices correspond to those in the On state. The black edges are the open edges that are adjacent to two On vertices, while the dashed red edges are the open edges that are either to the left of an L vertex or below a D vertex [Colour figure can be viewed at wileyonlinelibrary.com]

if a vertex is in state L , then the edge adjacent and to the left of it is open, while if a vertex is in state D , then the edge adjacent and down from it is open. All other edges are closed. We use ν_t to denote the left-down measure with parameter t .

Lemma 25. *If $0 \leq t \leq \frac{3}{8}$, then all components in \mathbf{G}_{ν_t} are finite almost surely.*

Proof. Let $z := 1 - \sqrt{1 - 2t}$. As $0 \leq t \leq \frac{3}{8}$ we have that $0 \leq z \leq \frac{1}{2}$. We start by taking a random subgraph of \mathbb{Z}^2 where every edge is open with probability z , independently of all other edges. We then further modify it as follows. For each vertex $v = (x, y)$ we look at the state of the edge e_1 from v to the vertex $(x - 1, y)$, and the state of the edge e_2 from v to the vertex $(x, y - 1)$. If at least one of e_1 or e_2 is closed, we do not change anything. However, if both e_1 and e_2 are open, with probability $\frac{1}{2}$ we close the edge e_1 , and otherwise we close the edge e_2 . We do this independently for every vertex v of \mathbb{Z}^2 .

It is easy to see that this is an equivalent formulation of ν_t , the left-down measure with parameter t . Indeed, to each vertex $v = (x, y)$ as above we may assign a state Off if both the edge $e_1(v)$ to the vertex to the left of v and the edge $e_2(v)$ to the vertex below v are closed, a state L if $e_1(v)$ is open and a state D if $e_2(v)$ is open. The probabilities of these three states are $(1 - z)^2 = 1 - 2t$, t and t respectively, and since the vertex states depend only on the pairwise disjoint edge sets $\{e_1(v), e_2(v)\}_{v \in \mathbb{Z}^2}$, they are independent of one another just as in the ν_t measure.

Thus we have coupled ν_t to the 0-independent bond percolation measure ξ on \mathbb{Z}^2 with edge-probability z . In this coupling we have that if an edge e is open in \mathbf{G}_{ν_t} , then it is also open in \mathbf{G}_ξ . As $z \leq 0.5$ we have that all components in \mathbf{G}_ξ are finite by the Harris-Kesten theorem, and so we also have that all of the components in \mathbf{G}_{ν_t} are finite too. ■

By considering an appropriate branching process it is possible to prove the stronger result that if $0 \leq t < \frac{1}{2}$, then almost surely all components in \mathbf{G}_{ν_t} are finite. We make no use of this stronger result

in this paper, so we omit its proof. It is also clear that when $t = \frac{1}{2}$, every vertex in \mathbf{G}_{v_i} is part of an infinite path consisting solely of steps to the left or steps downwards, and so percolation occurs in \mathbf{G}_{v_i} at this point.

Let us return to our original 1-independent measure μ , where every vertex is in state On, L or D . Recall that our aim is to show that all components have finite size in \mathbf{G}_μ . Consider removing all vertices in state L or D , and also any edges adjacent to these vertices. What is left will be a collection of components consisting only of edges between vertices in the On state, which we call the *On-sections*. The black edges in Figure 3 are the edges in the On-sections. As a vertex is On with probability $q < \theta_{\text{site}}(\mathbb{Z}^2)$, we have that almost surely every On-section is finite. Similarly, consider removing all edges in the On-sections. What is left will be a collection of edges adjacent to vertices in the L or D states. We call these components the *LD-sections*; the dashed red edges in Figure 3 are the edges in the *LD-sections*. As each vertex is in state L with probability $\frac{1}{2}(1 - q) \leq \frac{3}{8}$ and in state D with the same probability, Lemma 25 tells us that almost surely every *LD-section* is finite.

For each vertex v in state L orient the open edge to the left of it away from v , while for each vertex v in state D orient the open edge below it away from v . This gives a partial orientation of the open edges of \mathbf{G}_μ , in which every vertex in state L or D has exactly one edge oriented away from it, and vertices in state On have no outgoing edge. Furthermore, if v_1 is a vertex in the On state and v_2 is a vertex in the L or D state, then the edge between them is oriented from v_2 to v_1 . Since the *LD-sections* are almost surely finite, this implies the *LD* sections under this orientation consist of directed trees, each of which is oriented from the leaves to a unique root, which is in the On state. In particular, every *LD-section* attaches to at most one On-section. As such, almost surely every component in \mathbf{G}_μ consists of at most one On-section, and a finite number of finite *LD-sections* attached to it. Thus almost surely every component in \mathbf{G}_μ is finite. ■

3 | A GENERAL UPPER BOUND FOR $P_{1,\ell p}(\mathbb{Z} \times G)$

Let G be a finite connected graph. Set $v(G) := |V(G)|$. Recall that for any 1-independent bond percolation measure $\mu \in \mathcal{M}_{1,\geq p}(G)$, we have $\mu(\mathbf{G}_\mu \text{ is connected}) \geq f_{1,G}(p)$.

Theorem 26. *If p satisfies*

$$(f_{1,G}(p))^2 \geq \frac{1}{\alpha(1-\alpha)}(1-p)^{v(G)}, \quad (3.1)$$

for some $\alpha \in (0, 1/2]$, then for every $\ell \in \mathbb{N}$

$$f_{1,p_\ell \times G}(p) \geq ((1-\alpha)f_{1,G}(p))^\ell.$$

Proof. Consider an arbitrary measure $\mu \in \mathcal{M}_{1,\geq p}(\mathbb{Z} \times G)$. For any $n \in \mathbb{N}$, the restriction of μ to $[n] \times V(G)$ is a measure from $\mathcal{M}_{1,\geq p}(P_n \times G)$, and clearly all such measures can be obtained in this way. Furthermore, for every $n \in \mathbb{N}$, the restriction of μ to $\{n\} \times V(G)$ is a measure from $\mathcal{M}_{1,\geq p}(G)$, and in particular the subgraph of $(\mathbb{Z} \times \mathbf{G})_\mu$ induced by $\{n\} \times V(G)$ is connected with probability at least $f_{1,G}(p)$.

We consider the μ -random graph $(\mathbb{Z} \times \mathbf{G})_\mu$. For $n \geq 1$ let Y_n be the event that $[n] \times V(G)$ induces a connected subgraph. For $n \geq 2$, let X_n be the event that $[n-1] \times V(G)$ induces a connected subgraph and at least one vertex in $\{n\} \times V(G)$ is connected to a vertex in $\{n-1\} \times V(G)$. For $n = 1$, set X_1 to be

the trivially satisfied event occurring with probability 1. For $n \geq 1$, let V_n be the event that $\{n\} \times V(G)$ induces a connected subgraph, and for $n \geq 2$ let H_n be the event that at least one of the edges from $\{n-1\} \times V(G)$ to $\{n\} \times V(G)$ is present.

It easily follows that $X_n = Y_{n-1} \cap H_n$ and $X_n \cap V_n \subseteq Y_n$. From here, we obtain the following inclusions:

- (a) $(X_{n+1})^c \cap Y_n = (H_{n+1})^c \cap Y_n$,
- (b) $Y_n \cap Y_{n-1} \supseteq (V_n \cap X_n) \cap Y_{n-1}$, and
- (c) $(Y_n)^c \cap X_n \subseteq (V_n)^c \cap X_n$.

Now set

$$x_n := \mu \left(X_n^c \mid \bigcap_{m < n} Y_m \right) \quad \text{and} \quad y_n := \mu \left(Y_n^c \mid X_n \cap \left(\bigcap_{m < n} Y_m \right) \right).$$

We begin by establishing two inductive relations for the sequences x_n and y_n . First of all, using (a) and (b) we have,

$$\begin{aligned} x_{n+1} &= \frac{\mu((X_{n+1})^c \cap (\bigcap_{m \leq n} Y_m))}{\mu(\bigcap_{m \leq n} Y_m)} = \frac{\mu((H_{n+1})^c \cap (\bigcap_{m \leq n} Y_m))}{\mu(\bigcap_{m \leq n} Y_m)} \\ &\leq \frac{\mu((H_{n+1})^c)}{\mu(Y_n | (\bigcap_{m < n} Y_m))} \quad \text{by 1-independence} \\ &\leq \frac{(1-p)^{v(G)}}{\mu(V_n | (\bigcap_{m < n} Y_m)) - \mu((X_n)^c | (\bigcap_{m < n} Y_m))} \\ &\leq \frac{(1-p)^{v(G)}}{f_{1,G}(p) - x_n} \quad \text{by 1-independence.} \end{aligned} \quad (3.2)$$

Second, using (c),

$$\begin{aligned} y_n &= \frac{\mu((Y_n)^c \cap X_n \cap (\bigcap_{m < n} Y_m))}{\mu(X_n \cap (\bigcap_{m < n} Y_m))} \leq \frac{\mu((V_n)^c \cap (\bigcap_{m < n} Y_m))}{\mu(X_n \cap (\bigcap_{m < n} Y_m))} \\ &\leq \frac{\mu((V_n)^c)}{\mu(X_n | (\bigcap_{m < n} Y_m))} \quad \text{by 1-independence} \\ &\leq \frac{(1-f_{1,G}(p))}{1-x_n}. \end{aligned} \quad (3.3)$$

Now if (3.1) is satisfied, we claim that $x_n \leq \alpha f_{1,G}(p)$ for all n . Indeed $x_1 = 0$, and if $x_n \leq \alpha f_{1,G}(p)$, then by (3.3)

$$y_n \leq \frac{1-f_{1,G}(p)}{1-\alpha f_{1,G}(p)} = 1 - \frac{(1-\alpha)f_{1,G}(p)}{1-\alpha f_{1,G}(p)} < 1.$$

Furthermore, we have by (3.2) and (3.1) that

$$x_{n+1} \leq \frac{(1-p)^{v(G)}}{(1-\alpha)f_{1,G}(p)} \leq \alpha f_{1,G}(p),$$

so our claim follows by induction.

Finally, we have that

$$\mu(Y_\ell) = \prod_{i=1}^{\ell} (1 - x_i)(1 - y_i) > \left(\frac{(1 - \alpha)f_{1,G}(p)}{1 - \alpha f_{1,G}(p)} \right)^{\ell} (1 - \alpha f_{1,G}(p))^{\ell} = ((1 - \alpha)f_{1,G}(p))^{\ell}.$$

For any finite connected graph G , $f_{1,G}(p)$ is a nondecreasing function of p with $f_{1,G}(1) = 1$. Thus the function $(f_{1,G}(p))^2$ is also nondecreasing in p and attains a maximum value of 1 at $p = 1$. On the other hand, the function $4(1 - p)^{v(G)}$ is strictly decreasing in p and is equal to 4 at $p = 0$. Thus there exists a unique solution $p_{\star} = p_{\star}(G)$ in the interval $[0, 1]$ to the equation

$$(f_{1,G}(p))^2 = 4(1 - p)^{v(G)}. \quad (3.4)$$

Theorem 26 thus has the following immediate corollary.

Corollary 27. *Let G be a finite connected graph. Let $p_{\star} = p_{\star}(G)$ be as above. Then*

$$p_{1,\ell p}(\mathbb{Z} \times G) \leq p_{\star}.$$

Proof. Apply Theorem 26 with $\alpha = 1/2$. ■

4 | IMAGINARY LIMITS OF REAL CONSTRUCTIONS: A PRELIMINARY LEMMA

In this section we prove a lemma that we shall use in Sections 5 and 7. The lemma will allow us to use certain vertex-based constructions to create other 1-ipms that cannot be represented as vertex-based constructions (or would correspond to vertex-based constructions with “complex weights”).

Lemma 28. *Let G be a finite graph, and let $\mathcal{Q} := \{Q_H(\theta) : H \subseteq G\}$ be a set of polynomials with real coefficients, indexed by subgraphs of G . Given $\theta \in \mathbb{C}$, let μ_{θ} be the following function from subgraphs of G to \mathbb{C} :*

$$\mu_{\theta}(H) := Q_H(\theta).$$

Suppose there exists a nontrivial interval $I \subseteq \mathbb{R}$ such that, for all $\theta \in I$, the function μ_{θ} defines a 1-ipm on G . Suppose further that there exists a set $X \subseteq \mathbb{C}$ such that, for all $\theta \in X$ and all $H \subseteq G$, $\mu_{\theta}(H)$ is a nonnegative real number. Then μ_{θ} is a 1-ipm on G for all $\theta \in X$.

Proof. We start by proving that μ_{θ} is a measure on G for all $\theta \in X$. As $\mu_{\theta}(H)$ is a nonnegative real number for all $\theta \in X$ and all $H \subseteq G$, all that is left to prove is that

$$\left(\sum_{H \subseteq G} Q_H(\theta) \right) - 1 = 0. \quad (4.1)$$

The left hand side of (4.1) is a polynomial in θ with real coefficients, and is equal to zero for all θ in the interval I . By the fact that a nonzero polynomial over any field has only finitely many roots, the polynomial is identically zero and so (4.1) holds for all θ .

We now show that μ_θ is a 1-ipm on G for all $\theta \in X$. To do this we must show that the following holds true for all $\theta \in X$, for all $A, B \subseteq V(G)$ such that A and B are disjoint, and all G_1 and G_2 such that G_1 is a subgraph of $G[A]$ while G_2 is a subgraph of $G[B]$:

$$\mu_\theta (\mathbf{G}_{\mu_\theta} [A] = G_1, \mathbf{G}_{\mu_\theta} [B] = G_2) = \mu_\theta (\mathbf{G}_{\mu_\theta} [A] = G_1) \mu_\theta (\mathbf{G}_{\mu_\theta} [B] = G_2). \quad (4.2)$$

Both sides of (4.2) are polynomials in θ with real coefficients—the left hand side, for example, can be written as

$$\sum_{H \subseteq G: H[A]=G_1, H[B]=G_2} Q_H(\theta).$$

As μ_θ is a 1-ipm on G for all $\theta \in I$, we have that these two polynomials agree on I , and so must be the same polynomial. Thus (4.2) holds as required. ■

5 | THE LINE LATTICE \mathbb{Z}

In this section we prove Theorem 15 on the connectivity function of paths. Recall that, given $n \in \mathbb{N}_{\geq 2}$ and $p \in [0, 1]$, we let $\theta = \theta(p) := \frac{1+\sqrt{4p-3}}{2}$ and $p_n := \frac{1}{4} \left(3 - \tan^2 \left(\frac{\pi}{n+1} \right) \right)$. Let $g_n(\theta) := \sum_{j=0}^n \theta^j (1 - \theta)^{n-j}$.

We begin by constructing a measure $\nu_p \in \mathcal{M}_{1, \geq p}(P_n)$ as follows. Let us start with the case $p \geq \frac{3}{4}$. For each vertex of P_n , we set it to state 0 with probability θ , and set it to state 1 otherwise, and we do this independently for every vertex. Recall that for each $j \in [n]$ we write S_j for the state of vertex j ; in this construction, the states are independent and identically distributed random variables. We set the edge $\{j, j+1\}$ to be open if $S_j \leq S_{j+1}$, and closed otherwise. Thus, as $p = \theta + (1 - \theta)^2$, we have that each edge is open with probability p . Moreover $(\mathbf{P}_n)_{\nu_p}$ will be connected if and only if there exists some $j \in [n+1]$ such that $S_k = 0$ for all $k < j$, while $S_k = 1$ for all $k \geq j$. Therefore $(\mathbf{P}_n)_{\nu_p}$ is connected with probability $g_n(\theta)$. As this construction is vertex-based, it is clear that it is 1-independent.

When $p < \frac{3}{4}$ we have that θ is a complex number, and so the above construction is no longer valid. However, as discussed in Section 4, we will show that it is possible to extend this construction to all $p \in [p_n, 1]$. For each subgraph G of P_n , set $Q_G(\theta)$ to be the polynomial $\nu_p((\mathbf{P}_n)_{\nu_p} = G)$ for all $\theta \in [\frac{3}{4}, 1]$. The following claim, together with Lemma 28, shows that in fact ν_p is a 1-ipm on P_n for all $p \in [p_n, 1]$.

Claim 29. For all $p \in [p_n, \frac{3}{4})$ and all $G \subseteq P_n$ we have that $Q_G(\theta(p))$ is nonnegative real number.

Proof. We proceed by induction on n . When $n = 2$ we have that there are only two possible subgraphs of P_2 , which are P_2 itself and its complement $\overline{P_2}$. We have that $Q_{P_2}(\theta(p)) = p$ and $Q_{\overline{P_2}}(\theta(p)) = 1 - p$, so the claim holds as required for $n = 2$.

Let us now assume that $n > 2$ and that the claim is true for all cases from 2 up to $n - 1$. We split into two further subcases. We first deal with the case that $G = P_n$. We have that $Q_{P_n}(\theta(p)) = g_n(\theta)$. For $p < \frac{3}{4}$ we can write

$$g_n(\theta) = \frac{\theta^{n+1} - (1 - \theta)^{n+1}}{2\theta - 1}. \quad (5.1)$$

When $p < \frac{3}{4}$ we have that θ and $1 - \theta$ are complex conjugates, and also that $2\theta - 1$ is a pure imaginary number. Thus both the numerator and denominator of the above fraction are pure imaginary,

and so $g_n(\theta(p))$ is a real number for all $p < \frac{3}{4}$. By writing $\theta = re^{i\phi}$, where $r := \sqrt{1-p}$ and $\phi := \arctan\left(\sqrt{3-4p}\right)$, we can rewrite (5.1) as

$$g_n(\theta(p)) = \frac{2r^{n+1}}{\sqrt{3-4p}} \sin((n+1)\phi). \quad (5.2)$$

Now $p \in [p_n, \frac{3}{4})$ implies $0 < \phi \leq \arctan\left(\sqrt{3-4p_n}\right) = \frac{\pi}{n+1}$, which in turn gives $\sin((n+1)\phi) \geq 0$. Thus by (5.2) above, $g_n(\theta(p))$ is a nonnegative real number for all p in the interval $[p_n, \frac{3}{4})$, as required.

We now deal with the case that $G \neq P_n$. Let us consider the vertex-based construction from which $Q_G(\theta)$ was defined. As not every edge is present in G we have that there exists some $j \in [n-1]$ such $\{j, j+1\}$ is not an edge, and so $S_j = 1$ while $S_{j+1} = 0$. Note that if $j \geq 2$, then the edge $\{j-1, j\}$ is present in G regardless of the state of vertex $j-1$. Similarly, if $j \leq n-2$, then the edge $\{j+1, j+2\}$ is present in G regardless of the state of vertex $j+2$. If we write $G_1 := G[\{1, \dots, j-1\}]$ and $G_2 := G[\{j+2, \dots, n\}]$, then we have that

$$Q_G(\theta) = \theta(1-\theta)Q_{G_1}(\theta)Q_{G_2}(\theta). \quad (5.3)$$

Now, by induction, we have that $Q_{G_1}(\theta(p))$ and $Q_{G_2}(\theta(p))$ are positive real numbers for all $p \in [p_n, \frac{3}{4})$; to make this inductive step work we are using the fact that $(p_n)_{n \geq 2}$ forms an increasing sequence, and so $p \geq p_n$ implies that $p \geq p_s$ for all $s \leq n$. As $\theta(p)(1-\theta(p)) = 1-p$, we have that (5.3) is a positive real for all $p \in [p_n, \frac{3}{4})$, and so we have proven the claim. ■

Note that as this proof shows that $g_n(\theta(p_n)) = 0$, we have that the probability $(\mathbf{P}_n)_{v_{p_n}}$ is connected is equal to 0. As $v_{p_n} \in \mathcal{M}_{1, \geq p}(P_n)$ for all $p \leq p_n$, we have that $f_{1, P_n}(p) = 0$ for all $p \leq p_n$.

We now prove that this construction is optimal with respect to the connectivity function. Note that the following proof involves essentially following the proof of Theorem 26 when G consists of a single point and checking that the above construction is tight at every stage of this proof. Finally, we should emphasize that the main ideas in the construction of v_p and its analysis are due to Balister and Bollobás [4] (they considered slightly different probabilities for vertex states, setting $S_k = 0$ with probability q_k , where q_k is defined for $k \in [n]$ by $q_1 = 0$ and by the recurrence relation $q_k = \min\left(\frac{1-p}{1-q_{k-1}}, 1\right)$ for $k \geq 2$, which corresponds exactly to the equality case in inequality (5.5) below).

Proof of Theorem 15. The above construction discussed shows that

$$f_{1, P_n}(p) \leq \begin{cases} g_n(\theta) & \text{for } p \in [p_n, 1], \\ 0 & \text{for } p \in [0, p_n]. \end{cases}$$

It is clear that $f_{1, P_n}(p) \geq 0$ for all p , and so all that remains to show is that $f_{1, P_n}(p) \geq g_n(\theta(p))$ for all $p \in [p_n, 1]$.

Let $\mu \in \mathcal{M}_{1, \geq p}(P_n)$. For $k \in [n]$, let X_k be the event that the subgraph of $(\mathbf{P}_n)_\mu$ induced by the vertex set $[k]$ is connected, and let H_k be the event that the edge $\{k-1, k\}$ is not present in $(\mathbf{P}_n)_\mu$. Applying random sparsification as in Remark 3 if necessary, we may assume without loss of generality that for every k , the event H_k occurs with probability exactly $1-p$.

Let $q_2^\mu := \mu((X_2)^c) = 1 - p$, and for $k > 2$ let $q_k^\mu := \mu((X_k)^c | X_{k-1})$. We have that

$$\begin{aligned} q_k^\mu &= \frac{\mu(H_k \cap X_{k-1})}{\mu(X_{k-1})} \\ &\leq \frac{\mu(H_k \cap X_{k-2})}{\mu(X_{k-1})} \end{aligned} \quad (5.4)$$

$$\begin{aligned} &= \frac{\mu(H_k)\mu(X_{k-2})}{\mu(X_{k-1})} \quad \text{by 1-independence} \\ &\leq \frac{1-p}{1-q_{k-1}^\mu}. \end{aligned} \quad (5.5)$$

Note that $\mu(\mathbf{P}_n)_\mu = P_n = \prod_{i=2}^n (1 - q_i^\mu)$. Thus to show that the previous construction is optimal with respect to the connectivity function it is enough to show that equality holds for inequalities (5.4) and (5.5) when $\mu = \nu_p$. In the measure ν_p , we have that every edge is present with probability exactly p , thus $\nu_p(H_k) = 1 - p$ and so equality holds in (5.5). To prove that equality holds in (5.4), it is sufficient so show that

$$\nu_p(H_k \cap X_{k-1}) = \nu_p(H_k \cap X_{k-2}). \quad (5.6)$$

Both the left and right hand sides of (5.6) can be expressed as polynomials in $\theta(p)$, and so it is sufficient to show that equality holds for $p \geq \frac{3}{4}$, as that will show they are the same polynomial (and so equality holds for all $p \in [p_n, 1]$). Suppose that the event $(H_k \cap X_{k-2})$ occurs. As H_k has occurred we have that $S_{k-1} = 1$ while $S_k = 0$. As $S_{k-1} = 1$, we have that edge $\{k-2, k-1\}$ is open, regardless of S_{k-2} . Thus, as X_{k-2} has occurred we also have that X_{k-1} has occurred. Therefore $(H_k \cap X_{k-1})$ has also occurred, and so we are done. ■

We remark in similar fashion to the above proof that the following holds for any $\mu \in \mathcal{M}_{1, \geq p}(P_n)$:

$$\begin{aligned} \mu(X_n) &\geq \mu(X_{n-1}) - \mu(X_{n-1} \cap H_n) \\ &\geq \mu(X_{n-1}) - \mu(X_{n-2} \cap H_n) \\ &= \mu(X_{n-1}) - \mu(X_{n-2})\mu(H_n^c) \quad \text{by 1-independence} \\ &\geq \mu(X_{n-1}) - (1-p)\mu(X_{n-2}). \end{aligned}$$

Moreover, by once again considering what states of vertices can lead to the various events, we have that equality holds for all of the above inequalities when $\mu = \nu_p$. This leads us to another way to define $g_n(\theta(p))$: let $g_1(\theta(p)) := 1$, $g_2(\theta(p)) := p$, and for all $n \geq 3$ we have that

$$g_n(\theta(p)) = g_{n-1}(\theta(p)) - (1-p)g_{n-2}(\theta(p)).$$

We conclude this section with a proof of Theorem 11(i).

Proof of Theorem 11(i). For the upper bound we plug $f_{1, P_1}(p) = 1$ into equation (3.4), solve that equation to get $p_\star(P_1) = \frac{3}{4}$ and apply Corollary 27 to obtain $p_{1, \ell p}(\mathbb{Z}) \leq \frac{3}{4}$.

For the lower bound, let $p < \frac{3}{4}$ be fixed. As the sequence $(p_n)_{n \in \mathbb{N}}$ is monotone increasing and tends to $3/4$ as $n \rightarrow \infty$, there exists $N \in \mathbb{N}$ such that $p < p_N$. We showed in Theorem 15 that there exists a measure $\nu_{p_N} \in \mathcal{M}_{1, \geq p_N}(P_N)$ such that the probability $(\mathbf{P}_N)_{\nu_{p_N}}$ is connected is equal to zero.

We use this measure to create a measure $\nu \in \mathcal{M}_{1, \geq p}(\mathbb{Z})$. For each $i \in \mathbb{Z}$, we let the subgraphs $\mathbb{Z}_\nu[(i(N-1) + [N])]$ on horizontal shifts of P_N by $i(N-1)$ be independent identically distributed

random variables with distribution given by ν_{p_N} . This gives rise to a 1-independent model ν on \mathbb{Z} with edge-probability at least p (in fact at least p_N). Furthermore, all connected components of \mathbb{Z}_ν have size at most $2(N-1)-1$. In particular, $p_{1,\ell_p}(\mathbb{Z}) \geq p$. Since $p < \frac{3}{4}$ was chosen arbitrarily, this gives the required lower bound $p_{1,\ell_p}(\mathbb{Z}) \geq \frac{3}{4}$. ■

6 | THE LADDER LATTICE $\mathbb{Z} \times P_2$

In this section we construct a family of 1-ipms on segments of the ladder $\mathbb{Z} \times P_2$ with edge-probability close to $2/3$ for which with probability 1 there are no open left-right crossings. The idea of this construction is due to Walters and the second author [16] (though the technical work involved in rigorously showing the construction works is new).

Let us begin by giving an outline of our construction. We write the vertex set $V(P_N \times P_2)$ as $[N] \times [2]$. As in the case of the line lattice, we independently assign to each vertex (n, y) a random state $S_{(n,y)}$. If $n+y$ is even, then we let

$$S_{(n,y)} := \begin{cases} 2 & \text{with probability } p_n, \\ 0 & \text{with probability } 1 - p_n; \end{cases}$$

while if instead $n+y$ is odd, then we let

$$S_{(n,y)} := \begin{cases} 2 & \text{with probability } r_n, \\ 1 & \text{with probability } s_n, \\ 0 & \text{with probability } 1 - r_n - s_n. \end{cases}$$

Here $(p_n)_{n \in \mathbb{N}}, (r_n)_{n \in \mathbb{N}}, (s_n)_{n \in \mathbb{N}}$ are suitably chosen sequences of real numbers, ensuring that the $S_{(n,y)}$ are well-defined random variables. We then define a random spanning subgraph \mathbf{G}_μ of $G := P_N \times P_2$ from the random vertex states $S_{(n,y)}$: $(n, y) \in [N] \times [2]$ as follows:

- for each $n \in [N-1]$ and $y \in [2]$, the horizontal edge $\{(n, y), (n+1, y)\}$ is open in \mathbf{G}_μ if and only if $S_{(n,y)} \leq S_{(n+1,y)}$,
- for each $n \in [N]$, the vertical edge $\{(n, 1), (n, 2)\}$ is open in \mathbf{G}_μ if and only if $(S_{(n,1)} - S_{(n,2)})(1 - S_{(n,1)})(1 - S_{(n,2)}) = 0$.

Note the condition for a vertical edge $\{(n, 1), (n, 2)\}$ to be open can be rephrased as if and only if either $S_{(n,1)} = S_{(n,2)}$ or one of $S_{(n,1)}, S_{(n,2)}$ is equal to 1. So intuitively, the value of the $S_{(n,y)}$ must increase from left to right along open horizontal edges, and it must stay constant along open vertical edges unless one of the endpoints is in the special state 1, which allows free passage up or down.

Clearly the bond percolation measure μ associated to our random graph model \mathbf{G}_μ is a 1-ipm on the ladder $G = P_N \times P_2$ as it is vertex-based. By making a judicious choice of the sequences $(p_n)_{n \in \mathbb{N}}, (r_n)_{n \in \mathbb{N}}, (s_n)_{n \in \mathbb{N}}$ and taking N sufficiently large, one can ensure that in addition μ satisfies $d(\mu) \geq p$ and $\mu(\exists \text{ open left-right crossing}) = 0$. In particular, with this construction we prove the following result.

Theorem 30. Fix $p \in \left(\frac{1}{2}, \frac{2}{3}\right)$. Then there exists $N \in \mathbb{N}$ such that for all $n \geq N$,

$$p_{1,\times}(P_n \times P_2) \geq p.$$

Proof. Fix $p := \frac{2}{3} - \varepsilon$, with $\varepsilon \in (0, \frac{1}{6})$. We start by defining the sequences $(p_n)_{n \in \mathbb{N}}$, $(r_n)_{n \in \mathbb{N}}$ and $(s_n)_{n \in \mathbb{N}}$ iteratively as follows. We set $p_1 = r_1 = 1$ and $s_1 = 0$. Then for $n \in \mathbb{N}$, we let

$$\begin{aligned} p_{n+1} &= \begin{cases} 1 - \frac{1-p}{r_n+s_n} & \text{if } r_n + s_n \geq 1-p, \\ 0 & \text{otherwise;} \end{cases} \\ r_{n+1} &= \begin{cases} 1 - \frac{1-p}{p_n} & \text{if } p_n \geq 1-p, \\ 0 & \text{otherwise;} \end{cases} \\ s_{n+1} &= \begin{cases} \max \left\{ 1 - 2r_{n+1} + \frac{r_{n+1}-(1-p)}{p_{n+1}}, 0 \right\} & \text{if } p_{n+1} > 0, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Lemma 31. *The following hold for all $n \in \mathbb{N}$:*

- (i) $p_n, r_n \in [0, 1]$,
- (ii) $s_n \in [0, 1 - r_n]$,
- (iii) $p_{n+1} \leq p_n$,
- (iv) $r_{n+1} \leq r_n$,
- (v) $r_{n+1} + s_{n+1} \leq r_n + s_n$.

Proof. We prove the lemma by induction on n . By definition of our sequences, $p_1 = r_1 = 1 \geq p = p_2 = r_2$, $s_1 = 0$, and $0 < s_2 = \frac{(2p-1)(1-p)}{p} < 1 - p = r_1 + s_1 - r_2$, and thus (i)-(v) all hold in the base case $n = 1$.

Suppose now (i)-(v) hold for all $n \leq N$, for some $N \geq 1$. Since p_N and $r_N + s_N$ both lie in $[0, 1]$, the definition of p_{N+1} and r_{N+1} implies these also both lie in $[0, 1]$. This establishes (i) for $n = N + 1$. By construction, $s_{N+1} \geq 0$, and by the inductive hypotheses (ii) and (v), we have

$$s_{N+1} \leq r_N + s_N - r_{N+1} \leq 1 - r_{N+1},$$

whence (ii) holds for $n = N + 1$.

If $p_{N+2} = 0$, then $p_{N+2} \leq p_{N+1}$ trivially holds (since $p_{N+1} \geq 0$ by (i)). On the other hand, suppose $p_{N+2} = 1 - \frac{1-p}{r_{N+1}+s_{N+1}} > 0$. Then we have $r_{N+1} + s_{N+1} > 1 - p$, which by our inductive hypothesis (v) implies $r_N + s_N \geq r_{N+1} + s_{N+1} > 1 - p$. The definition of p_{N+1} then implies

$$p_{N+2} = 1 - \frac{1-p}{r_{N+1}+s_{N+1}} \leq 1 - \frac{1-p}{r_N+s_N} = p_{N+1},$$

as desired, establishing that (iii) holds for $n = N + 1$. Arguing in exactly the same way (using the inductive hypothesis (iii) instead of (v)), we obtain that $r_{N+2} \leq r_{N+1}$. Hence (iv) holds for $n = N + 1$.

Finally we consider (v) for $n = N + 1$, which is the most delicate part of the induction. We begin by recording two useful facts, the second of which we shall reuse later.

Claim 32. If $p_{N+2} = 0$ or $r_{N+2} = 0$, then $s_{N+2} = 0$.

Proof. If $p_{N+2} = 0$, then by construction $s_{N+2} = 0$ and so we are done. If $r_{N+2} = 0$, then by construction $p_{N+1} \leq 1 - p$, which by our inductive hypothesis (iii) implies $p_{N+2} \leq 1 - p$ and hence $s_{N+2} = \max \left\{ 1 - 2r_{N+2} + \frac{r_{N+2}-(1-p)}{p_{N+2}}, 0 \right\} = \max \left\{ 1 - \frac{1-p}{p_{N+2}}, 0 \right\} = 0$. ■

Claim 33. If p_{N+2} and r_{N+2} are both strictly positive, then for all $i \in \{2, \dots, N + 1\}$, we have $s_i > 0$.

Proof. Fix $i \in \{2, \dots, N+1\}$. By our inductive hypotheses (iii) and (iv) (which we have already established up to $n = N+1$) and since $i \geq 2$, we have $0 < p_{N+2} \leq p_i \leq p_2 = p$ and $0 < r_{N+2} \leq r_i \leq r_2 = p$. Since $r_{i+1} > 0$, we in fact have $p_i > 1 - p$. We also have that

$$1 - 2r_i + \frac{r_i - (1 - p)}{p_i} = \frac{1}{p_i} (p_i + r_i - 2r_i p_i - (1 - p)) =: \frac{1}{p_i} f(p_i, r_i).$$

Now for fixed $y \in [1/2, 1]$, the function $x \mapsto f(x, y)$ is a nonincreasing function of x . Thus if $r_i \geq 1/2$, we have

$$f(p_i, r_i) \geq f(p, r_i) = (2p - 1)(1 - r_i) > 0.$$

On the other hand for fixed $y \in (0, 1/2)$, the function $x \mapsto f(x, y)$ is strictly increasing in x . Therefore if $r_i < 1/2$, we have

$$f(p_i, r_i) > f(1 - p, r_i) = r_i(2p - 1) > 0.$$

In either case, $f(p_i, r_i) > 0$, and thus $s_i = \max\left(\frac{1}{p_i} f(p_i, r_i), 0\right) > 0$. ■

With these results in hand, we return to the proof of (v). If $s_{N+2} = 0$, then (v) follows immediately from (iv). Thus we may assume that $s_{N+2} > 0$, whence by Claim 32 $p_{N+2} > 0$ and $r_{N+2} > 0$. By Claim 33 and our inductive hypotheses (iii) and (iv), this implies that p_i, r_i and s_i are all strictly positive for $i \in \{2, 3, \dots, N+2\}$. By definition of our sequences we thus have for all $i \in [N+1]$ that

$$p_{i+1} = 1 - \frac{1 - p}{r_i + s_i}, \quad r_{i+1} = 1 - \frac{1 - p}{p_i}, \quad s_{i+1} = 1 - 2r_{i+1} + \frac{r_{i+1} - (1 - p)}{p_{i+1}}. \quad (6.1)$$

Combining these equations we obtain for $i \in \{2, \dots, N+1\}$ that:

$$p_{i+1} = 1 - \frac{p_i p_{i-1} (1 - p)}{p(p_{i-1} - p_i + 1) + p_i - 1}. \quad (6.2)$$

Claim 34. Under our assumption that $s_{N+2} > 0$, for all integers $i \in [N+1]$ we have

$$p_{i+1} = \frac{p_i - (1 - p)}{(2 - p)p_i - (1 - p)}.$$

Proof. Since $p_1 = 1$ and $p_2 = p$, our claim holds for $i = 1$. Suppose it holds for some $i \leq N$. Then by rearranging terms, we have

$$p_i = \frac{(1 - p)(1 - p_{i+1})}{1 - (2 - p)p_{i+1}}.$$

Substituting this into the formula for p_{i+2} given by (6.2), we see our claim holds for $i + 1$ as well. ■

It follows from Claim 34 and (6.1) that for all $i \in [N+1]$, we can write $r_{i+1} + s_{i+1}$ as a function of p_i :

$$r_{i+1} + s_{i+1} = 1 - r_{i+1} + \frac{r_{i+1} - (1 - p)}{p_{i+1}}$$

$$\begin{aligned}
&= \frac{p_i p(2-p) - (1-p)}{p_i - (1-p)} \\
&= p(2-p) - \frac{(1-p)^3}{p_i - (1-p)}.
\end{aligned} \tag{6.3}$$

For $p_i > (1-p)$ (which we recall holds since $r_{i+1} > 0$), the expression above is an increasing function of p_i . By our inductive hypothesis (iii) that $p_{N+1} \leq p_N$ it follows that $r_{N+2} + s_{N+2} \leq r_{N+1} + s_{N+1}$ and we have verified that (v) holds for $n = N + 1$. ■

Recall that $p = \frac{2}{3} - \epsilon$, for some fixed $\epsilon \in (0, \frac{1}{6})$.

Lemma 35. *We have that $p_n = r_n = s_n = 0$ for all $n \geq N_\epsilon$, where $N_\epsilon := \lceil 2\epsilon^{-1} \rceil$.*

Proof. Suppose first that there exists $m \in [N_\epsilon - 1]$ such that $r_m = 0$. Then $s_m = 0$, so $p_{m+1} = 0$ and $s_{m+1} = 0$ by construction and $r_{m+1} = 0$ by Lemma 31(iv). Lemma 31(iii)-(v) then implies $p_n = r_n = s_n = 0$ for all $n \in \mathbb{N}_{\geq m+1}$, as required.

Suppose instead that $r_n > 0$ for all $n \in [N_\epsilon - 1]$ and there exists some $m \in [N_\epsilon - 2]$ such that $p_m \leq 1 - p$. Then $r_{m+1} = 0$, and thus by the argument above, we have that $p_n = r_n = s_n = 0$ for all $n \in \mathbb{N}_{\geq m+2}$, as required.

Finally, suppose $p_n > 1 - p$ and $r_n > 0$ both hold for all $n \in [N_\epsilon - 2]$. By Claim 33, we have $s_n > 0$ for all $n \in \{2, \dots, N_\epsilon - 3\}$. This allows us in turn to apply Claim 34 to all n in this interval and to deduce that

$$\begin{aligned}
p_{n-1} - p_n &= p_{n-1} - \frac{p_{n-1} - (1-p)}{(2-p)p_{n-1} - (1-p)} \\
&= \frac{1}{(2-p)p_{n-1} - (1-p)} \left((2-p) \left(p_{n-1} - \frac{1}{2} \right)^2 + \frac{2-3p}{4} \right) \\
&\geq \frac{3\epsilon}{4}.
\end{aligned} \tag{6.4}$$

Recall that $p_1 = 1$. As such, it follows from inequality (6.4) that $p_n \leq 1 - (n-1)\frac{3\epsilon}{4}$ for all $n \in [N_\epsilon - 2]$.

In particular, as $N_\epsilon = \lceil 2\epsilon^{-1} \rceil$ and $\epsilon \in (0, \frac{1}{6})$, we have

$$p_{N_\epsilon-3} \leq 1 - \left(\frac{2}{\epsilon} - 4 \right) \frac{3\epsilon}{4} = -\frac{1}{2} + 3\epsilon < 0,$$

which is a contradiction. ■

Now let $N = N_\epsilon$ be the integer constant whose existence is given by Lemma 35 and construct the 1-ipm \mathbf{G}_μ on the graph $G = P_N \times P_2$ from independent random assignments of states $S_{(n,y)}$ to vertices (n, y) in $V(G) = [N] \times [2]$, as described at the beginning of this section.

We observe here that by Lemma 31(i) and (ii), the states $S_{(n,y)}$ are well-defined random variables for every $(n, y) \in [N] \times [2]$, and so μ is a well-defined 1-ipm. We recall here for the reader's convenience the state-based rules governing which edges are open in \mathbf{G}_μ :

- for each $n \in [N - 1]$ and $y \in [2]$, the horizontal edge $\{(n, y), (n + 1, y)\}$ is open if and only if $S_{(n,y)} \leq S_{(n+1,y)}$,
- for each $n \in [N]$, the vertical edge $\{(n, 1), (n, 2)\}$ is open if and only if either $S_{(n,1)} = S_{(n,2)}$ or one of $S_{(n,1)}, S_{(n,2)}$ is equal to 1.

So intuitively, the value of the $S_{(n,y)}$ must increase from left to right along open horizontal edges of \mathbf{G}_μ , and it must stay constant along open vertical edges of \mathbf{G}_μ unless one of the endpoints is in the special state 1 which allows free passage up or down.

Claim 36. We have that $d(\mu) \geq p$.

Proof. For $(n, y) \in [N-1] \times [2]$, consider the horizontal edge $\{(n, y), (n+1, y)\}$. If $n+y$ is even, then by definition of r_{n+1} ,

$$\mu(\{(n, y), (n+1, y)\} \in \mathbf{G}_\mu) = \mu(S_{(n,y)} \leq S_{(n+1,y)}) = r_{n+1} + (1 - r_{n+1})(1 - p_n) \geq p.$$

Similarly if $n+y$ is odd, then by definition of p_{n+1} ,

$$\mu(\{(n, y), (n+1, y)\} \in \mathbf{G}_\mu) = \mu(S_{(n,y)} \leq S_{(n+1,y)}) = p_{n+1} + (1 - p_{n+1})(1 - r_n - s_n) \geq p.$$

Finally, for a vertical edge $\{(n, 1), (n, 2)\}$, $n \in [N]$, we have

$$\begin{aligned} \mu(\{(n, 1), (n, 2)\} \in \mathbf{G}_\mu) &= \mu(S_{(n,1)} = S_{(n,2)} \text{ or } 1 \in \{S_{(n,1)}, S_{(n,2)}\}) \\ &= s_n + p_n r_n + (1 - p_n)(1 - r_n - s_n). \end{aligned}$$

Now, if $p_n = 0$, then $r_{n-1} \leq 1 - p$ by definition of p_n , whence $r_n \leq 1 - p$ by Lemma 31(iv), and so the expression above equals $1 - r_n \geq p$. On the other hand if $p_n \neq 0$, then by definition of s_n the expression above is at least p . Thus each horizontal edge and each vertical edge is open in \mathbf{G}_μ with probability at least p , and $d(\mu) \geq p$ as claimed. ■

Claim 37. There is no open path in \mathbf{G}_μ from $\{1\} \times [2]$ to $\{N\} \times [2]$.

Proof. By construction, $p_1 = r_1 = 1$, whence $S_{(1,1)} = S_{(1,2)} = 2$. Furthermore, by Lemma 35 and our choice of N , $p_N = r_N = s_N = 0$, whence $S_{(N,1)} = S_{(N,2)} = 0$.

Let N' be the largest $n \in [N]$ for which there exists an open path in \mathbf{G}_μ from $\{1\} \times [2]$ to $\{n\} \times [2]$. Let \mathcal{P} be such a path, and let $v_0 \in \{1\} \times [2]$, $v_1 \in \{2\} \times [2]$, $v_2, \dots, v_\ell \in \{n\} \times [2]$ be the vertices of \mathcal{P} traversed from left to right. Observe that in this ordering of the vertices of \mathcal{P} , every horizontal edge $\{(n, y), (n+1, y)\}$ of \mathcal{P} is traversed from left to right.

We claim that for all $i \in [\ell]$, we have $S_i \in \{1, 2\}$. Indeed, by construction $S_{v_0} = 2$. Suppose there exists some $1 \leq i < \ell$ such that $S_{v_j} \in \{1, 2\}$ for all $j < i$. If $S_{v_i} = 2$, then the edge $v_i v_{i+1}$ can be open in \mathbf{G}_μ only if $S_{v_{i+1}} \in \{1, 2\}$. What is more, $S_{v_{i+1}}$ can be equal to 1 if and only if $v_i v_{i+1}$ is a vertical edge. On the other hand, suppose $S_{v_i} = 1$. Then $v_{i-1} v_i$ was a vertical edge (since there is no edge both of whose endpoints are in state 1 and since horizontal edges are traversed from left to right by \mathcal{P}), and so $v_{i+1} = v_i + (1, 0)$. But then $v_i v_{i+1}$ open in \mathbf{G}_μ implies $S_{v_{i+1}} = 2$. Thus for every vertex v_i of \mathcal{P} , we have that S_{v_i} is indeed in state 1 or 2.

This implies in particular that $v_\ell \notin \{N\} \times [2]$ (since as we remarked above $S_{(N,1)} = S_{(N,2)} = 0$). Thus there is no open path in \mathbf{G}_μ from $\{1\} \times [2]$ to $\{N\} \times [2]$. ■

Thus μ is an element of $\mathcal{M}_{1, \geq p}(P_N \times P_2)$ for which

$$\mu(\exists \text{ open left-right crossing}) = 0.$$

Given $n \geq N$, we may extend μ to an element $\mu' \in \mathcal{M}_{1, \geq p}(P_n \times P_2)$ by letting every edge in $P_n \times P_2 \setminus P_N \times P_2$ be open independently at random with probability p . In this way we obtain a 1-independent bond percolation measure μ' on $P_n \times P_2$ with edge-probability p for which there almost surely are no open left-right crossings of $P_n \times P_2$, giving the required lower bound on $p_{1, \times}(P_n \times P_2)$. ■

We conclude this section by proving Theorem 11(ii), with the aid of Theorem 30.

Proof of Theorem 11(ii). Trivially, the 1-independent connectivity function of the path on 2 vertices P_2 (i.e., the graph consisting of a single edge) is $f_{1, P_2}(p) = p$. Thus the constant $p_\star(P_2)$ defined by equation (3.4) is the unique solution in $[0, 1]$ to the equation $x^2 = 4(1 - x)^2$, namely $p_\star(P_2) = \frac{2}{3}$. By Corollary 27, this implies $p_{1, \ell p}(\mathbb{Z} \times P_2) \leq \frac{2}{3}$.

For the lower bound, fix $p \in (\frac{1}{2}, \frac{2}{3})$. In the proof of Theorem 30, we showed there exist some integer $N \in \mathbb{N}$ and $\mu \in \mathcal{M}_{1, \geq p}(P_N \times P_2)$ such that

- (i) $\mu(\exists \text{ open left-right crossing}) = 0$;
- (ii) $\mu(\{(1, 1), (1, 2)\} \text{ and } \{(N, 1), (N, 2)\} \text{ are open}) = 1$.

We use this measure to create a measure $\nu \in \mathcal{M}_{1, \geq p}(\mathbb{Z} \times P_2)$. Let $G := \mathbb{Z} \times P_2$. For each $i \in \mathbb{Z}$, we let the subgraphs $\mathbf{G}_\nu[(i(N-1) + [N]) \times [2]]$ on horizontal shifts of the ladder $P_N \times P_2$ by $i(N-1)$ be independent identically distributed random variables with distribution given by μ . Thanks to property (ii) recorded above, the random subgraphs agree on the vertical rungs $\{1 + i(N-1)\} \times P_2$ of the ladder, and this gives rise to a bona fide 1-independent model ν on $\mathbb{Z} \times P_2$ with edge-probability p . Furthermore, property (i) implies all connected components in \mathbf{G}_ν have size at most $4(N-1) - 2 = 4N - 6$. In particular, $p_{1, \ell p}(\mathbb{Z} \times P_2) \geq p$. Since $p < \frac{2}{3}$ was chosen arbitrarily, this gives the required lower bound $p_{1, \ell p}(\mathbb{Z} \times P_2) \geq \frac{2}{3}$. ■

7 | COMPLETE GRAPHS

In this section we will prove Theorem 16. Recall that, given $n \in \mathbb{N}_{\geq 2}$ and $p \in [0, 1]$, we let $\theta = \theta(p) := \frac{1 + \sqrt{2p-1}}{2}$ and $p_n := \frac{1}{2} \left(1 - \tan^2 \left(\frac{\pi}{2n} \right) \right)$. Let $g_n(\theta) := \theta^n + (1 - \theta)^n$.

7.1 | An upper bound for $f_{1, K_n}(p)$

Before proving Theorem 16, let us give a simple vertex-based construction of a measure $\nu_p \in \mathcal{M}_{1, \geq p}(K_n)$ that shows $f_{1, K_n}(p) \leq g_n(\theta)$ for $p \geq \frac{1}{2}$. We call this measure the *Red-Blue construction*. We think of K_n as the complete graph on vertex set $[n]$, and we color each vertex Red with probability θ and color it Blue otherwise, and we do this independently for all vertices. The edge $\{i, j\} \in [n]^{(2)}$ is open if and only if i and j have the same color. As $p = \theta^2 + (1 - \theta)^2$, we have that each edge is present in $(\mathbf{K}_n)_{\nu_p}$ with probability p . Note that $(\mathbf{K}_n)_{\nu_p}$ will either be either a disjoint union of two cliques, in which case it is disconnected, or the complete graph K_n , in which case it is connected. This latter case occurs if and only if every vertex receives the same color, and so the probability that $(\mathbf{K}_n)_{\nu_p}$ is connected is equal to $g_n(\theta)$. As this construction is vertex-based, it is clear that it is 1-independent.

If $p < \frac{1}{2}$ then θ is a complex number, and so the Red-Blue construction is no longer valid. However, as discussed in Section 4, we will show that it is possible to extend this construction to all $p \in [p_n, 1]$. Given $j \in \{0, 1, \dots, n\}$, let

$$g_{n,j}(\theta) := \theta^j(1 - \theta)^{n-j} + \theta^{n-j}(1 - \theta)^j.$$

When $j = 0$ or $j = n$ we have that $g_{n,0}(\theta)$ and $g_{n,n}(\theta)$ are each equal to $g_n(\theta)$, and so we just write the latter instead. Given some $A \subseteq [n]$, let H_A be the disjoint union of a clique on A with a clique on $[n] \setminus A$. Note that when $A = \emptyset$ or $[n]$ we have that H_A is equal to $K_{[n]}$, and more generally that $H_A = H_{[n] \setminus A}$. For $p \in [0, 1]$, let μ_p be the following function on subgraphs G of K_n :

$$\mu_p(G) := \begin{cases} g_{n,|A|}(\theta(p)) & \text{if } G = H_A \text{ for some } A \subseteq [n], \\ 0 & \text{else.} \end{cases}$$

For $p \in \left[\frac{1}{2}, 1\right]$ this function matches the Red-Blue construction given above, and so by defining $\nu_p((\mathbf{K}_n)_{\nu_p} = G) := \mu_p(G)$ for all subgraphs $G \subseteq K_n$, we obtain a measure ν_p which is a 1-ipm defined without making reference to states of vertices. The following claim, together with Lemma 28, shows that in fact ν_p is a 1-ipm on K_n for all $p \in [p_n, 1]$.

Claim 38. For all $p \in \left[p_n, \frac{1}{2}\right]$ and all $j \in \{0, \dots, n\}$ we have that $g_{n,j}(\theta(p))$ is nonnegative real number.

Proof. Let us begin with the case $j = n$. As $p \leq \frac{1}{2}$, we have that θ and $1 - \theta$ are complex conjugates, and so $g_n(\theta(p))$ is a real number for all p in this range. By writing $\theta = re^{i\phi}$, where $r := \sqrt{\frac{1-p}{2}}$ and $\phi := \arctan\left(\sqrt{1-2p}\right)$, we can write

$$g_n(\theta(p)) = 2r^n \cos(n\phi). \quad (7.1)$$

Now $p \in \left[p_n, \frac{1}{2}\right]$ implies $0 \leq \phi \leq \frac{\pi}{2n}$, which in turn gives $\cos(n\pi) \geq 0$. By (7.1), it follows that $g_n(\theta(p))$ is a nonnegative real number for all $p \in \left[p_n, \frac{1}{2}\right]$, which proves the claim when $j = n$. For general $j \in \{0, \dots, n\}$, we have that

$$g_{n,j}(\theta(p)) = \begin{cases} \left(\frac{1-p}{2}\right)^j g_{n-2j}(\theta(p)) & \text{if } n \geq 2j, \\ \left(\frac{1-p}{2}\right)^{n-j} g_{2j-n}(\theta(p)) & \text{if } n \leq 2j. \end{cases}$$

Therefore the previous case of the claim shows that $g_{n,j}(\theta(p)) \in [0, 1]$ for all $p \in \left[p_n, \frac{1}{2}\right]$; at this stage we are using the fact that $(p_n)_{n \geq 2}$ forms an increasing sequence, and so $p \geq p_n$ implies that $p \geq p_s$ for all $s \leq n$. ■

Note that as this proof shows that $g_n(\theta(p_n)) = 0$, we have that the probability $(\mathbf{K}_n)_{\nu_{p_n}}$ is connected is equal to 0. As $\nu_{p_n} \in \mathcal{M}_{1, \geq p}(K_n)$ for all $p \leq p_n$, we have that $f_{1, K_n}(p) = 0$ for all $p \leq p_n$. We now prove that this construction is optimal with respect to the connectivity function.

7.2 | A lower bound on $f_{1, K_n}(p)$

Proof of Theorem 16. The previous constructions discussed show that

$$f_{1, K_n}(p) \leq \begin{cases} g_n(\theta) & \text{for } p \in [p_n, 1], \\ 0 & \text{for } p \in [0, p_n]. \end{cases}$$

It is clear that $f_{1,K_n}(p) \geq 0$ for all p , and so all that remains to show is that $f_{1,K_n}(p) \geq g_n(\theta)$ for $p \in [p_n, 1]$. We will prove this result by induction on n . The inequality is trivially true when $n = 2$, so let us assume that $n > 2$ and that the inequality is true for all cases from 2 up to $n - 1$. First, we note that $g_n(\theta) = g_j(\theta)g_{n-j}(\theta) - g_{n,j}(\theta)$ for all $j \in \{0, 1, \dots, n\}$. Thus, if we multiply both sides of this equation by $\binom{n}{j}$ and sum over all $j \in \{0, 1, \dots, n\}$, we have that

$$2^n g_n(\theta) = \left(\sum_{j=0}^n \binom{n}{j} g_j(\theta) g_{n-j}(\theta) \right) - 2, \quad (7.2)$$

Let $\mu \in \mathcal{M}_{1,\geq p}(K_n)$ and let C be the event that $(\mathbf{K}_n)_\mu$ is connected. Given $A \subseteq [n]$, let X_A be the event that $(\mathbf{K}_n)_\mu[A]$ and $(\mathbf{K}_n)_\mu[A^c]$ are each connected, where $A^c = [n] \setminus A$. Moreover, let Y_A be the event that $(\mathbf{K}_n)_\mu[A]$ and $(\mathbf{K}_n)_\mu[A^c]$ are each connected, and there are no edges between A and A^c in $(\mathbf{K}_n)_\mu$. For all $A \subseteq [n]$, we have that

$$\mu(C) \geq \mu(X_A) - \mu(Y_A). \quad (7.3)$$

Note that when $A = \emptyset$ or $A = [n]$, the above equation is trivially true due to the fact that $C, X_\emptyset, X_{[n]}, Y_\emptyset$ and $Y_{[n]}$ are all the same event. As μ is 1-independent we have that if A is a nonempty proper subset of $[n]$, then, by induction on n , we have

$$\mu(X_A) \geq g_{|A|}(\theta) g_{n-|A|}(\theta). \quad (7.4)$$

Note that here we are using the fact that $(p_n)_{n \geq 2}$ forms an increasing sequence, and so $p \geq p_n$ implies that $p \geq p_s$ for all $s \leq n$. We are also using the fact that $g_1(\theta) = 1$ for all $\theta \in [0, 1]$. We proceed by summing (7.3) over all nonempty proper subsets of $[n]$, and then applying (7.4) to obtain

$$\begin{aligned} (2^n - 2)\mu(C) &\geq \left(\sum_{A \subseteq [n]} g_{|A|}(\theta) g_{n-|A|}(\theta) \right) - 2g_0(\theta)g_n(\theta) \\ &\quad - \left(\sum_{A \subseteq [n]} \mu(Y_A) \right) + \mu(Y_\emptyset) + \mu(Y_{[n]}). \end{aligned} \quad (7.5)$$

We apply (7.2) and the fact that the events C, Y_\emptyset and $Y_{[n]}$ are all the same event to (7.5) to get

$$(2^n - 4)\mu(C) \geq (2^n - 4)g_n(\theta) + 2 - \left(\sum_{A \subseteq [n]} \mu(Y_A) \right). \quad (7.6)$$

Note that for all $A \subseteq [n]$, the events Y_A and Y_{A^c} are the same event, and so $\sum_{A \subseteq [n]} \mu(Y_A) = 2 \sum_{1 \in A \subseteq [n]} \mu(Y_A)$. Moreover, the set $\{Y_A : 1 \in A \subseteq [n]\}$ consists of pairwise disjoint events, and so $\sum_{1 \in A \subseteq [n]} \mu(Y_A) \leq 1$. Thus

$$\sum_{A \subseteq [n]} \mu(Y_A) \leq 2. \quad (7.7)$$

We apply (7.7) to (7.6) to obtain $(2^n - 4)\mu(C) \geq (2^n - 4)g_n(\theta)$. As $n > 2$, we have that $\mu(C) \geq g_n(\theta)$ and so we are done. ■

7.3 | A remark on $f_{k,K_n}(p)$ for $k \geq 2$

Clearly we can define $f_{k,G}(p)$ analogously to $f_{1,G}(p)$ for $k \in \mathbb{N}_0$. For $k = 0$, $f_{0,K_n}(p)$ is exactly the probability that an instance of the Erdős-Rényi random graph $\mathbf{G}_{n,p}$ contains a spanning tree. As far as we know, there is no nice closed form expression for this function.

In this section, we have computed $f_{1,K_n}(p)$ exactly, which is the other interesting case, as for $k \geq 2$ the connectivity problem is trivial.

Proposition 39. *For all $k, n \in \mathbb{N}_{\geq 2}$, we have that*

$$f_{k,K_n}(p) = \begin{cases} 0 & \text{if } p \leq 1 - \frac{2}{n}, \\ 1 - \frac{n(1-p)}{2} & \text{otherwise.} \end{cases}$$

Proof. For the lower bound, consider $\mu \in \mathcal{M}_{k,\geq p}(K_n)$. Since any subgraph of K_n with at least $\binom{n}{2} - (n-1)$ edges is connected, we can apply Markov's inequality to show that

$$1 - \mu(\{\text{connected}\}) \leq \mu(\{\exists \geq (n-1) \text{ closed edges}\}) \leq \frac{1}{n-1} \mathbb{E}_\mu \{\# \text{ closed edges}\} = \frac{n(1-p)}{2}.$$

For the upper bound, consider the random graph \mathbf{G} obtained as follows. Let $x := \frac{1-p}{2}$. With probability $\min(nx, 1)$, select a vertex $i \in [n] = V(K_n)$ uniformly at random, and let \mathbf{G} be the subgraph of K_n obtained by removing all edges incident with i . Otherwise, let \mathbf{G} be the complete graph K_n . It is easy to check that \mathbf{G} is a 2-independent model with edge-probability p and that \mathbf{G} is connected if and only if $\mathbf{G} = K_n$, an event which occurs with probability $1 - \min(1, nx) = \max(0, 1 - n(1-p)/2)$. ■

8 | CYCLES

8.1 | Linear programming for calculating $f_{1,G}(p)$

In this subsection we describe how we can represent the problem of finding $f_{1,G}(p)$, for any graph G , as a (possibly nonlinear) program.

Given a graph G on vertex set $[n]$, let $\mathcal{H} = \mathcal{H}(G)$ be the set of all labeled subgraphs of G . Throughout this section we treat these subgraphs as subsets of $E(G)$, and always imagine them to be on the full vertex set $[n]$. For each labeled subgraph of G we write

$$\mu(S) := \mu(S \subseteq \mathbf{G}_\mu) \quad \text{and} \quad \mu(\hat{S}) := \mu(\mathbf{G}_\mu = S).$$

Recall that for a function $\mu : \mathcal{H} \rightarrow \mathbb{R}_{\geq 0}$, we have $\mu \in \mathcal{M}_{1,\geq p}(G)$ if and only if the following three conditions all hold:

1. μ is a probability measure on labeled subgraphs of G ,
2. Every edge of G is open in \mathbf{G}_μ with probability at least p ,
3. Given nonempty $S, T \in \mathcal{H}$ such that S and T are supported on disjoint subsets of $[n]$, $\mu(S) \cdot \mu(T) = \mu(S \cup T)$.

As we are interested in determining $f_{1,G}(p)$, and as randomly deleting edges cannot increase the probability of being connected, we may assume that in fact every edge of G is open in \mathbf{G}_μ with probability exactly p (by applying random sparsification as in Remark 3 if necessary). We can thus rewrite the conditions above in the following way:

1. $\sum_{H \in \mathcal{H}} \mu(\hat{H}) = 1$,
2. For all edges $e \in E(G)$, we have that $\sum_{H \in \mathcal{H}} \mathbb{1}(e \in H) \mu(\hat{H}) = p$,
3. For all nonempty $S, T \in \mathcal{H}$, such that S and T are supported on disjoint subsets of $[n]$, we have that

$$\sum_{H \in \mathcal{H}} \mu(\hat{H}) \left(\mathbb{1}((S \cup T) \subseteq H) - \mathbb{1}(S \subseteq H) \mu(T) \right) = 0. \quad (8.1)$$

Let $A = A(G)$ be a matrix which has columns indexed by \mathcal{H} , and a row for each piece of information given by one of the above conditions. That is:

1. We have a row for the empty set such that $A_{\emptyset, H} := 1$.
2. We have a row for $e \in G$; the entry $A_{e, H} := \mathbb{1}(e \in H)$;
3. We have a row for each pair $S, T \in \mathcal{H} \setminus \{\emptyset\}$ supported on disjoint subsets of $[n]$; the entry $A_{\{S, T\}, H} := \mathbb{1}((S \cup T) \subseteq H) - \mu(T) \cdot \mathbb{1}(S \subseteq H)$.

Let $\mathbf{q} = \mathbf{q}(G)$ be a vector with indexing the same as the rows of A ; let $q_{\emptyset} := 1$, $q_e := p$ for $e \in G$, and $q_{\{S, T\}} := 0$ for each pair $S, T \in \mathcal{H} \setminus \{\emptyset\}$ supported on disjoint subsets of $[n]$. Then a vector \mathbf{w} , whose entries are indexed by \mathcal{H} , which satisfies $w_H \geq 0$ for all $H \in \mathcal{H}$, and also $A\mathbf{w} = \mathbf{q}$ corresponds precisely to a 1-ipm μ on G with $\mu\{e \text{ open}\} = p$ for all edges $e \in E(G)$.

Let \mathbf{c} be a vector indexed by \mathcal{H} defined by $c_H := \mathbb{1}(H \text{ is connected})$. Just to make it clear, we say that $H \in \mathcal{H}$ is connected if it contains a spanning tree of $[n]$. Then for a given value of p the vector $\mathbf{w}(p)$ satisfying $A\mathbf{w}(p) = \mathbf{q}$ and $\mathbf{c}^T \mathbf{w}$ minimal corresponds to a measure $\mu \in \mathcal{M}_{1, \geq p}(G)$ such that $\mu(H \text{ is connected}) = f_{1, G}(p)$.

Observe that for any graph with five vertices or fewer, any partition of the graph into two parts has that one part must have at most two vertices in it. In particular, if G is a graph on $[5]$, and S and T are nonempty subgraphs of G supported on disjoint subsets of $[5]$, then one of S and T must consist of precisely one edge of G . By choosing T to be this subgraph, we can always choose S and T for (8.1) so that $\mu(T) = p$. Thus for any choice of p , we can turn the problem of finding $f_{1, G}(p)$ into the following linear program:

$$a^* = \min_{\mathbf{w}} \mathbf{c}^T \mathbf{w} \quad \text{subject to } A\mathbf{w} = \mathbf{q}, \mathbf{w} \geq 0. \quad (8.2)$$

(Note that for graphs with six or more vertices, one may find S and T such that $\mu(T)$ (in (8.1)) is an unknown function of p , and thus the program is not linear; for example, this indeed is the case for C_6 .)

The duality theorem states that the asymmetric dual problem has the same optimal solution a^* :

$$a^* = \max_{\mathbf{x}} \mathbf{q}^T \mathbf{x} \quad \text{subject to } A^T \mathbf{x} \leq \mathbf{c} \text{ and } \mathbf{x} \geq 0. \quad (8.3)$$

One can easily solve the linear programs above for a specific value of p , for example using the software Maple, and the *LPSolve* function it contains. However we of course wish to find solutions for all values of $p \in [0, 1]$.

By writing $A = (a_{ij})$, $\mathbf{w} = (w_j)$, $\mathbf{c} = (c_j)$, $\mathbf{q} = (q_i)$ and $\mathbf{x} = (x_i)$ any solutions \mathbf{w} and \mathbf{x} must satisfy $\sum_j a_{ij} w_j = q_i$, $\sum_i a_{ij} x_i \leq c_j$, $x_i \geq 0$ and $w_j \geq 0$. Thus we have

$$\sum_i q_i x_i = \sum_i \left(\sum_j a_{ij} w_j \right) x_i = \sum_j \left(\sum_i a_{ij} x_i \right) w_j \leq \sum_j c_j w_j.$$

In particular for optimal solutions we have $\sum_i q_i x_i = \sum_j c_j w_j$ and so the inequality must be an equality, that is

$$\left(\sum_i a_{ij} x_i \right) w_j = c_j w_j, \quad \text{for all } j.$$

Consequently for each j we either have $w_j = 0$ or $\sum_i a_{ij} x_i = c_j$. Thus in our attempt to obtain a function for all p , it seems reasonable to look at an optimal solution for one value of p and see which w_j have been set to zero; assume for these indices that we always have $w_j = 0$ and attempt to directly solve the equations that result from this. This motivates the following method:

- Solve (8.2) with a specific value of p to obtain a solution $\mathbf{w}(p)$ and a set $J := \{j \in [|\mathbf{w}|] : w_j(p) = 0\}$.
- Solve the set of equations $\{(A\mathbf{w})_i = q_i, w_j = 0 : i \in [|\mathbf{w}|], j \in J\}$ to obtain functions of p for all w_k , $k \in [|\mathbf{w}|]$, which we write as $w'_k(p)$.
- Solve the set of equations $\{(A^T \mathbf{x})_i = c_i : i \in [|\mathbf{w}|] \setminus J\}$ to obtain functions of p for all x_k , $k \in [|\mathbf{w}|]$, which we write as $x'_k(p)$.
- Write $w^*(p) := \mathbf{c}^T \mathbf{w}'(p)$ and $x^*(p) := \mathbf{q}^T \mathbf{x}'(p)$.
- For a certain interval $P \subseteq [0, 1]$ of values of p , check that $(A^T \mathbf{x}')_i(p) \leq c_i$ and $w'_i(p) \geq 0$, for all $i \in [|\mathbf{w}|]$.

For the given interval P which works above, the conditions above ensure that the $\mathbf{w}'(p)$ and $\mathbf{x}'(p)$ obtained in this way are feasible solutions to (8.2) and (8.3) respectively. Thus if $w^*(p) = x^*(p)$, then by the duality theorem we have $f_{1,G}(p) = w^*(p)$. Furthermore, a measure μ on the subgraphs of G which is extremal is given directly by $\mathbf{w}'(p)$. In the following subsection we give, as examples, two results which are proved using the above method.

8.2 | The connectivity function of small cycles

In this subsection we prove Theorems 17 and 18 using the above method. Furthermore, the method gives us an extremal example in each case.

Proof of Theorem 17. For C_4 and $p \in \left[\frac{1}{2}, 1\right]$ an extremal construction is given by the measure μ , defined by

$$\mu(\hat{H}) = \begin{cases} 2p - 1 & \text{if } H = C_4; \\ \frac{p(1-p)}{2} & \text{if } H \text{ contains precisely two edges, which are adjacent;} \\ (1-p)^2 & \text{if } H \text{ contains precisely two edges, which are not adjacent;} \\ 0 & \text{otherwise.} \end{cases}$$

For C_4 and $p \in \left[0, \frac{1}{2}\right]$ an extremal construction is given by the measure μ , defined by

$$\mu(\hat{H}) = \begin{cases} 1 - 2p & \text{if } H \text{ is the empty graph;} \\ \frac{p(1-p)}{2} & \text{if } H \text{ contains precisely two edges, which are adjacent;} \\ p^2 & \text{if } H \text{ contains precisely two edges, which are not adjacent;} \\ 0 & \text{otherwise.} \end{cases}$$

■

We can in fact give a direct combinatorial proof of the lower bound in Theorem 17: for any $\mu \in \mathcal{M}_{1,\geq p}(C_4)$, we have by 1-independence that

$$\mu(\{\text{connected}\}) \geq \mu(\{12, 34 \text{ are open}\}) - \mu(\{23, 14 \text{ are closed}\}) \geq p^2 - (1-p)^2 = 2p - 1.$$

Together with the first of the constructions of measures μ above (which can be found by analyzing how the bound in the inequality above can be tight), this gives a second and perhaps more insightful proof of Theorem 17 than the one obtained from applying the linear optimisation method. However for the next result, on $f_{1,C_5}(p)$, we do not have a combinatorial proof, and our result relies solely on linear optimisation.

Proof of Theorem 18. For C_5 and $p \in \left[\frac{\sqrt{3}}{3}, 1\right]$ an extremal construction is given by the measure μ , defined by

$$\mu(\hat{H}) = \begin{cases} \frac{p(3p^2-1)}{3p-1} & \text{if } H = C_5; \\ \frac{p(1-p)(2p-1)}{5(3p-1)} & \text{if } H \text{ is missing precisely two edges, which are adjacent;} \\ \frac{p(1-p)^2}{5(3p-1)} & \text{if } H \text{ is missing precisely two edges, which are not adjacent;} \\ \frac{(2p-1)(1-p)^2}{3p-1} & \text{if } H \text{ is the empty graph;} \\ 0 & \text{otherwise.} \end{cases}$$

For C_5 and $p \in \left[0, \frac{\sqrt{3}}{3}\right]$ an extremal construction is given by the measure μ , defined by

$$\mu(\hat{H}) = \begin{cases} \frac{5p^3-5p^2-2p+2}{3p+2} & \text{if } H \text{ is the empty graph;} \\ \frac{p(1-3p^2)}{3p+2} & \text{if } H \text{ consists of precisely two edges, which are adjacent;} \\ \frac{p^3}{3p+2} & \text{if } H \text{ is missing precisely two edges, which are adjacent;} \\ \frac{p^2(1+p)}{3p+2} & \text{if } H \text{ is missing precisely two edges, which are not adjacent;} \\ 0 & \text{otherwise.} \end{cases}$$

8.3 | General bounds for cycles of length at least 6

We can use Markov's inequality to derive the following simple lower bound on $f_{1,C_n}(p)$ for $n \geq 6$.

Proposition 40. For $n \in \mathbb{N}$, with $n \geq 6$, and $p \in [0, 1]$, we have $f_{1,C_n}(p) \geq \frac{np-(n-2)}{2}$.

A small adjustment to this argument gives the following improvement for $n = 6$.

Proposition 41. For $p \in [0, 1]$ we have that $f_{1,C_6}(p) \geq -p^3 + 3p^2 - 1$.

Proof of Proposition 40. Let $\mu \in \mathcal{M}_{1,\geq p}(C_n)$. Note that \mathbf{G}_μ is connected if and only if it has at most one closed edge. Thus by Markov's inequality, we have

$$f_{1,C_n}(p) = 1 - \mu(\exists \geq 2 \text{ closed edges in } C_n)$$

$$\begin{aligned}
&\geq 1 - \frac{\mathbb{E}_\mu(\# \text{ closed edges in } C_n)}{2} \\
&= 1 - \frac{n(1-p)}{2} = \frac{np - (n-2)}{2}.
\end{aligned}$$

Proof of Proposition 41. Let X be the number of closed edges in \mathbf{G}_μ . Cyclically label the edges of C_6 as e_1, \dots, e_6 . Then by simple counting,

$$\begin{aligned}
2(1-p)^3 &= \mu(e_1, e_3, e_5 \text{ are closed}) + \mu(e_2, e_4, e_6 \text{ are closed}) \\
&\leq \mu(X=3) + \mu(X=4) + \mu(X=5) + 2\mu(X=6).
\end{aligned}$$

Now by simple counting again, linearity of expectation and the inequality above, we get:

$$\begin{aligned}
f_{1,C_6}(p) &= 1 - \mu(X \geq 2) = 1 - \frac{\mathbb{E}_\mu(X)}{2} \\
&\quad + \left(\frac{\mu(X=1) + \mu(X=3) + 2\mu(X=4) + 3\mu(X=5) + 4\mu(X=6)}{2} \right) \\
&\geq 1 - \frac{6(1-p)}{2} + (1-p)^3 = -p^3 + 3p^2 - 1.
\end{aligned}$$

9 | MAXIMIZING CONNECTIVITY

In this section, we derive our results for maximizing connectivity in 1-independent modes. First of all Theorem 16 allow us to easily determine the value of $F_{1,K_n}(p)$ and hence prove Theorem 21.

Proof of Theorem 21. Given a 1-independent model \mathbf{G} on K_n with edge-probability at least $1-p$, observe that the complement \mathbf{G}^c of \mathbf{G} in K_n is a 1-independent model in which every edge is open with probability at most p . Furthermore, \mathbf{G}^c is connected whenever \mathbf{G} fails to be connected. This immediately implies

$$1 - f_{1,K_n}(1-p) \leq F_{1,K_n}(p). \quad (9.1)$$

Furthermore, observe that the Red-Blue measure v_p we constructed to obtain the upper bound on $f_{1,K_n}(p)$ in the proof of Theorem 16 has the property that a v_p -random graph is connected if and only if its complement fails to be connected. This immediately implies that we have equality in (9.1). ■

For paths, a simple construction achieves the obvious upper bound for $F_{1,P_n}(p)$.

Proof of Theorem 20. For any measure $\mu \in \mathcal{M}_{1,\leq p}(P_n)$, we have by 1-independence that

$$\mu(\{\text{connected}\}) = \mu(\{P_n\}) \leq \mu\left(\bigcap_{1 \leq i \leq \lfloor \frac{n}{2} \rfloor} \{\text{the edge } \{2i-1, 2i\} \text{ is open}\}\right) \leq p^{\lfloor \frac{n}{2} \rfloor},$$

which implies $F_{1,P_n}(p) \leq p^{\lfloor \frac{n}{2} \rfloor}$. For the lower bound, we construct a 1-ipm as follows. For each integer i : $1 \leq i \leq n/2$, we assign a state On to the vertex $2i$ with probability p , and a state Off otherwise, independently at random. Then set an edge of P_n to be open if one of its endpoints is in state On,

and closed otherwise. This is easily seen to yield a 1-ipm μ on P_n in which every edge is open with probability p , and for which

$$\mu(\{\text{connected}\}) = \mu\left(\bigcap_{1 \leq i \leq \lfloor \frac{n}{2} \rfloor} \{\text{the vertex } 2i \text{ is in state On}\}\right) = p^{\lfloor \frac{n}{2} \rfloor}.$$

Thus $F_{1,p_n}(p) \geq p^{\lfloor \frac{n}{2} \rfloor}$, as claimed. \blacksquare

The case of cycles C_n appears to be slightly more subtle. For the 4-cycle, as in the previous section, we can give two proofs, one combinatorial and the other via linear optimisation.

Proof of Theorem 17. The theorem immediately follows from an application of the linear optimisation techniques from Section 8. Alternatively, we can obtain the upper bound by a direct argument. For any measure $\mu \in \mathcal{M}_{1,\leq p}(C_4)$, we have by 1-independence that

$$1 - \mu(\{\text{connected}\}) \geq \mu(\{\text{both } 12 \text{ and } 34 \text{ are closed}\}) \geq (1 - p)^2,$$

and, by a simple union bound and 1-independence,

$$\mu(\{\text{connected}\}) \leq \mu(\{\text{both } 12 \text{ and } 34 \text{ are open}\} \cup \{\text{both } 23 \text{ and } 14 \text{ are open}\}) \leq 2p^2.$$

Combining these two inequalities and using $1 - (1 - p)^2 = 2p - p^2$, we obtain

$$\mu(\{\text{connected}\}) \leq \min(2p^2, 2p - p^2),$$

which gives the claimed upper bound on $F_{1,C_4}(p)$.

For the lower bound, we give two different constructions, depending on the value of p . For $p \in [\frac{2}{3}, 1]$ consider the measure μ defined by

$$\mu(\hat{H}) = \begin{cases} p(3p - 2) & \text{if } H = C_4; \\ p(1 - p) & \text{if } H \text{ contains precisely three edges;} \\ 1 - p(2 - p) & \text{if } H \text{ is the empty graph with no edges;} \\ 0 & \text{otherwise.} \end{cases}$$

It is easily checked that $\mu \in \mathcal{M}_{1,\leq p}(C_4)$ and that $\mu(\{\text{connected}\}) = 1 - (1 - p(2 - p)) = 2p - p^2$, which is maximal for p in that range.

For $p \in [0, \frac{2}{3}]$, consider the measure μ defined by

$$\mu(\hat{H}) = \begin{cases} \frac{p^2}{2} & \text{if } H \text{ contains precisely three edges;} \\ \frac{p(2-3p)}{2} & \text{if } H \text{ contains precisely one edge;} \\ 1 - 4p(1 - p) & \text{if } H \text{ is the empty graph with no edges;} \\ 0 & \text{otherwise.} \end{cases}$$

Again, it is easily checked that $\mu \in \mathcal{M}_{1,\leq p}(C_4)$ and that $\mu(\{\text{connected}\}) = \mu(\{\geq 3 \text{ edges open}\}) = 2p^2$, which is maximal for p in that range. \blacksquare

Proof of Theorem 18. We simply apply the linear optimisation method from Section 8—here again we do not have a combinatorial proof. In addition to establishing the theorem, this gives us constructions of extremal 1-independent measures maximizing connectivity.

For $p \in \left[\frac{3}{5}, 1\right]$ an extremal construction is given by the measure μ , defined by

$$\mu(\hat{H}) = \begin{cases} \frac{p(5p-3)}{5-3p} & \text{if } H = C_5; \\ \frac{p(1-p^2)}{5-3p} & \text{if } H \text{ contains precisely four edges;} \\ \frac{3p^3-7p^2+5p-1}{5-3p} & \text{if } H \text{ contains precisely two edges, which are adjacent;} \\ \frac{2(1-p)^3}{5-3p} & \text{if } H \text{ contains precisely one edge;} \\ 0 & \text{otherwise.} \end{cases}$$

For $p \in \left[\frac{1}{2}, \frac{3}{5}\right]$ an extremal construction is given by the measure μ , defined by

$$\mu(\hat{H}) = \begin{cases} \frac{p^2}{3} & \text{if } H \text{ contains precisely four edges;} \\ \frac{p(2-3p)}{3} & \text{if } H \text{ contains precisely two edges, which are adjacent;} \\ \frac{p(2p-1)}{3} & \text{if } H \text{ contains precisely one edge;} \\ \frac{3-5p}{3} & \text{if } H \text{ is the empty graph with no edges;} \\ 0 & \text{otherwise.} \end{cases}$$

For $p \in \left[0, \frac{1}{2}\right]$ an extremal construction is given by the measure μ , defined by

$$\mu(\hat{H}) = \begin{cases} \frac{p^2(p+1)}{p+4} & \text{if } H \text{ contains precisely four edges;} \\ \frac{p^2(1-2p)}{p+4} & \text{if } H \text{ is missing precisely two edges, which are adjacent;} \\ \frac{p(p^2-3p+2)}{p+4} & \text{if } H \text{ contains precisely two edges, which are adjacent;} \\ \frac{5p^2-9p+4}{p+4} & \text{if } H \text{ is the empty graph with no edges;} \\ 0 & \text{otherwise.} \end{cases}$$

■

10 | PROOF OF THEOREM 1.12

Combining Corollary 27 with our results on 1-independent connectivity, much of Theorem 12 is immediate.

Proof of Theorem 12. For the lower bound in part (i), we note that $\mathbb{Z} \times C_n$ has the finite 2-percolation property. Thus, as described after the proof of Theorem 7, we have that $p_{1,\ell p}(\mathbb{Z} \times C_n) \geq 4 - 2\sqrt{3}$. For the upper bound in part (i), since the long paths critical probability is nondecreasing under the addition of edges, we have

$$p_{1,\ell p}(\mathbb{Z} \times C_n) \leq p_{1,\ell p}(\mathbb{Z} \times P_n) \leq p_{1,\ell p}(\mathbb{Z} \times P_2),$$

which is at most $2/3$ by Theorem 11(ii).

For the upper bounds (ii)-(iv) Theorem 12 follow directly from our results on 1-independent connectivity functions. For $G = K_3, C_4, C_5$, we plug in the value of $f_{1,G}(p)$ in equation (3.4), solve for $p_\star(G)$ and apply Corollary 27.

In part (v), we begin by noting that as we are considering an increasing nested sequence of graphs, the sequence $(p_{1,\ell p}(\mathbb{Z} \times K_n))_{n \in \mathbb{N}}$ is nonincreasing in $[0, 1]$ and hence tends to a limit as $n \rightarrow \infty$. For the lower bound in (v), observe that for any $n \in \mathbb{N}$ the graph $\mathbb{Z} \times K_n$ has the finite 2-percolation property—indeed for any finite k , the closure of a copy of $P_k \times K_n$ under 2-neighbor bootstrap percolation in $\mathbb{Z} \times K_n$ is equal to itself. We construct a 1-ipm μ on $\mathbb{Z} \times K_n$ as in Corollary 24 but with starting set $T_0 = \{0\} \times V(K_n)$ and hence $T_k = (\{k\} \times V(K_n)) \cup (\{-k\} \times V(K_n))$. It is easily checked that μ -almost surely, all components (and hence all paths) in a μ -random graph have length at most $5n$. Since by construction $d(\mu) = 4 - 2\sqrt{3}$, this proves

$$p_{1,\ell p}(\mathbb{Z} \times K_n) \geq 4 - 2\sqrt{3}$$

for all $n \in \mathbb{N}$. For the upper bound, we perform some simple analysis. By solving a quadratic equation, we see that

$$\left(\frac{1 + \sqrt{2p-1}}{2} \right)^2 > (1-p)$$

for all fixed $p \in \left(\frac{5}{9}, 1 \right)$. Then by Theorem 16, for any such fixed p and all n sufficiently large, we have that

$$(f_{1,K_n}(p))^2 > \left(\frac{1 + \sqrt{2p-1}}{2} \right)^{2n} > 4(1-p)^n = 4(1-p)^{v(K_n)}.$$

Thus $p_\star(K_n) < p$ for all n sufficiently large, which by Corollary 27 implies $p_{1,\ell p}(\mathbb{Z} \times K_n) < p$. ■

11 | OPEN PROBLEMS

11.1 | More tractable subclasses of 1-independent measures

The most obvious open problem about 1-independent percolation is of course whether the known lower and upper bounds on $p_{1,c}(\mathbb{Z}^2)$ can be improved. This problem is, we suspect, very hard in general. However, it may prove more tractable if we restrict our attention to a smaller family of measures.

Definition 42. Let G be a graph. A G -partition is a partitioned set $\sqcup_{v \in V(G)} \Omega_v$, with nonempty parts indexed by the vertices of G . A G -partite graph is a graph H on a G -partition $V(H) = \sqcup_{v \in V(G)} \Omega_v$ whose edges are a subset of the union of the complete bipartite graphs $\sqcup_{uv \in E(G)} \{\omega_u \omega_v : \omega_u \in \Omega_u, \omega_v \in \Omega_v\}$ corresponding to the edges of G .

Given a G -partite graph H on a G -partition $\sqcup_{v \in V(G)} \Omega_v$, we have a natural way of constructing 1-independent bond percolation models: given a family $\mathbf{X} = (S_v)_{v \in V(G)}$ of independent random variables with S_v taking values in Ω_v , the (H, \mathbf{X}) -random subgraph of G , denoted by $H[\mathbf{X}]$, is the random configuration on $E(G)$ obtained by setting uv to be open if and only if $S_u S_v \in E(H)$.

Definition 43. Let G be a graph. A measure $\mu \in \mathcal{M}_{1,\geq p}(G)$ is said to be vertex-based if there exist

- a G -partition $\sqcup_{v \in V(G)} \Omega_v$,
- an associated G -partite graph H , and
- a collection of independent random variables $(S_v)_{v \in V(G)}$ with S_v taking values in Ω_v ,

such that the (H, \mathbf{X}) -random subgraph $H[\mathbf{X}]$ has the same distribution as the μ -random graph \mathbf{G}_μ .

Let $\mathcal{M}_{\text{vb}, \geq p}(G)$ denote the collection of all vertex-based measures on G with edge-probability at least p .

Problem 44. Determine $\inf \left\{ p \in [0, 1] : \forall \mu \in \mathcal{M}_{\text{vb}, \geq p}(\mathbb{Z}^2), \mu(\{\text{percolation}\}) = 1 \right\}$.

Vertex-based measures arise naturally in renormalising arguments, and are thus a natural class of examples to consider. A special case of Problems 6 and 44 is obtained by further restricting our attention to the case where the Ω_v have bounded size.

Definition 45. A vertex-based measure μ on a graph G is N -uniformly bounded if it is as in Definition 43 above and in addition for each $v \in V(G)$, $|\Omega_v| \leq N$. Furthermore, a vertex-based measure μ on a graph G is uniformly bounded if it is N -uniformly bounded for some $N \in \mathbb{N}$.

Let $\mathcal{M}_{N-\text{ubvb}, \geq p}(G)$ and $\mathcal{M}_{\text{ubvb}, \geq p}(G)$ denote the collection of all vertex-based measures on G with edge-probability at least p that are N -uniformly bounded and uniformly bounded respectively.

Problem 46. (i) For $N \in \mathbb{N}$, determine

$$\inf \left\{ p \in [0, 1] : \forall \mu \in \mathcal{M}_{N-\text{ubvb}, \geq p}(\mathbb{Z}^2), \mu(\{\text{percolation}\}) = 1 \right\}.$$

(ii) Determine

$$\inf \left\{ p \in [0, 1] : \forall \mu \in \mathcal{M}_{\text{ubvb}, \geq p}(\mathbb{Z}^2), \mu(\{\text{percolation}\}) = 1 \right\}.$$

Finally, let us note that the second most obvious problem arising from our work, besides that of improving the bounds on $p_{1,c}(\mathbb{Z}^2)$, is arguably that of giving bounds on $p_{1,\ell p}(\mathbb{Z}^2)$ and closely related variants. Such problems, which correspond to new questions in extremal graph theory, are discussed in the subsections below. For these problems too we believe restrictions to the class of uniformly bounded vertex-based 1-ipms could be both fruitful and interesting in their own right.

11.2 | Harris critical probability for other lattices

Beyond \mathbb{Z}^2 , it is natural to ask about bounds on $p_{1,c}(G)$ for some of the other commonly studied lattices in percolation theory.

Problem 47. Give good bounds on the value of $p_{1,c}(G)$ when G is one of the eleven Archimedean lattices in the plane or the d -dimensional integer lattice \mathbb{Z}^d .

This problem is particularly interesting when G is the triangular lattice or the honeycomb lattice (two lattices for which the 0-independent Harris critical probability is known exactly), or the cubic integer lattice \mathbb{Z}^3 (which is important in applications). A challenge in all cases is finding constructions of non-percolating 1-independent measures with high edge-probability—indeed, our arsenal of constructions

for 1-independent percolation problems is so sparse that any new construction could be of independent interest.

In a different direction, we can observe that \mathbb{Z}^{d+1} contains a copy of \mathbb{Z}^d , whence the sequence $(p_{1,c}(\mathbb{Z}^d))_{d \in \mathbb{N}}$ is nonincreasing in $[0, 1]$ and converges to a limit. Balister and Bollobás asked for its value:

Problem 48. [Balister, Bollobás [4]] Determine $\lim_{d \rightarrow \infty} p_{1,c}(\mathbb{Z}^d)$.

Note that by Theorem 7 proved in this paper, this limit must be at least $4 - 2\sqrt{3}$.

11.3 | Other notions of 1-independent critical probabilities

Let G be an infinite, locally finite connected graph, and v_0 a fixed vertex of G . Given a bond percolation model μ on G , we let \mathbf{C}_{v_0} denote the connected component of \mathbf{G}_μ containing v_0 .

If μ is 0-independent, then $\mu(\{\text{percolation}\}) = 1$ if and only if $\mu(\{|\mathbf{C}_0| = \infty\}) > 0$. However this need not be true for a 1-independent measure. Indeed, consider the 1-ipm on \mathbb{Z}^2 obtained by taking the measure constructed in the proof of Theorem 7 to determine the state of the edges in the ℓ_∞ ball of radius 3 around the origin and setting every other edge to be open independently at random with probability $4 - 2\sqrt{3}$. Then in this model percolation occurs almost surely, but the origin is contained inside a component of order at most 28.

Thus in principle there are different edge-probability thresholds in 1-independent percolation on a graph G for percolation to occur *somewhere* with probability 1 and for it to occur *anywhere* with strictly positive probability. Indeed, if $p_{1,c}(\mathbb{Z}^2)$ were strictly less than $3/4$, then one could obtain examples of such a graph G by attaching a long path to the origin in \mathbb{Z}^2 .

Another critical edge-probability of interest is the *Temperley critical probability*, which in 0-independent percolation is the threshold p_T at which $\mathbb{E}|\mathbf{C}_v| = \infty$ for any vertex v (and every 0-independent measure with edge-probability $> p_T$). In general this threshold is different from the Harris critical probability. Again for 1-independent percolation we have that the threshold for *some* vertex $v \in V(G)$ to satisfy $\mathbb{E}|\mathbf{C}_v| = \infty$ and for the threshold for *all* vertices of G to satisfy this are different.

Problem 49. Given an infinite, locally finite connected graph G , determine the following four critical probabilities:

$$\begin{aligned} p_{1,T_1}(G) &:= \inf \left\{ p \in [0, 1] : \forall \mu \in \mathcal{M}_{1,\geq p}(G), \exists v \in V(G) : \mathbb{E}_\mu |\mathbf{C}_v| = \infty \right\}, \\ p_{1,T_2}(G) &:= \inf \left\{ p \in [0, 1] : \forall \mu \in \mathcal{M}_{1,\geq p}(G), \forall v \in V(G) : \mathbb{E}_\mu |\mathbf{C}_v| = \infty \right\}, \\ p_{1,H_1}(G) &:= \inf \left\{ p \in [0, 1] : \forall \mu \in \mathcal{M}_{1,\geq p}(G), \exists v \in V(G) : \mu(|\mathbf{C}_v| = \infty) > 0 \right\}, \\ p_{1,H_2}(G) &:= \inf \left\{ p \in [0, 1] : \forall \mu \in \mathcal{M}_{1,\geq p}(G), \forall v \in V(G) : \mu(|\mathbf{C}_v| = \infty) > 0 \right\}. \end{aligned}$$

It follows from their definition that these four critical probabilities satisfy

$$p_{1,T_1}(G) \leq p_{1,T_2}(G) \leq p_{1,H_2}(G) \quad \text{and} \quad p_{1,T_1}(G) \leq p_{1,H_1}(G) \leq p_{1,H_2}(G). \quad (11.1)$$

In general, these four critical probabilities are all different. Indeed, Balister and Bollobás showed in [4, Theorem 1.5] that there exists an infinite, locally finite connected graph G with $p_{1,H_1}(G) = \frac{1}{2}$.

For any p : $\frac{1}{2} < p < \frac{3}{4}$, we have shown in Theorem 15 that there exists N such that $f_{1,p_N}(p) = 0$. Attach one end of a path of length N to an arbitrary vertex of G to form a graph G_1 , and let v denote the other end of the path. Then there exist 1-ipm $\mu \in \mathcal{M}_{1,\geq p}(G_1)$ such that with probability 1 the component of v in a μ -random graph has order at most N , which is finite. Thus we have

$$p_{1,T_1}(G_1) \leq p_{1,H_1}(G_1) \leq p_{1,H_1}(G) = \frac{1}{2} < p \leq p_{1,T_2}(G_1) \leq p_{1,H_2}(G_1).$$

On the other hand consider a graph G_2 obtained from the line lattice by attaching to each vertex $i \in \mathbb{Z}$ a collection of $2^{|i|+2}$ leaves. Clearly, $p_{1,H_1}(G_2) = p_{1,H_1}(\mathbb{Z}) = 1$. Now consider a 1-ipm $\mu \in \mathcal{M}_{1,\geq \frac{3}{4}}(G_2)$. By Theorem 26 applied to $G = K_1$ and $\alpha = \frac{1}{2}$, for any path P of length i in G_2 , the μ -probability that all edges in P are open is at least $2^{-(i+1)}$. Thus for any $v_0 \in V(G_2)$, the expected size of $|C_{v_0}|$ is

$$\begin{aligned} \mathbb{E}_\mu |C_{v_0}| &= \sum_{v \in V(G_2)} \mu(\{v \in C_{v_0}\}) \geq \sum_{i \in \mathbb{Z}_{\geq 2}} \#\{v : \text{the path from } v_0 \text{ to } v \text{ has length } i\} 2^{-(i+1)} \\ &\geq \sum_{i \in \mathbb{Z}_{\geq 2}} 2^{i+1} 2^{-(i+1)} = \infty. \end{aligned}$$

Thus we have

$$p_{1,T_1}(G_2) \leq p_{1,T_2}(G_2) \leq \frac{3}{4} < 1 = p_{1,H_1}(G_2) = p_{1,H_2}(G_2).$$

Corollary 50. (i) *None of the inequalities in (11.1) may be replaced by an equality.*
(ii) *Neither $p_{1,T_2}(G) \leq p_{1,H_1}(G)$ nor the reverse inequality are true in general.* □

Observe that $p_{1,H_1}(G)$ is the 1-independent Harris critical probability $p_{1,c}(G)$ studied in this paper; given Corollary 50, we more precisely should call it the *first* Harris critical probability for 1-independent percolation. Our construction for the proof of Theorem 7 and the argument of Balister, Bollobás and Walters from [7] give the following bounds when $G = \mathbb{Z}^2$:

$$4 - 2\sqrt{3} \leq p_{1,T_1}(\mathbb{Z}^2) \leq p_{1,H_2}(\mathbb{Z}^2) \leq 0.8639.$$

Question 51. *Are any of the four critical probabilities from Problem 49 equal when $G = \mathbb{Z}^2$?*

Finally, note that Problem 49 asks, in essence, how much we can delay percolation phenomena relative to the 0-independent case by exploiting the local dependencies between the edges allowed by 1-independence. While perhaps less useful in applications, it is an equally natural and appealing extremal problem to ask how much we can use these local dependencies to instead hasten the emergence of an infinite connected component. Balister and Bollobás were the first to consider this problem in [4], which it would be remiss not to mention here.

Definition 52. Let G be an infinite, locally finite connected graph, and let $\mathcal{M}_{k,\leq p}(G)$ be as before the collection of k -ipms μ on G satisfying $\sup_{e \in E(G)} \mu\{e \text{ is open}\} \leq p$. The critical threshold for fast k -independent percolation on G is

$$p_{k,F}(G) := \inf \left\{ p \in [0, 1] : \exists \mu \in \mathcal{M}_{k,\leq p}(G) : \mu(\{\text{percolation}\}) = 1 \right\}.$$

Balister and Bollobás determined $p_{1,F}(G)$ when G is an infinite, locally finite tree, and also gave the simple general bounds

$$\frac{1}{(\mu_{\text{conn.}}(G))^2} \leq p_{1,F}(G) \leq (\theta_{\text{site}}(G))^2, \quad (11.2)$$

where $\mu_{\text{conn.}}(G)$ is the connective constant of G and $\theta_{\text{site}}(G)$ the critical value of the θ -parameter for site percolation on G . For the square integer lattice, this gives a lower bound on $p_{1,F}(\mathbb{Z}^2)$ of 0.1393 from known upper bounds on $\mu_{\text{conn.}}(\mathbb{Z}^2)$. In the other direction, we get a rigorous upper bound of $p_{1,F}(\mathbb{Z}^2)$ of 0.4618 and nonrigorous upper bound of 0.3515 from bounds and estimates for $\theta_{\text{site}}(\mathbb{Z}^2)$. This obviously leaves a big gap, which Balister and Bollobás asked to reduce.

Question 53. [Balister and Bollobás [4]] What is $p_{1,F}(\mathbb{Z}^2)$?

11.4 | Long paths critical probability

An obvious problem is to tighten the bounds in Theorem 12(v), which are not too far apart (compared to many of the other bounds on critical probabilities for 1-independent model).

Problem 54. Determine $\lim_{n \rightarrow \infty} p_{1,\ell_p}(\mathbb{Z} \times K_n)$ (which must be an element of $\left[4 - 2\sqrt{3}, \frac{5}{9}\right]$).

In a similar vein, the sequence $p_{1,\ell_p}(\mathbb{Z} \times P_n)$ is a nonincreasing function of n (since $\mathbb{Z} \times P_{n+1}$ contains $\mathbb{Z} \times P_n$ as a subgraph). In this paper, we have given constructions showing that for all integers $n \geq 3$,

$$4 - 2\sqrt{3} \leq p_{1,\ell_p}(\mathbb{Z} \times C_n) \leq p_{1,\ell_p}(\mathbb{Z} \times P_n) \leq \frac{2}{3} = p_{1,\ell_p}(\mathbb{Z} \times P_2).$$

Thus the sequence $(p_{1,\ell_p}(\mathbb{Z} \times P_n))_{n \in \mathbb{N}}$ tends to a limit in the interval $\left[4 - 2\sqrt{3}, \frac{2}{3}\right]$ as $n \rightarrow \infty$.

Problem 55. Determine

$$p_{1,\ell_p}(\mathbb{Z} \times P_\infty) := \lim_{n \rightarrow \infty} p_{1,\ell_p}(\mathbb{Z} \times P_n).$$

An in principle different but related problem is determining the value of the long paths critical probability in \mathbb{Z}^2 (which need not be equal to the quantity $p_{1,\ell_p}(\mathbb{Z} \times P_\infty)$ defined above).

Problem 56. Determine $p_{1,\ell_p}(\mathbb{Z}^2)$.

We can also ask for k -independent versions of the long paths critical probability. Defining $p_{k,\ell_p}(G)$ mutatis mutandis, it is straightforward to adapt our arguments and constructions from Section 5 to show the following result, which also follows directly from the work of Liggett, Schonmann and Stacey [24] on stochastic domination of 0-independent measures on \mathbb{Z} by k -independent ones.

Theorem 57. [Liggett, Schonman and Stacey [24]] For any $k \in \mathbb{N}_0$, we have

$$p_{k,\ell_p}(\mathbb{Z}) = 1 - \frac{k^k}{(k+1)^{k+1}},$$

with the convention that $0^0 = 1$.

Given k fixed, it is easy to construct a $3k$ -ipm μ on \mathbb{Z}^2 with $d(\mu) = 1 - \frac{2}{k}$ and no open path of length more than $(2k + 1)^2$. Indeed, build a random graph model as follows:

- begin with all edges of \mathbb{Z}^2 open;
- independently for each $(i, j) \in \mathbb{Z}^2$, choose $H_{ij} \in [k + 1]$ uniformly at random and then for all j' : $j(k + 1) - k \leq j' \leq j(k + 1) + k$, set the horizontal edge $\{(i(k + 1) + H_{ij} - 1, j'), (i(k + 1) + H_{ij}, j')\}$ to be closed;
- independently for each $(i, j) \in \mathbb{Z}^2$, choose $V_{ij} \in [k + 1]$ uniformly at random and then for all i' : $i(k + 1) - k \leq i' \leq i(k + 1) + k$, set the vertical edge $\{(i', j(k + 1) + V_{ij} - 1), (i', j(k + 1) + V_{ij})\}$ to be closed.

It is easy to check that this random graph model is $3k$ -independent, has edge probability at least $\frac{k^2}{(k+1)^2} \geq 1 - \frac{2}{k}$ and that every connected component has order at most $(2k + 1)^2$.

Corollary 58. For any fixed $k \in \mathbb{N}$,

$$p_{3k, \ell p}(\mathbb{Z}^2) \geq 1 - \frac{2}{k}. \quad \square$$

In particular we have $\lim_{k \rightarrow \infty} p_{k, \ell p}(\mathbb{Z}^2) = 1$ (and in fact a similar construction shows this remains true in \mathbb{Z}^d).

Finally, as in Section 11.3, we should observe that the almost sure existence of arbitrarily long open paths in a 1-independent model on G does not imply that for every $\ell \in \mathbb{N}$ every vertex of G has a strictly positive probability of being part of a path of length at least ℓ . Thus we may actually define a second long paths critical probability,

$$p_{1, \ell p_2}(G) := \inf \left\{ p \in [0, 1] : \forall \mu \in \mathcal{M}_{1, \geq p}(G), \forall v \in V(G), \forall \ell \in \mathbb{N}, \right. \\ \left. \mu(\exists \text{ open path from } v \text{ of length } \ell) > 0 \right\}.$$

Problem 59. Determine $p_{1, \ell p_2}(\mathbb{Z}^2)$.

Our construction in the proof of Theorem 7 shows that $p_{1, \ell p_2}(\mathbb{Z}^2) \geq 4 - 2\sqrt{3}$, and we know it is upper-bounded by $p_{1, H_2}(\mathbb{Z}^2) \leq 0.8639$. As in Section 11.1, it may be fruitful to study the long paths critical constant when one restricts one's attention to a smaller class of 1-ipms. In particular, by considering the class of uniformly bounded vertex-based measures, one is led to the following intriguing problem in graph theory.

Given an n -uniformly bounded \mathbb{Z}^2 -partite graph H with partition $\sqcup_{v \in \mathbb{Z}^2} \Omega_v$. A *transversal subgraph* of H is a subgraph of H induced by a set of distinct representatives S for the parts of H , that is, a set of vertices of H such that $|S \cap \Omega_v| = 1$ for all $v \in \mathbb{Z}^2$. The *G-partite density* of H is

$$d_G(H) := \inf \left\{ \frac{e(H[\Omega_u \sqcup \Omega_v])}{|\Omega_u| \cdot |\Omega_v|} : uv \in E(\mathbb{Z}^2) \right\}.$$

Question 60. Suppose H is an n -uniformly bounded \mathbb{Z}^2 -partite graph in which in every transversal subgraph the connected component containing the origin is...

- ... of size at most C , for some constant $C \in \mathbb{N}$.
- ... finite.

How large can $d_G(H)$ be?

This question can be viewed as a problem from extremal multipartite graph theory. Plausibly some tools from that area, in particular the work of Bondy, Shen, Thomassé and Thomassen [12] and Pfender [28], could be brought to bear on it.

11.5 | Connectivity function

We determined in Sections 8 and 9 the connectivity function $f_{1,C_n}(p)$ for cycles C_n of length at most 5. It is natural to ask what happens for longer cycles.

Problem 61. Determine $f_{1,C_n}(p)$ for $n \in \mathbb{N}_{\geq 6}$.

As mentioned in Section 8, the problem of finding $f_{1,C_6}(p)$ is nonlinear. Nevertheless, one can use software, such as Maple and its contained NLPsolve function, to try to estimate the answer. This suggests the following:

- The threshold at which $f_{1,C_6}(p)$ becomes nonzero is approximately $p = 0.59733$;
- For p just above this threshold, the best “asymmetric” (see the next subsection for a definition) measure is *better* than the best “symmetric” measure; e.g. at $p = 0.62$ we have $f_{1,C_6}(0.62)$ is approximately 0.007, but is as high as 0.11 when restricted to “symmetric” measures.

More generally, one can ask what happens in cycles if we have higher dependency or if we try to maximize connectivity rather than minimize.

Problem 62. Determine $f_{k,C_n}(p)$ for all $p \in [0, 1]$, $k \in \mathbb{N}$ and integers $n \geq k + 2$.

Problem 63. Determine $F_{k,C_n}(p)$ for all $p \in [0, 1]$, $k \in \mathbb{N}$ and integers $n \geq k + 2$.

Beyond paths, cycles and complete graphs, the 1-independent connectivity problem is perhaps most natural to study in the hypercube graph Q_n and in the $n \times n$ toroidal grid $C_n \times C_n$. Progress on either of these would likely lead to progress on other problems in 1-independent percolation as well.

Problem 64. Determine $f_{1,Q_n}(p)$ for all $n \geq 3$.

Problem 65. Determine $f_{1,C_n \times C_n}(p)$ for all $n \geq 3$.

In a different direction, we can ask whether the extremal measures attaining $f_{1,G}(p)$ can be required to have “nice” properties. For C_4 and $p \in [0, 1/2]$ another extremal construction for $f_{1,C_4}(p)$ is given by the measure μ , defined by

$$\mu(\hat{H}) = \begin{cases} 1 - 2p & \text{if } H \text{ is the empty graph;} \\ p(1 - p) & \text{if } H \text{ is } \{12, 14\} \text{ or } \{23, 34\}; \\ p^2 & \text{if } H \text{ contains precisely two edges, which are not adjacent;} \\ 0 & \text{otherwise.} \end{cases}$$

Motivated by the above, we call a measure $\mu \in \mathcal{M}_{1,\geq p}(G)$ *symmetric* if for any pair of labeled subgraphs S and T of G such that there exists an automorphism of G mapping S to T , then $\mu(\hat{S}) = \mu(\hat{T})$. Note that the above measure is an example of a nonsymmetric extremal construction for $f_{1,C_4}(p)$, whereas the measure given at the end of Section 8.1 is symmetric. This leads to the following question.

Question 66. For any G and any $p \in [0, 1]$, does there always exist a symmetric measure $\mu \in \mathcal{M}_{1, \geq p}(G)$ which achieves $f_{1,G}(p)$?

If the program for solving $f_{1,G}(p)$ attained via our method in Section 8.1 is linear, then the answer for G is yes (see the appendix for a proof of this fact).

Another natural question is when the extremal connectivity can be attained by vertex-based measures.

Question 67. For which G and which p does there exist a vertex-based measure $\mu \in \mathcal{M}_{1, \geq p}(G)$ which achieves $f_{1,G}(p)$?

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APPENDIX

Let S_n be the symmetric group on n elements. For a graph G let $\Gamma(G)$ be the automorphism group of G , $\Gamma(G) := \{\sigma \in S_n : \sigma(i)\sigma(j) \in E(G) \text{ if and only if } ij \in E(G), \text{ for all } i, j \in [n]\}$. Enumerate the elements of $\Gamma(G)$ as σ_i , $i \in [|\Gamma(G)|]$, and for $H \subseteq G$ write H_{σ_i} for its image under σ_i . Recall μ is symmetric if for all $H \subseteq G$ and for all i, j we have $\mu(\hat{H}_{\sigma_i}) = \mu(\hat{H}_{\sigma_j})$.

Theorem 68. *Let G be a graph with vertex set $[n]$ such that the optimisation problem for $f_{1,G}(p)$ is linear. Suppose that μ is a nonsymmetric measure which achieves $f_{1,G}(p)$ for some value of p . Then for this same value of p there exists another μ' which is symmetric and also achieves $f_{1,G}(p)$.*

Proof. Let μ be the nonsymmetric measure which achieves $f_{1,G}(p)$. For all $H \subseteq G$ define

$$\mu'(\hat{H}) := \frac{1}{|\Gamma(G)|} \sum_{j=1}^{|\Gamma(G)|} \mu(\hat{H}_{\sigma_j}).$$

First note that we have the following:

$$\begin{aligned} \mu'(S_{\sigma_j}) &= \sum_{H \subseteq G} \mathbb{1}(S_{\sigma_j} \subseteq H) \mu'(\hat{H}) \\ &= \frac{1}{|\Gamma(G)|} \sum_{H \subseteq G} \sum_{i=1}^{|\Gamma(G)|} \mathbb{1}(S_{\sigma_j} \subseteq H_{\sigma_i}) \mu'(\hat{H}) \\ &= \frac{1}{|\Gamma(G)|} \sum_{H \subseteq G} \sum_{i=1}^{|\Gamma(G)|} \mathbb{1}(S_{\sigma_i} \subseteq H) \mu'(\hat{H}) \\ &= \frac{1}{|\Gamma(G)|} \sum_{i=1}^{|\Gamma(G)|} \mu(S_{\sigma_i}). \end{aligned} \tag{A1}$$

The first and final equalities follow by definition. The second equality follows by summing through each automorphism of H and the fact that $\mu'(\hat{H}_{\sigma_i}) = \mu'(\hat{H}_{\sigma_j})$ for all i, j . The third equality follows by swapping automorphisms of H to automorphisms of S , which again works since $\mu'(\hat{H}_{\sigma_i}) = \mu'(\hat{H}_{\sigma_j})$.

Now note that if S is the empty graph or a single edge, then $\mu(S_{\sigma_i}) = \mu(S_{\sigma_j})$ for all i, j and thus we obtain $\mu'(S_{\sigma_i}) = \mu(S_{\sigma_i})$ for all i . It easily follows that μ' is a measure with edge-probability p . We must show $\mu'(S) \cdot \mu'(T) = \mu'(S \cup T)$ for all S, T which are labeled nonempty subgraphs of G supported on disjoint subsets of vertices. If the optimisation problem is linear, then without loss of generality we have $\mu'(T) = p$, and so this follows by linearity and (A1). It remains to show that μ' also achieves $f_G(p)$. Again this follows easily since

$$\begin{aligned} \mu'(\text{a } \mu'\text{-random graph is connected}) &= \sum_{H \subseteq G} \mathbb{1}(H \text{ is connected}) \mu'(H) \\ &= \frac{1}{|\Gamma(G)|} \sum_{H \subseteq G} \sum_{i=1}^{|\Gamma(G)|} \mathbb{1}(H_{\sigma_i} \text{ is connected}) \mu'(H_{\sigma_i}) \\ &= \mu(\text{a } \mu\text{-random graph is connected}), \end{aligned}$$

where the second equality follows by summing through each automorphism of H , and the third since H_{σ_i} is connected if and only if H_{σ_j} is connected, for all i, j . ■