http://www.diva-portal.org

This is the published version of a paper published in Journal of Differential Equations.

Citation for the original published paper (version of record):
Lundström, N L. (2022)
Growth of subsolutions to fully nonlinear equations in halfspaces
Journal of Differential Equations, 320: 143-173
https://doi.org/10.1016/j.jde.2022.02.048

Access to the published version may require subscription.
N.B. When citing this work, cite the original published paper.

Permanent link to this version:
http://urn.kb.se/resolve?urn=urn:nbn:se:umu:diva-193056

# Growth of subsolutions to fully nonlinear equations in halfspaces 

Niklas L.P. Lundström<br>Department of Mathematics and Mathematical Statistics, Umeå University, SE-90187 Umeå, Sweden<br>Received 8 May 2021; revised 22 February 2022; accepted 23 February 2022


#### Abstract

We characterize lower growth estimates for subsolutions in halfspaces of fully nonlinear partial differential equations on the form $$
F\left(x, u, D u, D^{2} u\right)=0
$$ in terms of solutions to ordinary differential equations built upon assumptions on $F$. Using this characterization we derive several sharp Phragmen-Lindelöf-type theorems for certain classes of well known PDEs. The equation need not be uniformly elliptic nor homogeneous and we obtain results both in case the subsolution is bounded or unbounded. Among our results we retrieve classical estimates in the halfspace for $p$-subharmonic functions and extend those to more general equations; we prove sharp growth estimates, in terms of $k$ and the asymptotic behavior of $\int_{0}^{R} C(s) d s$, for subsolutions of equations allowing for sublinear growth in the gradient of the form $C(|x|)|D u|^{k}$ with $k \geq 1$; we establish a Phragmen-Lindelöf theorem for weak subsolutions of the variable exponent $p$-Laplace equation in halfspaces, $1<p(x)<\infty, p(x) \in C^{1}$, of which we conclude sharpness by finding the "slowest growing" $p(x)$-harmonic function together with its corresponding family of $p(x)$-exponents. The paper ends with a discussion of our results from the point of view of a spatially dependent diffusion problem. © 2022 The Author(s). Published by Elsevier Inc. This is an open access article under the CC BY license (http://creativecommons.org/licenses/by/4.0/).


MSC: 35B40; 35B50; 35B53; 35D40; 35J25; 35J60; 35J70
Keywords: Phragmen Lindelöf; Non standard growth; Variable exponent; Quasi linear; Non homogeneous; Sub linear

[^0]
## 1. Introduction

We consider fully nonlinear nonhomogeneous elliptic partial differential equations (PDE) in nondivergence form,

$$
F\left(x, u, D u, D^{2} u\right)=0,
$$

in halfspaces of $\mathbb{R}^{n}$ for $n \geq 1$. Here, $D u$ is the gradient, $D^{2} u$ the Hessian, $F: \mathbb{R}^{n} \times \mathbb{R} \times \mathbb{R}^{n} \times$ $\mathbb{S}^{n} \rightarrow \mathbb{R}$ in which $\mathbb{S}^{n}$ is the set of symmetric $n \times n$ matrices equipped with the positive semidefinite ordering; for $X, Y \in \mathbb{S}^{n}$, we write $X \leq Y$ if $\langle(X-Y) \xi, \xi\rangle \leq 0$ for all $\xi \in \mathbb{R}^{n}$. Without loss of generality we fix the halfspace to $\mathbb{R}_{+}^{n}:=\left\{x \in \mathbb{R}^{n}: x_{n}>0\right\}$ and assume the following:

Degenerate ellipticity holds, i.e. $F(x, u, p, X) \geq F(x, v, p, Y)$ whenever $u \geq v, X \leq Y$, as well as the growth condition

$$
-F(x, 0, p, X) \leq \Phi(|x|,|p|)+\Lambda\left(x_{n}\right) \operatorname{Tr}\left(X^{+}\right)-\lambda\left(x_{n}\right) \operatorname{Tr}\left(X^{-}\right)
$$

whenever $x, p \in \mathbb{R}^{n}, X \in \mathbb{S}^{n}, X=X^{+}-X^{-}, X^{+} \geq 0, X^{-} \geq 0$ and $X^{+} X^{-}=0$. Here, $\Phi:(0, \infty) \times(0, \infty) \rightarrow(-\infty, \infty)$ is continuous, nonincreasing in its first argument and $\lambda, \Lambda:(0, \infty) \rightarrow(0, \infty)$ are functions such that $\lambda$ is nonincreasing and $\Lambda$ is nondecreasing.

Concerning $\Phi$ we will also need the following assumption:
Either $\Phi$ is nonnegative and it holds, for all $\epsilon, t>0$, that (interpreting $1 / 0=\infty$ )

$$
\int_{0}^{\epsilon} \frac{d s}{\Phi(t, s)}=\infty
$$

or $\Phi$ is nonpositive, $-\Phi$ satisfies $(\star \star \star)$ and (2.1) has a continuous solution in $\mathbb{R}_{+}$.
Under assumptions ( $\star \star$ ) $-(\star \star \star)$ we characterize the growth of viscosity subsolutions of $(\star)$ in halfspaces in terms of solutions to ordinary differential equations (ODE) (Theorem 2.1) which are built upon functions $\Phi, \lambda$ and $\Lambda$ in ( $\star \star$ ). Using this characterization we are able to derive sharp growth estimates of Phragmen-Lindelöf-type once the solutions of the ODEs are sufficiently understood. Indeed, to apply Theorem 2.1 one needs to (1) find functions $\Phi, \lambda$ and $\Lambda$ to ensure assumptions ( $\star \star$ ) and ( $\star \star \star$ ), (2) solve the corresponding ODEs given in (2.1) and (3) find the limit in Theorem 2.1. An estimate is obtained if this limit is positive. Theorem 2.1 applies both in case the subsolution is bounded or unbounded, and it can be used to find such border.

In Section 3 we apply Theorem 2.1 to derive sharp estimates for subsolutions of some well known PDEs of which the corresponding ODEs can be solved explicitly. For example, we retrieve the classical Phragmen-Lindelöf theorem in halfspaces for $p$-subharmonic functions by Lindqvist [30] and show in addition that it holds also for equations of $p$-Laplace type with lower order terms and vanishing ellipticity. We obtain sharp lower estimates of the growth, in terms of $k \geq 1$ and the asymptotic behavior of $\int_{0}^{R} C(s) \lambda^{-1}(s) d s$, for subsolutions of equations with sublinear growth in the gradient such as

$$
-P_{\lambda, \Lambda}^{-}\left(D^{2} u\right)+C(|x|)|D u|^{k}=0
$$

in which $P_{\lambda, \Lambda}^{-}$is a Pucci operator (definition recalled below) and $C(t)$ is a nonincreasing function. These results reveal e.g. the border determining if a subsolution must grow to infinity or not in terms of $C(t)$ and $k$, see Corollary 3.1 and estimate (3.9). Moreover, Theorem 2.1 applies to nonhomogeneous PDEs including the variable exponent $p$-Laplace equation

$$
\nabla \cdot\left(|D u|^{p(x)-2} D u\right)=0
$$

and we prove a sharp Phragmen-Lindelöf theorem for weak subsolutions of this equation whenever $1<p(x)<\infty$ is $C^{1}$ regular (Theorem 3.3). It turns out that the growth estimate heavily depends on whether the subsolution ever exceeds $x_{n}$ (distance to boundary) or not. We conclude sharpness by finding the "slowest growing" $p(x)$-harmonic function in the halfspace, for a given ellipticity bound, together with its corresponding family of $p(x)$ exponents (Remark 3.4). In the geometric setting of halfspaces, these results sharpen some results of Adamowicz [1].

The proof of Theorem 2.1 relies on comparison with certain classical supersolutions of ( $\star$ ) which we construct in Lemma 2.2 using solutions of the aforementioned ODEs. We stress generality by pointing out that with the validity of Theorem 2.1 at hand, growth estimates for subsolutions to certain PDEs not considered in Section 3 can be proved mainly by estimating solutions of first order ODEs and limits.

We end the paper by discussing the problem under investigation from the point of view of a diffusion problem. Indeed, in Section 4 we briefly discuss, through the application of spatially dependent diffusion, why parts of the results presented in Theorem 3.3 should hold.

We remark that our main results allow for ellipticity to blow up at infinity as $\lambda\left(x_{n}\right)$ may vanish and $\Lambda\left(x_{n}\right)$ may explode at infinity. Moreover, the Osgood-type condition in ( $\star \star \star$ ) is necessary to ensure that subsolutions must continue to grow. Indeed, for the strong maximum principle, see Julin [23], Lundström-Olofsson-Toivanen [34] and the remarks below Theorem 2.1. Furthermore, assumption ( $\star \star$ ) can be written, with $\lambda=\lambda\left(x_{n}\right)$ and $\Lambda=\Lambda\left(x_{n}\right)$,

$$
-F(x, 0, p, X) \leq \Phi(|x|,|p|)-\mathcal{P}_{\lambda, \Lambda}^{-}(X) \quad \text { whenever } \quad x, p \in \mathbb{R}^{n}, X \in \mathbb{S}^{n}
$$

where $\mathcal{P}_{\lambda, \Lambda}^{-}(X)=-\Lambda \operatorname{Tr}\left(X^{+}\right)+\lambda \operatorname{Tr}\left(X^{-}\right)$is the Pucci maximal operator, $X=X^{+}-X^{-}$with $X^{+} \geq 0, X^{-} \geq 0$ and $X^{+} X^{-}=0$. In particular, if $X \in \mathbb{S}^{n}$ has eigenvalues $e_{1}, e_{2}, \ldots, e_{n}$ the Pucci extremal operators $\mathcal{P}_{\lambda, \Lambda}^{+}$and $\mathcal{P}_{\lambda, \Lambda}^{-}$with ellipticity $0<\lambda \leq \Lambda$ are defined by

$$
\mathcal{P}_{\lambda, \Lambda}^{+}(X):=-\lambda \sum_{e_{i} \geq 0} e_{i}-\Lambda \sum_{e_{i}<0} e_{i} \quad \text { and } \quad \mathcal{P}_{\lambda, \Lambda}^{-}(X):=-\Lambda \sum_{e_{i} \geq 0} e_{i}-\lambda \sum_{e_{i}<0} e_{i}
$$

For properties of the Pucci operators see e.g. Caffarelli-Cabre [10] or Capuzzo-Dolcetta-Vitolo [11]. We remark also that the above assumption ( $\star \star$ ) is implied by the standard ellipticity assumption

$$
\begin{equation*}
\lambda \operatorname{Tr}(Y) \leq F(x, u, p, X)-F(x, u, p, X+Y) \leq \Lambda \operatorname{Tr}(Y) \tag{1.1}
\end{equation*}
$$

whenever $Y$ is positive semi-definite, together with

$$
\begin{equation*}
-F(x, 0, p, 0) \leq \Phi(|x|,|p|) \quad \text { whenever } \quad x, p \in \mathbb{R}^{n} \tag{1.2}
\end{equation*}
$$

Observe also that ( $(\star$ ) allows for nonlinear degenerate elliptic operators which do not satisfy (1.1). For example, operators of the form

$$
F(X)=-\Lambda\left(\sum_{i=1}^{n} \Gamma\left(\mu_{i}^{+}\right)\right)+\lambda\left(\sum_{i=1}^{n} \Psi\left(\mu_{i}^{-}\right)\right)
$$

where $\mu_{i}, i=1, \ldots, n$, are the eigenvalues of the matrix $X \in \mathbb{S}^{n}$ and $\Gamma, \Psi:[0, \infty) \rightarrow[0, \infty)$ are continuous and nondecreasing functions such that $\Gamma(s) \leq s \leq \Psi(s)$, see Capuzzo-DolcettaVitolo [11].

The Phragmén-Lindelöf principle and results of Phragmén-Lindelöf type, which has connections to elasticity theory (Horgan [21], Quintanilla [40], Leseduarte-Carme-Quintanilla [29]), have been frequently studied during the last century. To mention some papers (without giving a complete summary), Ahlfors [2] extended results from Phragmén-Lindelöf [39] to the upper halfspace of $\mathbb{R}^{n}$, Gilbarg [16], Serrin [41] and Herzog [19] considered more general elliptic equations of second order. Miller [37] considered uniformly elliptic operators in nondivergence form and unbounded domains contained in cones. Kurta [28] and Jin-Lancaster [22] estimated growth of bounded solutions of quasilinear equations, the later used solutions to boundary value problems, while Vitolo [42] considered elliptic equations in sectors. Capuzzo-Dolcetta-Vitolo [11] and Armstrong-Sirakov-Smart [3] considered fully nonlinear equations, the later in certain Lipschitz domains, and Koike-Nakagawa [27] established Phragmén-Lindelöf theorems for subsolutions of fully nonlinear elliptic PDEs with unbounded coefficients and inhomogeneous terms. Adamowicz [1] studied subsolutions of the variable exponent $p$-Laplace equation, while Bhattacharya [8] and Granlund-Marola [17] considered infinity-harmonic functions. Lindqvist [30] established Phragmén-Lindelöf's theorem for $n$-subharmonic functions when the boundary is an $m$-dimensional hyperplane in $\mathbb{R}^{n}, 0 \leq m \leq n-1$, which was extended to $p$-subharmonic functions, $n-m<p \leq \infty$, in Lundström [33]. We also mention that recently, Braga-Moreira [7] showed that nonnegative solutions of a generalized $p$-Laplace equation in the upper halfplane, vanishing on $\left\{x_{n}=0\right\}$, is $u(x)=x_{n}$ (modulo normalization) and Lundström-Singh [35] proved a similar result for $p$-harmonic functions in planar sectors as well as a sharp Phragmen-Lindelöf theorem. Lundberg-Weitsman [31] studied the growth of solutions to the minimal surface equation over domains containing a halfplane. The spatial behavior of solutions of the Laplace equation on a semi-infinite cylinder with dynamical nonlinear boundary conditions was investigated in Leseduarte-Carme-Quintanilla [29]. Phragmén-Lindelöf theorems for plurisubharmonic functions on cones was proved by Momm in [38] while Bhattacharya-Mohammed considered $k$ Hessian equations with lower order terms [9]. Finally, we mention that recently, local estimates such as a sharp Harnack inequality (Julin [23]), boundary Harnack inequalities (Avelin-Julin [6]) as well as strong maximum and minimum principles (Lundström-Olofsson-Toivanen [34]) were established for fully nonlinear PDEs covered by the class of equations considered here.

### 1.1. Preliminaries

For a point $x \in \mathbb{R}^{n}$ we use the notation $x=\left(x_{1}, x_{2}, \ldots x_{n-1}, x_{n}\right)=\left(x^{\prime}, x_{n}\right)$. By $\Omega$ we denote a domain, that is, an open connected set. For a set $E \subset \mathbb{R}^{n}$ we let $\bar{E}$ denote the closure and $\partial E$ the boundary of $E$. By $c$ we denote a positive constant not necessarily the same at each occurrence. We write $A \precsim B$ if there exists $c$ such that $A \leq c B$.

A function $u: \Omega \rightarrow \mathbb{R}$ is a classical subsolution (supersolution) to ( $\star$ ) in $\Omega$ if it is twice differentiable in $\Omega$ and satisfies $F\left(x, u, D u, D^{2} u\right) \leq 0\left(F\left(x, u, D u, D^{2} u\right) \geq 0\right)$. If the inequality holds strict then $u$ is a strict classical subsolution (supersolution), and if equality holds then it is a classical solution.

We choose to present our main results for viscosity subsolutions, of which we recall the definition below in case $F: \mathbb{R}^{n} \times \mathbb{R} \times \mathbb{R}^{n} \times \mathbb{S}^{n} \rightarrow \mathbb{R}$ is a continuous function (which is not necessary for our results).

The following definition is from Crandall-Ishii-Lions [13]: An upper semicontinuous (USC) function $u: \Omega \rightarrow \mathbb{R}$ is a viscosity subsolution if for any $\varphi \in C^{2}(\Omega)$ and any $x_{0} \in \Omega$ such that $u-\varphi$ has a local maximum at $x_{0}$ it holds that

$$
F\left(x_{0}, u\left(x_{0}\right), D \varphi\left(x_{0}\right), D^{2} \varphi\left(x_{0}\right)\right) \leq 0
$$

A lower semicontinuous (LSC) function $u: \Omega \rightarrow \mathbb{R}$ is a viscosity supersolution if for any $\varphi \in$ $C^{2}(\Omega)$ and any $x_{0} \in \Omega$ such that $u-\varphi$ has a local minimum at $x_{0}$ it holds that

$$
F\left(x_{0}, u\left(x_{0}\right), D \varphi\left(x_{0}\right), D^{2} \varphi\left(x_{0}\right)\right) \geq 0 .
$$

A continuous function is a viscosity solution if it is both a viscosity sub- and a viscosity supersolution.

Let $u$ be a subsolution and $v$ a supersolution to $(\star)$ and let $a$ and $b$ be constants. As $(\star)$ is not necessarily homogeneous, $a+b u$ and $a+b v$ may fail as sub- and supersolutions. However, degenerate ellipticity guarantees that $u-c$ is a subsolution, and $u+c$ is a supersolution whenever $c \geq 0$.

We will not discuss the validity of a general comparison principle for viscosity solutions of ( $\star$ ) since we only need the possibility to compare viscosity subsolutions to classical supersolutions which is possible. Indeed, let $\Omega$ be a bounded domain, $u$ a viscosity subsolution and $v$ a classical strict supersolution in $\Omega, u \leq v$ on $\partial \Omega$ and suppose that $u \geq v$ somewhere in $\Omega$. By USC the function $u-v$ attains a maximum at some point $x_{0} \in \Omega$. Since $v \in C^{2}(\Omega), u-v$ has a maximum at $x_{0}$ and $u$ is a viscosity subsolution it follows by definition of viscosity solutions that

$$
\begin{equation*}
F\left(x_{0}, u\left(x_{0}\right), D v\left(x_{0}\right), D^{2} v\left(x_{0}\right)\right) \leq 0 \tag{1.3}
\end{equation*}
$$

But since $v$ is a classical strict supersolution we have $F\left(x, v(x), D v(x), D^{2} v(x)\right)>0$ whenever $x \in \Omega$, and as $u\left(x_{0}\right) \geq v\left(x_{0}\right)$ it follows from degenerate ellipticity that $F\left(x_{0}, u\left(x_{0}\right), D v\left(x_{0}\right)\right.$, $\left.D^{2} v\left(x_{0}\right)\right) \geq F\left(x_{0}, v\left(x_{0}\right), D v\left(x_{0}\right), D^{2} v\left(x_{0}\right)\right)>0$. This contradicts (1.3) and hence we have proved the following lemma:

Lemma 1.1. Let $\Omega$ be a bounded domain, $u \in U S C(\bar{\Omega})$ a viscosity subsolution and $v \in \operatorname{LSC}(\bar{\Omega})$ $a$ viscosity supersolution of $(\star)$ in $\Omega$ satisfying $u \leq v$ on $\partial \Omega$. Assume degenerate ellipticity. If either $u$ is a strict classical subsolution, or $v$ is a strict classical supersolution, then $u<v$ in $\Omega$.

Neither the choice of viscosity solutions nor the assumption that $F$ is continuous are necessary for our results. Many other definitions of "weak solutions" can be considered, whenever more appropriate for the equation, as long as such weak subsolutions of ( $\star$ ) are USC and can be compared to classical strict supersolutions of ( $\star$ ). In particular, our proof relies on construction of


Fig. 1. Geometric definitions and constructions.
a classical strict supersolution to $(\star)$ and comparison with this barrier function. What is needed is the validity of the following simple comparison result:

Lemma 1.2. Let $\Omega$ be a bounded domain, $u \in U S C(\bar{\Omega})$ a subsolution (in some weak sense) and $v$ a classical strict supersolution to $(\star)$ in $\Omega$, continuous on $\bar{\Omega}$. If $u \leq v$ on $\partial \Omega$ then $u \leq v$ in $\Omega$.

## 2. Characterizing growth in terms of solutions to ordinary differential equations

We will estimate the growth of subsolutions to $(\star)$ in terms of solutions $f:[0, \infty) \rightarrow \mathbb{R}$ to the following initial value problems, originating from assumption ( $\star \star$ ): If $\Phi(t, s) \geq 0$ for all $t, s \in \mathbb{R}_{+}$then we will make use of solutions to

$$
\begin{equation*}
\frac{d f}{d t}=-\frac{\Phi(t, f(t))}{\lambda(t)}-K(R) \frac{\Lambda(t)}{\lambda(t)} f(t), \quad t>0 \quad \text { with } \quad f(0)=v \tag{2.1}
\end{equation*}
$$

in which $v>0$ and $R>0$. Through the paper, we will by $f_{v, R}=f_{\nu, R}(t)$ denote the solution of (2.1) with $K(R)=n / \gamma(R)$, in which $\gamma(R)$ appears in the domain defined in (2.2) below. Further, we denote by $f_{v}=f_{v}(t)$ the solution of (2.1) with $K(R) \equiv 0$. If $\Phi(t, s) \leq 0$ for all $t, s \in \mathbb{R}_{+}$ then we use instead solutions of (2.1) but with $\lambda(t)$ replaced by $\Lambda(t)$ in the first term on the right hand side of (2.1). We allow ourselves to simplify notation according to $\lambda=\lambda(\cdot), \Lambda=\Lambda(\cdot), K=$ $K(R)$ and $\gamma=\gamma(R)$ whenever appropriate.

If $\Phi$ satisfies the Osgood-type condition $(\star \star \star)$ then $\Phi(t, s) \rightarrow 0$ as $s \rightarrow 0$, and, for any $v>0$, the solutions $f_{v, R}$ and $f_{v}$ will be positive. This plays a role in our main results, as pointed out in the remarks made below Theorem 2.1. In Section 3 Figs. 2 and 3 several solutions of (2.1) are plotted for some choices of $\Phi$.

To proceed we define, for a nondecreasing function $\gamma=\gamma(R)>0$ and $n \geq 1$, the domain

$$
\begin{equation*}
D(R):=\left\{x \in \mathbb{R}_{+}^{n}: \sum_{i=1}^{n-1} x_{i}^{2}+\left(x_{n}+\gamma\right)^{2}<(R+\gamma)^{2}\right\} \tag{2.2}
\end{equation*}
$$

see Fig. 1. Finally, for a subsolution $u$ and for $R>0$ we define

$$
M(R)=\sup _{\partial D(R)} u
$$

and

$$
M^{\prime}(R)=\liminf _{h \rightarrow 0^{+}} \frac{M(R)-M(R-h)}{h} .
$$

The following theorem characterizes a sharp lower growth estimate of subsolutions to $(\star)$ in terms of solutions $f_{\nu, R}$ and $f_{\nu}$ of the $\operatorname{ODE}$ (2.1):

Theorem 2.1. Suppose that ( $\star \star)$ and $(\star \star \star)$ hold and let u be a subsolution of $(\star)$ in $\mathbb{R}_{+}^{n}$ satisfying

$$
\limsup _{x \rightarrow y} u(x) \leq 0 \quad \text { for all } \quad y \in \partial \mathbb{R}_{+}^{n}
$$

Then either $u \leq 0$ in $\mathbb{R}_{+}^{n}$ or $M(R)$ is increasing and it holds that

$$
\liminf _{R \rightarrow \infty} \frac{M^{\prime}(R)}{f_{v}(R)} \geq \liminf _{R \rightarrow \infty} \frac{f_{v, R}(R)}{f_{v}(R)}
$$

where $v$ satisfies $u(\bar{x}) \geq \int_{0}^{\bar{x}_{n}} f_{v}(t) d t$ for some $\bar{x}$ on the $x_{n}$-axis.
Using Theorem 2.1 an "explicit" growth estimate can be found by estimating the limit $f_{v, R}(R) / f_{v}(R)$ as $R \rightarrow \infty$. In Section 3 we will consider certain PDEs for which we can solve the ODE (2.1) explicitly - estimate the limit - and thereby prove several Phragmen-Lindelöftype theorems. Let us note that if we can prove

$$
\liminf _{R \rightarrow \infty} \frac{M^{\prime}(R)}{f_{v}(R)}>0
$$

then $M(R)-M\left(R_{0}\right) \geq c \int_{R_{0}}^{R} f_{v}(t) d t$ whenever $R>R_{0}$ for some positive $R_{0}, c$ and thus

$$
\liminf _{R \rightarrow \infty} \frac{M(R)}{\int_{0}^{R} f_{v}(t) d t}>0
$$

Hence, if the integral

$$
\int_{0}^{\infty} f_{v}(t) d t
$$

converges, then subsolutions may be bounded, but if the integral diverges, then subsolutions must grow to infinity and the conclusion of Theorem 2.1 takes the form of classical PhragmenLindelöf theorems.

We remark that the assumption " $\bar{x}$ lies on the $x_{n}$-axis" is only for notational simplicity; we may translate coordinates otherwise. Note also that Theorem 2.1 holds whenever $\mathbb{R}_{+}^{n}$ is replaced (in the theorem and in (2.2)) with $\Omega \subset \mathbb{R}_{+}^{n}$, and that Theorem 2.1 gives a growth estimate for any
initial condition $v>0$ in (2.1) as long as $u(\bar{x}) \geq \int_{0}^{\bar{x}_{n}} f_{v}(t) d t$ for some $\bar{x}$. The best estimate corresponds to the largest $v$. Moreover, it can be realized from the proof of Lemma 2.2 that the assumption " $\lambda$ nonincreasing and $\Lambda$ nondecreasing" can be replaced by the slightly weaker assumption that $\lambda / \Lambda$ is nonincreasing and $\lambda$ is nonincreasing ( $\Lambda$ is nondecreasing) when $\Phi \geq 0(\Phi \leq 0)$. Finally, it will be clear from the proof that we also have $M(R)-M(R-h) \geq \int_{R}^{R+h} f_{v, R}(t) d t$ for $R>\bar{x}_{n}$ and any $h \in(0, R)$. We realize that $M(R)$ must increase as long as $f_{v, R}>0$, which happens whenever $\Phi$ satisfies the Osgood-type condition ( $\star \star \star$ ). Otherwise, the strong maximum principle does not hold and a positive subsolution of ( $\star$ ) may stop growing and attain an interior maximum, see Julin [23] or Lundström-Olofsson-Toivanen [34, Remark 4.3] for a counterexample.

Concerning sharpness of Theorem 2.1 we consider the function

$$
\begin{equation*}
u(x)=\int_{0}^{x_{n}} f_{v}(t) d t \tag{2.3}
\end{equation*}
$$

vanishing on $\partial \mathbb{R}_{+}^{n}$, depending only on $x_{n}$ with derivative $u_{x_{n}}(x)=f_{v}\left(x_{n}\right)$. This function satisfies $M^{\prime}(R)=f_{v}(R)$. In case $\Phi \geq 0$ it holds that $u_{x_{n} x_{n}}(x)=f_{v}^{\prime}\left(x_{n}\right)=-\lambda^{-1}\left(x_{n}\right) \Phi\left(x_{n}, f_{v}\left(x_{n}\right)\right)$ and hence we obtain $\lambda(x) u^{\prime \prime}(x)+\Phi\left(|x|, f_{v}(x)\right)=0$ when $n=1$. When $n \geq 2$ we obtain, for example, that

$$
\begin{equation*}
-F\left(x, D u, D^{2} u\right)=\lambda\left(x_{n}\right) \Delta u+\bar{\Phi}\left(f_{v}\left(x_{n}\right)\right)=0 \tag{2.4}
\end{equation*}
$$

for some function $\bar{\Phi}=\bar{\Phi}(s)$. In case $\Phi \leq 0$ the same holds but with $\lambda$ replaced by $\Lambda$. Thus, the function defined in (2.3) is a classical solution of an equation of type ( $\star$ ) satisfying ( $\star \star$ ) and ( $\star \star \star$ ) as well as the remaining assumptions in Theorem 2.1. In conclusion, when the limit in Theorem 2.1 is positive then the growth estimate cannot be improved (in case $\Phi(t, s)$ is independent of $t$ when $n \geq 2$ ), ignoring the shape of $D(R)$ and the value of the limit.

Concerning the shape of $D(R)$ we note the following. If $n=1$ then $D(R)=(0, R)$ independent of $\gamma$, but if $n \geq 2$ then $\gamma=c R$ implies that the spherical segment $D(R)$ preserves its geometric proportions for all $R>0$. If $\gamma(R) / R$ is increasing then $D(R)$ expands faster in the $x^{\prime}$-direction, implying slightly weaker estimates since $\partial D(R)$, on which supremum is taken, becomes larger. Observe that if the problem is considered in $\Omega \subset \mathbb{R}_{+}^{n}$ this might be of minor importance, especially if e.g. $\Omega$ is bounded in $x^{\prime}$-directions or contained in a cone with apex at the origin. There is not much of a gain to consider $\gamma(R) / R$ decreasing since $D(R)$ still expands at rate $R$ in $x^{\prime}$-directions.

The proof of Theorem 2.1 relies on comparison arguments and the following construction of a classical strict supersolution of $(\star)$.

Lemma 2.2. Suppose that ( $\star \star$ ) and ( $\star \star \star$ ) hold, let $R>0$ and put

$$
\Xi_{R}(x)=\sqrt{\sum_{i=1}^{n-1} x_{i}^{2}+\left(x_{n}+\gamma\right)^{2}}-\gamma=\left|\left(x^{\prime}, x_{n}+\gamma\right)\right|-\gamma
$$

in which $\gamma=\gamma(R)$ is from (2.2). Then for any $v>0$ the function

$$
V_{v, R}(x)=\int_{0}^{\Xi_{R}(x)} f_{v, R}(t) d t
$$

is a strict classical supersolution of $(\star)$ in $D(R)$.
Proof of Lemma 2.2. For notational simplicity we set $\Xi=\Xi_{R}(x), V=V_{v, R}(x)$ and $f(t)=$ $f_{v, R}(t)$. Differentiating yields

$$
\frac{\partial V}{\partial x_{i}}=\frac{x_{i}}{\left|\left(x^{\prime}, x_{n}+\gamma\right)\right|} f(\Xi), \quad 1 \leq i \leq n-1, \quad \frac{\partial V}{\partial x_{n}}=\frac{x_{n}+\gamma}{\left|\left(x^{\prime}, x_{n}+\gamma\right)\right|} f(\Xi) .
$$

It follows that

$$
\begin{equation*}
|D V|=f(\Xi) \tag{2.5}
\end{equation*}
$$

The second derivatives become

$$
\frac{\partial^{2} V}{\partial x_{i}^{2}}=\left(\frac{x_{i}}{\left|\left(x^{\prime}, x_{n}+\gamma\right)\right|}\right)^{2} f^{\prime}(\Xi)+\left(\frac{1}{\left|\left(x^{\prime}, x_{n}+\gamma\right)\right|}-\frac{x_{i}^{2}}{\left|\left(x^{\prime}, x_{n}+\gamma\right)\right|^{3}}\right) f(\Xi)
$$

for $1 \leq i \leq n-1$, and

$$
\frac{\partial^{2} V}{\partial x_{n}^{2}}=\left(\frac{x_{n}+\gamma}{\left|\left(x^{\prime}, x_{n}+\gamma\right)\right|}\right)^{2} f^{\prime}(\Xi)+\left(\frac{1}{\left|\left(x^{\prime}, x_{n}+\gamma\right)\right|}-\frac{\left(x_{n}+\gamma\right)^{2}}{\left|\left(x^{\prime}, x_{n}+\gamma\right)\right|^{3}}\right) f(\Xi),
$$

giving

$$
\operatorname{Tr}\left(D^{2} V\right)=f^{\prime}(\Xi)+\frac{n-1}{\left|\left(x^{\prime}, x_{n}+\gamma\right)\right|} f(\Xi) .
$$

We assume from here on that $\Phi \geq 0$. By construction we then have from (2.1) that $f^{\prime}(t)=$ $-\frac{\Phi(t, f(t))}{\lambda(t)}-K \frac{\Lambda(t)}{\lambda(t)} f(t)$ and hence

$$
\operatorname{Tr}\left(D^{2} V\right)=-\frac{\Phi(\Xi, f(\Xi))}{\lambda(\Xi)}-K \frac{\Lambda(\Xi)}{\lambda(\Xi)} f(\Xi)+\frac{n-1}{\left|\left(x^{\prime}, x_{n}+\gamma\right)\right|} f(\Xi) .
$$

We decompose $D^{2} V=D^{2} V^{+}-D^{2} V^{-}$so that

$$
\begin{aligned}
\operatorname{Tr}\left(D^{2} V^{+}\right) & =\frac{n-1}{\left|\left(x^{\prime}, x_{n}+\gamma\right)\right|} f(\Xi) \quad \text { and } \\
\operatorname{Tr}\left(D^{2} V^{-}\right) & =\frac{\Phi(\Xi, f(\Xi))}{\lambda(\Xi)}+K \frac{\Lambda(\Xi)}{\lambda(\Xi)} f(\Xi) .
\end{aligned}
$$

Utilizing the structure assumption ( $\star \star$ ), the fact that $V \geq 0$ and using (2.5) give

$$
\begin{align*}
F\left(x, V, D V, D^{2} V\right) \geq & F\left(x, 0, D V, D^{2} V\right) \\
\geq & -\Phi(|x|, f(\Xi))-\Lambda\left(x_{n}\right) \frac{n-1}{\left|\left(x^{\prime}, x_{n}+\gamma\right)\right|} f(\Xi) \\
& +\frac{\lambda\left(x_{n}\right) \Phi(\Xi, f(\Xi))}{\lambda(\Xi)}+K \frac{\lambda\left(x_{n}\right) \Lambda(\Xi)}{\lambda(\Xi)} f(\Xi) \\
\geq & -\Lambda\left(x_{n}\right) \frac{n-1}{\left|\left(x^{\prime}, x_{n}+\gamma\right)\right|} f(\Xi)+K \frac{\lambda\left(x_{n}\right) \Lambda(\Xi)}{\lambda(\Xi)} f(\Xi) \tag{2.6}
\end{align*}
$$

since $\lambda(\Xi) \leq \lambda\left(x_{n}\right)$ and $\Phi(|x|, f(\Xi)) \leq \Phi(\Xi, f(\Xi))$ hold. This last statement follows since $x_{n} \leq \Xi_{R}(x) \leq|x|$ by geometry, see Fig. 1, and functions are nonincreasing by assumption.

To show that $V$ is a strict classical supersolution we need $F\left(x, V, D V, D^{2} V\right)>0$ and by (2.6) and the fact that $f(\Xi)>0$ it suffices to ensure

$$
\frac{n-1}{\left|\left(x^{\prime}, x_{n}+\gamma\right)\right|}<K \frac{\lambda\left(x_{n}\right) \Lambda(\Xi)}{\Lambda\left(x_{n}\right) \lambda(\Xi)} .
$$

Observing that $\frac{\lambda\left(x_{n}\right)}{\Lambda\left(x_{n}\right)} \geq \frac{\lambda(\Xi)}{\Lambda(\Xi)}$ holds since $\lambda / \Lambda$ is nonincreasing, it suffices to ensure

$$
\frac{n-1}{\left|\left(x^{\prime}, x_{n}+\gamma\right)\right|}<K .
$$

We know that $\gamma \leq\left|\left(x^{\prime}, x_{n}+\gamma\right)\right|$ in $\mathbb{R}_{+}^{n}$ so it is enough to have

$$
\frac{n-1}{\gamma(R)}<K
$$

which holds since we have $K=n / \gamma(R)$.
If $\Phi \leq 0$ then by construction $f^{\prime}(t)=-\frac{\Phi(t, f(t))}{\Lambda(t)}-K \frac{\Lambda(t)}{\lambda(t)} f(t)$ and we obtain

$$
\begin{align*}
\operatorname{Tr}\left(D^{2} V^{+}\right) & =-\frac{\Phi(\Xi, f(\Xi))}{\Lambda(\Xi)}+\frac{n-1}{\left|\left(x^{\prime}, x_{n}+\gamma\right)\right|} f(\Xi), \\
\operatorname{Tr}\left(D^{2} V^{-}\right) & =K \frac{\Lambda(\Xi)}{\lambda(\Xi)} f(\Xi) \tag{2.7}
\end{align*}
$$

and thus instead of (2.6) we end up with

$$
\begin{align*}
F\left(x, V, D V, D^{2} V\right) \geq & F\left(x, 0, D V, D^{2} V\right) \\
\geq & -\Phi(|x|, f(\Xi))+\frac{\Lambda\left(x_{n}\right)}{\Lambda(\Xi)} \Phi(\Xi, f(\Xi)) \\
& -\Lambda\left(x_{n}\right) \frac{n-1}{\left|\left(x^{\prime}, x_{n}+\gamma\right)\right|} f(\Xi)+\lambda\left(x_{n}\right) K \frac{\Lambda(\Xi)}{\lambda(\Xi)} f(\Xi) \\
\geq & -\Lambda\left(x_{n}\right) \frac{n-1}{\left|\left(x^{\prime}, x_{n}+\gamma\right)\right|} f(\Xi)+\lambda\left(x_{n}\right) K \frac{\Lambda(\Xi)}{\lambda(\Xi)} f(\Xi) . \tag{2.8}
\end{align*}
$$

Here, the last inequality holds since $x_{n} \leq \Xi_{R}(x) \leq|x|, \Lambda$ is nondecreasing and $-\Phi \geq 0$ is nondecreasing in its first argument so that

$$
-\Phi(|x|, f(\Xi)) \geq \frac{\Lambda\left(x_{n}\right)}{\Lambda(\Xi)}(-\Phi(\Xi, f(\Xi)))
$$

To ensure that $V$ is a strict supersolution we see from (2.8) that it remains to show that

$$
\frac{n-1}{\left|\left(x^{\prime}, x_{n}+\gamma\right)\right|}<K \frac{\lambda\left(x_{n}\right)}{\Lambda\left(x_{n}\right)} \frac{\Lambda(\Xi)}{\lambda(\Xi)}
$$

and we are thus back in the same situation as in the case $\Phi \geq 0$. The proof of Lemma 2.2 is complete.

Proof of Theorem 2.1. Let $u$ be as in the statement of the theorem and denote with $\nu_{0}$ the initial condition in (2.1) for which we want to prove the growth estimate. Let $R>0, v>0$ and put $V:=V_{v, R}(x)$, where $V_{\nu, R}(x)$ is the strict supersolution in $D(R)$ guaranteed by Lemma 2.2.

If $V \geq M(R)$ on $\partial D(R)$ then, if $\Phi$ is nonnegative, it follows that $f_{v} \leq v$ and we obtain equality by decreasing $\nu$. If $\Phi$ is nonpositive we note that $f_{v, R}^{\prime}(t) \leq \widetilde{\Phi}\left(f_{v, R}(t)\right)$ for some $\widetilde{\Phi}(s)>$ 0 satisfying ( $\star \star \star$ ). Thus

$$
\int_{v}^{f_{v, R}(t)} \frac{d s}{\widetilde{\Phi}(s)} \leq t
$$

which implies that $f_{v, R}(t) \rightarrow 0$ as $v \rightarrow 0$ for all $t \in[0, R]$. Therefore, we obtain equality by decreasing $v$ also in this case. If $V<M(R)$ on $\partial D(R)$ then we increase $v$. If this does not help, (note that we may have $V \leq A$ on $\partial D(R)$, for all $\nu$, all $R$, and some $A>0$ ), then we lift the supersolution by adding a nonnegative constant. Indeed, for $C \geq 0$ it follows from degenerate ellipticity that also $V+C$ is a strict supersolution. We conclude that

$$
V+C \geq C \quad \text { on } \quad\left\{x_{n}=0\right\} \quad \text { and } \quad V+C=M(R) \quad \text { on } \quad \partial D(R) \cap \mathbb{R}_{+}^{n} .
$$

We clarify that if $C>0$ then we have taken $v>\nu_{0}$. It follows that

$$
\limsup _{x \rightarrow z} u(x) \leq V(z)+C \quad \text { for all } \quad z \in \partial D(R)
$$

and the weak comparison principle in Lemma 1.1 implies that $u \leq V+C$ in $D(R)$.
We next conclude that $v \geq v_{0}$. In particular, assume $v<\nu_{0}$. By assumption and by the above we have $V(\bar{x}) \geq u(\bar{x}) \geq \int_{0}^{\bar{x}_{n}} f_{\nu_{0}}(t) d t$ for some $\bar{x} \in D(R) \cap\left\{\bar{x}^{\prime}=0\right\}$, but on the other hand

$$
V(\bar{x})=\int_{0}^{\Xi_{R}(\bar{x})} f_{v, R}(t) d t=\int_{0}^{\bar{x}_{n}} f_{v, R}(t) d t<\int_{0}^{\bar{x}_{n}} f_{v_{0}, R}(t) d t \leq \int_{0}^{\bar{x}_{n}} f_{v_{0}}(t) d t
$$

where the last inequality follows since $f_{v, R} \leq f_{v}$. Hence, we have a contradiction and we therefore conclude $v \geq v_{0}$.

Now let $R>\bar{x}_{n}, h \in(0, R)$ and note that by the comparison principle

$$
\begin{aligned}
M(R)-M(R-h) & \geq V(0, \ldots, 0, R)-V(0, \ldots, 0, R-h) \\
& =\int_{R-h}^{R} f_{\nu, R}(t) d t \geq \int_{R-h}^{R} f_{\nu_{0}, R}(t) d t>0 .
\end{aligned}
$$

It follows that $M(R)$ is increasing and by taking the limit we see that

$$
M^{\prime}(R)=\liminf _{h \rightarrow 0^{+}} \frac{M(R)-M(R-h)}{h} \geq \liminf _{h \rightarrow 0^{+}} \frac{\int_{R-h}^{R} f_{v_{0}, R}(t) d t}{h}=f_{v_{0}, R}(R)
$$

Hence

$$
\liminf _{R \rightarrow \infty} \frac{M^{\prime}(R)}{f_{\nu_{0}}(R)} \geq \liminf _{R \rightarrow \infty} \frac{f_{\nu_{0}, R}(R)}{f_{\nu_{0}}(R)}
$$

which completes the proof of the theorem.

## 3. Applications to some well known equations

In this section we apply Theorem 2.1 to some PDEs for which we can find solutions of the ODE in (2.1), estimate the limit

$$
\liminf _{R \rightarrow \infty} \frac{f_{v, R}(R)}{f_{v}(R)}
$$

and conclude explicit growth estimates. We begin with the simplest case $\Phi \equiv 0$, including e.g. the famous $p$-Laplace equation, proceed with PDEs having sublinear growth in the gradient according to $\Phi(t, s)=C(t) s^{k}$ for $k \geq 1$ and end by the variable exponent $p$-Laplace equation, which satisfies assumption ( $\star \star$ ) with $\Phi(s)=C(t) s|\log s|$.

When stating corollaries for specific classes of PDEs we would sometimes prefer to infer other types of "weak" solutions than viscosity solutions whenever such are more suitable or more commonly used for such equations in the literature. As the equivalence of different kinds of "weak" solutions often is a nontrivial problem we will in some cases avoid going into these details, but this should not make things unclear. The reason is that we only use comparison between "weak" subsolutions and classical strict supersolutions - i.e. Lemma 1.2.

### 3.1. The case $\Phi(s) \equiv 0$

In this simple case the ODE (2.1) reduces to

$$
\frac{d f}{d t}=-K(R) \frac{\Lambda(t)}{\lambda(t)} f(t), \quad t>0 \quad \text { with } \quad f(0)=v
$$

and hence $f_{\nu, R}(t)=v e^{-K(R)} \int_{0}^{t} \Lambda(s) \lambda^{-1}(s) d s$ and $f_{v}(t) \equiv v$. We obtain the limit

$$
\begin{equation*}
\liminf _{R \rightarrow \infty} \frac{M^{\prime}(R)}{f_{v}(R)} \geq \liminf _{R \rightarrow \infty} \frac{f_{v, R}(R)}{f_{v}(R)}=\lim _{R \rightarrow \infty} e^{-K(R) \int_{0}^{R} \Lambda(s) \lambda^{-1}(s) d s} \tag{3.1}
\end{equation*}
$$

which is positive if $K(R) \int_{0}^{R} \Lambda(s) \lambda^{-1}(s) d s \precsim 1$. Recall from the definitions in (2.1) that $K(R)=$ $n / \gamma(R)$. This forces us to choose the function $\gamma(R)$ in the definition of $D(R)$, given by (2.2), so that

$$
\begin{equation*}
\int_{0}^{R} \frac{\Lambda(s)}{\lambda(s)} d s \precsim \gamma(R) . \tag{3.2}
\end{equation*}
$$

Following the remark just below Theorem 2.1 we see that (3.1) then implies

$$
\begin{equation*}
\liminf _{R \rightarrow \infty} \frac{M(R)}{R}>0 \tag{3.3}
\end{equation*}
$$

and we thus retrieve the classical form of the Phragmen-Lindelöf theorem. If the PDE is uniformly elliptic, i.e. $\Lambda / \lambda=$ constant, then according to (3.2) we can pick $\gamma(R)=R$ and thereby $D(R)$ preserves its geometric proportions for all $R>0$, which also agrees with the classical Phragmen-Lindelöf theorem. If ellipticity blows up at infinity, i.e. $\Lambda(R) / \lambda(R) \rightarrow \infty$, then the loss in estimate (3.3) comes only in the shape of $D(R)$ - it expands faster in $x^{\prime}$-directions since we need to take a larger $\gamma(R)$ according to (3.2).

Concerning sharpness of (3.3) we note that if $\Phi \equiv 0$ then (2.3) yields

$$
\int_{0}^{x_{n}} f_{v}(t) d t=v x_{n}
$$

which clearly hits the bottom of (3.3).
Following ( $\star$ ) and ( $\star \star$ ) we see that (3.3) holds, e.g., for subsolutions of the quasilinear equations

$$
\begin{equation*}
-\sum_{i, j=1}^{n} A_{i j}(x) \frac{\partial^{2} u}{\partial x_{i} \partial x_{j}}+f(x, u, D u)=0, \tag{3.4}
\end{equation*}
$$

corresponding to $F(x, r, p, X)=-\operatorname{Tr}(A(x) X)+f(x, u, D u)$, where $A(x) \in \mathbb{S}^{n}$ satisfies $\lambda\left(x_{n}\right) \operatorname{Tr}(Y) \leq \operatorname{Tr}(A(x) Y) \leq \Lambda\left(x_{n}\right) \operatorname{Tr}(Y)$ for all $Y \geq 0$, and

$$
\begin{equation*}
P_{\lambda, \Lambda}^{-}\left(D^{2} u\right)+f(x, u, D u)=0, \tag{3.5}
\end{equation*}
$$

whenever $f(x, u, D u) \geq 0$ is nondecreasing in $u$. One such PDE is the following $p$-Laplace equation, $p \in(1, \infty)$, with lower order terms

$$
\begin{equation*}
-\nabla \cdot\left(|D u|^{p-2} D u\right)+f(x, u, D u)=0 . \tag{3.6}
\end{equation*}
$$

Indeed, with

$$
-F\left(x, u, D u, D^{2} u\right)=\Delta u+(p-2) \Delta_{\infty} u-\frac{f(x, u, D u)}{|D u|^{p-2}}
$$

where $\Delta_{\infty} u=\left\langle D^{2} u \frac{D u}{|D u|}, \frac{D u}{|D u|}\right\rangle$ denotes the infinity Laplace operator, we see that $F$ satisfies (1.2) with $\Phi \equiv 0$ and

$$
\min \{1, p-1\} \operatorname{Tr}(Y) \leq F(x, u, D u, X)-F(x, u, D u, X+Y) \leq \max \{1, p-1\} \operatorname{Tr}(Y),
$$

whenever $Y \geq 0$. Hence $F$ satisfies (1.1) with $\lambda=\min \{1, p-1\}$ and $\Lambda=\max \{1, p-1\}$ and Theorem 2.1 applies.

Recalling that Lemma 1.2 holds for weak solutions (defined in the usual way) to $p$-Laplace type problems or that viscosity solutions and weak solutions are equivalent for some $p$-Laplace type equations, see e.g. Juutinen-Lindqvist-Manfredi [25], Julin-Juutinen [24] and MedinaOchoa [36], we retrieve and generalize the well known Phragmen-Lindelöf result in Lindqvist [30] in the setting of a halfspace $\mathbb{R}_{+}^{n}$.

### 3.2. The case $\Phi(t, s)=C(t) s^{k}$

We now consider equations satisfying $(\star \star)$ with $\Phi(t, s)=C(t) s^{k}$ where $k \in \mathbb{R}$ is a constant such that $k \geq 1$. Note that such $\Phi$ satisfies $(\star \star \star)$ and hence Theorem 2.1 implies that $M(R)$ is increasing. To derive exact growth estimates we observe that the ODE in (2.1) yields

$$
\frac{d f}{d t}=-\frac{C(t) f^{k}}{\lambda(t)}-K(R) \frac{\Lambda(t)}{\lambda(t)} f, \quad t>0 \quad \text { with } \quad f(0)=v
$$

As $C$ is nonincreasing (by assumptions on $\Phi$ ) and $\Lambda$ is nondecreasing we can replace the above equation with the separable ODE

$$
\frac{d f}{d t}=-A(t)\left(f^{k}+\widetilde{K} f\right)
$$

where $A(t)=C(t) / \lambda(t)$ and $\widetilde{K}=\frac{K \Lambda(R)}{C(R)}=\frac{n \Lambda(R)}{C(R) \gamma(R)}$. This is possible since solutions of this ODE will approach zero faster as $t$ increases and hence it creates a lower bound on the limit in Theorem 2.1. To find the solution for $k>1$ we observe that

$$
\frac{1}{\widetilde{K}} \int_{\nu}^{f(t)}\left(\frac{1}{y}-\frac{y^{k-2}}{y^{k-1}+\widetilde{K}}\right) d y=-\int_{0}^{t} A(s) d s
$$

and

$$
\frac{1}{k-1}\left[\log y^{k-1}-\log \left(y^{k-1}+\widetilde{K}\right)\right]_{v}^{f(t)}=-\widetilde{K} \int_{0}^{t} A(s) d s
$$

Thus

$$
f_{v, R}(t)= \begin{cases}v e^{-(1+\widetilde{K}) f_{0}^{t} A(s) d s} & \text { if } k=1 \\ \widetilde{K}^{\frac{1}{k-1}}\left(e^{(k-1) \widetilde{K} \int_{0}^{t} A(s) d s}\left(\frac{\widetilde{K}}{v^{k-1}}+1\right)-1\right)^{\frac{1}{1-k}} & \text { if } k>1,\end{cases}
$$

and by solving $\frac{d f}{d t}=-\frac{C(t) f^{k}}{\lambda(t)}$ with $f(0)=v$ we also obtain

$$
f_{v}(t)= \begin{cases}v e^{-\int_{0}^{t} A(s) d s} & \text { if } k=1  \tag{3.7}\\ \left((k-1) \int_{0}^{t} A(s) d s+v^{1-k}\right)^{\frac{1}{1-k}} & \text { if } k>1\end{cases}
$$

The limit in Theorem 2.1 becomes, for $k=1$,

$$
\liminf _{R \rightarrow \infty} \frac{M^{\prime}(R)}{f_{v}(R)} \geq \liminf _{R \rightarrow \infty} \frac{f_{v, R}(R)}{f_{v}(R)}=\lim _{R \rightarrow \infty} e^{-\widetilde{K} \int_{0}^{R} A(s) d s}
$$

and for $k>1$,

$$
\liminf _{R \rightarrow \infty} \frac{M^{\prime}(R)}{f_{v}(R)} \geq \liminf _{R \rightarrow \infty} \frac{f_{v, R}(R)}{f_{v}(R)}=\lim _{R \rightarrow \infty}\left(\frac{\widetilde{K}(k-1) \int_{0}^{R} A(s) d s+\widetilde{K} v^{1-k}}{e^{\widetilde{K}(k-1) \int_{0}^{R} A(s) d s}\left(\widetilde{K} v^{1-k}+1\right)-1}\right)^{\frac{1}{k-1}}
$$

Let's observe that if $\widetilde{K}(R) \int_{0}^{R} A(s) d s \precsim 1$, which is obtained by taking

$$
\begin{equation*}
\frac{\Lambda(R)}{C(R)} \int_{0}^{R} A(s) d s \precsim \gamma(R), \tag{3.8}
\end{equation*}
$$

then the limits are positive and we obtain

$$
\begin{equation*}
\liminf _{R \rightarrow \infty} \frac{M^{\prime}(R)}{f_{v}(R)}>0 \tag{3.9}
\end{equation*}
$$

Thus, we may derive several Phragmen-Lindelöf-type results using Theorem 2.1, whose form will depend on the exponent $k$ and the functions $C, \lambda$ and $\Lambda$. For example, using (3.7)-(3.9) we have proved:

Corollary 3.1. Suppose that ( $\star \star)$ holds with $\Phi(t, s)=C(t) s^{k}, k \geq 1$. Let $u$ be a subsolution of $(\star)$ in $\mathbb{R}_{+}^{n}$ satisfying

$$
\limsup _{x \rightarrow y} u(x) \leq 0 \quad \text { for all } \quad y \in \partial \mathbb{R}_{+}^{n}
$$

Assume also that $u(\bar{x})>0$ for some $\bar{x}$ on the $x_{n}$-axis. Then the following is true, with $A(t)=$ $C(t) / \lambda(t)$ :
(i) If $\int_{0}^{R} A(s) d s \precsim R^{\alpha(k-1)}$ for $\alpha \geq 0, k>1$ and $\frac{\Lambda(R)}{C(R)} R^{\alpha(k-1)} \precsim \gamma(R)$ then

$$
\liminf _{R \rightarrow \infty} \frac{M^{\prime}(R)}{R^{-\alpha}}>0 \quad \text { implying } \quad \liminf _{R \rightarrow \infty} \frac{M(R)}{R^{1-\alpha}}>0
$$

(ii) If $\int_{0}^{R} A(s) d s \precsim R$ and $\frac{\Lambda(R)}{C(R)} R \precsim \gamma(R)$ then

$$
\begin{aligned}
& \liminf _{R \rightarrow \infty} \frac{M^{\prime}(R)}{e^{-R}}>0 \quad \text { if } k=1, \quad \liminf _{R \rightarrow \infty} \frac{M^{\prime}(R)}{R^{-\frac{1}{k-1}}}>0 \quad \text { if } k \in(1,2), \\
& \liminf _{R \rightarrow \infty} \frac{M(R)}{\log (R)}>0 \quad \text { if } k=2 \quad \text { and } \quad \liminf _{R \rightarrow \infty} \frac{M(R)}{R^{\frac{k-2}{k-1}}}>0 \quad \text { if } 2<k .
\end{aligned}
$$

(iii) If $\int_{0}^{R} A(s) d s \precsim \log (R), \frac{\Lambda(R)}{C(R)} \log (R) \precsim \gamma(R)$ and $k=1$ then

$$
\liminf _{R \rightarrow \infty} \frac{M(R)}{\log (R)}>0
$$

We remark that in Corollary 3.1 we only summarize some examples of growth estimates that take simple forms - the reader may return to conclusion (3.9) for the general case. Note also that all conclusions in Corollary 3.1 are independent of $\nu$, meaning that we only have to use arbitrary small $v>0$ to prove them. Therefore, since $f_{v}$ and $f_{v, R}$ are nonincreasing functions in this case, we only need that the assumption $\Phi(t, s)=C(t) s^{k}$ holds for arbitrary small $s$.

Conclusion (i) takes the form of the classical Phragmen-Lindelöf theorem and when $\alpha=0$ it applies e.g. when

$$
\begin{equation*}
\Phi(|x|, s)=C(|x|) s^{k}=\frac{c}{(1+|x|)^{a}} s^{k}, \tag{3.10}
\end{equation*}
$$

$k \geq 1, a>1$, ellipticity $\lambda=$ constant,$\Lambda=$ constant and $R^{a} \precsim \gamma(R)$. Conclusion (ii) holds e.g. when $A=$ constant, $\Lambda / C=$ constant and $\gamma(R) \equiv R$. We observe that the exponent $k$ in $\Phi(t, s)=C(t) s^{k}$ has a borderline value at $k=2$. Namely, if $k \in[1,2)$ then subsolutions may be bounded, but if $k \in[2, \infty)$ then any subsolution must grow to infinity. As already mentioned in Section 2 such border is, beyond the assumptions in Corollary 3.1, characterized by convergence/divergence of

$$
\int_{0}^{\infty} f_{v}(t) d t .
$$

Conclusion (iii) holds e.g. when $a=1$ in (3.10), $R \log (R) \precsim \gamma(R)$ and $\Lambda=$ constant.
We further remark that upper bounds on $\int_{0}^{R} A(s) d s$ have played an important role for related results in the literature, see e.g. Gilbarg [16], Hopf [20] and Vitolo [42], and that PhragmenLindelöf theorems for similar equations in more general domains but with $k=1$ and $k=2$ are proved by Capuzzo-Dolcetta-Vitolo [11] and Koike-Nakagawa [27].


Fig. 2. The derivative $f_{v}$ in (3.7) (left panel) and the solution $\int_{0}^{x} f_{\nu}(t) d t$ (right panel). Right panel: Dashed curve is bounded, thin solid curve approaches infinity at speed $\log (x)$ while dashed-dot curve approaches infinity at speed $x^{8 / 9}$ (see (ii) in Corollary 3.1). Thick solid curve approaches infinity at speed $x^{2 / 3}$ (see (i) in Corollary 3.1 with $\alpha=1 / 3$ ). In all simulations, $v=5$. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

As in the case $\Phi \equiv 0$ the results in this subsection apply to PDEs of type (3.4) and (3.5) but now with relaxed assumption on $f$, namely $f(x, u, D u) \geq-C(|x|)|D u|^{k}$. In case of the $p$ Laplace type equation (3.6), $1<p<\infty$, the growth condition on the lower order terms becomes $f(x, u, D u) \geq-C(|x|)|D u|^{k+p-2}$.

When $C(t) \equiv 1$ then the function in (2.3), with $f_{v}$ from (3.7), is a classical solution of (2.4) ensuring sharpness. If $\lambda=$ constant we find explicitly that

$$
u(x)=\int_{0}^{x_{n}} f_{v}(t) d t=\lambda \begin{cases}v\left(1-e^{-\lambda^{-1} x_{n}}\right) & \text { if } k=1 \\ \log \left(\lambda^{-1} x_{n} v+1\right) & \text { if } k=2 \\ \frac{v^{2-k}}{2-k}\left(1-\left((k-1) \lambda^{-1} x_{n} v^{k-1}+1\right)^{\frac{2-k}{1-k}}\right) & \text { otherwise }\end{cases}
$$

Fig. 2 shows the solution $\int_{0}^{x_{n}} f_{v}(t) d t$ for some values of $k, v$ and different functions $\lambda(t)$.

### 3.3. The case $\Phi(t, s)=C(t) s|\log s|$ : variable exponent p-Laplace equation

We set $\Phi(t, s)=C(t) s|\log s|$ and obtain the ODE

$$
\frac{d f}{d t}=-\frac{C(t) f|\log f|}{\lambda(t)}-K(R) \frac{\Lambda(t)}{\lambda(t)} f, \quad t>0 \quad \text { with } \quad f(0)=v .
$$

By the same argument as in the case $\Phi=C(t) s^{k}$ we replace the above ODE with

$$
\frac{d f}{d t}=-A(t)(f|\log f|+\widetilde{K} f)
$$

where $A(t)=C(t) / \lambda(t)$ and $\widetilde{K}=\frac{K \Lambda(R)}{C(R)}=\frac{n \Lambda(R)}{C(R) \gamma(R)}$. This equation separates, when $0<f \leq 1$, to

$$
\log (\widetilde{K}-\log v)-\log (\widetilde{K}-\log f)=-\int_{0}^{t} A(s) d s
$$

Thus

$$
f_{v, R}(t)=e^{\tilde{K}\left(1-e^{\int_{0}^{t} A(s) d s}\right)} \nu^{e_{0}^{t} A(s) d s}
$$

and since $f$ must be nonincreasing this holds for $0<v \leq 1$. If $v>1$ the solution takes a similar form, namely

$$
f_{\nu, R}(t)= \begin{cases}e^{\widetilde{K}\left(e^{-\int_{0}^{t} A(s) d s}-1\right)} \nu^{e^{-\int_{0}^{t} A(s) d s}} & \text { if } 0 \leq t<t_{0},  \tag{3.11}\\ e^{\widetilde{K}\left(1-e^{\int_{t_{0}}^{t} A(s) d s}\right)} & \text { if } t_{0} \leq t,\end{cases}
$$

where $t_{0}$ is such that $f_{v, R}\left(t_{0}\right)=1$. Moreover,

$$
f_{v}(t)= \begin{cases}v^{e^{\rho_{0}^{t} A(s) d s}} & \text { if } 0<v \leq 1,  \tag{3.12}\\ v^{-\int_{0}^{t} A(s) d s} & \text { if } 1<v .\end{cases}
$$

The limit in Theorem 2.1 becomes, for $0<v \leq 1$,

$$
\begin{equation*}
\liminf _{R \rightarrow \infty} \frac{M^{\prime}(R)}{f_{v}(R)} \geq \liminf _{R \rightarrow \infty} \frac{f_{v, R}(R)}{f_{v}(R)}=\lim _{R \rightarrow \infty} e^{\tilde{K}\left(1-e^{\int_{0}^{R} A(s) d s}\right)} \tag{3.13}
\end{equation*}
$$

which is positive if $\widetilde{K} e^{\int_{0}^{R} A(s) d s} \precsim 1$. Therefore, we have to pick $\frac{\Lambda(R)}{C(R)} e^{R} A(s) d s \precsim \gamma(R)$ to achieve a growth estimate.

When $v>1$ we know that $f_{v, R}$ in (3.11) stays above 1 if

$$
(\widetilde{K}+\log \nu) e^{-\int_{0}^{t} A(s) d s}>\widetilde{K}
$$

which needs at least $\frac{\Lambda(R)}{C(R)} e_{0}^{R} A(s) d s \precsim \gamma(R)$. In such case

$$
\left.\liminf _{R \rightarrow \infty} \frac{M^{\prime}(R)}{f_{v}(R)} \geq \liminf _{R \rightarrow \infty} \frac{f_{v, R}(R)}{f_{v}(R)}=\lim _{R \rightarrow \infty} e^{\widetilde{K}\left(e^{-\int_{0}^{R} A(s) d s}-1\right.}\right) \geq \lim _{R \rightarrow \infty} e^{-\widetilde{K}}>0
$$

as $\frac{\Lambda(R)}{C(R)} \precsim \gamma(R)$. If the solution $f_{\nu, R}$ decreases to 1 then the limit can be estimated as in (3.13) since $f_{v, R}$ then follows the expression for $v \in(0,1]$ with $v=1$.

In summary, since $\Phi(t, s)=C(t) s|\log s|$ satisfies $(\star \star \star)$ we can conclude that for a subsolution $u$ satisfying the assumptions in Theorem 2.1 with $\Phi(t, s)=C(t) s|\log s|$, the following is true when $\frac{\Lambda(R)}{C(R)} e^{\int_{0}^{R} A(s) d s} \precsim \gamma(R)$, denoting $\check{u}(x)=\int_{0}^{x_{n}} f_{\nu}(s) d s$ :

- If $u \geq \check{u}$ somewhere on the $x_{n}$-axis and $v \in(0,1)$, then $M(R)$ may be bounded but

$$
\begin{equation*}
\liminf _{R \rightarrow \infty} \frac{M^{\prime}(R)}{v^{e_{0}^{R} A(s) d s}}>0 \tag{3.14}
\end{equation*}
$$

- If $u \geq \check{u}$ somewhere on the $x_{n}$-axis and $v \geq 1$, then $M(R)$ approaches infinity according to

$$
\begin{equation*}
\liminf _{R \rightarrow \infty} \frac{M^{\prime}(R)}{v^{-\int_{0}^{R} A(s) d s}}>0 \quad \text { implying } \quad \liminf _{R \rightarrow \infty} \frac{M(R)}{R}>0 \tag{3.15}
\end{equation*}
$$

Thus we retrieve the classical form of a Phragmen-Lindelöf theorem if the subsolution exceeds $\check{u}$ with $v \geq 1$, but if the subsolution only exceeds $\check{u}$ with $v<1$, it may grow very slowly and need not approach infinity. The border at $\nu=1$ originates from the fact that $\Phi(t, 1)=0$ and thus $f_{1} \equiv 1$, while $\Phi(t, s)>0$ for all other positive $s$ implying $f_{v} \rightarrow 1$ if $v>1$. Moreover, $\Phi(t, s) \rightarrow 0$ as $s \rightarrow 0$ and thus $f_{v} \rightarrow 0$ if $v \in(0,1)$.

Concerning sharpness we observe that $\check{u}(x)=\int_{0}^{x_{n}} f_{v}(s) d s$, in which $f_{v}$ is from (3.12) with $A(s)=\lambda^{-1}(s)$, solves the $\operatorname{PDE}(2.4)$ with $\Phi(t, s)=s|\log s|$, i.e.

$$
\begin{equation*}
\lambda\left(x_{n}\right) \Delta u+|D u||\log | D u| |=0 . \tag{3.16}
\end{equation*}
$$

When $\lambda=$ constant we find the explicit expression

$$
\check{u}(x)=\int_{0}^{x_{n}} f_{v}(s) d s=\lambda \begin{cases}-E_{i}(\log v)+E_{i}\left(e^{\lambda^{-1} t} \log v\right) & \text { if } 0<v<1  \tag{3.17}\\ x_{n} & \text { if } v=1 \\ E_{i}(\log v)-E_{i}\left(e^{-\lambda^{-1} t} \log v\right) & \text { if } 1<v\end{cases}
$$

where $E_{i}$ is the Exponential integral. See Fig. 3 (upper row) for some illustrations of the functions $f_{\nu}$ in (3.12) and $\check{u}$ in (3.17).

### 3.3.1. Variable exponent p-Laplace equation

The $p(x)$-Laplace equation in a domain $\Omega \subset \mathbb{R}^{n}$, which often serves as a model example for PDEs with nonstandard growth, yields

$$
\begin{equation*}
\nabla \cdot\left(|D u|^{p(x)-2} D u\right)=0 . \tag{3.18}
\end{equation*}
$$

The function $p: \Omega \rightarrow(1, \infty)$ is usually called a variable exponent. If $p=$ constant then this equation is the classical $p$-Laplace equation and if $p=2$ it's the famous Laplace equation. Apart from interesting theoretical considerations such equations arise in the applied sciences, for instance in fluid dynamics, see e.g. Diening-Růžička [14], in image processing, see e.g. Chen-Levine-Rao [12] and in electro-rheological fluids, see e.g. Harjulehto-Hästö-Lê-Nuortio [18] to which we also refer the reader for a recent survey and further references.

We recall the following standard definition of weak solutions of (3.18): A function $u \in$ $W_{l o c}^{1, p(x)}(\Omega)$ is a weak (sub)solution of (3.18) if


Fig. 3. Functions $f_{v}$ in (3.12) and $\check{u}$ in (3.17) (upper row), $f_{\mathcal{v}}$ in (3.26) and $\check{u}$ in (3.27) (lower row). In all simulations, $\lambda=\Lambda=1$. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

$$
\int_{\Omega}|D u|^{p(x)-2}\langle D u, D \psi\rangle d x(\leq)=0
$$

for all (nonnegative) $\psi \in C_{0}^{\infty}(\Omega)$. Similarly, $u$ is a weak supersolution if $-u$ is a weak subsolution. A function which is both a weak subsolution and a weak supersolution is called a weak solution. An (USC/LSC) weak (sub/super)solution is called a $p(x)$-(sub/super)harmonic function. We also note that $u \in W_{l o c}^{1,1}(\Omega)$ is $p(x)$-harmonic in $\Omega$ if it is a local minimizer of the energy

$$
\int_{\Omega} \frac{1}{p(x)}|D u|^{p(x)} d x
$$

where $1<p(x)<\infty$.

To proceed we define the operator

$$
\Delta_{p(x)} u:=\Delta u+(p(x)-2) \Delta_{\infty} u+\log |D u|\langle D p, D u\rangle,
$$

where $\Delta_{\infty} u=\left\langle D^{2} u \frac{D u}{|D u|}, \frac{D u}{|D u|}\right\rangle$ denotes the infinity Laplace operator. We note that $\Delta_{p(x)} u \geq 0$ implies

$$
\begin{equation*}
-\widehat{F}\left(x, D u, D^{2} u\right):=\Lambda(x) \operatorname{Tr}\left(D^{2} u^{+}\right)-\lambda(x) \operatorname{Tr}\left(D^{2} u^{-}\right)+|D p\|D u\| \log | D u \| \geq 0 \tag{3.19}
\end{equation*}
$$

with $\lambda(x)=\min \{1, p(x)-1\}$ and $\Lambda(x)=\max \{1, p(x)-1\}$. This suggests that $p(x)-$ subharmonic functions should be viscosity subsolutions of $\widehat{F}=0$, which is the case. Indeed, following the proof in Julin [23], which expands on Juutinen-Lukkari-Parviainen [26], we can conclude the following slightly generalized version of [23, Lemma 5.2]:

Lemma 3.2. Suppose that $p(x)$ is $C^{1}\left(\mathbb{R}_{+}\right), 1<p(x)<\infty, \lambda(x)=\min \{1, p(x)-1\}$ and $\Lambda(x)=$ $\max \{1, p(x)-1\}$. If $u$ is $p(x)$-subharmonic in a domain $\Omega \in \mathbb{R}^{n}, \varphi \in C^{2}(\Omega)$ is such that $\varphi\left(x_{0}\right)=$ $u\left(x_{0}\right)$ at $x_{0} \in \Omega$ and $\varphi \geq u$ then

$$
\widehat{F}\left(x_{0}, D \varphi\left(x_{0}\right), D^{2}\left(\varphi\left(x_{0}\right)\right)\right) \leq-\Delta_{p(x)} \varphi\left(x_{0}\right) \leq 0 .
$$

To obtain a PDE satisfying the required assumptions we redefine $\widehat{F}$ by replacing ellipticity with $\lambda\left(x_{n}\right) \leq \min \{1, p(x)-1\}$ nonincreasing, $\Lambda\left(x_{n}\right) \geq \max \{1, p(x)-1\}$ nondecreasing and also by replacing the nonhomogeneous term with $\Phi(|x|, s) \geq|D p| s|\log s|$, where $\Phi(|x|, s)$ is nonincreasing in $|x|$. In particular, we can take

$$
\begin{align*}
& \lambda(t)=p_{\lambda}(t):=\inf _{y: y_{n} \leq t} \min \{1, p(y)-1\}, \quad \Lambda(t)=p_{\Lambda}(t):=\sup _{y: y_{n} \leq t} \max \{1, p(y)-1\} \quad \text { and } \\
& \Phi(t, s)=\|D p\|_{\infty, t} s|\log s| \quad \text { where } \quad\|f\|_{\infty, t}=\sup _{y:|y| \geq t}|f(y)| \tag{3.20}
\end{align*}
$$

By the above reasoning we can conclude that a weak USC subsolution (a $p(x)$-subharmonic function) to the variable exponent $p$-Laplace equation is a viscosity subsolution of a PDE of type ( $\star$ ) satisfying ( $\star \star$ ) and ( $\star \star \star$ ). We can therefore conclude that deductions (3.14) and (3.15) hold for $p(x)$-subharmonic functions whenever $p(x)$ is $C^{1}\left(\mathbb{R}_{+}\right)$and $1<p(x)<\infty$.

We summarize our findings in the following theorem yielding Phragmen-Lindelöf-type results, of which some are sharp, for weak solutions of the variable exponent $p$-Laplace equation:

Theorem 3.3. Suppose that $p(x)$ is $C^{1}\left(\mathbb{R}_{+}^{n}\right), 1<p(x)<\infty$, and let $u$ be $p(x)$-subharmonic in $\mathbb{R}_{+}^{n}$ satisfying

$$
\limsup _{x \rightarrow y} u(x) \leq 0 \quad \text { for all } \quad y \in \partial \mathbb{R}_{+}^{n}
$$

Then $u$ is a viscosity subsolution of an equation of type ( $\star$ ) satisfying ( $\star \star$ ) and $(\star \star \star$ ) with $\Phi, \lambda=$ $p_{\lambda}$ and $\Lambda=p_{\Lambda}$ as in (3.20). Moreover, if $\frac{p_{\Lambda}(R)}{\|D p\|_{\infty, R}} \exp \left(\int_{0}^{R} \frac{\|D p\|_{\infty, s}}{p_{\lambda}(s)} d s\right) \precsim \gamma(R)$ and $\check{u}(x)=$ $\int_{0}^{x_{n}} f_{v}(s) d s$ with $f_{v}$ from (3.12) then the following is true:

- If $u \geq \check{u}$ somewhere on the $x_{n}$-axis for $v \in(0,1)$ then

$$
\liminf _{R \rightarrow \infty} \frac{M^{\prime}(R)}{\left.\nu^{\exp \left(\int_{0}^{R} \frac{\|D D\| \infty, s}{P_{\lambda}(s)}\right.} d s\right)}>0 .
$$

- If $u \geq \check{u}$ somewhere on the $x_{n}$-axis for $v \geq 1$ then

$$
\liminf _{R \rightarrow \infty} \frac{M(R)}{R}>0
$$

We thus retrieve the classical form of a Phragmen-Lindelöf theorem if the subsolution exceeds $\check{u}$ with $v \geq 1$, in particular if it exceeds $x_{n}$. On the other hand, if the subsolution only exceeds $\check{u}$ with $v<1$, then Theorem 3.3 states that it may grow very slowly and be bounded. The sharpness in the case $v \geq 1$ follows by observing, e.g., that

$$
\begin{equation*}
u(x)=c x_{n} \quad \text { is } p(x) \text {-harmonic with } \quad p(x)=M_{0}+\sum_{i=1}^{n-1} M_{i} x_{i}^{2}, \tag{3.21}
\end{equation*}
$$

whenever $c \geq 1, M_{0}>1$ and $M_{i}$, for $i \in[1, n-1]$, are constants. It is worth observing that the conclusion

$$
\liminf _{R \rightarrow \infty} \frac{M(R)}{R}>0
$$

follows also in the case $v \in(0,1)$ if $\int_{0}^{R} \frac{\|D p\|_{\infty, s}}{p_{\lambda}(s)} d s \precsim 1$ since then $\liminf _{R \rightarrow \infty} M^{\prime}(R)>0$. This holds e.g. if the exponent satisfies $p^{-}<p(x)$ and $\|D p\|_{\infty, s} \precsim s^{-k}$ for some constants $p^{-}, k>1$; a natural conclusion since these assumptions force the equation toward the constant exponent $p$ Laplace equation far away from the origin.

Versions of Theorem 3.3 are possible to derive from (3.14) and (3.15); e.g., it may be useful to replace the norm in (3.20) by $\|D p\|_{\infty, x_{n}}$ where $\|f\|_{\infty, t}=\sup _{y: y_{n}=t}|f(y)|$. Then, if $\|D p\|_{\infty, x_{n}} \neq 0$ for all $x_{n}>0$ we may divide (3.19) by $\|D p\|_{\infty, x_{n}}$ and conclude, for $\lambda\left(x_{n}\right) \leq$ $\frac{\min \{1, p(x)-1\}}{\|D p\|_{\infty}, x_{n}}$ nonincreasing and $\Lambda\left(x_{n}\right) \geq \frac{\max \{1, p(x)-1\}}{\|D p\|_{\infty}, x_{n}}$ nondecreasing, that Theorem 3.3 holds with $\frac{p_{\Lambda}(R)}{\|D p\|_{\infty, R}}$ replaced by $\Lambda(R)$ and $\frac{\|D p\|_{\infty, s}}{p_{\lambda}(s)}$ replaced by $\lambda^{-1}(s)$. In particular, in the case $v \in(0,1]$ the conclusion then reads

$$
\begin{equation*}
\liminf _{R \rightarrow \infty} \frac{M^{\prime}(R)}{v^{\int_{0}^{R} \lambda^{-1}(s) d s}}>0 . \tag{3.22}
\end{equation*}
$$

We build sharpness of this result in Remark 3.4 below in which we find a family of exponents for which the solution in (3.17), which satisfies $M^{\prime}(R)=v^{e^{R_{0} \lambda^{-1}(s) d s}}$, is $p(x)$-harmonic.

We further remark that Theorem 3.3 sharpens some results of Adamowicz [1] in the geometric setting of halfspaces, and the $C^{1}$-assumption on $p(x)$ should be replaceable with locally Lipschitz continuity by approximation arguments. Furthermore, the reader may recall the remarks made below deductions (3.14) and (3.15) and also note that contrary to the results in the former subsection, for $\Phi(t, s)=C(t) s^{k}$, the growth estimates here depend on $v$. Finally, the main results
in this section should hold also when the equation involves a "sink" term - in particular, for PDEs of the type $-\nabla \cdot\left(|D u|^{p(x)-2} D u\right)+f(x, u, D u)=0$ when $f(x, u, D u) \geq 0$ is nondecreasing in $u$.

Our estimates may not be optimal when $\log |D u|\langle D p, D u\rangle$ is negative since then we lose information by our choice of $\Phi$. We can improve by taking $\phi(s) \equiv 0$, but we still lose information when subsolutions gradients are not "close to perpendicular" to $D p$. This motivates us to derive better estimates under assumptions excluding e.g. the solution in (3.21). We do so by studying a nonpositive $\Phi$; the case $\Phi(s)=-s|\log s|$, in the next section.

We proceed by proving the following result, in which we find the "slowest growing" $p(x)$ harmonic function, for a given ellipticity bound, and the corresponding family of exponents.

Remark 3.4. The function $\check{u}(x)=\int_{0}^{x_{n}} f_{v}(s) d s$, in which $f_{v}$ is from (3.12) with $A(s)=\lambda^{-1}(s)$, is $p(x)$-harmonic with exponent

$$
\check{p}(x)=1+M e^{-\int_{0}^{x_{n}} \lambda^{-1}(s) d s} \quad \text { if } v \in(0,1] \quad \text { and } \quad \check{p}(x)=1+M e^{\int_{0}^{x_{n}} \lambda^{-1}(s) d s} \quad \text { if } v>1,
$$

whenever $M \in \mathbb{R}_{+}$is a constant.
The function $\check{u}(x)$ is the slowest growing $p(x)$-harmonic function in the sense of version (3.22) of Theorem 3.3. In particular, any $p(x)$-subharmonic function with exponent $p(x) \in$ $C^{1}\left(\mathbb{R}_{+}^{n}\right), 1<p(x)<\infty, \lambda\left(x_{n}\right) \leq \frac{\min \{1, p(x)-1\}}{\|D p\|_{\infty, x}}$ and $\frac{\max \{1, p(x)-1\}}{\|D p\|_{\infty, x}} \leq \Lambda\left(x_{n}\right)$, satisfying

$$
\limsup _{x \rightarrow y} u(x) \leq 0 \quad \text { for all } \quad y \in \partial \mathbb{R}_{+}^{n}
$$

that exceeds $\check{u}$ somewhere on the $x_{n}$-axis satisfies

$$
\liminf _{R \rightarrow \infty} \frac{M^{\prime}(R)}{\check{u}\left(R e_{n}\right)}=\liminf _{R \rightarrow \infty} \frac{M^{\prime}(R)}{v^{e_{0}^{R} \lambda^{-1}(s) d s}}>0 .
$$

Finally, if $\lambda$ is constant then

$$
\check{u}(x)=\lambda \begin{cases}-E_{i}(\log v)+E_{i}\left(e^{\lambda^{-1} x_{n}} \log v\right) & \text { if } 0<v<1, \\ x_{n} & \text { if } v=1\end{cases}
$$

where $E_{i}$ is the Exponential integral.
Proof. Since $\check{u}$ depends only on $x_{n}$ and solves (3.16) the first statement follows if we prove that the variable exponent $p(x)$-Laplace equation, with exponent $\check{p}(x)=1+M e^{\mp \int_{0}^{x_{n}} A(s) d s}$, reduces to the PDE (3.16) in one dimension. Without derivatives in $x^{\prime}$-directions we have

$$
\begin{equation*}
\Delta_{p(x)} u(x)=(p(x)-1) u_{x_{n} x_{n}}^{\prime \prime}(x)+\log \left|u_{x_{n}}^{\prime}(x)\right| p_{x_{n}}^{\prime}(x) u_{x_{n}}^{\prime}(x)=0 . \tag{3.23}
\end{equation*}
$$

Observe that the exponent $\check{p}(x)$ is the unique family of $C^{1}\left(\mathbb{R}_{+}^{n}\right)$ solutions to the ODE

$$
p_{x_{n}}^{\prime}(x)=\mp(p(x)-1) \lambda^{-1}\left(x_{n}\right)
$$

and substituting this equality into (3.23) yields $u_{x_{n} x_{n}}^{\prime \prime}(x) \mp \lambda^{-1}\left(x_{n}\right) \log \left|u_{x_{n}}^{\prime}(x)\right| u_{x_{n}}^{\prime}(x)=0$ where the " - " sign is for $v \in(0,1]$ when $\log \left|\check{u}_{x_{n}}^{\prime}(x)\right|<0$. Thus

$$
\lambda\left(x_{n}\right) u_{x_{n} x_{n}}^{\prime \prime}(x)+|\log | u_{x_{n}}^{\prime}(x) \mid u_{x_{n}}^{\prime}(x)=0
$$

which is (3.16) in one dimension.
To prove the second statement we need to ensure that a weak subsolution of the variable exponent $\check{p}$-Laplace equation, $\check{p}(x)=1+M e^{-\int_{0}^{x_{n}} \lambda(s)^{-1} d s}$ for some $M$, of which $\check{u}$ is a solution, is a viscosity subsolution of $(\star)$ where $(\star \star)$ holds with the same $\Phi(s)$ and $\lambda(t)$ as in version (3.22) of Theorem 3.3. To do so we observe that, recalling Lemma 3.2, any $p(x)$-subharmonic function is viscosity solution of

$$
\Delta_{p(x)} u=\Delta u+(p(x)-2) \Delta_{\infty} u+\log |D u|\langle D p, D u\rangle \geq 0,
$$

and hence of

$$
|D p\|D u\| \log | D u \|+\max \{1, p(x)-1\} \operatorname{Tr}\left(D^{2} u^{+}\right)-\min \{1, p(x)-1\} \operatorname{Tr}\left(D^{2} u^{-}\right) \geq 0 .
$$

Inserting $\check{p}(x)=1+M e^{-\int_{0}^{x_{n}} \lambda(s)^{-1} d s},|D \check{p}|=(\check{p}(x)-1) \lambda^{-1}\left(x_{n}\right)$ and assuming that $1<\check{p}(x) \leq$ 2 , which we may by taking $M \in(0,1]$, we see that

$$
\left|D u\|\log \mid D u\|+\frac{\lambda\left(x_{n}\right)}{M e^{-\int_{0}^{x_{n}} \lambda^{-1}(s) d s}} \operatorname{Tr}\left(D^{2} u^{+}\right)-\lambda\left(x_{n}\right) \operatorname{Tr}\left(D^{2} u^{-}\right) \geq 0 .\right.
$$

This is a PDE satisfying ( $\star \star$ ) with $\Phi(s)=s|\log s|$ and $\lambda(t)$ as in version (3.22) of Theorem 3.3.
It remains to show that $\check{p}$ satisfies

$$
\lambda\left(x_{n}\right) \leq \frac{\min \{1, \check{p}(x)-1\}}{\|D \check{p}\|_{\infty, x}} .
$$

This holds with equality since

$$
\check{p}(x)-1=M e^{-\int_{0}^{x_{n}} \lambda^{-1}(s) d s}, \quad\|D \check{p}\|_{\infty, x}=\left|\check{p}^{\prime}(x)\right|=-\lambda\left(x_{n}\right)^{-1} M e^{-\int_{0}^{x_{n}} \lambda^{-1}(s) d s}
$$

and we have assumed $M \in(0,1]$. The proof is complete.

### 3.4. The case $\Phi(s)=-s|\log s|$

In this case the ODE (2.1) becomes (we skip $t$-dependence in $\Phi$ for simplicity)

$$
\frac{d f}{d t}=\frac{f|\log f|}{\Lambda(t)}-K(R) \frac{\Lambda(t)}{\lambda(t)} f, \quad t>0 \quad \text { with } \quad f(0)=v .
$$

By the same argument as in the former cases we replace this ODE by

$$
\begin{equation*}
\frac{d f}{d t}=\Lambda^{-1}(t)(f|\log f|-\widehat{K} f) \tag{3.24}
\end{equation*}
$$

which separates. As in the case $\Phi(t, s)=C(t) s|\log s|$ we obtain, with $\widehat{K}=K \frac{\Lambda^{2}(R)}{\lambda(R)}=\frac{n \Lambda^{2}(R)}{\lambda(R) \gamma(R)}$,

$$
f_{\nu, R}(t)=e^{-\widehat{K}\left(1-e^{-\int_{0}^{t} \Lambda^{-1}(s) d s}\right)} \nu^{e^{-\int_{0}^{t} \Lambda^{-1}(s) d s}}
$$

when $0<v \leq 1$. If $v>1$ then

$$
f_{\nu, R}(t)= \begin{cases}e^{-\widehat{K}\left(e^{\int_{0}^{t} \Lambda^{-1}(s) d s}-1\right)} \nu^{e^{t_{0}^{t} \Lambda^{-1}(s) d s}} & \text { if } 0 \leq t<t_{0}  \tag{3.25}\\ e^{-\widehat{K}\left(1-e^{-\int_{t_{0}}^{t} \Lambda^{-1}(s) d s}\right)} & \text { if } t_{0} \leq t\end{cases}
$$

where $t_{0}$ is such that $f_{v, R}\left(t_{0}\right)=1$. Moreover,

$$
f_{v}(t)= \begin{cases}v^{e^{-\int_{0}^{t} \Lambda^{-1}(s) d s}} & \text { if } 0<v \leq 1,  \tag{3.26}\\ v^{e_{0}^{t} \Lambda^{-1}(s) d s} & \text { if } 1<v\end{cases}
$$

The limits become, for $0<v \leq 1$,

$$
\liminf _{R \rightarrow \infty} \frac{M^{\prime}(R)}{f_{v}(R)} \geq \liminf _{R \rightarrow \infty} \frac{f_{v, R}(R)}{f_{v}(R)} \geq \lim _{R \rightarrow \infty} e^{-\widehat{K}}
$$

and we only need $\widehat{K} \precsim 1$. When $v>1$ we know that $f_{\nu, R}$ in (3.25) stays above 1 if

$$
(\log v-\widehat{K}) e^{\int_{0}^{t} \Lambda^{-1}(s) d s}>-\widehat{K}
$$

which forces us to take $\widehat{K}<\log \nu$. Then

$$
\liminf _{R \rightarrow \infty} \frac{M^{\prime}(R)}{f_{v}(R)} \geq \liminf _{R \rightarrow \infty} \frac{f_{v, R}(R)}{f_{v}(R)}=\lim _{R \rightarrow \infty} e^{-\widehat{K}\left(e^{\int_{0}^{R} \Lambda^{-1}(s) d s}-1\right)}
$$

and we need also $\widehat{K} e^{\int_{0}^{R} \Lambda^{-1}(s) d s} \precsim 1$ to achieve a growth estimate.
We have defined $\widehat{K}(R)=\frac{n \Lambda(R)^{2}}{\lambda(R) \gamma(R)}$ in this case. However, from (3.24) we realize that $f_{\nu, R}$ is nondecreasing if $\widehat{K} \leq|\log \nu|$. This means that

$$
\frac{d f}{d t}=\Lambda^{-1}(t)(f|\log f|-\widehat{K} f) \geq 0, \quad t>0
$$

Now, we let $f_{\nu, R}$ solve this ODE in place of (2.1) and in the proof of Lemma 2.2 we replace (2.7) with

$$
\begin{aligned}
& \operatorname{Tr}\left(D^{2} V^{+}\right)=-\Lambda(\Xi)^{-1} \Phi(\Xi, f(\Xi))-\Lambda(\Xi)^{-1} \widehat{K} f(\Xi)+\frac{n-1}{\left|\left(x^{\prime}, x_{n}+\gamma\right)\right|} f(\Xi), \\
& \operatorname{Tr}\left(D^{2} V^{-}\right)=0
\end{aligned}
$$

By tracing the remaining part of the proof of Lemma 2.2 we realize that it is enough to pick $\widehat{K}(R)=\frac{n \Lambda(R)}{\gamma(R)}$.

As in the former situation the solution of (2.4) with $\Phi(s)=-s|\log s|$ can be calculated analytically when $\Lambda=$ constant :

$$
\check{u}(x)=\int_{0}^{x_{n}} f_{v}(s) d s=\Lambda \begin{cases}E_{i}(\log v)-E_{i}\left(e^{-\Lambda^{-1} t} \log v\right) & \text { if } 0<v<1  \tag{3.27}\\ x_{n} & \text { if } v=1 \\ -E_{i}(\log v)+E_{i}\left(e^{\Lambda^{-1} t} \log v\right) & \text { if } 1<v\end{cases}
$$

See Fig. 3 (lower row) for functions $f_{v}$ in (3.26) and $\check{u}$ in (3.27).
Now, using the calculations above (3.19) we see that $\Delta_{p(x)} u \geq 0$ implies

$$
\max \{1, p(x)-1\} \operatorname{Tr}\left(D^{2} u^{+}\right)-\min \{1, p(x)-1\} \operatorname{Tr}\left(D^{2} u^{-}\right)+|D p \| D u| \cos \theta \log |D u| \geq 0
$$

where $\theta=\theta(x)$ is the angle between $D u$ and $D p$. Assume $|D p \| \cos \theta|>0$ and divide the PDE with this factor to obtain

$$
\Lambda\left(x_{n}\right) \operatorname{Tr}\left(D^{2} u^{+}\right)-\lambda\left(x_{n}\right) \operatorname{Tr}\left(D^{2} u^{-}\right)+\frac{\cos \theta}{|\cos \theta|}|D u| \log |D u| \geq 0
$$

where $\lambda\left(x_{n}\right) \leq \frac{\min \{1, p(x)-1\}}{|D p p \| \cos \theta|}$ and $\Lambda\left(x_{n}\right) \geq \frac{\max \{1, p(x)-1\}}{|D p| \| \cos \theta \mid}$ for some nonincreasing function $\lambda$ and nondecreasing function $\Lambda$. Assuming $\cos \theta \log |D u| \leq 0$ leads to

$$
\Lambda\left(x_{n}\right) \operatorname{Tr}\left(D^{2} u^{+}\right)-\lambda\left(x_{n}\right) \operatorname{Tr}\left(D^{2} u^{-}\right)-|D u||\log | D u| | \geq 0
$$

and we can apply the results from this section, in particular (3.26)-(3.27) and Lemma 3.2, to obtain:

Corollary 3.5. Suppose that $p(x)$ and $u$ are as in Theorem 3.3. Let $\theta=\theta(x)$ be the angel between $D p$ and Du and assume that

$$
|D p \| \cos \theta|>0 \quad \text { and } \quad \cos \theta \log |D u| \leq 0
$$

hold in $\mathbb{R}_{+}^{n}$ (in a suitable weak sense if $u$ is not $C^{1}$ with $|D u| \neq 0$ ). Then $u$ is a subsolution of an equation of type $(\star)$ satisfying $(* \star)$ with $\Phi(s)=-s|\log s|, \lambda\left(x_{n}\right) \leq \frac{\min \{1, p(x)-1\}}{|D p \| \cos \theta|}$ and $\Lambda\left(x_{n}\right) \geq$ $\frac{\max \{1, p(x)-1\}}{|D p| \| \cos \theta \mid}$ for some nonincreasing function $\lambda$ and nondecreasing function $\Lambda$. If $\frac{n \Lambda(R)}{|\log \nu|}<\gamma(R)$ and $\breve{u}(x)=\int_{0}^{x_{n}} f_{v}(s) d s$ with $f_{v}$ from (3.26) then the following is true:

- If $u \geq \check{u}$ somewhere on the $x_{n}$-axis, $v \in(0,1]$ then

$$
\liminf _{R \rightarrow \infty} \frac{M(R)}{R}>0
$$

- If $u \geq \check{u}$ somewhere on the $x_{n}$-axis, $v>1$ and $\Lambda(R) e^{\int_{0}^{R} \Lambda^{-1}(s) d s} \precsim \gamma(R)$, then

$$
\liminf _{R \rightarrow \infty} \frac{M^{\prime}(R)}{v^{\int_{0}^{R} \Lambda^{-1}(s) d s}}>0 \quad \text { implying } \quad \liminf _{R \rightarrow \infty} \frac{M(R)}{\int_{0}^{R} v^{e^{\int_{0}^{t} \Lambda^{-1}(s) d s}} d t}>0
$$

which yields, if $\Lambda=$ constant and $E_{i}$ is the Exponential integral,

$$
\liminf _{R \rightarrow \infty} \frac{M(R)}{E_{i}\left(e^{\Lambda^{-1} R} \log \nu\right)-E_{i}(\log \nu)}>0 .
$$

In the one dimensional case Corollary 3.5 shows that if we know that the exponent $p(x)$ is increasing, $p^{\prime}>0$, and that the subsolution satisfies $0<u^{\prime}<1$, then $\liminf _{R \rightarrow \infty} u(R) / R>0$. Similarly, if we know that $p^{\prime}<0$ and $1<u^{\prime}$ then $\liminf _{R \rightarrow \infty} u(R) / \nu^{e^{\int_{0}^{R} \Lambda^{-1}(s) d s}}>0$. These estimates are much stronger than the growth estimates that can be derived from Theorem 3.3 in this situation. The improvements can be visualized by comparing the right panels in Fig. 3; the upper right panel corresponds to Theorem 3.3 while the lower right panel corresponds to the results in Corollary 3.5.

## 4. Connections with nonlinear diffusion problems

We follow the presentation in Lundström [32] and let $u$ denote the density of some quantity in equilibrium, $\Omega$ be a domain and $E \subset \Omega$ a $C^{1}$-domain so that the divergence theorem can be applied. Due to the equilibrium, the net flux of $u$ through $\partial E$ is zero, that is

$$
\oint_{\partial E}\langle\boldsymbol{F}, \boldsymbol{n}\rangle d s=0,
$$

where $\boldsymbol{F}$ denotes the flux density, $\boldsymbol{n}$ the normal to $\partial E$ and $d s$ is the surface measure. The divergence theorem gives

$$
\int_{E} \nabla \cdot \boldsymbol{F} d x=\oint_{\partial E}\langle\boldsymbol{F}, \boldsymbol{n}\rangle d s=0 .
$$

Since $E$ was arbitrary, we conclude

$$
\begin{equation*}
\nabla \cdot \boldsymbol{F}=0 \quad \text { in } \quad \Omega \tag{4.1}
\end{equation*}
$$

In many situations it is physically reasonable to assume that the flux vector $\boldsymbol{F}$ and the gradient $\nabla u$ are related by a power-law of the form

$$
\begin{equation*}
\boldsymbol{F}=-c|D u|^{q} D u \tag{4.2}
\end{equation*}
$$

for some factor $c$ and exponent $q$, which may depend on space as well. One reason is that flow is usually from regions of higher concentration to regions of lower concentration. From this assumption, with $q=p-2$, and from (4.1), we obtain the $p$-Laplace equation

$$
\nabla \cdot\left(|D u|^{p-2} D u\right)=0 \quad \text { in } \quad \Omega .
$$

The linear case $p=2$ in (4.2) arises as a physical law in the following: If $u$ denotes a chemical concentration, then it is the well known Fick's law of diffusion, if $u$ denotes a temperature, then it is Fourier's law of heat conduction, if $u$ denotes electrostatic potential, it is Ohm's law of electrical conduction, and if $u$ denotes pressure, then it is Darcy's law of fluid flow through a porous media. A problem involving the nonlinear case $p \neq 2$ is fast/slow diffusion of sandpiles, see Aronsson-Evans-Wu [4]. In that case $p$ is very large and $u$ models the height of a sandpile. If $|D u|>1+\delta$ for some $\delta>0$, then $|D u|^{p-2}$ is very large, and hence the transport of sand is also large, and if $|D u|<1-\delta$, then $|D u|^{p-2}$ is very small. Therefore, when adding sand particles to a sandpile, they accumulate as long as the slope of the pile does not exceed one. If the slope exceeds one, then the sand becomes unstable and instantly slides. Other application in which (4.2) arises with $p \neq 2$ is Hele-Shaw flow of power-law fluids (Aronsson-Janfalk [5], Fabricius-Manjate-Wall [15]) and electro-rheological fluids (Harjulehto-Hästö-Lê-Nuortio [18]). When properties of the quantity under investigation depend on space we may model it by a variable exponent $p=p(x)$ in (4.2) and thus enter equations of type (3.18) studied in Section 3.

We will now discuss the problem under investigation from the point of a diffusion problem. Indeed, we will briefly explain, through spatially dependent diffusion, why parts of our results presented in Section 3 for the variable exponent $p$-Laplace equation hold true. Suppose that $u$ denotes the density of some quantity at equilibrium in the $n$-dimensional halfspace $\left\{x_{n}>0\right\}$ and that (4.2) holds with a variable exponent $p\left(x_{n}\right), 1<p\left(x_{n}\right)<\infty$. Assume also that $u=0$ on the boundary $x_{n}=0$, and at some $x_{n}=a>0$ we assume that $u(x)>0$. We conclude that then $u$ satisfies the $p(x)$-Laplace equation (3.18) in the halfspace and that our results apply. We simplify by further assuming that concentration $u(x)$ is independent of $x^{\prime}$-directions. Since $|D u|$ must be positive there is a flux of $u$, independent of $x^{\prime}$, flowing perpendicular through the plane at $x_{n}=a$ toward the boundary $x_{n}=0$. Due to the equilibrium, the flux must be independent also of $x_{n}$ and is therefore constant through the halfspace. Since the problem is herefrom independent of $x^{\prime}$, we drop the index and write in the following $x=x_{n}$.

Suppose that $p(x)$ is decreasing. As the flux of $u$, given by assumption (4.2), is constant, the concentration $u$ must be convex (upwards) if $|D u|=u^{\prime}>1$. Indeed, if $u^{\prime}>1$ near the boundary we locally have that (4.2) yields flux $\mathbf{F}=-c\left(u^{\prime}\right)^{p(x)-1}$ and since $p(x)-1>0$ is decreasing it follows that $u^{\prime}$ must be increasing. A similar reasoning explains that if $u^{\prime}=1$ somewhere then the flux $\mathbf{F}=c$ implying $u(x)=x$, and if $u^{\prime}<1$ then $u$ must be concave. Fig. 4 (left) shows examples of how the concentration $u(x)$ may depend on $x$ for two different decreasing exponents. We remark that if $p(x)$ becomes very large near the boundary then $u^{\prime}$ must be very close 1 there, otherwise the flux becomes zero or infinity - that is fast/slow diffusion (red solid curve). Similarly, if $p(x)$ comes close to 1 as we move into the domain then $u^{\prime}$ must grow fast if $u^{\prime}$ ever was larger than 1 along the curve in order to keep the flux constant (green dashed curve). Finally, we realize that if $p(x)$ becomes constant then $u^{\prime}$ becomes constant (recall that $u(x)=c x$ is $p$-harmonic when $p=$ constant ).

Suppose now instead that $p(x)$ is decreasing. Reasoning as in the former case we realize that we may switch our conclusions made near the boundary in the former case with those made further away into the domain. Thus fast/slow diffusion may occur away from the boundary and in such a case the slope of $u(x)$ must approach 1 . If $p(x)$ approaches 1 near the boundary then $u^{\prime}$ must explode there, see Fig. 4 (right).

The above reasoning agrees with our mathematical results. Indeed, the border at $u=x$ corresponds to the two cases $v \in(0,1)$ and $v \geq 1$ in Theorem 3.3. When $v \in(0,1)$ we are below


Fig. 4. Examples of how the concentration $u$ may depend on $x$ for decreasing exponents (left) and increasing exponent (right). The slope explodes or vanish as $p(x) \rightarrow 1$ : The green dashed curves correspond to an exponent $p(x)$ that approaches 1 as $x$ increases (left) and as $x \rightarrow 0$ (right). The slope approaches 1 as $p(x) \rightarrow \infty$, i.e. fast/slow diffusion: The red solid curves correspond to an exponent $p(x)$ that becomes very large as $x \rightarrow 0$ (left) and as $x$ increases (right). (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)
$u=x$ and subsolutions of the variable $p(x)$-Laplacian may grow very slowly according to the Theorem. If $v \geq 1$ on the other hand, then Theorem 3.3 implies the stronger growth

$$
\liminf _{R \rightarrow \infty} \frac{M(R)}{R}>0 .
$$

A similar comment holds for Corollary 3.5. Moreover, returning to (3.23) and Remark 3.4 we find that the one-dimensional $p(x)$-Laplace equation yields

$$
\Delta_{p(x)} u(x)=(p(x)-1) u^{\prime \prime}(x)+\log \left|u^{\prime}(x)\right| p^{\prime}(x) u^{\prime}(x)=0 .
$$

Recalling (3.17) and (3.27) we also realize that with the decreasing exponent

$$
p(x)=1+M e^{-A x}
$$

where $M>0, A>0$ are constants, the solution yields

$$
u(x)=\frac{1}{A} \begin{cases}-E_{i}(\log v)+E_{i}\left(e^{A x} \log v\right) & \text { if } v \neq 1  \tag{4.3}\\ x & \text { if } v=1\end{cases}
$$

Similarly, with the increasing exponent

$$
p(x)=1+M e^{A x}
$$

the solution yields

$$
u(x)=\frac{1}{A} \begin{cases}E_{i}(\log v)-E_{i}\left(e^{-A x} \log v\right) & \text { if } v \neq 1  \tag{4.4}\\ x & \text { if } v=1\end{cases}
$$

With $A=\lambda=\Lambda=1$, solution curves for decreasing exponent in (4.3) are plotted in Fig. 3 (upper right) (below line $u=x$ ) and (lower right) (above line $u=x$ ), and solution curves for increasing exponent in (4.4) are plotted in Fig. 3 (upper right) (above line $u=x$ ) and (lower right) (below line $u=x$ ). Compare the structure of these curves to those in Fig. 4 with properties of the exponent $p(x)$ in mind.

## Acknowledgment

I would like to thank an anonymous reviewer for valuable comments and suggestions which really helped to improve arguments and correct mistakes. This work was partially supported by the Swedish Research Council grant 2018-03743.

## References

[1] T. Adamowicz, Phragmén-Lindelöf theorems for equations with nonstandard growth, Nonlinear Anal., Theory Methods Appl. 97 (2014) 169-184.
[2] L. Ahlfors, On Phragmén-Lindelöf's principle, Trans. Am. Math. Soc. 41 (1937) 1-8.
[3] S.N. Armstrong, B. Sirakov, C.K. Smart, Singular solutions of fully nonlinear elliptic equations and applications, Arch. Ration. Mech. Anal. 205 (2) (2012) 345-394.
[4] G. Aronsson, L.C. Evans, Y. Wu, Fast/slow diffusion and growing sandpiles, J. Differ. Equ. 131 (2) (1996) 304-335.
[5] G. Aronsson, U. Janfalk, On Hele-Shaw flow of power-law fluids, Eur. J. Appl. Math. 3 (4) (1992) 343-366.
[6] B. Avelin, V. Julin, A Carleson type inequality for fully nonlinear elliptic equations with non-Lipschitz drift term, J. Funct. Anal. 272 (8) (2017) 3176-3215.
[7] J.E.M. Braga, D. Moreira, Classification of nonnegative g-harmonic functions in half-spaces, Potential Anal. (2020) 1-19.
[8] T. Bhattacharya, On the behaviour of infinity-harmonic functions on some special unbounded domains, Pac. J. Math. 219 (2) (2005) 237-253.
[9] T. Bhattacharya, A. Mohammed, Maximum principles for k-Hessian equations with lower order terms on unbounded domains, J. Geom. Anal. 31 (4) (2021) 3820-3862.
[10] L.A. Caffarelli, X. Cabre, Fully Nonlinear Elliptic Equations, American Mathematical Society Colloquium Publications, vol. 43, American Mathematical Society, Providence, RI, 1995.
[11] I. Capuzzo-Dolcetta, A. Vitolo, A qualitative Phragmén-Lindelöf theorem for fully nonlinear elliptic equations, J. Differ. Equ. 243 (2) (2007) 578-592.
[12] Y. Chen, S. Levine, M. Rao, Variable exponent, linear growth functionals in image restoration, SIAM J. Appl. Math. 66 (4) (2006) 1383-1406.
[13] M.G. Crandall, H. Ishii, P.-L. Lions, User's guide to viscosity solutions of second order partial differential equations, Bull. Am. Math. Soc. 27 (1992) 1-67.
[14] L. Diening, M. Růžička, Strong solutions for generalized Newtonian fluids, J. Math. Fluid Mech. 7 (2005) 413-450.
[15] J. Fabricius, S. Manjate, P. Wall, On pressure-driven Hele-Shaw flow of power-law fluids, preprint, 2021.
[16] D. Gilbarg, The Phragmén-Lindelöf theorem for elliptic partial differential equations, J. Ration. Mech. Anal. 1 (1952) 411-417.
[17] S. Granlund, N. Marola, Phragmén-Lindelöf theorem for infinity harmonic functions, Commun. Pure Appl. Anal. 14 (1) (2016) 127-132.
[18] P. Harjulehto, P. Hästö, U.V. Lê, M. Nuortio, Overview of differential equations with non-standard growth, Nonlinear Anal., Theory Methods Appl. 72 (12) (2010) 4551-4574.
[19] J.O. Herzog, Phragmen-Lindelöf theorems for second order quasi-linear elliptic partial differential equations, Proc. Am. Math. Soc. 15 (5) (1964) 721-728.
[20] E. Hopf, Remark on a preceding paper of D. Gilbarg, J. Ration. Mech. Anal. 1 (1952) 419-424.
[21] C.O. Horgan, Decay estimates for boundary-value problems in linear and nonlinear continuum mechanics, in: Mathematical Problems in Elasticity, in: Ser. Adv. Math. Appl. Sci., vol. 38, World Sci. Publ., River Edge, NJ, 1996, pp. 47-89.
[22] Z. Jin, K. Lancaster, A Phragmén-Lindelöf theorem and the behavior at infinity of solutions of non-hyperbolic equations, Pac. J. Math. 211 (1) (2003) 101-121.
[23] V. Julin, Generalized Harnack inequality for nonhomogeneous elliptic equations, Arch. Ration. Mech. Anal. 2 (216) (2015) 673-702.
[24] V. Julin, P. Juutinen, A new proof for the equivalence of weak and viscosity solutions for the p-Laplace equation, Commun. Partial Differ. Equ. 37 (5) (2012) 934-946.
[25] P. Juutinen, P. Lindqvist, J.J. Manfredi, On the equivalence of viscosity solutions and weak solutions for a quasilinear equation, SIAM J. Math. Anal. 33 (3) (2001) 699-717.
[26] P. Juutinen, T. Lukkari, M. Parviainen, Equivalence of viscosity and weak solutions for the $p(x)$-Laplacian, Ann. Inst. Henri Poincaré, Anal. Non Linéaire 27 (6) (2010) 1471-1487, Elsevier Masson.
[27] S. Koike, K. Nakagawa, Remarks on the Phragmén-Lindelöf theorem for viscosity solutions of fully nonlinear PDEs with unbounded ingredients, Electron. J. Differ. Equ. 2009 (2009) [electronic only].
[28] V.V. Kurta, Phragmén-Lindelöf theorems for second-order quasilinear elliptic equations (in Russian) Ukr. Mat. Zh. 44 (10) (1992) 1376-1381; translation in Ukr. Math. J. 44 (10) (1992) 1262-1268, 1993.
[29] M. Leseduarte, M. Carme, R. Quintanilla, Phragmén-Lindelöf alternative for the Laplace equation with dynamic boundary conditions, J. Appl. Anal. Comput. 7 (4) (2017) 1323-1335.
[30] P. Lindqvist, On the growth of the solutions of the differential equation $\nabla \cdot\left(|\nabla u|^{p-2} \nabla u\right)=0$ in $n$-dimensional space, J. Differ. Equ. 58 (1985) 307-317.
[31] E. Lundberg, A. Weitsman, On the growth of solutions to the minimal surface equation over domains containing a halfplane, Calc. Var. Partial Differ. Equ. 54 (4) (2015) 3385-3395.
[32] N.L.P. Lundström, p-harmonic functions near the boundary, Doctoral Thesis, Umeå, ISSN 1102-8300, ISBN 978-91-7459-287-0, 2011.
[33] N.L.P. Lundström, Phragmén-Lindelöf theorems and p-harmonic measures for sets near low-dimensional hyperplanes, Potential Anal. 44 (2016) 313-330.
[34] N.L.P. Lundström, M. Olofsson, O. Toivanen, Strong maximum principle and boundary estimates for nonhomogeneous elliptic equations, arXiv preprint arXiv:2005.03338, 2020.
[35] N.L.P. Lundström, J. Singh, Estimates of p-harmonic functions in planar sectors, arXiv preprint arXiv:2111.02721, 2021.
[36] M. Medina, P. Ochoa, On viscosity and weak solutions for non-homogeneous p-Laplace equations, Adv. Nonlinear Anal. 8 (2017) 468-481.
[37] K. Miller, Extremal barriers on cones with Phragmen-Lindelöf theorems and other applications, Ann. Mat. Pura Appl. 90 (1) (1971) 297-329.
[38] S. Momm, A Phragmén-Lindelöf theorem for plurisubharmonic functions on cones in $\mathbb{C}^{n}$, Indiana Univ. Math. J. 41 (3) (1992) 861-867.
[39] E. Phragmén, E. Lindelöf, Sur une extension d'un principe classique de l'analyse et sur quelques propriétés des functions monogénes dans le voisinage d'un point singulier, Acta Math. 31 (1) (1908) 381-406.
[40] R. Quintanilla, Some theorems of Phragmén-Lindelöf type for nonlinear partial differential equations, Publ. Mat. 37 (1993) 443-463.
[41] J. Serrin, On the Phragmén-Lindelöf principle for elliptic differential equations, J. Ration. Mech. Anal. 3 (1954) 395-413.
[42] A. Vitolo, On the Phragmén-Lindelöf principle for second-order elliptic equations, J. Math. Anal. Appl. 300 (1) (2004) 244-259.


[^0]:    E-mail address: niklas.lundstrom@umu.se.
    https://doi.org/10.1016/j.jde.2022.02.048
    0022-0396/© 2022 The Author(s). Published by Elsevier Inc. This is an open access article under the CC BY license (http://creativecommons.org/licenses/by/4.0/).

