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A Monte Carlo-type simulation toolbox for Solar System small body dynamics: application to the October Draconids

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Abstract

We present the current status and first results from a Monte Carlo-type simulation toolbox for Solar System small body dynamics. We also present fundamental methods for evaluating the results of this type of simulations using convergence criteria. The calculations consider a body in the Solar System with a mass loss mechanism that generates smaller particles. In our application the body, or parent body, is a comet and the mass loss mechanism is a sublimation process. In order to study mass propagation from parent bodies to Earth, we use the toolbox to sample the uncertainty distributions of relevant comet parameters and to find the resulting Earth influx distributions. The initial distributions considered represent orbital elements, sublimation distance, cometary and meteoroid densities, comet and meteoroid sizes and cometary surface activity. Simulations include perturbations from all major planets, radiation pressure and the Poynting-Robertson effect. In this paper we present the results of an initial software validation performed by producing synthetic versions of the 1933, 1946, 2011 and 2012 October Draconids meteor outbursts and comparing them with observational data and previous models. The synthetic meteor showers were generated by ejecting and propagating material from the recognized parent body of the October Draconids; the comet 21P/Giacobini-Zinner. Material was ejected during 17 perihelion passages between 1866 and 1972. Each perihelion passage was sampled with 50 clones of the parent body, all producing meteoroid streams. The clones were drawn from a multidimensional Gaussian distribution on the orbital elements, with distribution variances proportional to observational uncertainties. In the simulations, each clone ejected 8,000 particles. Each particle was assigned an individual weight proportional to the mass loss it represented. This generated a total of 6.7 million test particles, out of which 43 thousand entered the Earth’s Hill sphere during 1900-2020 and were considered encounters. The simulation reproduces the predictions and observations of the 1933, 1946, 2011 and 2012 October Draconids, including the unexpected but measured deviation of the meteoroid mass index from a power law in 2012 as compared to 2011. We show that when convergence is sufficient in the simulation, the fraction between two encountered mass distributions is independent of the assumed input mass distribution. Finally, we predict an outburst for the 2018 October Draconids with a peak on October 8-9 that could be up to twice as large as the 2011 and 2012 outbursts.

Keywords: Small body dynamics, Meteoroids, Comets, Numerical simulation, 21P/Giacobini-Zinner, October Draconids

1. Introduction

Every day the Earth’s atmosphere is bombarded by billions of dust-sized particles and larger pieces of material from space. The objects of a size between 100 microns and 1 metre moving in interplanetary space are by definition called meteoroids \[ \text{[1]} \]. The phenomena they give rise to in the atmosphere, in particular the visible streaks of light commonly seen on the night sky, are called meteors. A meteor shower occurs when the Earth passes through a stream of meteoroids released from a parent body, which is usually a comet. In addition to the meteor showers, there is a continuous flux of sporadic meteors caused by meteoroids from old streams that have dissipated over time due to orbital instabilities. This sporadic population consists of meteoroids perturbed over such long time-scales that distinction of their original orbits is generally impossible. By con-
In contrast, the shower meteors are due to meteoroid streams still in a young stage of dissipation. Orbital associations between the meteoroids can be made, concluding that they belong to the same meteoroid stream. The collection of all meteoroids in the Solar System is sometimes called the meteoritic complex [e.g. 2, 3] or more recently the meteoroid complex [e.g. 4, 5].

In spite of the fact that the underlying orbital dynamics are well understood, it is still an open question how much extraterrestrial material enters the Earth’s atmosphere [6]. Meteoroids and dust entering the atmosphere take part in physical and chemical processes important for a wide range of phenomena, such as the formation of clouds at 15-25 km altitude responsible for ozone destruction in the polar regions and mid-latitude ice clouds at 75-85 km which are possible tracers of global climate change [7]. Characterization of dust trails and meteoroid streams is also highly relevant for models like the European Space Agency (ESA) Interplanetary Meteoroid Environment for Exploration (IMEX) project [8] to assess the dust impact hazard to spacecraft. For example, the ESA communications satellite Olympus lost pointing control in 1993, probably due to the impact of a Perseid meteoroid [9]. For comet and asteroid interaction missions, such as the NASA space probe Deep Impact [10], JAXA’s Hayabusa 2 and future asteroid deflection missions, accurately estimating debris dynamics will be vital. These space missions include ejecting an impactor towards an asteroid or a comet. The impact event will release a cloud of material that in a worst-case scenario may propagate to the Earth, damage satellite infrastructure, or even generate damaging impact events on Earth. Proper risk assessment can for example be performed using numerical simulations of the distributions of impact scenarios. Numerical simulations in this area need to be developed to test and validate physical models, ranging from theories of dust ejection and ice sublimation of comets [11] to the chaotic mechanism supplying the Solar System with new comets [12].

Numerical simulations of cometary dust trails and the dynamics of meteoroid streams is an active research area. In 1999 Asher et al. [13] provided the first observational demonstration of the importance of resonant behaviour in meteoroid streams by explaining the 1998 Leonid fireball outburst using numerical simulations of the long-term fine structure within the dust trails released from comet 55P/Tempel-Tuttle. Previous numerical simulations concentrated on finding years of increased activity by short-term stream modelling [e.g 14, 15, 16, 17]. This work paved the way for quantitative predictions of meteor showers, in contrast to the methods used up until then, which were largely based on roughly estimated properties of the comet.
with uncertainties or confidence intervals. The methods currently reviewed and adapted include long-term stable symplectic Hamiltonian integrators, such as the Mercury6 software [25]; short-term efficient but accurate integrators such as the ESA NEOPROP2; instability (or chaos) estimation calculations such as Lyapunov indicators (L1) or Mean Exponential Growth of Nearby Orbits (MENGO) [26]; software for generating realistic measurement uncertainties by Bayesian inversion theory [24]; software for generating arbitrary particle creation scenarios (covered in Section 2.4); and finally a general coordinating software unifying all the theory in a rigorous manner ensuring proper statistics (described in section 2.5). To maximize the contribution to the current research area we are developing the software as a modular toolbox designed for reuse. This paper covers its current status and first results. Using standard terms for software release life cycles, we are currently in an alpha phase of the release life cycle of developed modules, which is the first phase where software testing can be performed. We have employed several structural (so-called white-box) testing techniques that are not presented here and have moved on to additional functionality (black-box) testing techniques. The results referred to as initial validation presented here are the first black-box tests. During an alpha phase, software functions and capabilities are still being added and the results and method should be viewed with that in mind. The software toolbox has been described in detail in [21]. In Section 2, we review the main functionality and in the remainder of the paper we present a case study applying the toolbox to the comet 21P/Giacobini-Zinner, its meteoroid stream and the associated October Draconid meteor shower observed on Earth.

The goal of this case study is to examine how dust and meteoroid ejecta of different masses from comet 21P/Giacobini-Zinner propagate to the Earth and compare these simulations with ground-based measurements during the 2011 and 2012 October Draconid meteor showers. More specifically, we investigate the unexpected deviation of the October Draconid meteoroid mass index from a power law in 2012, a feature not present in 2011 [28–29]. To explore this feature of the 2011 and 2012 October Draconids, we set up a multiple trail simulation of 21P/Giacobini-Zinner as described in Section 3. The particular choice of the October Draconids was not arbitrary; it is a widely observed meteor shower suitable for initial validation purposes (as outlined in Section 4). The main results of the case study are presented in Section 5.1–5.3 where we compare the simulation outcomes with previously published simulations and observations. Finally, in Section 5.4 we predict an outburst for the 2018 October Draconids with a peak on October 8–9 that could be up to twice as large as the 2011 and 2012 outbursts.

2. Method

This section briefly reviews the developed software and its current state. A more detailed description of the logistics of the calculations including an outline of implemented algorithms has been reported by [27]. The main novelty of the current approach is the introduction of a statistical perspective all the way from simulation input to the output of results. Many authors already generate statistical clones of comets to increase the validity of the simulated meteoroid streams [e.g. 22–23]. Our method aims to supply simple ways to implement proper data paths from initial input distributions of variables to output data distributions and resulting statistics. To accomplish this aim we have covered in detail many components of the simulation methodology, only a few of which are covered below and in the appendices. A flowchart of the iterative part of the toolbox is displayed in Figure 1.

2.1. Modular development

One of the most challenging tasks when designing software is broad functionality without loosing usability [30]. To maximize the functionality of all the developed software we have chosen to proceed in a modular fashion, where every significant self-contained part of the analysis and simulation is made into an independent module. One module acts as a master program that has the ability to call several other independent modules. However, the modules can also run independently of the master program. The scientific investigator can therefore link modules in different orders, choose to distribute data between modules in different ways and tailor case studies in a powerful fashion. This also allows for easy black-box testing of the individual modules when changes are made. In Section 2.2 and 2.3 we introduce the two main modules that were used. Two other modules were replaced by existing software as described in Section 2.5 due to the toolbox being in an alpha phase.

2.2. Monte Carlo Association Statistics module

Monte Carlo Association Statistics or MCAS is developed to manage a Monte Carlo iteration process to
generate bodies that produce small particles in the Solar System. An illustration of the MCAS functionality is shown in Figure 1. This module is the previously mentioned coordinating software that calls the different modules to perform actions such as ejecting meteoroids from their parent body, propagating the meteoroid stream and calculating initial distributions. MCAS itself handles the data flow, manages the output distributions and checks simulation convergence criteria. The parent bodies are generated from probability distributions of some variable or set of variables. In its current stage of development, we have selected the most relevant cometary properties from which to make distributions: this includes orbital elements, critical sublimation radius, parent body mass, parent body size, particle mass distribution and surface activity.

In this module some data filtering decisions have to be made, one of which is the particles to be considered as meteors in the simulation. A usual simplification is to track the nodes of the test particle orbits (instead of keeping track of the positions of the test particles) and consider all nodes that encounter the Earth as meteors. This is not a valid approximation if it is not certain that enough time has passed for the gravitational perturbations to have spread the stream of represented particles over the entire orbit. We have therefore not used this approximation in this short time scale initial validation, but instead chosen to regard test particles passing within 1 Hill radius of the Earth to be considered meteors. This is preferrable, since entering the Hill sphere indicates gravitational dominance of the Earth over the particle and greatly increases the probability of an actual meteor event occurring.

2.3. Probability formulation

In this section we address the probability formulation of the simulation. We will use a very basic concept from statistical simulations,

$$P(X) = \frac{\#\text{Simulations containing } X}{\# \text{Simulations}},$$

where we want to find the probability $P$ of an event $X$. As a practical example, we can ask “What is the Earth encounter probability as a function of ejection time (comet perihelion passage) and detection time (Solar longitude)?”. This is one possible way to give a definition of an event set $X$. We would then find all simulations in which a non-zero meteoroid flux occurred during these intervals. If this turned out to be true for 99 out of 100 simulations, we are 99% sure that a flux will occur given our input set of possible scenarios. As seen
from the example, the probability in equation 1 does not address the flux rate unless stated in the event description of X, only the probability that flux will occur. To examine the fluxes we can take two approaches: either we calculate the mean flux over the simulations in which the flux occurred, or we calculate the mean flux over all simulations. In our illustrations we have calculated the mean flux over all simulations using

\[ E(X) = \frac{\sum \text{Value of } X \text{ in simulation } i}{\# \text{ Simulations}}, \]  

(2)
as we find this is a more intuitive way to examine the data. Thus, we have generated two different types of maps, one that describes the probability of an event occurring and one that describes this event’s magnitude in terms of the mean flux.

2.4. Parent Body Ejector module

The Parent Body Ejector PBE module is designed to make it easy to generate long-term initial conditions. The module currently provides implemented sublimation models from \[31, 32, 33\]. After reviewing the different models for sublimation given in \[11\], we opted to use the one described in \[32\] as it is the one with the least ambiguity in derivation, provides a large parameter space and is very easily applied to 21P/Giacobini-Zinner. The PBE is designed to take all parent body parameters as input together with a configuration file. From this it generates as output the complete simulation state (Sun, planets, comet, meteoroid stream, etc.) once ejection of particles is complete. Thus for each generated clone of 21P/Giacobini-Zinner the PBE module generates additional ejected particles for each perihelion passage. As only relatively short time spans should be integrated using this module, we have so far implemented only two different integration methods. These are the Bulirsch-Stoer method and a sympletic 8-part Hamiltonian split method.

The ejection of particles from 21P/Giacobini-Zinner was performed using the Bulirsch-Stoer method. This method combines the three ideas of Richardson extrapolation, rational function extrapolation and the modified midpoint method \[34\]. This technique is designed for differential equations containing smooth functions and for differential equations without singular points inside the integration interval. Thus it is suited for integrating Hamiltonian systems with slight electromagnetic perturbations. We chose this method due to the short orbital time of 21P/Giacobini-Zinner and since the Bulirsch-Stoer method is widely recognized as one of the best ways to calculate high-accuracy solutions to ordinary differential equations with minimal computational time and is easy to implement \[35\].

The 8-part Hamiltonian split concept and order determination description was implemented following \[36, 37\] as described in \[27, sect. 30\]. It was not used in the current case study.

2.5. Replacements

Due to the alpha-state of the software toolbox, we have implemented Gaussian orbital uncertainties proportional to the uncertainties given by the JPL for the initial distribution representing 21P/Giacobini-Zinner’s orbit instead of a general initial distribution from Bayesian inversion theory based on observations. Also, the Celestial Mechanics Simulator, the CMS module, was not stable enough to apply in this case study. We substituted a modified version of mercury6 \[25\], configured to use a Hybrid sympletic/Bulirsch-Stoer integrator. mercury6 \[25\] is a general-purpose dynamical astronomy software designed for N-body integrations. More specifically, it is designed to propagate objects in the gravitational field of a large central body and allows for examination of close encounters, integration of mass-less test particles and the additions of non-gravitational forces. mercury6 \[25\] is written in Fortran 77. The modifications we made allowed for the software to consider test particles with different densities and sizes and to compute the radiation pressure and the Poynting-Robertson effect following \[38\]. We also enhanced the data output function of mercury6 to generate additional orbital data regarding close encounters.

3. Simulation parameters

<table>
<thead>
<tr>
<th>Physical parameters</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>21P radius $R_c$</td>
<td>1 000 m</td>
</tr>
<tr>
<td>21P activity factor $\Psi$</td>
<td>0.05</td>
</tr>
<tr>
<td>21P bulk density $\sigma$</td>
<td>1000 kg/m$^3$</td>
</tr>
<tr>
<td>21P surface absorption factor $g$</td>
<td>0.1</td>
</tr>
<tr>
<td>Particle bulk density $\rho$</td>
<td>1000 kg/m$^3$</td>
</tr>
<tr>
<td>Adiabatic acceleration $\theta$</td>
<td>2</td>
</tr>
<tr>
<td>Sublimation energy $H$</td>
<td>1.88 x 10$^6$ J/kg</td>
</tr>
<tr>
<td>Free drag coefficient $\xi$</td>
<td>2</td>
</tr>
<tr>
<td>Relative molecular mass $\mu$</td>
<td>20 x 1.661 x 10$^{-24}$ g</td>
</tr>
</tbody>
</table>

Table 2: Physical parameter input variables for the simulation, data from \[20, 21, 31, 32, 34, 36\]. Parameter symbols are defined as in \[29\] where their usage in the simulation algorithms is further described.

Simulations of the October Draconid meteor shower have been performed by, e.g., \[20\] and \[28\] to predict
and explain the recent outbursts in 2011 and 2012, respectively. Both of these meteor outbursts were observed visually and reported to the International Meteor Organisation (IMO). The outbursts were also detected and studied using radar systems sensitive to fainter-than-visual meteors, produced by lower-mass meteoroids [43, 44, 28, 29].

To generate the data needed, a set of 17 simulations was set up parallel to each other. Each process considered one perihelion passage and its connected meteoroid stream. One simulation considered one set of perihelion passages of 21P/Giacobini-Zinner each, ranging from 1866 to 1972. All the clones of 21P/Giacobini-Zinner were generated from Gaussian orbital uncertainties equal to the uncertainties given in the JPL small body database [39]. The simulations after and including the passage of 1900, except the years 1920 and 1953, used orbital elements from the JPL small body database. These orbital elements are calculated from observations. The two passages in the years 1920 and 1953 were, however, not observed and therefore no data is available. Thus, these passages had to be numerically estimated by propagating to elements of the closest observation. The same is true for the passages prior to 1900. We have calculated orbital elements from observations, however, at year 1900

<table>
<thead>
<tr>
<th>Orbital parameters</th>
<th>Mean</th>
<th>Standard deviation</th>
<th>Distribution type</th>
</tr>
</thead>
<tbody>
<tr>
<td>21P semi-major axis $a$ at year 1900</td>
<td>3.4702 AU</td>
<td>0.01 AU</td>
<td>Normal</td>
</tr>
<tr>
<td>21P eccentricity $e$ at year 1900</td>
<td>0.7316</td>
<td>0.005</td>
<td>Normal</td>
</tr>
<tr>
<td>21P inclination $i$ at year 1900</td>
<td>29.8295°</td>
<td>0.05°</td>
<td>Normal</td>
</tr>
<tr>
<td>21P argument of perihelion $\omega$ at year 1900</td>
<td>171.0457°</td>
<td>0.05°</td>
<td>Normal</td>
</tr>
<tr>
<td>21P longitude of ascending node $\Omega$ at year 1900</td>
<td>198.1360°</td>
<td>0.05°</td>
<td>Normal</td>
</tr>
<tr>
<td>21P perihelion passage time $t_p$ at year 1900</td>
<td>2415351.9965 JD</td>
<td>0.01 JD</td>
<td>Normal</td>
</tr>
</tbody>
</table>

Table 1: Input orbital distributions for 21P/Giacobini-Zinner at year 1900 [39].

The Parent Body Ejector (PBE) module is designed to take as input distributions with widths proportional to measurement uncertainties, if such are given. To reduce the computation time in this initial case study we restricted the measurement uncertainties to orbital parameters. Physical parameters such as critical sublimation radius, parent body mass and parent body size were all set to typical values for 21P/Giacobini-Zinner found in literature, e.g [45], and given in Table 2. The time step of mercury6 [25] was set to one day and the time step of PBE was set to 16 days. PBE utilized a Bulirsch-Stoer algorithm and mercury6 [25] was configured to use the hybrid symplectic integrator, both including electromagnetic forces. Finally, the mass distribution for ejection of particles was set to a logarithmic uniform distribution between $10^{-9}$ kg to $10^{-1}$ kg, which provides good sampling of the space we wish to examine. In Appendix A we show that if analysis is limited to the space where convergence is sufficient, any distribution that samples the input space and the output space well is a good input distribution. As long as convergence is sufficient, it is possible to apply a realistic input distribution to derive a realistic encountered distribution without additional simulations. With sufficient convergence one can also calculate input distribution invariants, and as such the logarithmically uniform distribution is a good choice even if not physically realistic.

This set of simulations took six days to compute on
a standard dual-core desktop PC. The complete set of input parameters (not including system parameters covered in [27]) are available in Tables 1 and 2.

4. Initial validation

Figure 2: Histogram of the stream-Earth encounter probability for 1933 in the October Draconids simulation.

Figure 3: Histogram of the stream-Earth encounter probability for 1946 in the October Draconids simulation.

A common method to validate software is to make back-predictions on data which is already well-established both numerically and observationally. The October Draconids is famous for its very strong 1946 outburst, which had a zenith hourly rate (ZHR) that according to estimations at the time reached up to 10,000 [46]. The October Draconid shower has been extensively simulated [e.g. 47, 20, 48, 49] and there are also plenty of observations available [e.g. 28, 29, 43, 44, 50, 51, 52, 53, 54, 55, 56], making it a very good candidate for initial software validation.

As in the simulations by [20], our simulations of the October Draconids should show a meteor outburst on Earth during October 9, 1933, and October 10, 1946, both at ≈ 197° in J2000.0 Solar longitude.

The resulting synthetic October Draconids for the years 1933 and 1946 are illustrated in Figures 2 and 3. These figures show probability distributions representing the probability of the meteoroid stream released by 21P/Giacobini-Zinner encountering the Earth, as calculated by equation 1. In Figures 2 and 3, we are therefore comparing the observational data with the probability distribution of outbursts occurring rather than comparing it with the simulated flux rates.

The probability distribution acts like a potential field for flux, where there is no probability no flux can occur. The higher the probability the larger is the fraction of all simulations that supplied flux to that area. These simulated data correspond well with previous simulations [20] and we regard the initial validation as accomplished. The comparisons are summarized in Table 3.

<table>
<thead>
<tr>
<th>October Draconids</th>
<th>Model [20]: mean Solar longitude</th>
</tr>
</thead>
<tbody>
<tr>
<td>1933</td>
<td>196.9990°</td>
</tr>
<tr>
<td>1946</td>
<td>196.9956°</td>
</tr>
<tr>
<td><strong>Our model</strong>: Solar longitude interval including 90% of encounters</td>
<td></td>
</tr>
<tr>
<td>1933</td>
<td>196.4562° to 197.2371° mean 197.0132°</td>
</tr>
<tr>
<td>1946</td>
<td>196.6894° to 197.1730° mean 197.0241°</td>
</tr>
<tr>
<td><strong>Observation</strong> [57, 58]: Solar longitude of observations</td>
<td></td>
</tr>
<tr>
<td>1933</td>
<td>196.9° to 197.05°</td>
</tr>
<tr>
<td>1946</td>
<td>197°</td>
</tr>
</tbody>
</table>

Table 3: Comparison between the Vaubaillon model [20], this model, and observations of the 1933 and 1946 October Draconids [57, 58]. All Solar longitudes have been converted to the standard equinox of J2000.0.
5. Results

In summary, 821 of the 850 propagated meteoroid streams contained encounters with Earth before 2020. In total, 6,714,499 test particles were propagated and out of them 43,637 encountered the Earth. The low number of Earth encounters is due to the fact that we do not only track the nodes of the particles and require their orbits to cross that of the Earth, also require the particles themselves to enter the sphere of Earth’s gravitational influence (Section 2). In Appendix A we discuss simulation convergence and the application of particle weights to represent additional virtual particles. In Appendix B we introduce the development of convergence criteria and present estimator standard deviations. We find only moderate statistical certainty for the estimators due to limited processor resources in the case study. Based on the current computation time and hardware, to decrease the uncertainty of the results to a quarter of the current values would demand approximately three additional months of computing. However, the results are sufficient to demonstrate the significant trends in the data and the capabilities of the described methodology.

5.1. 2011 October Draconids

Figure 4 illustrates the calculated probability of stream encounter together with measurements from the Shigaraki middle and upper atmosphere (MU) radar in Japan [43] and the IMO [41]. In Figure 4 the ZHR:s have been normalized so that the highest ZHR for each measurement system is equal to 1. The IMO database ZHR, the MU radar ZHR, and the simulation fluxes are not directly comparable. The IMO ZHR was derived from a visual observation rate compiled from a distribution of observers active during different time intervals. The MU radar is a single measurement instrument where the rate we have adopted here is an instrument-specific ZHR calculated by the hourly rate of meteors classified as October Draconids and compensated by the radiant elevation with $\gamma = 1$ [29]. Also, as was discussed in detail in Section 4 and Appendix A, we cannot derive a realistic ZHR from our simulation flux without the use of a realistic input mass distribution. Instead we show probability distributions of the simulations, as calculated by equation 1, and compare to the normalized ZHR:s. The different measures are quite disparate, but their respective normalized curves are still representative of the shower activity and thus interesting to compare with the simulated results.

Figures 5 and 6 show the 2011 October Draconid Dust trail contribution map for the simulated mean Earth encounter rate and Earth encounter probability, respectively. Both of these plots have ejection time (perihelion passage) on the vertical axis and detection time (Solar longitude) on the horizontal axis. Here, the mean encounter rates are calculated as described in Section 2.3 and it should be considered that this flux originates from a logarithmically uniform initial mass distribution. One interesting feature obvious in Figure 5 is a high encounter rate coming from the meteoroid stream formed during the year 1972. However, from Figure 6 it is at the same time clear that the probability of occurrence of this high encounter rate is relatively low compared to the probability of encounters from other trails. That the probability is low, but the encounter rate is high, indicates that for many of the simulated 1972 dust trails, particles did not encounter the Earth at all, but that in the few simulations where Earth encounters occurred, the number of particles is high. The peak of the elevated encounter rates from the 1972 trail is displaced in Solar longitude from the central peak generated by trails from between 1900 and 1920. This displacement corresponds to around 4 to 8 hours after the main October Draconid peak. It should therefore have been distin-

\[\text{http://www.imo.net/live/draconids2011/}\] (accessed September 11, 2016)
Figure 5: Dust trail contribution map for the simulated Earth encounter rate. The map is formed by taking the mean of all particle encounters during the 2011 October Draconids produced by the clones of 21P/Giacobini-Zinner drawn from a multidimensional Gaussian distribution representing its orbital uncertainty. Each row represents one perihelion passage.

Figure 6: Dust trail contribution map for the simulated Earth encounter probability. The map is formed by taking the fraction of all simulations producing encounters during the 2011 October Draconids. Each row represents one perihelion passage.
guishable from the flux of other dust trails in observational data. If a secondary peak was clearly observed, this could in principle be used to narrow down the initial distributions of the simulation as all initial conditions that did not produce flux could be excluded. If a secondary peak was not observed, however, then the initial distributions could still in principle be narrowed down, but in a conjugate way where the parts of the initial distribution that produced the flux in the simulation are potentially wrong. Such statements are only true if we are assured that the current scenario was insensitive to all other parameters not part of the initial distributions and that the model included all major effects. This can be done in the future by small sets of simulations with parameter variation and sensitivity analysis, but we have not implemented such methods yet.

Unfortunately, at the location of the Canadian Meteor Orbit Radar (CMOR), which provided a large collection area for Draconids during other parts of the activity profile, the radiant had descended below the horizon at the time. Thus, the time when the 1972 encounter rate increase may have appeared was outside their observation window. In MU radar data, there is one trailing observation which if converted to flux could be indicative of a second peak [43]. Also in the IMO visual data, there is a trailing single observation at 195.382° Solar longitude, which when converted to ZHR/flux, indicates an increased activity. However due to the low number of observations in the ZHR calculation of the IMO and MU, these detections are associated with large uncertainties. The combination of these two independent occurrences provides a somewhat strengthened indication that the increased activity after the main peak in the simulations of the 2011 October Draconids actually occurred, but unfortunately no firm conclusions can be drawn from the current data. Finally, it should also be noted that there are no uncertainties given for the 21P/Giacobini-Zinner orbit were too large, but it could also very well be because visual measurements were mostly not conducted during the Solar longitudes between 194.9° and 195.4°. The MU radar campaign lasted between 195.3° and 196.9°, so also in this case the discrepancy may very well be an observation bias: the fluxes are normalized and we have not calculated a commonly calibrated flux between the measurement sources and our simulation. Also, as before, Figures 8 and 9 contain the 2012 synthetic October

5.2. 2012 October Draconids

To complete the comparison between the 2011 and 2012 events we next present an examination of the 2012 October Draconids. As in the presentation of the 2011 October Draconids in Section 5.1, Figure 7 illustrates the calculated probability of stream encounter together with measurements from the MU radar [29] and the IMO [4]. Here, the simulations provide a somewhat wider probability distribution than the measurement profiles. Again, this could be because the assumed initial uncertainties in 21P/Giacobini-Zinner orbit were too large, but it could also very well be because visual measurements were mostly not conducted during the Solar longitudes between 194.9° and 195.4°. The MU radar campaign lasted between 195.3° and 196.9°, so also in this case the discrepancy may very well be an observation bias: the fluxes are normalized and we have not calculated a commonly calibrated flux between the measurement sources and our simulation. Also, as before, Figures 8 and 9 contain the 2012 synthetic October


Figure 8: Dust trail contribution map for the simulated Earth encounter rate. The map is formed by taking the mean of all particle encounters during the 2012 October Draconids produced by the clones of 21P/Giacobini-Zinner drawn from a distribution representing its orbital uncertainty. Each row represents one perihelion passage.

Figure 9: Dust trail contribution map for the simulated Earth encounter probability. The map is formed by taking the fraction of all simulations producing encounters during the 2012 October Draconids. Each row represents one perihelion passage.
Draconids trail composition. Even though we have not used a realistic input mass distribution, the combination of the probability distribution, which can be seen as the areas with potential flux in Figure 9, together with the mean flux distribution in Figure 8, gives strong indications that the majority of the flux observed by the IMO and the MU radar originates from the 1966 trail and not from the 1972 trail. Such a distinction can be made due to the separation in encounter time of the 1966 and 1972 trails in Figures 8 and 9. This is the same line of reasoning as in Section 5.1 with the possible 1972 trail encounter during the 2011 outburst. Again, a more proper definition of the initial distribution may change the result and the connection between initial distributions and the resulting synthetic meteor outbursts should be investigated further.

5.3. Mass difference

Figure 10: The fraction of normalized Earth encountered mass distributions between the simulated October Draconids of 2011 and 2012, i.e. the \( f(m) = \Phi(m) - 1 \) distribution described in equation [A.12].

The input mass distribution of all iterations is a logarithmically uniform function where an equal number of particles in each mass bin is ejected. This provides good opportunities to examine how large a fraction of each bin encounters the Earth, as well as to easily compare bins to each other. As previously mentioned and discussed further in Appendix A and Appendix B, this is also a valid input distribution on which to do year-to-year comparisons. Without using any calibration process or fitting of distributions to observations as done in [18], we can still infer information about the meteor shower and its mass content by investigating in a way independent of input distribution the ratio between the encountered mass distributions during different occurrences of the meteor shower.

To accomplish a mass transfer efficiency comparison, we first select all the simulated Earth encounters in 2011 and 2012, create mass distributions for both years and then normalize them. Finally, we calculate the fraction between the normalized mass distributions from both years and subtract one to centre the resulting values around zero for clarity, resulting in Figure 10. Step by step, we first calculate the normalized mass distribution, i.e. the relative amounts of different masses in one occurrence. Then we calculate the fraction between the two normalized occurrences, i.e. examining the differences in relative amounts between two occurrences. The final centering around zero is purely a visual guideline to help interpret the data.

The graph in Figure 10 indicates that there was a clear difference between the two years in relative mass propagation efficiency from 21P/Giacobini-Zinner to the Earth. In 2011, the visual magnitude meteoroids in the mass range between \( 10^{-6} \) kg to \( 10^{-4} \) kg are up to 30-40% more numerous than in 2012, when compared to the other rates for 2011. In 2012 radar-sized meteoroids in the mass range \( 10^{-9} \) kg to \( 10^{-6} \) kg dominated. This is in line with the observations and simulations presented by [28]. In addition, we also see a higher abundance of the 0.1 to 0.01 kg meteoroids in the 2012 outburst. But since the number of bright meteorites from such large meteoroids is quite low in observations, a slightly elevated encounter rate may have been difficult to detect due to low-number statistics.

Looking at not only the visual-to-radar-flux observation ratios, the calculated mean radar cross-section (RCS) magnitude of the October Draconids recorded by the MU radar in 2011 was -17.1 dBsm while the mean RCS magnitude in 2012 was -24.3 dBsm, where the unit of RCS is expressed in decibel relative to a square metre (dBsm) and 0 dBsm is equal 1 m². This indicated brighter Draconid head echos in 2011 than in 2012 among the radar detected sample [29]. For meteoroids of the same entry velocity (as those belonging to the same meteor shower), the head echo RCS is directly proportional to meteoroid mass [59]. This indicates that meteoroids with lower masses, i.e. fainter meteors, dominated the 2012 outburst, and is also consistent with the simulation results.

Similar simulations have been performed before to distinguish the same kind of differences. For example, the 2005 October Draconids outburst that initially came as a surprise [60] was later reproduced in simulations by modifying the mass sampling range [47].
5.4. Summary and prediction

In previous sections we have focused on simulated results of the October Draconids for four specific outbursts/years, namely 1933, 1946, 2011 and 2012. To give a broader view of the available data, Figure 11 shows the mean yearly flux distribution together with markers indicating which years contained an activity increase of more than two times the mean flux for the entire simulation. The 1946 outburst is by far the strongest in the simulations. This plot shows that the simulations also cover several additional peaks in the previous activity of the October Draconids, such as the 2005 outburst observed by CMOR [60], as well as a prediction of an outburst in the year 2018. From examining the Solar longitudes of the 2018 October Draconids simulation, we predict that the shower will peak in intensity on October 8-9, corresponding to 195.4°. Without the use of a realistic input mass distribution, however, e.g. as outlined in Appendix A, ZHR rates for the expected visual and radar measurements cannot be calculated from this data.

6. Conclusion

We have presented the current status and first results of a Monte Carlo-type simulation toolbox for Solar System small body dynamics. We have also presented fundamental methods for evaluating the results from this type of simulation. We successfully used the toolbox to investigate mass propagation in the October Draconids from comet 21P/Giacobini-Zinner to Earth. Simulations included perturbations from all major planets, radiation pressure and the Poynting-Robertson effect. Initial validation of the software was performed by simulating the 1933, 1946, 2011 and 2012 October Draconids. Material was ejected from comet 21P/Giacobini-Zinner during seventeen perihelion passages between 1866 and 1972 and propagated forward in time. Each perihelion passage was sampled with 50 clones that produced meteoroid streams. The clones were sampled from a multidimensional Gaussian distribution on the orbital elements proportional to given uncertainties. These orbital clones were then given characteristic physical values from measurements of 21P/Giacobini-Zinner found in literature. Each clone ejected 8,000 particles. We reproduced results of previous simulations and observations of the 1933, 1946, 2011 and 2012 October Draconids, including the unexpected and measured deviation of the meteor mass index from a power law in the 2012 October Draconids, a feature not present in the 2011 October Draconids. The convergence analysis outlined in Appendix B shows that we reached moderate convergence sufficient to present the simulation results. However, it would be desirable to extend the simulations and reach higher convergence by scaling up the number of Monte Carlo iterations by about a factor of at least 4². In addition to reproducing previous observations we also predict an October Draconids outburst in 2018 with an activity peak on October 8-9 that could be up to twice as large as the 2011 and 2012 outbursts.
Appendix A. Convergence and distributions

This section covers the general idea behind the application of convergence in interpretation of the results and the theoretical treatment of independent measures. Here, we show that once convergence is sufficient, yearly mass distribution comparisons are independent of the assumed input mass distribution. This conclusion has been derived using a simple examination of the simulation methodology so it could be applied to the other input/output distributions and measures as well.

One of the most important aspects of a Monte Carlo type simulation is that one can choose how much importance to assign to different areas in the output space and confirm sufficient convergence in selected areas only. The reason for doing so is primarily to confirm that results are not statistical artifacts due to insufficient sampling, but it also provides significant computational benefits, as we shall show here.

Assume we have a space representing the input variables and a distribution on this space. This input distribution is a probability distribution describing the probability that the input variable will have a certain value. Let us also consider a space of output variables and an output distribution on this space. The output distribution is a mean over the simulations or a fraction of simulations giving a certain output configuration. For example, let us use as an input variable the semi-major axis of the comet and assume that the distribution on this axis is Gaussian. Then in each iteration the value for the comet semi-major axis will be drawn from this distribution. An output variable of interest may be the number of particles that encounter the Earth during the year 2011 in the mass interval 10^{-8} kg to 10^{-7} kg, i.e. the space of positive natural numbers. Then the output distribution on this variable will be the fraction of simulations that produced a number of encountered particles. The output distribution could also be the mean over all simulations describing the number of encountered particles as a function of mass. Which of these two or other alternatives is a question of definition and application.

Our software can be seen as a “black box” function T that takes the input variable space and its associated distributions and generates an output distribution on the associated output variable space. This function does not need to be well-behaved in any way, it need not be continuous, one to one, or possess any other regularly assumed property of functions. Ideally, we would like to have an analytical version of T so that, given any input, we would know the output. Since this is not possible, we instead have to sample the input variable space in a Monte Carlo fashion according to the input distributions, propagate every point sampled through a numeric version of T (our software toolbox) and examine how the output distribution is affected.

However the space of output variables is vast and, depending on the time scales, so is the size of the output distribution too. Thus we need to make some compromise to take into account limited computing power as it would be impractical to examine the entity of the possible output variables. We do this by picking a smaller subset Q of our output variable space and examine convergence solely in this space. When sufficient convergence is reached, Monte Carlo sampling can be stopped. In the current case study simulation, we picked as a subset the logarithmic mass distribution between 10^{-8} kg and 10^{-1} kg of all particles that entered the Earth’s Hill sphere. Let us call the Earth encountered mass distribution on this subset µ(m) ∀ m ∈ Q. This distribution is defined as the mean of every Monte Carlo iteration output encountered mass distribution. If we call the individual simulation distributions µ(m) we can define µ(m) as the mean over all n simulations µ(m) = n^{-1} \sum_{i=1}^{n} µ(m). We also found that several other subsets have sufficient convergence under this criterion, such as the mean yearly encounter rates for the Earth. As long as analysis is not performed outside of the subsets that have converged, the results are stable and representative. One also has to consider the method for analysis inside the regions of convergence as transformations of these regions can affect statistical results. For example, in the case of using a logarithmic mass space for sampling and convergence control, it would not be sensible to perform histogram type analyses on the linear mass space. The probability of sampling and convergence rates are skewed in the linear space and thus the resulting histogram can be misleading. By probability of sampling we refer to how frequently each region is sampled in a random sampling scheme. One of the fundamental properties of Monte Carlo-sampling is that the sampling probability is equal to that of the scenario probability, which is different from grid methods [61]. Using a Monte Carlo-sampling one will get a sampling resolution proportional to the likelihood of the scenarios, but with grid methods the sampling reso-
olution is uniform and static over the scenarios. Nevertheless, when dealing with continuous distributions, we can never have complete sampling as this would require an uncountably infinite number of samples.

Even though we stated that $T$ does not need to be a well-behaved function, we can deduce if it is well-behaved in certain subsets. The interesting subsets to examine would be the input and output mass distributions. Let us denote the subset of initial mass variables as $P$ where the initial mass distribution is $\eta(m) \forall m \in P$. We are interested in the transformation of $\eta \mapsto \mu$ through $T$. First, we identify that after drawing a particle from $P$ according to $\eta$, this mapping from $\eta$ to $\mu$ occurs solely through celestial dynamics. The reason for this is, first of all, that the resulting mass distribution at the Earth is calculated through the positions of the particles in space, i.e. they become part of the distribution if they hit the Earth. Furthermore, since we do not consider the particles interacting with each other, adding more particles of any one kind will not influence any other particle’s end state of hitting or not hitting the Earth.

The main reasoning is that: if sufficient convergence has been reached we can assume that any additional particles will travel along already sufficiently sampled paths and not generate new information. To clarify, instead of adding another particle, we can simply take an existing particle that is representative of the new one and change the existing weight $w$ to $w + 1$ instead. This means that adjusting the weights controls how many “additional virtual particles” we have considered of a certain kind. Modifying weights to alter entire initial distributions must be done when information is close to complete, i.e. sufficient convergence and sampling. This is because we can only change already acquired information through weights.

When considering the specific problem of parent bodies and ejected particles this becomes somewhat more complicated. Consider picking two identical initial conditions for the generation of a meteoroid stream and then simulating these two streams. This can be seen as the same as doing one simulation of the meteoroid stream with twice as many particles being ejected during the simulation. This is true thanks to particle ejections being stochastic by nature and only as long as the resulting frequency is in relation to the ejection frequency. As such, convergence in the output distribution of many simulations is directly connected to sufficient particle ejection numbers in the individual simulations as well as the sampling of the simulation initial conditions. If we have sufficient sampling of the initial conditions of the simulations as well as sufficient particle ejection numbers, additional particles or simulations will not generate new information. This explanation, here derived for an ensemble of simulations, is analogue to the explanation for a single simulation in [18] that each test particle can be assigned a statistical weight which can be changed after the simulation has been completed.

One should, however, always modify weights with great care. If we do not have sufficient convergence or sampling, we will with the modification of weights certainly alter the initial distribution, but we will not necessarily do so consistently throughout the entire initial distributions as we can only affect the actual values that were sampled. For example, we can adversely affect other stochastic processes, such as the ejection direction and timing. Or if we are in a multidimensional simulation scenario, changing of weights may even affect the initial distribution of the comet orbital elements. Of course, if it is possible to conclude that some initial distributions do not matter, ensuring sufficient sampling of these conditions can be disregarded. However the only way to distinguish the importance is through repeated simulations, followed by sensitivity and stability analysis, and the application of convergence control. Here we generalize the discussion by an approximation based on sufficient convergence and to distributions of simulations. We have not implemented any processes, such as particle-particle collisions, that can change the mass of the particles during the propagation. This means that each distribution value in $\eta$ that is associated to a mass $m \in P$ will only affect the corresponding distribution value $\mu$ that is associated to the mass $m \in Q$. The magnitude of this transition from $\eta$ to $\mu$ depends on the rest of the simulation. This implies that we can equate $P$ and $Q$ and view $T$ in this subset as a function defined as

$$\mu(m) = T(\eta(m)) = \eta(m)\alpha(m,T) \forall m \in Q. \quad (A.1)$$

Here $\alpha$ is a transfer coefficient reflecting all the dynamics of the simulated scenario. It depends on the method of simulation $T$, the mass we are examining $m$, and the initial distributions for all the other input variables except for the initial mass distribution. We can also restrict $\alpha$ to account solely for some subset of the output variables, such as examining only the encountered mass distribution in a specific year. The approximation sign is an indication that this is only an equal relation if we have infinite samples. If and only if we have sufficient convergence can we say that equation (A.1) is a valid approximation due to the previous discussion of statistical weights. We discuss measuring convergence in Appendix B.
The above implies that if we can use equation [A.1] then given any known initial mass distribution $\eta$ and its known output distribution $T(\eta) = \mu$, we can find the result $\mu'$ of another arbitrary input mass distribution $\eta'$ without computation of $T(\eta')$. Instead of computation of $T$ we may use a relative fraction

$$\frac{\eta'(m)}{\eta(m)} = \kappa(m), \quad (A.2)$$

and through equation [A.1] we can derive the output distribution $\mu'$ as

$$\mu'(m) = T(\eta'(m)) = \eta'(m)\alpha(m, T) = \kappa(m)\eta(m)\alpha(m, T) = \kappa(m)\mu(m). \quad (A.3)$$

As seen above, the distribution $\kappa(m)$ is analogous to the assigning of statistical weights. This is the main reason why we have chosen to use a logarithmic mass distribution. Our intention was to sample the logarithmic mass space in an efficient manner within a finite range, at a desirable resolution, and we can afterward find any other resulting distribution in this domain without performing another simulation. Even though it would be worthwhile, it is very difficult to find a realistic $\eta'$ since this depends on several factors: the particle abundance in the ices of the comet, the particle size and mass distributions, the cometary material compositions, and so on. In this study we therefore only compare mass distributions in two distinct years. The conclusion here shows that the comparison between two mass distributions are valid even without a realistic initial distribution.

From now on we will omit the mass dependence notation on the distributions. Let us consider two encountered mass distributions $\mu_1$ and $\mu_2$ during two different years. Coming back to the previous notation in equation [A.1] the coefficient $\alpha(m, T)$ would only include the mass transfer to the year for $\mu_1$. Both of these encountered mass distributions came from the same initial mass distribution $\eta$. Our goal is to examine the relative difference between the mass distributions. Thus we can start by looking at the mass distribution fraction $\mu_1/\mu_2$,

$$\Phi = \frac{\mu_1}{\mu_2}, \quad (A.4)$$

Following the same logic as in equation [A.3] let us assume that we know a realistic mass distribution $\eta'$ and thus we know that the corresponding realistic encountered distributions can be found as

$$\eta' = \kappa, \quad (A.5)$$

$$\mu'_1 = \kappa \mu_1, \quad (A.6)$$

$$\mu'_2 = \kappa \mu_2. \quad (A.7)$$

Let us now examine how this affects the year-to-year comparison for the realistic mass distribution fraction $\Phi'$,

$$\Phi' = \frac{\mu'_1}{\mu'_2} = \frac{\kappa \mu_1}{\kappa \mu_2} = \frac{\mu_1}{\mu_2} = \Phi. \quad (A.8)$$

We may conclude that the measure of the mass distribution fraction $\Phi$ is independent of the initial distribution at sufficient convergence. The goal, however, was to examine the internal relative difference between two meteor showers and this can be done by a fraction of their normalized distributions as

$$\hat{\Phi} = \frac{\mu_1}{\mu_2} = \frac{\int \mu_1}{\int \mu_2}, \quad (A.9)$$

that relates to a realistic input distribution as

$$\frac{\mu'_1}{\mu'_2} = \hat{\Phi}' = \frac{\kappa \mu_1}{\kappa \mu_2} = \frac{\mu_1}{\mu_2} = \hat{\Phi}, \quad (A.10)$$

where $c$ is a constant defined by

$$c = \frac{\int \kappa \mu_2}{\int \kappa \mu_1}. \quad (A.11)$$

The measure $\hat{\Phi}$ is therefore affected by the choice of input distribution but only by a scalar preserving its general structure. The distribution $\hat{\Phi}$ is centred around one as a higher abundance in $\mu_1$ is represented by a fraction value greater than one, while a higher abundance in $\mu_2$ results in a fraction value less than one. To clearly illustrate the difference in encountered mass distributions, we plot the distribution

$$f(m) = \hat{\Phi}(m) - 1, \quad (A.12)$$

as it is centred around zero while preserving the shape of the distribution that is independent of the initial mass distribution and thus representative of a realistic case, although scaled.
As a final remark it should be noted that it is possible to miss important features of a simulated meteor shower by choosing too small a space \( Q \), sampling the initial conditions too sparsely, or simply by not using strict-enough convergence controls.

### Appendix B. Convergence criteria

The convergence of a simulation is a measure of its quality. Implementing a convergence measure is relatively straightforward in Monte Carlo simulations. First, we start by defining the standard statistical ensemble and define the sample variance as

\[
E = \langle R \rangle, \quad \text{(B.1)}
\]

and the ensemble variance,

\[
\text{var}(R) = \sigma^2 = \langle (R - E)^2 \rangle. \quad \text{(B.2)}
\]

If we denote a sample of \( R \) as \( R_n \), define the sample mean as

\[
\bar{R}_n = \frac{1}{n} \sum_{i=1}^{n} R_i, \quad \text{(B.3)}
\]

and define the sample variance as

\[
S_n^2 = \frac{1}{n-1} \sum_{i=1}^{n} (R_i - \bar{R}_n)^2, \quad \text{(B.4)}
\]

we end up with the exact measures of the moments of the random variable \( R \) present in the system. However the sample mean and variance do not coincide with the ensemble mean and variance. The sample moments will oscillate around the ensemble moments, but there is no guarantee that they will coincide unless we reach an infinite number of samples, presenting a problem in practical applications. It is therefore common practice to use the central limit theorem and interpret the sample moments as random variables as well, and thereafter relate them to the ensemble moments as

\[
\langle \bar{R}_n \rangle = E, \quad \text{(B.5)}
\]

\[
\text{var}(\bar{R}_n) = \frac{\sigma^2}{n}. \quad \text{(B.6)}
\]

This means that \( \bar{R}_n \) can be an estimator that is normally distributed around \( E \) with variance \( \sigma^2/n \). Note that the root mean square of the estimator scales with \( \frac{1}{\sqrt{n}} \), which indicates that to decrease the root mean square by a factor of two we need to increase \( n \) by a factor of four. This can be used to predict the number of needed iterations left during simulations. \( \text{var}(\bar{R}_n) \) is an indication of the uncertainty between of the estimator \( \bar{R}_n \) and the ensemble counterpart \( E \).

In Appendix A we referred to the output mass distribution \( \mu(m) \) and defined it as \( \mu(m) = \frac{1}{\sum_{i=1}^{n} \mu(m)} \) where \( \mu(m) \) are the individual simulation results. Since \( \mu(m) \) will be discrete in practical applications, it is logical to define \( M_i \) as a mass interval \([m_i - \Delta m, m_i + \Delta m]\) and then use \( \mu(M_i) \) as the sample mean, i.e.

\[
\bar{R}_n(M_i) = \mu(M_i), \quad \text{(B.7)}
\]

where we have used the superscript \( n \) to denote how many simulations we include in the calculation of \( \mu \).

Now all the tools are in place to define the convergence of the simulation. As we have a low number of iterations, we cannot use Chebyshev’s inequality to calculate confidence intervals. This would overestimate the intervals too much. We should instead, in this case, give an approximation of the estimator standard deviation or variance given in equation [B.6]. This can be done in a multitude of ways. Our measure of interest is the function in equation [A.12]. Statistics based on this function are not subject to the central limit theorem, as it is not an ensemble moment, but a fraction of two distribution functions, and thus equation [B.6] is not valid for calculating the estimator variance. However another way to estimate the \( n f(m) \) estimator variance \( \text{var}(n f(m)) \) is to use the jackknife method [62]. Here we have defined \( n f(m) \) exactly as in equation [A.12] but with the \( n \) dependence on \( \mu_1(m) \) and \( \mu_2(m) \). This method is based on re-sampling of data to find the variance and bias of an estimator. To find the jackknife variance estimation we first find the mean of each year by excluding data point \( j \) as

\[
\bar{\mu}_j = \frac{1}{n} \sum_{k=1, k \neq j}^{n} \mu(M_i). \quad \text{(B.8)}
\]

We can then define the \( j \) excluded estimator \( n f \) as

\[
\bar{R}_n(M_i) = \bar{\mu}_j = \frac{\mu_1}{n} \int \bar{\mu}_j \frac{\mu_2}{n} - 1. \quad \text{(B.9)}
\]
Then we estimate the estimator variance by

$$\text{var}(\bar{R}_u(M)) \approx \frac{n - 1}{n} \sum_{i=1}^{n} \left( \bar{R}_u(M_i) - \bar{R}_u(M) \right)^2,$$

(B.10)

where the complete estimator (no excluded samples) is now defined as

$$\bar{R}_u(M) = \frac{n \mu_1(M)}{n \mu_2(M)} \int \frac{n \mu_1(M)}{n \mu_2(M)} = 1.$$

(B.11)

Figure B.12 shows the result of using the jackknife method to estimate the variance of the mass comparison described in Section 5.3. As the uncertainties of the individual years 2011 and 2012 accumulate when the variance of $f(m)$ of equation (A.12) is estimated, the result in Figure B.12 appears quite uncertain. But the most important features are still visible within the uncertainties. It should be noted that this case study was performed with the aim of developing methods, demonstrating underlying concepts, and showcasing the software toolbox. In the future, when we have more processing power available, it will be desirable to use at least $4^2$ times the current number of iterations, i.e. to perform $12 \cdot 50 = 600$ additional iterations. This will decrease the yearly estimator variances to approximately a quarter of their current values according to the iteration dependence of the estimator variance in equation (B.6). This would take around three months of computation with the currently available resources and could not be completed within the time frame of the current work.

**References**