

The codegree threshold of K_4^-

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Abstract

The codegree threshold $\text{ex}_2(n, F)$ of a 3-graph F is the minimum $d = d(n)$ such that every 3-graph on n vertices in which every pair of vertices is contained in at least $d + 1$ edges contains a copy of F as a subgraph. We study $\text{ex}_2(n, F)$ when $F = K_4^-$, the 3-graph on 4 vertices with 3 edges. Using flag algebra techniques, we prove that if n is sufficiently large, then

$$\text{ex}_2(n, K_4^-) \leq \frac{n+1}{4}.$$

This settles in the affirmative a conjecture of Nagle [Congressus Numerantium, 1999, pp. 119–128]. In addition, we obtain a stability result: for every near-extremal configuration G , there is a quasirandom tournament T on the same vertex set such that G is $o(n^3)$ -close in the edit distance to the 3-graph $C(T)$ whose edges are the cyclically oriented triangles from T . For infinitely many values of n , we are further able to determine $\text{ex}_2(n, K_4^-)$ exactly and to show that tournament-based constructions $C(T)$ are extremal for those values of n .

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1 | INTRODUCTION

Interest in the extremal theory of hypergraphs dates back to Turán's celebrated 1941 paper [36]. However, despite significant efforts from the research community, the problem of determining the Turán density of a given hypergraph F is open in all but a small number of cases — see, for example, Keevash's survey of the field [18]. The difficulty of the problem has led researchers to investigate a number of other notions of extremal density, notably the codegree density, which is studied in this paper. Before we describe our results in detail, we introduce some basic notation.

1.1 | Notation

Given a set A , we write $A^{(r)}$ for the collection of all unordered r -tuples from A . Write $[n]$ for the set $\{1, 2, \dots, n\}$. A *3-graph* or *triple system* is a pair $G = (V, E)$, where $V = V(G)$ is a set of vertices and $E = E(G) \subseteq V^{(3)}$ is a collection of unordered triples, which constitute the edges of G . We set $v(G) := |V(G)|$ and $|G| := |E(G)|$. For notational convenience, we shall often identify a 3-graph with its edge-set, and write $x_1 x_2 \dots x_r$ to denote the set $\{x_1, x_2, \dots, x_r\}$. Given two 3-graphs G and G' on a common vertex set V , their *edit distance* $|G \Delta G'|$ is the size of the symmetric difference of their edge-sets. The *link graph* of a vertex x in a 3-graph G is

$$G_x := (V(G) \setminus \{x\}, \{yz : xyz \in E(G)\}),$$

and the *joint neighbourhood* of a pair $\{x, y\}$ is

$$G_{xy} := \{z : xyz \in E(G)\}.$$

A *subgraph* of G is a 3-graph H with $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. If G does not contain a copy of F as a subgraph, we say that G is *F-free*. The *Turán number* $\text{ex}(n, F)$ of a non-empty 3-graph F is the maximum number of edges in an F -free 3-graph on n vertices, and its *Turán density* is the limit $\pi(F) := \lim_{n \rightarrow \infty} \text{ex}(n, F) / \binom{n}{3}$ (this is easily shown to exist). In this paper we shall be interested in variants of the Turán number and the Turán density.

The *codegree* of a pair $xy \in V(G)^{(2)}$ is $d(x, y) := |G_{xy}|$, the number of edges of G containing the pair xy . The *minimum codegree* of G , which we denote by $\delta_2(G)$, is the minimum of $d(x, y)$ over all pairs $xy \in V(G)^{(2)}$. The *codegree threshold* $\text{ex}_2(n, F)$ of a non-empty 3-graph F is the maximum of $\delta_2(G)$ over all F -free 3-graphs on n vertices. A probabilistic averaging argument [24] yields that the limit

$$\pi_2(F) := \lim_{n \rightarrow \infty} \frac{\text{ex}_2(n, F)}{n - 2}$$

exists; this quantity is called the *codegree density* of F . Another straightforward averaging argument shows that $0 \leq \pi_2(F) \leq \pi(F) \leq 1$, and it is known that $\pi_2(F) \neq \pi(F)$ in general (see also Section 1.3).

In addition to 3-graphs, we shall also need to consider tournaments in this paper. An *oriented graph* is an ordinary graph together with an orientation of its edges. A *tournament* is an orientation of a complete graph. In an oriented graph O , we denote by $N_O^-(x)$ and $N_O^+(x)$ the *in-neighbourhood* and *out-neighbourhood* of a vertex x , respectively, that is, the collection of y such that the edge $\{x, y\}$ is oriented into x (as \vec{yx}) and out of x (as \vec{xy}), respectively. We write $d_O^-(x) := |N_O^-(x)|$ and

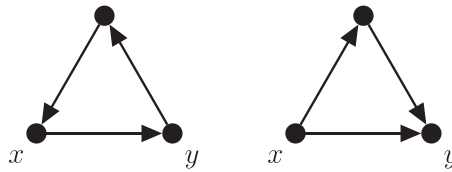


FIGURE 1 Configurations in a tournament T on the left and right counted by $C_T(x, y)$ and $R_T(x, y)$, respectively.

$d_O^+(x) := |N_O^+(x)|$ for the *in-degree* and *out-degree* of x . Further, we write $d_O^-(x, Y)$ and $d_O^+(x, Y)$ for the number of in- and out-edges of x , respectively, with the other endpoint being inside the set Y . Note that when the oriented graph O is clear from the context, we omit the subscript in the in- and out-degree notation. For a tournament T and an arc $\vec{xy} \in E(T)$, we write $C_T(x, y)$ to denote the number of cyclically oriented triangles in T that contains both x and y , that is, the number of vertices $z \in V(T)$ such that both \vec{yz} and \vec{zx} are arcs in T . Similarly, we let

$$R_T(x, y) := |\{z \in V(T) : \{\vec{xz}, \vec{zy}\} \in E(T)\}|.$$

See Figure 1 for an illustration of the configurations counted by these two quantities. Note that $R_T(x, y)$ is equal to the number of cyclically oriented triangles containing both of x and y in the tournament T' obtained from T by reversing the orientation of \vec{xy} .

Motivated by the relation between tournaments and K_4^- -free 3-graphs described in Construction 1.2 below, we define a notion of *codegree* for tournaments: given a tournament T , we define its minimum codegree to be

$$\delta_2(T) := \min_{xy \in V(T)^{(2)}} C_T(x, y).$$

We shall use standard Landau notation throughout this paper: for functions $f, g : \mathbb{N} \rightarrow \mathbb{R}_{\geq 0}$ we write $f = o(g)$ if $f(n)/g(n) \rightarrow 0$ as $n \rightarrow \infty$, $f = O(g)$ if $\limsup_{n \rightarrow \infty} |f(n)/g(n)|$ is finite, and $f = \Omega(g)$ if $g = O(f)$. Finally, we write $f = \Theta(g)$ if $f = \Omega(g)$ and $f = O(g)$ both hold.

1.2 | Results

In this paper we study the codegree threshold of $K_4^- := ([4], \{123, 124, 134\})$. This is the unique (up to isomorphism) 3-graph on 4 vertices with 3 edges, or, alternatively, this is the complete 3-graph on 4 vertices with one edge removed. From the perspective of Turán-type problems, the 3-graph K_4^- is the smallest non-trivial 3-graph. Moreover K_4^- -free 3-graphs have a nice interpretation in terms of their link graphs: a 3-graph is K_4^- -free if and only if its link graphs are triangle-free. Determining the codegree threshold for K_4^- can thus be viewed as a 3-graph generalisation of the minimum degree version of Mantel's theorem. In 1999, Nagle [25] made the following conjecture (see also [5]).

Conjecture 1.1 (Nagle). $\pi_2(K_4^-) = 1/4$.

The lower bound in Nagle's conjecture comes from an old construction due to Erdős and Hajnal [7].

Construction 1.2 (Erdős–Hajnal tournament construction). Given a tournament T on the vertex set V , define a 3-graph $C(T)$ on the same vertex set by setting $E(C(T))$ to consist of the elements of $V^{(3)}$ that induce a cyclically oriented triangle in T .

It can be easily checked that no 4-vertex tournament contains more than two cyclically oriented triangles, whence $C(T)$ is a K_4^- -free 3-graph. Also note that $\delta_2(T) = \delta_2(C(T))$. Moreover, if the tournament T is chosen uniformly at random, then $\delta_2(T) = n/4 + o(n)$ with high probability.

In this paper we settle Nagle's conjecture in the affirmative.

Theorem 1.3 (Codegree density). $\pi_2(K_4^-) = 1/4$.

Our proof of Theorem 1.3 relies on flag algebra techniques: using the semi-definite method of Razborov [27], we establish an asymptotic identity (Lemma 2.8) between densities of subgraphs on at most seven vertices in K_4^- -free 3-graphs, from which we deduce Nagle's conjecture. Further, by analysing this identity, we are able to show that all near-extremal configurations look like the random tournament construction described above. To make this more precise, let us recall one of the many equivalent definitions of a *quasirandom* tournament (for other forms, see Chung and Graham [3]).

Definition 1.4. A tournament T on $[n]$ is δ -*quasirandom* if for every pair of sets $X, Y \subseteq [n]$ we have

$$\sum_{x \in X} |d^+(x, Y) - d^-(x, Y)| \leq \delta n^2.$$

Theorem 1.5 (Stability). Let G be a K_4^- -free 3-graph on $[n]$ with $\delta_2(G) \geq n/4 - o(n)$. Then there exists a $o(1)$ -quasirandom tournament T on $[n]$ such that $|G \Delta C(T)| = o(n^3)$.

Finally, we are able to show that tournament-based constructions are extremal for large n and to calculate the exact value of the codegree threshold for infinitely many n . A *skew Hadamard matrix* of order n is an $n \times n$ square matrix A with ± 1 entries such that (i) $AA^t = nI_n$ and (ii) $A + A^t = 2I_n$. Here I_n denotes the $n \times n$ identity matrix and A^t denotes the transpose of the matrix A .

Theorem 1.6 (Codegree threshold). For all n sufficiently large,

$$\text{ex}_2(n, K_4^-) \leq \left\lfloor \frac{n+1}{4} \right\rfloor.$$

Further, for all k sufficiently large if there exists a skew Hadamard matrix of order $4k+4$, then for $n = 4k+3$ and $n = 4k+2$ we have equality in the equation above. Moreover, for all k sufficiently large if $n = 4k+3$ and $\text{ex}_2(n, K_4^-) = \lfloor \frac{n+1}{4} \rfloor$, then every extremal 3-graph for that value of n is given by an Erdős–Hajnal tournament construction and there exists a skew Hadamard matrix of order $n+1 = 4k+4$.

In [38], Seberry (née Wallis) conjectured a strengthening of Hadamard's conjecture, stating that there exists a skew Hadamard matrix for any order n divisible by 4. It is well known that every (not necessarily skew) Hadamard matrix must have order 1, 2 or a multiple of 4. Seberry's conjecture is known to hold for $n < 276$, see [39], and for all values of n that are of the form $n = 2^t \prod_{i \in I} (q_i + 1)$,

where $t \in \mathbb{Z}_{\geq 0}$, $I \neq \emptyset$ and q_i is a prime power congruent to 3 modulo 4 for every $i \in I$, see [34, Theorem 4.1].

Corollary 1.7. *If Seberry's conjecture is true, then for all n sufficiently large*

$$\text{ex}_2(n, K_4^-) = \begin{cases} \left\lfloor \frac{n+1}{4} \right\rfloor & \text{if } n \equiv 2, 3 \pmod{4}, \\ \left\lfloor \frac{n+1}{4} \right\rfloor \text{ or } \left\lfloor \frac{n-3}{4} \right\rfloor & \text{if } n \equiv 0, 1 \pmod{4}. \end{cases}$$

1.3 | Background

In the late 1990s, Nagle [25] and Czygrinow and Nagle [5] made conjectures on the values of the codegree densities $\pi_2(K_4^-)$ and $\pi_2(K_4)$, respectively, where K_4 denotes the complete 3-graph with 4 vertices. Mubayi [23] determined the codegree density of the Fano plane, and Keevash and Zhao [19] later extended Mubayi's work to other projective geometries. The precise codegree threshold of the Fano plane was determined for large enough n by Keevash [17] using hypergraph regularity, and DeBiasio and Jiang [6] later gave a second, regularity-free proof of the same result. Mubayi and Zhao [24] established a number of theoretical properties of codegree density, while Falgas-Ravry [9] gave evidence that codegree density problems for complete 3-graphs may not be stable in general. Falgas-Ravry, Marchant, Pikhurko and Vaughan [11] for their part determined the codegree threshold of the 3-graph $F_{3,2} = \{abc, abd, abe, cde\}$ for all n sufficiently large. Finally, Lo and Zhao [20] determined the asymptotic order of $1 - \pi_2(K_t^{(3)})$ as $t \rightarrow \infty$, where $K_t^{(3)}$ denotes the complete 3-graph on t vertices.

In this paper, we add a new example to this scant list of known non-trivial codegree densities by showing $\pi_2(K_4^-) = 1/4$. As the smallest non-trivial 3-graph from the perspective of Turán-type problems, K_4^- has received extensive attention from researchers in the area. Its Turán density is not known, but is conjectured by Mubayi [22] to be $2/7 = 0.2857 \dots$, with the lower bound coming from a recursive construction of Frankl and Füredi [13]. Matthias [21] and Mubayi [22] proved upper bounds on $\pi(K_4^-)$, before the advent of Razborov's flag algebra framework [26], and in particular his semi-definite method, led to computer-aided improvements by Razborov [27] and Baber and Talbot [1], with the current best upper bound for $\pi(K_4^-)$ being $0.2868 \dots$, see [12].

In addition, 'smooth' variants of the Turán density problem for K_4^- have been studied. Given $\delta > 0$, the δ -linear density of a 3-graph G is the minimum edge density attained by an induced subgraph of G on at least $\delta v(G)$ vertices. Erdős and Sós [8] asked whether there is $\delta > 0$ such that every large enough 3-graph with positive δ -linear density contains a copy of K_4^- . Füredi observed, however, that the tournament construction $C(T)$ of Erdős and Hajnal [7] described in the previous section with T chosen at random gives a negative answer to this question: a density of more than $1/4$ is required for the existence of a K_4^- -subgraph. Glebov, Král' and Volec [15] showed this $1/4$ lower bound is tight, using flag algebraic techniques amongst other ingredients in their proof. More recently, Reiher, Rödl and Schacht [31] reproved Glebov, Král' and Volec's result and established the edge density at which weakly quasirandom 3-graphs must contain a copy of K_4^- , for various notions of 'weakly quasirandom'. The extremal problem for K_4^- under both a codegree and a smoothness assumption had been studied earlier by Kohayakawa, Rödl and Szemerédi (see [25, 31]).

1.4 | Organisation of the paper

In Section 2 we introduce our setting of the flag algebra framework, establish the key flag algebraic identity (Lemma 2.8) and prove Theorem 1.3. In Section 3, we extract some information about near-extremal configurations from the flag algebraic identity in order to show that their structure must be close to that of a random tournament construction (Theorem 1.5).

After studying the relation between tournaments and skew Hadamard matrices in Section 4, we devote Section 5 to determining the codegree threshold of K_4^- and prove Theorem 1.6. We conclude the paper with some remarks and open problems presented in Section 6.

2 | FLAG ALGEBRAS AND THE CODEGREE DENSITY OF K_4^-

Our proof of Theorem 1.3 uses the flag algebra framework introduced by Razborov in [26], and in particular the semi-definite method first deployed by Razborov in [27]; see also [1, 11, 12, 14] for expositions of the basic ideas. Such an approach is by now well established in extremal hypergraph theory, and since a treatment of the general theory of flag algebras is outside the scope of this article, we content ourselves here with giving brisk definitions of some of the standard terms and concepts of flag algebras that we shall use, and refer an interested reader to the papers cited above for further details and discussion.

Let \mathcal{F} denote the set of all non-isomorphic finite K_4^- -free 3-graphs, and let \mathcal{F}_k denote the subset of \mathcal{F} consisting of all non-isomorphic k -vertex K_4^- -free 3-graphs. A *type* σ is an element of \mathcal{F} together with a labelling of its vertices, that is, a bijection from $[v(\sigma)]$ to $V(\sigma)$. For a fixed type σ , we define the set \mathcal{F}^σ to be the collection of all (up to σ -preserving isomorphism) finite K_4^- -free 3-graphs with a fixed embedding of σ . The elements of \mathcal{F}^σ are referred to as σ -flags. A σ -flag can be thought of as a 3-graph on a partially labelled vertex set, with the labelled vertices inducing a copy of σ . Note that any 3-graph can be viewed as an \emptyset -flag, where \emptyset is the empty type on 0 vertices. Analogously to the unlabelled case, we let \mathcal{F}_k^σ denote the set of all k -vertex σ -flags. For a σ -flag F , we define its *root* to be the fixed embedding of σ in F .

Fix now an ℓ -vertex type σ . Given two σ -flags F and G , we let $p(F, G)$ to be the probability that a random extension of the root of G by $v(F) - \ell$ unlabelled vertices of G yields a σ -flag isomorphic to F . In the degenerate case $v(F) > v(G)$, we define $p(F, G) = 0$. We refer to $p(F, G)$ as the σ -flag density of F in G . Further, given three σ -flags F_1, F_2 and G such that $v(G) = v(F_1) + v(F_2) - \ell$, we let $p(F_1, F_2, G)$ denote the probability that a randomly chosen set of $v(F_1) - \ell$ unlabelled vertices of G extends the root of G to a σ -flag isomorphic to F_1 , while the remaining $v(F_2) - \ell$ unlabelled vertices extend the root of G to a σ -flag isomorphic to F_2 .

We are now ready to describe the *flag algebra* \mathcal{A}^σ . Informally, \mathcal{A}^σ is obtained by taking $\mathbb{R}\mathcal{F}^\sigma$, that is, the vector space of all formal finite linear combinations of elements of \mathcal{F}^σ , and for every $F \in \mathcal{F}^\sigma$ and $k \geq v(F)$ quotienting out the relation

$$\sum_{G \in \mathcal{F}_k^\sigma} p(F, G) \cdot G = F.$$

We then define a multiplication in \mathcal{F}^σ by setting

$$F_1 \times F_2 := \sum_{G \in \mathcal{F}_{v(F_1)+v(F_2)-\ell}^\sigma} p(F_1, F_2, G) \cdot G \quad \text{for every } F_1, F_2 \in \mathcal{F}^\sigma,$$

which then uniquely extend to the whole set \mathcal{A}^σ by [26, Lemma 2.4]. For brevity, we write \mathcal{A} to denote \mathcal{A}^\emptyset . The averaging operator $\llbracket \cdot \rrbracket_\sigma : \mathcal{A}^\sigma \rightarrow \mathcal{A}$ is defined by

$$\llbracket F \rrbracket_\sigma = p_F^\sigma \cdot F^\emptyset,$$

where F^\emptyset is the unlabelled 3-graph obtained from the σ -flag F by forgetting about its embedding of σ , and p_F^σ is the probability that a random injection from $V(\sigma)$ to $V(F^\emptyset)$ yields an embedding of σ such that the resulting σ -flag is isomorphic to F .

We now relate flag algebras to asymptotic properties of 3-graphs. We say that a sequence of K_4^- -free 3-graphs $(H_n)_{n \in \mathbb{N}}$ with $\nu(H_n) \rightarrow \infty$ is *convergent* if $\lim_{n \rightarrow \infty} p(F, H_n)$ exists for every $F \in \mathcal{F}$. Since $p(F, H_n) \in [0, 1]$ for every pair (F, H_n) , Tychonoff's theorem implies that every sequence $(H_n)_{n \in \mathbb{N}}$ has a convergent subsequence. For each convergent sequence of graphs $(H_n)_{n \in \mathbb{N}}$, we define a function $\phi : \mathcal{A} \rightarrow \mathbb{R}$, which we call the *limit* of $(H_n)_{n \in \mathbb{N}}$, by letting

$$\phi(F) := \lim_{n \rightarrow \infty} p(F, H_n) \quad \text{for every } F \in \mathcal{F},$$

and extending it to the rest of \mathcal{A} by linearity. For a given type σ , we define convergence and limits for σ -flag sequences analogously.

Let $\text{Hom}^+(\mathcal{A}, \mathbb{R})$ be the set of all algebra homomorphisms ϕ from \mathcal{A} to \mathbb{R} such that $\phi(F) \geq 0$ for every $F \in \mathcal{F}$. By construction, for every convergent sequence of 3-graphs $(H_n)_{n \in \mathbb{N}}$ the associated limit ϕ is an element of $\text{Hom}^+(\mathcal{A}, \mathbb{R})$. Similarly, given a type σ we let $\text{Hom}^+(\mathcal{A}^\sigma, \mathbb{R})$ denote the set of all algebra homomorphisms ψ from \mathcal{A}^σ to \mathbb{R} with $\psi(F^\sigma) \geq 0$ for every $F^\sigma \in \mathcal{F}^\sigma$.

Observe that given a fixed embedding of σ in a 3-graph H_n , we have a map $\psi_n : F \mapsto p(F, H_n)$ sending a σ -flag F to its σ -flag density in H_n , which extends by linearity to a map $\mathbb{R}^{\mathcal{F}^\sigma} \rightarrow \mathbb{R}$. For a sequence of 3-graphs $(H_n)_{n \in \mathbb{N}}$ converging to the limit $\phi \in \text{Hom}^+(\mathcal{A}, \mathbb{R})$ and σ a type such that $\phi(\llbracket \sigma \rrbracket_\sigma) > 0$, we let (\mathbf{P}_n^σ) be the sequence of probability distributions on such maps $\psi_n : \mathbb{R}^{\mathcal{F}^\sigma} \rightarrow \mathbb{R}$ after turning H_n into a σ -flag by selecting an embedding of σ in H_n uniformly at random. Similarly, let \mathbf{P}_ϕ^σ be the unique probability distribution on $\text{Hom}^+(\mathcal{A}^\sigma, \mathbb{R})$ satisfying

$$\mathbb{E}_{\psi \sim \mathbf{P}_\phi^\sigma} \psi(f) = \frac{\phi(\llbracket f \rrbracket_\sigma)}{\phi(\llbracket \sigma \rrbracket_\sigma)} \quad \forall f \in \mathcal{A}^\sigma. \quad (2.1)$$

Razborov proved the existence and uniqueness of \mathbf{P}_ϕ^σ in [26, Theorem 3.5], and further that the sequence of probability distributions $(\mathbf{P}_n^\sigma)_{n \in \mathbb{N}}$ converges weakly to \mathbf{P}_ϕ^σ as $n \rightarrow \infty$ in [26, Theorem 3.12].

2.1 | Tight paths

A crucial step in our proof of the stability result stated in Theorem 1.5 will be constructing an auxiliary orientation of almost all pairs of the vertices of a K_4^- -free 3-graph H with large codegree. In order to do so, we shall use a relation on the vertex pairs of H associated with what is known as *tight connectivity*.

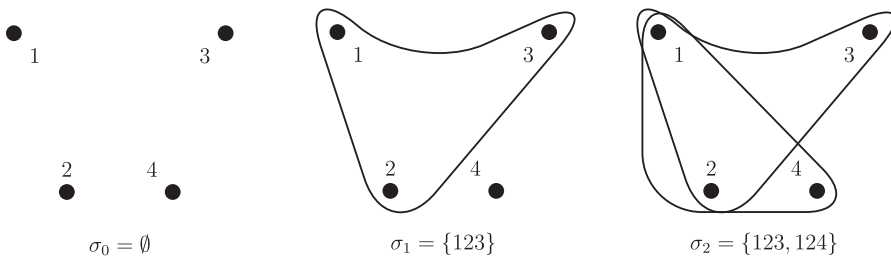


FIGURE 2 The 4-vertex types σ_0, σ_1 and σ_2 .

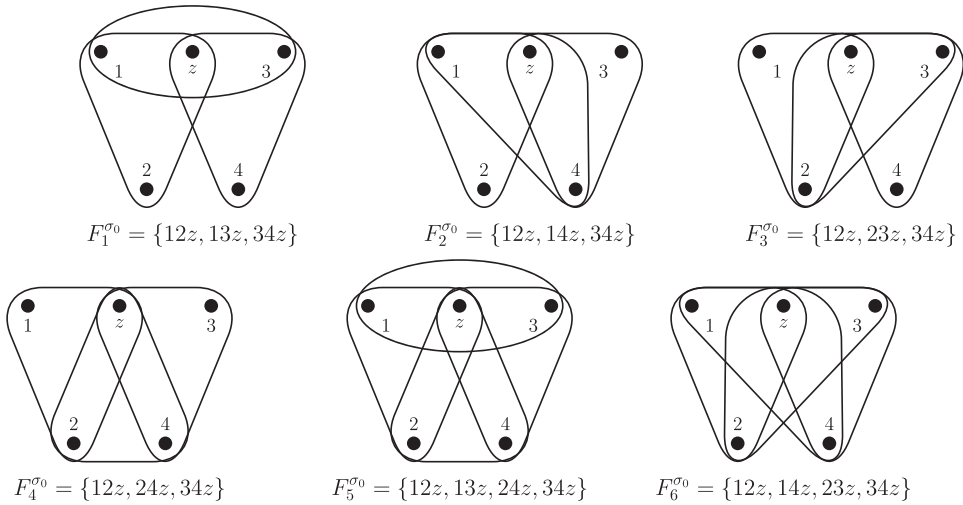


FIGURE 3 The set of σ_0 -flags $\mathcal{P}_3^{\sigma_0}$.

Definition 2.1. We say that a pair of vertices $\{a, b\}$ is *tightly connected* to a pair $\{c, d\}$ by a path of length ℓ if there exists a sequence of distinct $\ell + 2$ vertices $v_1, v_2, \dots, v_{\ell+2}$ and ℓ edges e_1, e_2, \dots, e_ℓ such that $e_i = \{v_i, v_{i+1}, v_{i+2}\}$ for every $i \in [\ell]$, $\{a, b\} \subseteq e_1$ and $\{c, d\} \subseteq e_\ell$.

Having a fixed orientation of a given pair of vertices $\{a, b\}$ and a tight path P of length ℓ connecting the pair $\{a, b\}$ to a pair $\{c, d\}$ allows us to propagate the orientation of $\{a, b\}$ to $\{c, d\}$ in a rather natural way: there is a unique orientation of the remaining two pairs contained in the first edge e_1 of P so that the three pairs in e_1 form a cyclically oriented triangle, which fixes the orientation of one pair contained in the next edge of P . Repeating the procedure ℓ times along the edges of P yields an orientation for the pair $\{c, d\}$.

Let us now introduce a flag algebra notion, which we will use to capture tight connectivity in the proof of Theorem 1.5. Fix σ_0, σ_1 and σ_2 to be the 4-vertex types depicted in Figure 2. Note that, up to labelling of the vertices, σ_0, σ_1 and σ_2 are the only K_4^- -free 4-vertex types. Let $\mathcal{P}_3^{\sigma_0}, \mathcal{P}_3^{\sigma_1}$ and $\mathcal{P}_3^{\sigma_2}$ be the sets of 5-vertex σ_0 -, σ_1 - and σ_2 -flags depicted in Figures 3–5, respectively. A straightforward inspection of these flags yields the following.

Observation 2.2. For any $i \in \{0, 1, 2\}$ and any $F^{\sigma_i} \in \mathcal{P}_3^{\sigma_i}$, the pair $\{1, 2\}$ is tightly connected to the pair $\{3, 4\}$ in F^{σ_i} by a path of length 3.

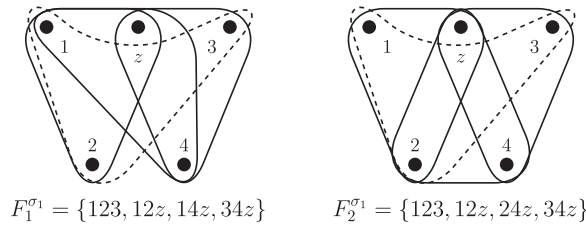


FIGURE 4 The set of σ_1 -flags $\mathcal{P}_3^{\sigma_1}$. The edge induced by the labelled vertices 1,2,3,4 is represented by a dashed curve.

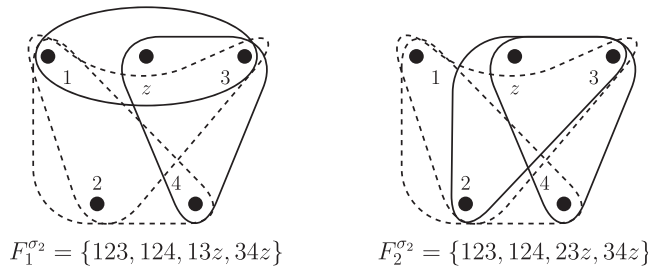


FIGURE 5 The set of σ_2 -flags $\mathcal{P}_3^{\sigma_2}$. The two edges induced by the labelled vertices 1,2,3,4 are represented by dashed curves.

For $i \in \{0, 1, 2\}$, we define $\mathcal{P}_i \in \mathcal{A}$ to be the following non-labelled expression:

$$\mathcal{P}_i := \left\| \left(16 \cdot \sum_{F \in \mathcal{P}_3^{\sigma_i}} F - \sum_{F \in \mathcal{F}_5^{\sigma_i}} F \right)^2 \right\|_{\sigma_i} = \left\| \left(15 \cdot \sum_{F \in \mathcal{P}_3^{\sigma_i}} F - \sum_{F \in \mathcal{F}_5^{\sigma_i} \setminus \mathcal{P}_3^{\sigma_i}} F \right)^2 \right\|_{\sigma_i}.$$

Note that \mathcal{P}_0 , \mathcal{P}_1 and \mathcal{P}_2 can be expressed as a linear combination of the elements of \mathcal{F}_6 . We now relate the expressions \mathcal{P}_i to the existence of (short) tight paths.

Proposition 2.3. *Let $(H_n)_{n \in \mathbb{N}}$ be a convergent sequence of K_4^- -free 3-graphs and let $\phi \in \text{Hom}^+(\mathcal{A}, \mathbb{R})$ be its limit. If $\phi(\mathcal{P}_0) = \phi(\mathcal{P}_1) = \phi(\mathcal{P}_2) = 0$ and $n \in \mathbb{N}$, then for all but $o(v(H_n)^4)$ quadruples of vertices $a, b, c, d \in V(H_n)$ the pair $\{a, b\}$ is tightly connected to $\{c, d\}$ by a path of length at most 3.*

Proof. Let $a, b, c, d \in V(H_n)$ be a quadruple of distinct vertices and E' the set of edges in H_n the quadruple induces. Since H_n is K_4^- -free, there are, up to symmetry, exactly four possibilities:

- 1) $E' = \emptyset$, 2) $E' = \{abc\}$, 3) $E' = \{abc, abd\}$, and 4) $E' = \{abc, bcd\}$.

In the last case, the edges abc and bcd already form a tight path of length 2 between the pairs $\{a, b\}$ and $\{c, d\}$.

For the remaining three cases, our aim will be to find a suitable additional vertex $x \in V(H_n)$ such that the subgraph of H_n induced by $\{a, b, c, d, x\}$ contains a tight path from $\{a, b\}$ to $\{c, d\}$ of

length 3. Based on E' , we break down the analysis into Claims 2.4–2.6. As each case can be treated in a very similar manner, we only deal with the case $E' = \emptyset$ in detail.

Claim 2.4. There are $o(v(H_n)^4)$ choices of a, b, c, d such that $\{a, b, c, d\}$ induces no edge and $\{a, b\}$ is not tightly connected to $\{c, d\}$ by a path of length 3.

Proof. Without loss of generality, we may assume $\phi(\llbracket \sigma_0 \rrbracket_{\sigma_0}) > 0$ as otherwise the total number of choices of $\{a, b, c, d\}$ inducing no edge in H_n is $o(v(H_n)^4)$. Let $\mathbf{P}_n^{\sigma_0}$ be the probability distribution on mappings from $\mathbb{R}^{\mathcal{F}^{\sigma_0}}$ to \mathbb{R} that arise from picking a copy of σ_0 in H_n uniformly at random, that is, choosing a quadruple of vertices Q uniformly at random conditioned on the event that Q induces no edges in H_n .

Let ψ_n be drawn according to the probability distribution $\mathbf{P}_n^{\sigma_0}$, that is, fix a copy of σ_0 in H_n uniformly at random, and let a, b, c and d be the vertices of H_n corresponding to the σ_0 vertices 1–4, respectively. Recall that $\psi_n(F)$, where $F \in \mathcal{F}_5^{\sigma_0}$, corresponds to the probability that a random vertex $x \in V(H_n) \setminus \{a, b, c, d\}$ extends $\{a, b, c, d\}$ into a σ_0 -flag isomorphic to F . In particular, $\sum_{F \in \mathcal{F}_5^{\sigma_0}} \psi_n(F) = 1$.

Clearly, the pair $\{1, 2\}$ is tightly connected to $\{3, 4\}$ by a path of length 3 in any $F \in \mathcal{P}_3^{\sigma_0}$. In fact, a straightforward inspection reveals that the set $\mathcal{P}_3^{\sigma_0}$ describes all the possibilities for such a path in a K_4^- -free setting. Therefore, the number of tight paths of length 3 between $\{a, b\}$ and $\{c, d\}$ in H_n is equal to $(v(H_n) - 4) \cdot \sum_{F \in \mathcal{P}_3^{\sigma_0}} \psi_n(F)$. Motivated by Construction 1.2, we aim to prove the claim by showing that this expression is equal to $(1 - o(1)) \cdot v(H_n)/16$ with probability $1 - o(1)$.

Suppose to the contrary that there is $\varepsilon_0 > 0$ and an infinite subsequence $(n_k)_{k \in \mathbb{N}}$ of \mathbb{N} such that for every $k \in \mathbb{N}$ there are (at least) $\varepsilon_0 \cdot v(H_{n_k})^4$ copies of σ_0 in H_{n_k} such that the corresponding ψ_{n_k} satisfies the two following equivalent inequalities:

$$\left| \sum_{F \in \mathcal{P}_3^{\sigma_0}} \psi_{n_k}(F) - \frac{1}{16} \right| \geq \varepsilon_0 \quad \Longleftrightarrow \quad \left(\sum_{F \in \mathcal{P}_3^{\sigma_0}} \psi_{n_k}(F) - \frac{1}{16} \cdot \sum_{F \in \mathcal{F}_5^{\sigma_0}} \psi_{n_k}(F) \right)^2 \geq (\varepsilon_0)^2.$$

Fix such a ψ_{n_k} . The definition of the flag-algebra multiplication on \mathcal{F}^{σ_0} in turn yields that

$$\psi_{n_k} \left(\left(\sum_{F \in \mathcal{P}_3^{\sigma_0}} F - \sum_{F \in \mathcal{F}_5^{\sigma_0}} \frac{1}{16} \cdot F \right)^2 \right) \geq (\varepsilon_0)^2 - o(1).$$

As $(\mathbf{P}_{n_k}^{\sigma_0})_{k \in \mathbb{N}}$ weakly converges to the probability distribution $\mathbf{P}_\phi^{\sigma_0}$ associated with ϕ , we have

$$\lim_{k \rightarrow \infty} \mathbb{E}_{\psi_{n_k} \sim \mathbf{P}_{n_k}^{\sigma_0}} \left[\psi_{n_k} \left(\left(\sum_{F \in \mathcal{P}_3^{\sigma_0}} F - \sum_{F \in \mathcal{F}_5^{\sigma_0}} \frac{1}{16} \cdot F \right)^2 \right) \right] = \mathbb{E}_{\psi \sim \mathbf{P}_\phi^{\sigma_0}} \left[\psi \left(\left(\sum_{F \in \mathcal{P}_3^{\sigma_0}} F - \sum_{F \in \mathcal{F}_5^{\sigma_0}} \frac{1}{16} \cdot F \right)^2 \right) \right].$$

Clearly, the limit on the left-hand side must be at least $(\varepsilon_0)^3$; however, the expectation on the right-hand side is by (2.1) equal to $\frac{1}{256} \cdot \frac{\phi(\mathcal{P}_0)}{\phi(\llbracket \sigma_0 \rrbracket_{\sigma_0})} = 0$, a contradiction. \square

As $\phi(\mathcal{P}_1) = \phi(\mathcal{P}_2) = 0$, an analogous argument for embeddings of σ_1 and σ_2 in $(H_n)_{n \in \mathbb{N}}$ using the respective probability distributions $(\mathbf{P}_n^{\sigma_1}) \rightarrow \mathbf{P}_\phi^{\sigma_1}$ and $(\mathbf{P}_n^{\sigma_2}) \rightarrow \mathbf{P}_\phi^{\sigma_2}$ yields the following.

Claim 2.5. There are $o(v(H_n)^4)$ choices of a, b, c, d such that $\{a, b, c, d\}$ induces one edge abc and $\{a, b\}$ is not tightly connected to $\{c, d\}$ by a path of length 3.

Claim 2.6. There are $o(v(H_n)^4)$ choices of a, b, c, d such that $\{a, b, c, d\}$ induces edges $\{abc, abd\}$ and $\{a, b\}$ is not tightly connected to $\{c, d\}$ by a path of length 3.

This concludes the proof of Proposition 2.3. \square

2.2 | Proof of $\pi_2(K_4^-) = 1/4$

We shall prove Theorem 1.3 by establishing a specific identity between subgraph densities of 7-vertex K_4^- -free 3-graphs that holds for any $\phi \in \text{Hom}^+(\mathcal{A}, \mathbb{R})$. This identity is obtained by an application of the semi-definite method of Razborov and some computer-aided flag algebra calculations. Before we can state this identity, we need to introduce a few more definitions.

For each $n \in \mathbb{N}$, let T_n be a uniformly random tournament on $[n]$. It is straightforward to check that the sequence $(C(T_n))_{n \in \mathbb{N}}$ is an almost surely convergent sequence of 3-graphs. Indeed, as $n \rightarrow \infty$, the density of any fixed F in $C(T_n)$ almost surely converges to the probability that $F = C(T)$ when T is a random tournament on $V(F)$ in which pairs are oriented randomly and independently of each other. Let ϕ_T denote the limit of $(C(T_n))_{n \in \mathbb{N}}$ and let

$$\mathcal{E}_T := \{F \in \mathcal{F} : \phi_T(F) > 0\}.$$

We call the elements of \mathcal{E}_T *tournament-realizable*. The definition of \mathcal{E}_T readily yields the following.

Observation 2.7. $F \in \mathcal{E}_T$ if and only if there is a tournament T_F on $V(F)$ such that $F = C(T_F)$.

We now continue by introducing the remaining flag algebra notation. Let $\mathcal{G} := \mathcal{F} \setminus \mathcal{E}_T$ be the set of K_4^- -free 3-graphs that are not tournament-realizable. For every $k \in \mathbb{N}$, we set $\mathcal{G}_k := \mathcal{G} \cap \mathcal{F}_k$. Let τ be the unique 2-vertex type consisting of two labelled vertices 1 and 2. Let N and E be the 3-vertex 3-graphs with 0 and 1 edge, respectively, that is, the (unlabelled) *Non-edge* and *Edge*, and similarly N^τ and E^τ the unique 3-vertex τ -flags with 0 and 1 edge, respectively. A codegree assumption $\delta_2(H) \geq (c - o(1)) \cdot v(H)$ implies in the flag algebra language the statement that $\phi^\tau(E^\tau) \geq c$ with probability 1, where ϕ^τ is drawn from \mathbf{P}_ϕ^τ . Equivalently, $\phi^\tau(\frac{1-c}{c} \cdot E^\tau - N^\tau) \geq 0$ with probability 1.

Let $\iota_1, \iota_2, \dots, \iota_6$ be the six 5-vertex types depicted in Figure 6. For every $i \in [6]$, we let $k_i := |\mathcal{F}_6^{\iota_i}|$, and fix an arbitrary enumeration $F_1^{\iota_i}, F_2^{\iota_i}, \dots, F_{k_i}^{\iota_i}$ of the elements of $\mathcal{F}_6^{\iota_i}$. Note the particular enumeration that we will be using in the whole paper is the one used in our ancillary computer programmes. Similarly, we fix an enumeration of $\mathcal{F}_5^{\sigma_i}$ for $i \in \{0, 1, 2\}$.

For $i \in [6]$, we let e_i be the k_i -dimensional vector from $(\mathcal{A}^{\iota_i})^{k_i}$ whose j th coordinate is equal to $F_j^{\iota_i}$. By a straightforward computer search, we have $k_1 = 191$, $k_2 = 173$, $k_3 = 148$, $k_4 = 135$, $k_5 = 124$ and $k_6 = 95$ (this can be also checked using Flagmatic, see, for example, [12, 37]).

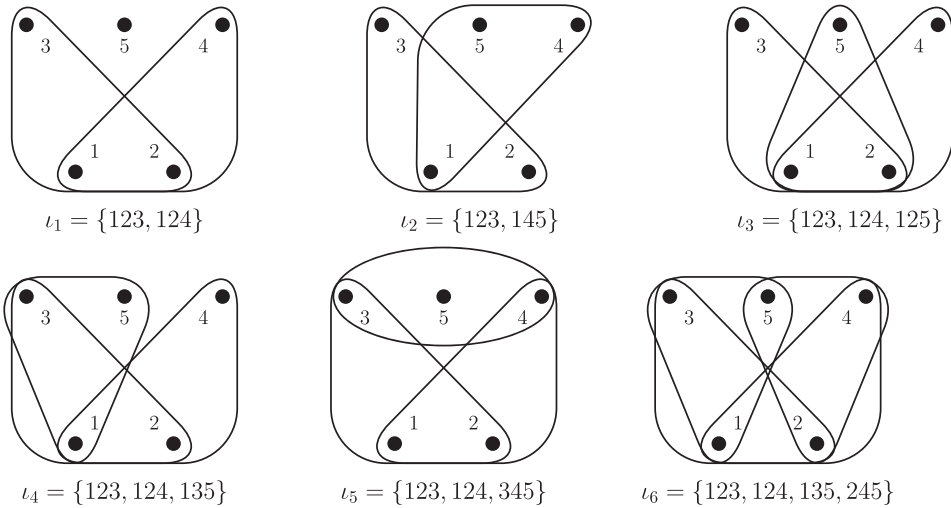


FIGURE 6 The 5-vertex types ι_i for $i \in [6]$ used in the proof of Lemma 2.8.

Next, it is easy to check that $\phi_T(\llbracket \iota_2 \rrbracket_{\iota_2}) = \phi_T(\llbracket \iota_5 \rrbracket_{\iota_5}) = 0$. On the other hand, for $i \in \{1, 3, 4, 6\}$ we have $\phi_T(\llbracket \iota_i \rrbracket_{\iota_i}) > 0$, and so the probability distribution $\mathbf{P}_{\phi_T}^{\iota_i}$ on $\text{Hom}^+(\mathcal{A}^i, \mathbb{R})$ is well defined. Note that in this case $\mathbf{P}_{\phi_T}^{\iota_i}$ is a purely atomic measure, and $|\text{supp}(\mathbf{P}_{\phi_T}^{\iota_i})| \leq 2^{10}$ (since the right-hand side is equal to the number of 3-graphs on a labelled set of 5 vertices).

For any proof of $\pi_2(K_4^-) \leq 1/4$, the support of $\mathbf{P}_{\phi_T}^{\iota_i}$ restricts arguments used in the proof that involve ι_i -flags. In particular, if such a proof uses $\llbracket f^2 \rrbracket_{\iota_i} \geq 0$ for some $f \in \mathcal{A}^i$, then clearly we must have $\psi^{\iota_i}(f) = 0$ for all $\psi^{\iota_i} \in \text{supp}(\mathbf{P}_{\phi_T}^{\iota_i})$.

Recall that $F_1^{\iota_i}, F_2^{\iota_i}, \dots, F_{k_i}^{\iota_i}$ is the fixed enumeration of $\mathcal{F}_6^{\iota_i}$. We set d_i to be the dimension of the subspace of \mathbb{R}^{k_i} generated by

$$\left\{ x \in \mathbb{R}^{k_i} \mid \exists \psi^{\iota_i} \in \text{supp}(\mathbf{P}_{\phi_T}^{\iota_i}) \text{ such that } x_j = \psi^{\iota_i}(F_j^{\iota_i}) \text{ for all } j \in [k_i] \right\}.$$

By inspection of the tournaments whose cyclically oriented triangles give rise to a given type ι_i under the Erdős–Hajnal construction, we have that $d_2 = d_5 = 0$, $d_4 = d_6 = 1$, $d_1 = 4$ and $d_3 = 6$. Indeed, up to reversing the orientation of all the edges there are exactly 4, 0, 6, 1, 0 and 1 distinct tournaments on [5] realising the types $\iota_1, \iota_2, \iota_3, \iota_4, \iota_5$ and ι_6 , respectively. Each such a tournament J corresponds to a different kind of embedding of ι_j in ϕ_T , which we denote by H^J , and determines a different probability distribution on $\mathcal{F}_6^{\iota_j}$ given by extending H^J by a single random vertex of ϕ_T . By the definition of ϕ_T , this is in a one-to-one correspondence with adding a new vertex x to J that has all its edges to $V(J)$ oriented randomly, and considering the random variable $C(J \cup \{x\})$. Moreover, when viewing each such a probability distribution as a vector in $\mathbb{R}^{\mathcal{F}_6^{\iota_j}}$, for every $j \in [6]$ the set of the d_j vectors corresponding to all such distributions on $\mathcal{F}_6^{\iota_j}$ is linearly independent.

Finally, yet another straightforward computer search yields $|\mathcal{F}_6^\tau| = 1643$. Moreover, exactly 167 elements $F \in \mathcal{F}_6^\tau$ are such that swapping the labels of τ (that is, relabelling the vertex 1 to 2 and vice versa) yields a τ -flag isomorphic to F . Now observe that the expression $\llbracket F \times (3E^\tau - N^\tau) \rrbracket_\tau$

is invariant under such a swap of the labels of τ for every $F \in \mathcal{F}^\tau$. Therefore, there are exactly $(1643 + 167)/2 = 905$ pairwise different expressions of the form $\llbracket F \times (3E^\tau - N^\tau) \rrbracket_\tau$ where $F \in \mathcal{F}_6^\tau$.

Let $\mathcal{D} := \{\llbracket F \times (3E^\tau - N^\tau) \rrbracket_\tau \mid F \in \mathcal{F}_6^\tau\}$. The discussion from the previous paragraph yields that $|\mathcal{D}| = 905$. Also note that $\phi(D) \geq 0$ for every $D \in \mathcal{D}$ where ϕ is the limit of a sequence of K_4^- -free 3-graphs $(H_n)_{n \in \mathbb{N}}$ with $\liminf_{n \rightarrow \infty} \frac{\delta_2(H_n)}{v(H_n)} \geq \frac{1}{4}$, and every $D \in \mathcal{D}$ can be expressed as a linear combination of the elements of \mathcal{F}_7 .

We are now ready to state the main lemma of this section.

Lemma 2.8. *There exist*

- (1) *positive rationals c_0, c_1, c_2, w_G for every $G \in \mathcal{G}_7$, and u_D for every $D \in \mathcal{D}$,*
- (2) *rational matrices I_i of sizes $(k_i - d_i) \times k_i$ for $i \in [6]$ and*
- (3) *positive definite rational matrices $Q_i > 0$ of sizes $(k_i - d_i) \times (k_i - d_i)$ for $i \in [6]$,*

such that the following identity holds in the theory of K_4^- -free 3-graphs:

$$N - 3E = \underbrace{\sum_{D \in \mathcal{D}} u_D \cdot D}_{\text{CODEGREE}} + \underbrace{\sum_{i \in \{0,1,2\}} c_i \cdot P_i}_{\text{TIGHT-PATH}} + \underbrace{\sum_{i \in [6]} \llbracket e_i^T I_i^T Q_i I_i e_i \rrbracket_{i_i}}_{\text{POSITIVE-DEFINITE}} + \underbrace{\sum_{G \in \mathcal{G}_7} w_G \cdot G}_{\text{SLACK}}. \quad (2.2)$$

Proof. This is a standard flag algebra computation in the theory of K_4^- -free 3-graphs, which was performed with the aid of a computer. Files containing the rational matrices I_1, \dots, I_6 and Q_1, \dots, Q_6 , the rationals (u_D) , c_0, c_1 and c_2 , enumerations of \mathcal{F}_7 and $\mathcal{F}_6^{i_i}$ as well as verifications of the positive definiteness of the matrices Q_i and the flag algebraic identity claimed by Lemma 2.8 can be found at <http://honza.ucw.cz/proj/codeg-k4e/> as well as on the arXiv version of this paper; see the Appendix for details. \square

With Lemma 2.8 in hand, the remainder of the proof of Theorem 1.3 is straightforward.

Proof of Theorem 1.3. Suppose for a contradiction that $\pi_2(K_4^-) > 1/4$. Then there exist $\varepsilon_0 > 0$ and a sequence of K_4^- -free 3-graphs $(H_n)_{n \in \mathbb{N}}$ with $v(H_n) \rightarrow \infty$ such that

$$\liminf_{n \rightarrow \infty} \frac{\delta_2(H_n)}{v(H_n)} \geq 1/4 + \varepsilon_0.$$

By compactness, there exists a convergent subsequence $(H'_n)_{n \in \mathbb{N}}$ of $(H_n)_{n \in \mathbb{N}}$. Let ϕ_0 denote its limit. A double-counting argument yields that $\phi_0(E) = \phi_0(\llbracket E^\tau \rrbracket_\tau) \geq 1/4 + \varepsilon_0$. Therefore,

$$\phi_0(N - 3E) = \phi_0(N + E) - 4 \cdot \phi_0(E) \leq 1 - 4 \cdot \left(\frac{1}{4} + \varepsilon_0\right) = -4\varepsilon_0 < 0. \quad (2.3)$$

On the other hand, our codegree assumption yields that almost surely $\phi_0^\tau(3E^\tau - N^\tau) \geq 4\varepsilon_0$, where ϕ_0^τ is a random homomorphism drawn from $\mathbf{P}_{\phi_0}^\tau$. Therefore, for any non-negative $(u_D)_{D \in \mathcal{D}}$,

$$\phi_0\left(\sum_{D \in \mathcal{D}} u_D \cdot D\right) = \sum_{D \in \mathcal{D}} u_D \cdot \phi_0(D) \geq 0.$$

As the evaluation of the remaining three summands in (2.2) is non-negative for any $\phi \in \text{Hom}^+(\mathcal{A}, \mathbb{R})$, Lemma 2.8 yields that $\phi_0(N - 3E) \geq 0$ contradicting (2.3). \square

3 | STABILITY OF CONSTRUCTION 1.2

In order to relate K_4^- -free 3-graphs to Construction 1.2, we establish the following proposition.

Proposition 3.1. *Let $H = (V, E)$ be an n -vertex 3-graph such that:*

- (i) *every 7-vertex subgraph of H is tournament-realizable,*
- (ii) *for all but $o(n^4)$ choices of $a, b, c, d \in V$ the pair $\{a, b\}$ is tightly connected to $\{c, d\}$ by a path of length at most 3 and*
- (iii) *all but $o(n^2)$ pairs of vertices of H have codegree at least $n/4 - o(n)$.*

Then there is an oriented graph \vec{G} on the vertex set V such that:

- (1) *\vec{G} has at least $n^2/2 - o(n^2)$ arcs,*
- (2) *if $u, v, w \in V$ span a transitive triangle in \vec{G} , then $uvw \notin E$ and*
- (3) *for all but $o(n^2)$ arcs \vec{uv} in \vec{G} there are at least $n/4 - o(n)$ vertices $w \in V$ such that $\{u, v, w\}$ induces a cyclically oriented triangle in \vec{G} .*

Proof. We shall use the assumption on the existence of tight paths of length at most 3 in order to define an orientation for almost every pair of vertices $\{u, v\} \subseteq V^{(2)}$. For two vertices u and v , we write $u \rightarrow v$ to denote the fact that we place an arc in \vec{G} that goes from u to v .

By Assumption (ii), there exists a pair $\{a, b\} \subseteq V$ such that $\{a, b\}$ is tightly connected to $\{c, d\}$ by a path of length at most 3 for all but $o(n^2)$ pairs $\{c, d\} \subseteq V$. Fix such a pair $\{a, b\}$.

We start by placing an arc $a \rightarrow b$ in \vec{G} . Now, for every pair $\{c, d\}$ tightly connected to $\{a, b\}$ by a path of length at most 3, we define the orientation of $\{c, d\}$ in \vec{G} as follows: let P_{cd} be an arbitrarily chosen tight path between $\{a, b\}$ and $\{c, d\}$ of length at most 3, and let S_{cd} denote the set of all pairs of vertices that are contained in some edge of P_{cd} . In other words, S_{cd} is the 2-shadow of P_{cd} . As we have noted in Section 2.1, there exists a unique orientation O of the pairs in S_{cd} such that $a \rightarrow b$, and every edge of P_{cd} induces a cyclically oriented triangle. We orient the pair $\{c, d\}$ in \vec{G} according to its orientation in O . Note that by our choice of $\{a, b\}$, the oriented graph resulting from this procedure will satisfy Property (1).

Let us first show that the orientation \vec{G} we have just described is well defined, that is, that the orientation of a pair $\{c, d\}$ does not depend on the particular choice of the tight path P_{cd} . Indeed, given any two tight paths P_{cd} and P'_{cd} from $\{a, b\}$ to $\{c, d\}$ of length at most 3, the subgraph induced by $V(P_{cd}) \cup V(P'_{cd})$ has at most 6 vertices. Therefore, it is tournament-realizable by Assumption (i) on H . Let S'_{cd} be the set of pairs of vertices contained in some edge of $E(P_{cd}) \cup E(P'_{cd})$. As in the previous paragraph, there exists a unique orientation of the pairs in S'_{cd} such that $a \rightarrow b$ and every edge of $E(P_{cd}) \cup E(P'_{cd})$ induces a cyclically oriented triangle (the existence is guaranteed by the tournament realizability, the uniqueness by the tightness of the paths). In particular, P_{cd} and P'_{cd} define the same orientation for $\{c, d\}$ in \vec{G} .

Next, we claim that if three vertices $u, v, w \in V$ span a transitive triangle in \vec{G} , then $uvw \notin E(H)$. The argument is very similar to the one in the previous paragraph. Without loss of generality, suppose that the arcs on $\{u, v\}$ and $\{u, w\}$ are oriented as $u \rightarrow v$ and $u \rightarrow w$ in \vec{G} . Let P_{uv} and P_{uw} be (some) tight paths of length at most 3 from $\{a, b\}$ to $\{u, v\}$ and $\{u, w\}$, respectively. Since the

induced subgraph $F := H[V(P_{uv}) \cup V(P_{uw})]$ has at most 7 vertices, it is tournament-realizable by Assumption (i). In particular, there is a tournament T_F with $a \rightarrow b$ realising F . The presence of the tight paths implies that $u \rightarrow v$ and $u \rightarrow w$ in T_F , so $uvw \notin E(F)$ and hence also $uvw \notin E(H)$.

It remains to establish Property (3). Fix $\varepsilon_0 > 0$ and suppose for a contradiction there were at least $\varepsilon_0 \cdot n^2$ arcs such that each of them is contained in fewer than $(1/4 - \varepsilon_0)n$ cyclically oriented triangles in \vec{G} . Since the arc density of \vec{G} is $1 - o(1)$, at least $\varepsilon_0/2 \cdot n^2$ of these arcs must be contained in at least $(3/4 + \varepsilon_0/2)n$ transitive triangles in \vec{G} . By Property (2) of \vec{G} that we have established in the previous paragraph, these transitive triangles correspond to non-edges of H . Thus we have found $(\varepsilon_0/2) \cdot n^2$ pairs in H with codegree at most $(1/4 - \varepsilon_0/2)n$ contradicting Assumption (ii). \square

The main result of this section is the following flag algebra version of Theorem 1.5.

Theorem 3.2. *Let $(H_n)_{n \in \mathbb{N}}$ be a sequence of K_4^- -free 3-graphs. If*

$$\liminf_{n \rightarrow \infty} \frac{\delta_2(H_n)}{n} = \pi_2(K_4^-) = \frac{1}{4},$$

then $(H_n)_{n \in \mathbb{N}}$ converges to ϕ_T .

Proof. Fix any sequence $(H_n)_{n \in \mathbb{N}}$ of K_4^- -free 3-graphs with $\delta_2(H_n) \geq n/4 - o(n)$. By compactness, pass to a subsequence $(H'_n)_{n \in \mathbb{N}}$ convergent to some limit ϕ . Let V_n denote the vertex set of H'_n and set $v_n := |V_n|$. Our aim is to use Proposition 3.1 and assign to each element of the sequence a nearly-complete oriented graph. However, we will first ‘clean-up’ the sequence (H'_n) using standard regularity tools.

As established in the proof of Theorem 1.3, $\phi(N - 3E) \geq 0$. On the other hand, our codegree assumption yields $\phi(N - 3E) \leq 0$, and hence $\phi(N - 3E) = 0$. Now Properties 1–4 from Lemma 2.8 guarantee that the ϕ -evaluation of each of the four summands in (2.2) is non-negative. In particular, $\phi(N - 3E) = 0$ and Identity (2.2) together imply that the ϕ -evaluation of all four summands on the right-hand side of (2.2) must be equal to 0.

Recall that \mathcal{G}_7 is the set of all 7-vertex 3-graphs that are not tournament-realizable. Since $w_G > 0$ for every $G \in \mathcal{G}_7$ (Property 1 in Lemma 2.8), the ‘SLACK’ summand in (2.2) evaluating to zero under ϕ implies that $\phi(G) = 0$ for every $G \in \mathcal{G}_7$. Therefore, there are at most $o(v_n^7)$ induced 7-vertex subgraphs in H'_n that are not tournament-realizable. Applying the induced version of the Hypergraph Removal Lemma of Rödl and Schacht [33, Theorem 6], we can add or remove $o(v_n^3)$ edges in H'_n for each $n \in \mathbb{N}$ in order to obtain a modified sequence of 3-graphs $(H''_n)_{n \in \mathbb{N}}$ such that in fact every induced 7-vertex subgraph of H''_n is tournament-realizable. (Note this implies in particular that H''_n remains K_4^- -free for every $n \in \mathbb{N}$.)

Since $|E(H''_n) \Delta E(H'_n)| = o(v_n^3)$, the sequence $(H''_n)_{n \in \mathbb{N}}$ is also convergent and its limit is ϕ . Thus it is enough to show that the limit of $(H''_n)_{n \in \mathbb{N}}$ is equal to ϕ_T . Also, by construction, the codegree assumption on H'_n and averaging imply that all but $o(v_n^2)$ pairs of vertices in H''_n have codegree at least $v_n/4 - o(v_n)$.

As we noted above, the ‘TIGHT-PATH’ summand in (2.2) must ϕ -evaluate to zero. Since $c_i > 0$ for every $i \in \{0, 1, 2\}$ (Property 1 in Lemma 2.8), this implies that $\phi(P_0) = \phi(P_1) = \phi(P_2) = 0$. Proposition 2.3 then yields that $\{a, b\}$ is tightly connected to $\{c, d\}$ by a path of length at most 3 for all but $o(v_n^4)$ choices of a, b, c, d .

We have thus verified in the three paragraphs above that the elements of $(H''_n)_{n \in \mathbb{N}}$ satisfy Conditions (i)–(iii) of Proposition 3.1. Applying it, we assign to each 3-graph H''_n an oriented graph \vec{G}_n on V_n satisfying Properties (1)–(3) from Proposition 3.1.

Now, every triple of vertices $\{u, v, w\}$ inducing an edge in H''_n either corresponds to a cyclically oriented triangle of \vec{G}_n , or contains one of the $o(v_n^2)$ pairs that do not span an arc in \vec{G}_n . Since all but $o(v_n^2)$ pairs from $V(H''_n)$ have codegree $v_n/4 + o(v_n)$, the density of cyclically oriented triangles in \vec{G}_n must be at least $1/4 - o(1)$.

On the other hand, a simple application of the Cauchy–Schwarz Inequality (see (4.1) below) shows that the density of cyclically oriented triangles in any oriented graph is at most $1/4 + o(1)$. Therefore, for all but $o(v_n^3)$ triples $\{u, v, w\} \subseteq V(H''_n)$, $\{u, v, w\}$ induces a cyclically oriented triangle in \vec{G}_n if and only if it induces an edge in H''_n .

Let T_n be obtained from \vec{G}_n by adding the $o(v_n^2)$ missing arcs with arbitrary orientations. Clearly, Property (3) of \vec{G}_n from Proposition 3.1 does transfer to T_n , and hence, $(T_n)_{n \in \mathbb{N}}$ is a sequence of $o(1)$ -quasirandom tournaments by a result of Coregliano and Razborov [4, Theorem 3.2]. A classical result of Chung and Graham [3, Theorem 1 (P1)] states that for any fixed tournament S , the proportion of $v(S)$ -vertex subsets of V_n inducing in T_n a copy of S is asymptotically equal to the expected density of S in a random tournament. In particular, for any 3-graph F , the proportion of $v(F)$ -vertex subsets of V_n inducing in $C(T_n)$ a copy of F is asymptotically $\phi_T(F)$. In other words, ϕ_T is the limit of $(C(T_n))_{n \in \mathbb{N}}$. As \vec{G}_n differs from T_n on $o(v_n^2)$ arcs, we have $|E(C(T_n)) \Delta E(H''_n)| = o(v_n^3)$. Hence, $\phi = \phi_T$. It follows that ϕ_T is the only possible accumulation point of the original sequence $(H_n)_{n \in \mathbb{N}}$, so (by compactness again) the sequence $(H_n)_{n \in \mathbb{N}}$ converges to ϕ_T . \square

4 | TOURNAMENTS WITH LARGE CODEGREE AND HADAMARD MATRICES

In this section, we relate tournaments T that have $\delta_2(T)$ close to $v(T)/4$ to skew Hadamard matrices. Recall from the introduction that a *skew Hadamard matrix* of order n is an $n \times n$ square matrix A with ± 1 entries such that (i) $AA^t = nI_n$ and (ii) $A + A^t = 2I_n$. Here I_n denotes the $n \times n$ identity matrix and A^t denotes the transpose of the matrix A .

Let $t(n)$ be the largest minimum codegree in an n -vertex tournament, that is,

$$t(n) := \max\{\delta_2(T) : T \text{ a tournament on } [n]\}.$$

Proposition 4.1. $t(n) \leq \lfloor \frac{n+1}{4} \rfloor$ for every $n \in \mathbb{N}$.

Proof. Let $n = 4k + r$ with $r \in \{0, 1, 2, 3\}$, and let T be a tournament on $[n]$. Set $u := \lfloor \frac{n-1}{2} \rfloor \cdot \lceil \frac{n-1}{2} \rceil$. Double-counting and applying the Cauchy–Schwarz inequality, we have that the number $|C(T)|$ of cyclically oriented triangles in T satisfies:

$$2|C(T)| + \binom{n}{3} = \sum_{x \in [n]} d^-(x)d^+(x) \leq nu. \quad (4.1)$$

Since $\binom{n}{2} \cdot \delta_2(T) \leq 3 |C(T)|$, rearranging the terms in (4.1) yields the following estimate:

$$\delta_2(T) \leq \left\lfloor \frac{3 |C(T)|}{\binom{n}{2}} \right\rfloor \leq \left\lfloor \frac{3u}{n-1} - \frac{n-2}{2} \right\rfloor. \quad (4.2)$$

The right-hand side of (4.2) is equal to $k+1$ if $r=3$, and k otherwise. \square

The proof above implies that for $n \equiv 3 \pmod{4}$, the upper bound of $\frac{n+1}{4}$ on $\delta_2(T)$ can only be attained if T is highly regular. Explicitly, we have the following corollary.

Corollary 4.2. *If T is an n -vertex tournament with $\delta_2(T) = \frac{n+1}{4}$, then $d^-(v) = d^+(v) = \frac{n-1}{2}$, $C_T(v, w) = \frac{n+1}{4}$ and $R_T(v, w) = \frac{n-3}{4}$ for every distinct $v, w \in V(T)$.*

Proof. Since $|C(T)|$ meets the upper bound from Proposition 4.1, we must have equality in both estimates (4.1) and (4.2). Note that $3u/(n-1) - (n-2)/2$ is exactly $k+1$, without rounding. In particular, every vertex has both in-degree and out-degree exactly $\frac{n-1}{2}$, and every pair of vertices has codegree exactly $\frac{n+1}{4}$. Fix a pair $\{v, w\} \in V(T)^{(2)}$, and assume without loss of generality it is oriented as \vec{vw} in T . Since $d^-(v) + d^+(w) = n-1$, we have by the inclusion-exclusion principle that

$$R_T(v, w) = (n-2) - d^-(v) - d^+(w) + C_T(v, w) = (n-2) - (n-1) + \frac{n+1}{4} = \frac{n-3}{4}. \quad \square$$

When is the upper bound in Proposition 4.1 tight? For $n \equiv 3 \pmod{4}$, this question is very closely related to the existence of a skew Hadamard matrix of order $n+1$. As we have mentioned in the introduction, skew Hadamard matrices are known to exist for infinitely many orders, and conjectured to exist for any order divisible by four.

Proposition 4.3. *For $n \equiv 3 \pmod{4}$, $t(n) = \frac{n+1}{4}$ if and only if there exists a skew Hadamard matrix of order $n+1$.*

Proof. Fix $n = 4k+3$ and T a tournament on $[n]$ with $\delta_2(T) = \lfloor \frac{n+1}{4} \rfloor$. By Corollary 4.2, all in- and out-degrees in T are equal to $2k+1$ and all codegrees are equal to $k+1$. An old result of Reid and Brown [28, Theorems 1 and 2] yields that such a tournament exists if and only if there exists a skew Hadamard matrix of order $4k+4$. For the sake of completeness, we sketch how one can obtain a skew Hadamard matrix from such a tournament (that is, the ‘only if’ direction). Note that all the steps can also be applied in the reverse order to establish the ‘if’ direction.

Let A be the adjacency matrix of T with $A_{ii} = 0$, $A_{ij} = +1$ if $\vec{ij} \in T$ and $A_{ij} = -1$ if $\vec{ji} \in T$. Now fix a pair $\{i, j\} \in [n]^{(2)}$ with $i < j$ and, say, $\vec{ji} \in E(T)$. Consider the $2 \times (n-2)$ -submatrix $M := A(\{i, j\}, [n] - \{i, j\})$. By Corollary 4.2, this submatrix has exactly $k+1$ columns $(1, -1)^T$, corresponding to the exactly $k+1$ vertices v with $\vec{iv}, \vec{vj} \in E(T)$. The row of M indexed by i has a further $2k+1 - (k+1) = k$ entries equal to 1, so the column $(1, 1)^T$ appears in M exactly k times. Likewise, the column $(-1, -1)^T$ appears exactly k times in M , and finally, the column $(-1, +1)^T$ appears $R_T(i, j) = k$ times. Thus, the scalar product of the rows i and j in A is equal to $-(k+1) + k + k - k = -1$.

Now append to A a row indexed by 0 with all entries set to 1, and then a column indexed by 0 whose first entry is 0 and all of whose other entries are set to -1 . This yields an $(n+1) \times (n+1)$ matrix C satisfying $C = -C^t$ (so C is skew-symmetric) and $CC^t = nI_{n+1}$ (for the latter, observe that any pair of distinct rows is orthogonal). Consider now the $(n+1) \times (n+1)$ matrix $H := C + I_{n+1}$. Then H is a square matrix with ± 1 entries such that $H + H^t = C + C^t + 2I_{n+1} = 2I_{n+1}$ and

$$HH^t = (C + I_{n+1})(C^t + I_{n+1}) = CC^t + I_{n+1} + C + C^t = (n+1)I_{n+1}.$$

Thus H is a skew Hadamard matrix of order $n+1$. □

Clearly, knowing the exact value of $t(4k+3)$ for some $k \in \mathbb{N}$ allows us to also give good bounds on $t(n)$ for n close to $4k+3$. We summarise our knowledge of $t(n)$ in the following proposition.

Proposition 4.4. *Fix a non-negative integer k . If there exists a skew Hadamard matrix of order $4k+4$, then*

- (1) $k+1 \geq t(4k+4) \geq k$,
- (2) $t(4k+3) = k+1$,
- (3) $t(4k+2) = k$ and
- (4) $k \geq t(4k+1) \geq k-1$.

Proof. All four upper bounds on $t(n)$ follow from Proposition 4.1, so we only need to establish the lower bounds.

Suppose that there exists a skew Hadamard matrix of order $4k+3$. Proposition 4.3 establishes the lower bound in (2). Let T_k be an optimal tournament on $4k+3$ vertices. Deleting an arbitrary vertex from T_k yields a $(4k+2)$ -vertex tournament T_k^- satisfying $\delta_2(T_k^-) \geq k$, proving (3). Similarly, deleting two arbitrary vertices from T_k yields a tournament T_k^- on $4k+1$ vertices with $\delta_2(T_k^-) \geq k-1$, proving (4).

It remains only to show that $t(4k+4) \geq k$. Fix an arbitrary vertex $x \in V(T_k)$, and let T_k^+ be a $(4k+4)$ -vertex tournament constructed from T_k in the following way: add a new vertex y to T_k , orient the arc xy arbitrarily and give each pair vy with $v \in V(T_k) \setminus \{x\}$ the orientation opposite to the orientation of vx (that is, if $\vec{vx} \in T_k$, then $\vec{vy} \in T_k^+$, and if $\vec{xv} \in T_k$, then $\vec{vy} \in T_k^+$).

We claim that $\delta_2(T_k^+) \geq k$. By the assumption on T_k , it is enough to check that all the pairs containing y are in at least k cyclically oriented triangles. Firstly, consider a pair vy for some $v \in V(T_k) \setminus \{x\}$. Since vy is oriented in the opposite direction to vx , there are $R_{T_k}(v, x) = k$ cyclically oriented triangles in T_k^+ containing v and y , as desired. Finally, consider the pair xy . Without loss of generality, we oriented it as \vec{xy} , so every one of the $2k+1$ in-neighbours of x is an out-neighbour of y . Therefore, $C_{T_k^+}(x, y) = 2k+1 > k$, which concludes the proof. □

5 | THE CODEGREE THRESHOLD OF K_4^-

We now turn our attention to determining the exact value of $\text{ex}_2(n, K_4^-)$ for infinitely many n . The main result of this section is the following.

Theorem 5.1. *Let G be a K_4^- -free 3-graph on n vertices and with $\delta_2(G) = n/4 - o(n)$. Then there is a tournament T on $V(G)$ such that $|G| \leq |C(T)|$. Moreover, if $|G| = |C(T)|$, then $G = C(T)$.*

Before proving Theorem 5.1, let us apply it to determine the codegree threshold of K_4^- .

Proof of Theorem 1.6. Theorem 5.1 implies that tournament constructions have the largest number of edges over all near-extremal constructions. In particular if G is a K_4^- -free 3-graph on n vertices and n is sufficiently large, then there exists a tournament T on n vertices such that

$$\delta_2(G) \leq \left\lfloor \frac{3|G|}{\binom{n}{2}} \right\rfloor \leq \left\lfloor \frac{3|C(T)|}{\binom{n}{2}} \right\rfloor \leq \left\lfloor \frac{n+1}{4} \right\rfloor, \quad (5.1)$$

where the last inequality follows from (4.2). This gives the claimed upper bound on $\text{ex}_2(n, K_4^-)$. Proposition 4.4 implies that we have equality for $n = 4k + 3$ and $n = 4k + 2$ if there exists a skew Hadamard matrix of order $4k + 4$.

Conversely, note that by Proposition 4.3, for $n = 4k + 3$ the last inequality in (5.1) is an equality if and only if there exists a skew Hadamard matrix of order $4k + 4$ and T is an n -vertex tournament with $\delta_2(T) = t(n)$. Furthermore, by the codegree regularity for such tournaments observed in Corollary 4.2, we have $\lfloor \frac{3|G|}{\binom{n}{2}} \rfloor = \lfloor \frac{3|C(T)|}{\binom{n}{2}} \rfloor$ only if $|G| = |C(T)|$. By the ‘moreover’ part of Theorem 1.6 and (5.1), it follows that $\delta_2(G) = \frac{n+1}{4}$ only if $G = C(T)$. In particular, for $n = 4k + 3$ sufficiently large, if $\text{ex}_2(n, K_4^-) = \frac{n+1}{4}$, then every codegree-extremal 3-graph for that value of n is given by an Erdős–Hajnal tournament construction, and there exists skew Hadamard matrices of order $n + 1$. \square

5.1 | Proof of Theorem 5.1

Throughout the remainder of this section, we assume that $n_0 \in \mathbb{N}$ is sufficiently large, and $n \geq n_0$. Let G be a K_4^- -free 3-graph on $[n]$ with $\delta_2(G) = \frac{n}{4} - o(n)$, and T a tournament on $[n]$ that maximises $|G \cap C(T)|$ (or, equivalently, minimises $|G \setminus C(T)|$). The definition of T implies, amongst other things, that we cannot increase the size of the intersection $|G \cap C(T)|$ by reversing the orientation of an arc \vec{xy} in T . In terms of the joint neighbourhoods of $\{x, y\}$, this means

$$|(G \cap C(T))_{xy}| \geq |G_{xy} \cap \{z : \vec{xz}, \vec{zy} \in T\}|. \quad (5.2)$$

We begin our proof by showing that the symmetric difference of G and $C(T)$ is small.

Claim 5.2. $|G \Delta C(T)| = o(n^3)$.

Proof. By Theorem 1.5, there exists a tournament T' such that

$$|G \setminus C(T)| \leq |G \setminus C(T')| \leq |G \Delta C(T')| = o(n^3).$$

The second of the two inequalities in (4.2) shows that $C(T)$ and $C(T')$, being tournament constructions, can have at most $\binom{n}{3}/4 + O(n^2)$ edges each. On the other hand, the codegree condition on G tells us that G must have at least $\binom{n}{2} \cdot \frac{n/4 - o(n)}{3} = \binom{n}{3}/4 - o(n^3)$ edges. Therefore, when n is

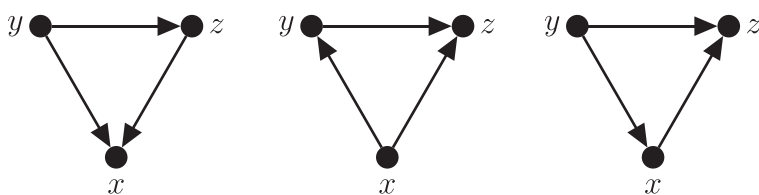


FIGURE 7 The arcs \vec{yz} in the sets B_x^- , B_x^+ and B_x^{\mp} .

sufficiently large, we have

$$|C(T) \setminus G| = |C(T)| - |G| + |G \setminus C(T)| = o(n^3),$$

and hence also $|G \Delta C(T)| = o(n^3)$. \square

Next, we show that T must be a quasirandom tournament.

Claim 5.3. T is $o(1)$ -quasirandom.

Proof. Since $|G \Delta C(T)| = o(n^3)$, $|G| = \binom{n}{3}/4 + o(n^3)$ and $\delta_2(G) = n/4 - o(n)$, there are only $o(n^2)$ pairs of vertices x and y that have $|n/4 - C_T(x, y)| = \Omega(n)$. Therefore, T is $o(1)$ -quasirandom by [4, Theorem 3.2]. \square

Let $B := G \setminus C(T)$, and $M := C(T) \setminus G$ be the 3-graphs consisting of the *bad* and *missing* triples, respectively. Our aim is to show that $|M| \geq |B|$. Before we can do so, we need to prove some auxiliary results on the degrees and codegrees in B and M .

Fix an arbitrary vertex $x \in V(G)$, and partition the arcs in T corresponding to the pairs from B_x into three oriented graphs as follows (see also Figure 7):

$$B_x^- := \{\vec{yz} \in T : xyz \in G, \{y, z\} \subseteq N_T^-(x)\},$$

$$B_x^+ := \{\vec{yz} \in T : xyz \in G, \{y, z\} \subseteq N_T^+(x)\} \text{ and}$$

$$B_x^{\mp} := \{\vec{yz} \in T : xyz \in G, y \in N_T^-(x), z \in N_T^+(x)\}.$$

Our aim in the next few claims is to show that all these three oriented graphs contain only $o(n^2)$ arcs, which (since x is arbitrary) will imply that all links graphs in B are sparse (Corollary 5.12). Let X^- be the collection of those vertices $y \in N_T^-(x)$ that have out-degree $\Omega(n)$ in B_x^- .

Claim 5.4. $|X^-| = o(n)$.

Proof. Fix $y \in X^-$ and let $Y^+ := N_{B_x^-}^+(y)$ denote the out-neighbourhood of y in B_x^- . We now consider the effect of reversing the orientation of $\vec{yx} \in T$. Let T^R be the resulting tournament. For each $w \in Y^+$, the triple xyw now belongs to $G \cap C(T^R)$, hence $|(G \cap C(T))_{xy}| \geq |Y^+| = \Omega(n)$ by (5.2).

Clearly, the joint neighbourhood $(G \cap C(T))_{xy}$ is a subset of $N_T^+(x) \cap N_T^-(y)$. Also note that $Y^+ \subseteq N_T^-(x) \cap N_T^+(y)$. Since both Y^+ and $(G \cap C(T))_{xy}$ have sizes $\Omega(n)$, the $o(1)$ -quasirandomness

of T (see [3, Theorem 1 (P8)]) yields that

$$\frac{1}{2} \cdot |Y^+| \cdot |(G \cap C(T))_{xy}| - o(n^2) = \Omega(n^2)$$

pairs $(w, z) \in Y^+ \times (G \cap C(T))_{xy}$ are oriented as \vec{wz} in T .

For each such pair, we have that $ywz \in C(T)$. On the other hand, $y\vec{w} \in B_x^-$ and thus $xyw \in G$. Since $xyz \in G$ and G is K_4^- -free, we have $ywz \in C(T) \setminus G = M$. We conclude that each vertex $y \in X^-$ is incident to at least $\Omega(n^2)$ missing triples of this form. Summing over all $y \in X^-$, we have

$$|X^-| \times \Omega(n^2) = |\{ywz \in M : xyz \in G \cap C(T), y\vec{w} \in B_x^-, y \in X^-\}| \leq |G \Delta C(T)| = o(n^3),$$

which, after division by $\Omega(n^2)$, yields the claimed bound $|X^-| = o(n)$. \square

A symmetric argument yields that $|X^+| = o(n)$, where X^+ is the set of vertices $z \in N_T^+(x)$ with in-degree at least $\Omega(n)$ in B_x^+ .

By definition, every vertex outside of $X^- \cup X^+$ has out-degree $o(n)$ in B_x^- and in-degree $o(n)$ in B_x^+ . Therefore, the number of arcs in B_x^- and in B_x^+ is $o(n^2)$.

Corollary 5.5. $|B_x^-| = o(n^2)$ and $|B_x^+| = o(n^2)$.

We now turn our attention to the cross-arcs B_x^\mp . We begin with the following simple claim.

Claim 5.6. For every $y\vec{z} \in B_x^\mp$ and every w such that $z\vec{w}, w\vec{y} \in T$, at least one of the triples wyz, wxy, wxz is in M .

Proof. Clearly, $wyz \in C(T)$. If $x\vec{w} \in T$, then also $wxy \in C(T)$, while if $w\vec{x} \in T$, then $wxz \in C(T)$. Either way, the set $xyzw$ induces two triples in $C(T)$. Since $xyz \in G \setminus C(T)$ and G is K_4^- -free, at least one of wyz, wxy, wxz must be in $C(T) \setminus G = M$. \square

Let X_2^- be the collection of $y \in N_T^-(x)$ having a linear out-degree in B_x^\mp , that is, $d_{B_x^\mp}^+(y) = \Omega(n)$. We shall focus on ‘typical’ elements of X_2^- by moving to a subset $Y \subseteq X_2^-$, where $y \in Y$ if and only if the following three properties are satisfied:

- (i) y is incident with $o(n)$ arcs in $B_x^- \cup B_x^+$,
- (ii) $|d^+(y, N_T^+(x)) - d^-(y, N_T^+(x))| = o(n)$ and
- (iii) $|M_y| = o(n^2)$.

First of all, we show that the set Y is not much smaller than X_2^- .

Claim 5.7. $|Y| = |X_2^-| - o(n)$.

Proof. By Corollary 5.5, only $o(n)$ vertices of X_2^- can be incident to linearly many arcs in $B_x^- \cup B_x^+$ (in fact, the vertices in X_2^- are by definition incident to no arc in B_x^+). Further, the $o(1)$ -quasirandomness of T yields that only $o(n)$ of vertices of X_2^- can fail to satisfy Property (ii). Finally, $|M| = o(n^3)$ by Claim 5.2, hence there can be only $o(n)$ vertices $y \in [n]$ with a quadratic degree in M , that is, having $|M_y| = \Omega(n^2)$. \square

For each $y \in Y$, let Z_{xy} be the collection of vertices $z \in N_T^+(x)$ such that $\vec{yz} \in B^+(x)$. Note that $|Z_{xy}| = \Omega(n)$ by definition of $X_2^- \supseteq Y$. Next, we define W_{xy} to be the collection of $w \in N_T^+(x)$ such that $\vec{wy} \in T$. By definition, $W_{xy} = C(T)_{xy}$. Moreover, Property (ii) of $y \in Y$ yields that $|W_{xy}| \geq d_{B_x^+}^+(y) - o(n) = \Omega(n)$.

By the $o(1)$ -quasirandomness of T , there are $\Omega(n^2)$ pairs $(w, z) \in W_{xy} \times Z_{xy}$ that are oriented as \vec{zw} in T . For such pairs, Claim 5.6 yields that $|\{wxy, wyz\} \cap M| \geq 1$. We use this fact to prove that the codegree of xy in M must be large.

Claim 5.8. $|M_{xy}| \geq |W_{xy}| - o(n)$ for every $y \in Y$.

Proof. Given a vertex $y \in Y$, let $W'_{xy} \subseteq W_{xy}$ consist of those vertices $w \in W_{xy}$ satisfying:

- (a) $|M_{wy}| = o(n)$ and
- (b) $d_T^-(w, Z_{xy}) \geq |Z_{xy}|/2 - o(n)$.

By Property (iii) of Y , only $o(n)$ vertices in W_{xy} can have linear degree in M_y . Further, the $o(1)$ -quasirandomness of T yields that only $o(n)$ vertices in W_{xy} have in-degree from Z_{xy} below $|Z_{xy}|/2 - o(n)$. Hence, $|W'_{xy}| = |W_{xy}| - o(n)$.

However, for each $w \in W'_{xy}$, there are $\Theta(|Z_{xy}|) = \Omega(n)$ vertices $z \in Z_{xy}$ such that we have both $\vec{zw} \in T$ (by (b)) and $wyz \notin M$ (by (a)). Therefore, $wxy \in M$ by Claim 5.6 (or more precisely the consequence of Claim 5.6 noted above Claim 5.8), and thus $|M_{xy}| \geq |W'_{xy}| = |W_{xy}| - o(n)$ as claimed. \square

As the next step in the proof, we show that most of the codegree of such pairs xy , where $y \in Y$, lies inside B .

Claim 5.9. $|B_{xy} \cap N_T^+(x)| = d(x, y) - o(n)$ for every $y \in Y$.

Proof. Let $y \in Y$. Clearly, $|C(T)_{xy}| - |M_{xy}| + |B_{xy}| = d(x, y)$. Since $C(T)_{xy} = W_{xy}$, Claim 5.8 readily yields that $|B_{xy}| = d(x, y) - o(n)$. Moreover, since y is incident with $o(n)$ arcs in B_x^- and B_x^+ (by Property (i) in the definition of Y), the vast majority of the codegree of the pair xy must come from arcs incident to y that lie in B_x^+ . Since $y \in N_T^-(x)$, the other endpoints of those arcs are from $N_T^+(x)$. \square

Since most of the codegree of xy is in B , $G \cap C(T)$ cannot have many edges containing xy .

Corollary 5.10. $|(G \cap C(T))_{xy}| = o(n)$ for every $y \in Y$.

We are now ready to prove that $|B_x^+| = o(n^2)$. By symmetry, it is enough to show that only $o(n)$ vertices from $N_T^-(x)$ can be incident to linearly many edges of B_x^+ .

Claim 5.11. $|X_2^-| = o(n)$.

Proof. Suppose for a contradiction that $|X_2^-| = \Omega(n)$. Thus, by Claim 5.7, we have $|Y| = \Omega(n)$. By Claim 5.9 and Property (i) of Y , at least $d(x, y) - o(n) \geq n/4 - o(n)$ of the elements $z \in B_{xy}$ correspond to arcs $\vec{yz} \in B_x^+$ with $z \in N_T^+(x)$. We conclude that there are at least $|Y| \cdot (n/4 - o(n)) = \Omega(n^2)$ arcs from $Y \subseteq N_T^-(x)$ to $N_T^+(x)$.

Let $X_2^+ \subseteq N_T^+(x)$ be the set of vertices with in-degree $\Omega(n)$ in B_x^+ . Clearly, $|X_2^+| = \Omega(n)$. Arguments analogous to those in the proofs of Claims 5.7–5.9 yield the existence of a subset $Z \subseteq X_2^+$ of size $|X_2^+| - o(n)$ such that every $z \in Z$ satisfies:

- $|d^+(z, N_T^-(x)) - d^-(z, N_T^-(x))| = o(n)$,
- $|(G \cap C(T))_{xz}| = o(n)$ and
- $|M_{xz}| \geq |U_{xz}| - o(n)$, where U_{xz} is the collection of vertices $u \in N_T^-(x)$ with $\vec{zu} \in T$.

Now observe that the Property (ii) of $y \in Y$ can be rewritten as $|W_{xy}| = |N_T^+(x)|/2 - o(n)$. Thus for all $y \in Y$ we have

$$|N_T^+(x)| - |B_{xy} \cap N_T^+(x)| = |N_T^+(x) \setminus B_{xy}| \geq |C(T)_{xy}| = |W_{xy}| \geq \frac{|N_T^+(x)|}{2} - o(n).$$

Rearranging the terms and applying Claim 5.9 and the codegree assumption yield

$$|N_T^+(x)| \geq 2|B_{xy} \cap N_T^+(x)| + o(n) \geq 2d(x, y) + o(n) \geq n/2 - o(n).$$

A symmetric argument for a vertex $z \in Z$ yields that $|N_T^-(x)| \geq n/2 - o(n)$, whence both $N_T^+(x)$ and $N_T^-(x)$ have size $n/2 + o(n)$.

By Corollary 5.10, only $o(n^2)$ pairs $(y, z) \in Y \times N_T^+(x)$ extend x to an edge $xyz \in G \cap C(T)$. On the other hand, by Property (i) of Y , for $y \in Y$ most of the codegree of the pair xy in B , which is at least $n/4 - o(n)$ by Claim 5.9, must be from arcs $\vec{yz} \in T$ with $z \in N_T^+(x)$. Since there are $n/4 + o(n)$ such arcs \vec{yz} by Property (ii) of Y and since, as proved above, $|N_T^+(x)| = n/2 + o(n)$, we conclude that all but $o(n)$ vertices from $N_T^+(x) \cap N_T^+(y)$ lie in B_{xy} .

Consider now the tournament T' obtained from T by reversing the orientations of all the arcs \vec{yx} , where $y \in Y$, and all the arcs \vec{xz} , where $z \in Z$. This has the following effect on ‘good’ and ‘bad’ triples:

- (1) all the ‘bad pairs’ $yz \in B_x$ with $(y, z) \in Y \times Z$ become ‘good’ pairs, that is, $xyz \in C(T') \cap G$,
- (2) all the ‘good pairs’ $yz \in (C(T) \cap G)_x$ with $y \in Y$ and $z \in N_T^+(x)$ or $y \in N_T^-(x)$ and $z \in Z$ become ‘bad’ pairs with respect to $C(T')$ and
- (3) all other ‘good’ triples (that is, all apart those corresponding to pairs from (2) above) from $C(T) \cap G$ remain in $C(T') \cap G$.

Note that there might be some new missing triples $xyz \in C(T') \setminus (G \cup C(T))$ but they are irrelevant for our argument. Our aim is to show that $|G \cap C(T')| \geq |G \cap C(T)| + \Omega(n^2)$ (thereby contradicting our assumption that T maximises $|G \cap C(T)|$).

Indeed, by $o(1)$ -quasirandomness of T , there are at least $|Y| \cdot |Z|/2 - o(n^2) = \Omega(n^2)$ arcs $\vec{yz} \in T$ with $(y, z) \in Y \times Z$. Moreover, all but $o(n^2)$ of those pairs yz lie in B_x .

On the other hand, for all $y \in Y$ and $z \in Z$, Corollary 5.10 and an analogous argument for Z show $|(G \cap C(T))_{xy}| = o(n)$ and $|(G \cap C(T))_{xz}| = o(n)$, and hence

$$|C(T') \cap G| - |C(T) \cap G| = \Omega(n^2) - |Y| \cdot \max_{y \in Y} |(G \cap C(T))_{xy}| - |Z| \cdot \max_{z \in Z} |(G \cap C(T))_{xz}| = \Omega(n^2).$$

This contradicts our assumption that T was a best fit tournament. Thus $|X_2^-| = o(n)$, as claimed. \square

A symmetric argument yields that only $o(n)$ vertices from $N_T^+(x)$ can have a linear degree B_x^+ . Therefore, $|B_x^+| = o(n^2)$. Since our choice of x was arbitrary, this latter result together with Corollary 5.5 allows us to conclude that all the link graphs of B must be sparse.

Corollary 5.12. *For every $x \in [n]$, we have $|B_x| = o(n^2)$.*

We now show that T must be asymptotically balanced and that M has low vertex degrees, that is, that for every vertex x the in-degree and the out-degree of x in T are almost equal, and that there are few triples of M containing x .

Claim 5.13. For every $x \in [n]$, $|d_T^+(x) - d_T^-(x)| = o(n)$ and $|M_x| = o(n^2)$.

Proof. The link graph G_x has minimum degree $n/4 - o(n)$, which it inherits from $\delta_2(G) = n/4 - o(n)$. In particular, $|G_x| \geq n^2/8 - o(n^2)$. On the other hand, $|B_x| = o(n^2)$ by Corollary 5.12. Finally, the $o(1)$ -quasirandomness of T yields that

$$|C(T)_x| = \frac{1}{2} \cdot d_T^+(x)(n-1-d_T^+(x)) + o(n^2) \leq \frac{n^2}{8} + o(n^2).$$

Since

$$|G_x| = |C(T)_x| - |M_x| + |B_x|,$$

both parts of the statement follows. \square

We now show that for every pair of distinct vertices x, y the codegree of xy in B is small.

Claim 5.14. For every pair of distinct vertices x, y in $[n]$, we have $|B_{xy}| = o(n)$.

Proof. Fix $x \in [n]$. Suppose that there exists a vertex y such that $|B_{xy}| = \Omega(n)$, and, without loss of generality, suppose that $\vec{xy} \in T$. Let us divide the vertices in B_{xy} into three sets:

$$B_{xy}^+ := \{z \in B_{xy} : \{\vec{xz}, \vec{yz}\} \subseteq T\},$$

$$B_{xy}^- := \{z \in B_{xy} : \{\vec{zx}, \vec{zy}\} \subseteq T\} \text{ and}$$

$$B_{xy}^\pm := B_{xy} \setminus (B_{xy}^- \cup B_{xy}^+) = \{z \in B_{xy} : \{\vec{xz}, \vec{zy}\} \subseteq T\}.$$

It is enough to show that all three sets have size $o(n)$.

Suppose first that $|B_{xy}^+| = \Omega(n)$. Let W_{xy}^- be the collection of $w \in [n]$ such that $\{\vec{wx}, \vec{wy}\} \subseteq T$. By Claim 5.13 and the inclusion-exclusion principle, we have

$$|W_{xy}^-| \geq n - 2 - d^+(x) - d^+(y) + |B_{xy}^+| \geq \Omega(n).$$

By $o(1)$ -quasirandomness of T , there are at least $\frac{1}{2}|W_{xy}^-| \cdot |B_{xy}^+| - o(n^2)$ pairs $(w, z) \in W_{xy}^- \times B_{xy}^+$ such that $\vec{zw} \in T$. For each such pair, both wxz and wyz lie in $C(T)$. However, since $xyz \in B$

and G is K_4^- -free, at least one of the triples wxz, wyz must lie in M , and hence,

$$|M_x| + |M_y| \geq \frac{|W_{xy}^-| \cdot |B_{xy}^+|}{2} - o(n^2) = \Omega(n^2),$$

contradicting the vertex-degree bound in M established in Claim 5.13. Therefore, $|B_{xy}^+| = o(n)$. A symmetric argument yields that $|B_{xy}^-| = o(n)$.

Finally, we turn our attention to the collection B_{xy}^\pm . Suppose, for the sake of a contradiction, that $|B_{xy}^\pm| = \Omega(n)$. Then, by (5.2), we have also $|(G \cap C(T))_{xy}| = \Omega(n)$. Fix a pair of vertices $(w, z) \in (G \cap C(T))_{xy} \times B_{xy}^\pm$.

If $\vec{z}\vec{w} \in T$, then wxz induces a cyclically oriented triangle in T which must be a missing triple; indeed, note that $xyw, xyz \in G$. Similarly if $\vec{w}\vec{z} \in T$, then $ywz \in M$. Analogously to the case of bounding $|B_{xy}^+|$, x or y must have its degree in M being at least

$$\frac{|B_{xy}^\pm| \cdot |(G \cap C(T))_{xy}|}{2} = \Omega(n^2)$$

contradicting Claim 5.13. \square

With these results in hand, we are now ready to prove the crucial claim needed to finish the proof of Theorem 5.1.

Claim 5.15. For every bad edge $E_B \in B$ there are at least $n/4 - o(n)$ missing edges $E_M \in M$ such that $|E_B \cap E_M| = 2$.

Proof. Let $E_B = xyz \in B$ be an arbitrary bad edge. Without loss of generality, the pairs from E_B are oriented as $\vec{x}\vec{y}, \vec{y}\vec{z}, \vec{x}\vec{z}$ in T . Now consider a vertex $w \in C(T)_{xz}$.

By definition, we have $\vec{z}\vec{w}, \vec{w}\vec{x} \in T$. If $\vec{w}\vec{y} \in T$, then $\{wxz, wyz\} \subseteq C(T)$, and since G is K_4^- -free and $xyz \in G$, at least one of the triples wxy, wxz must lie in M . Similarly, if $\vec{y}\vec{w} \in T$, then $\{wxz, wxy\} \subseteq C(T)$ and at least one of the triples lies in M . In either case, each $w \in C(T)_{xz}$ yields a triple containing w that lies in M and intersects E_B in two vertices. However, by the codegree assumption on G and Claim 5.14, we have

$$|C(T)_{xz}| \geq |G_{xy}| - |B_{xy}| = n/4 - o(n)$$

choices of such a vertex w . This finishes the proof of the claim. \square

Let us now finish the proof of Theorem 5.1. Provided that n is sufficiently large, Claim 5.15 yields at least $|B| \cdot \frac{n}{5}$ pairs $(E_B, E_M) \in B \times M$ with $|E_B \cap E_M| = 2$. Moreover, each $E_M = xyz \in M$ can feature in at most $|B_{xy}| + |B_{yz}| + |B_{xz}|$ of these pairs, which by Claim 5.14 is at most $o(n)$. Therefore,

$$o(n) \cdot |M| \geq |B| \cdot \frac{n}{5},$$

thus if there is at least one bad edge, then $|M| > |B|$. We conclude that $|C(T)| \geq |(G)|$ with equality only if $G = C(T)$. \square

6 | CONCLUDING REMARKS

6.1 | Better lower bounds for $\text{ex}_2(n, K_4^-)$?

We have shown in Theorem 1.6 that $\text{ex}_2(n, K_4^-) \leq \lfloor \frac{n+1}{4} \rfloor$, and that this upper bound is tight when $n = 4k + 3$ or $4k + 2$ and there exists a skew Hadamard matrix of order $4k + 4$. Short of proving Seberry's (and hence Hadamard's) conjecture, what is the best general lower bound one can give on $\text{ex}_2(n, K_4^-)$?

Taking a tournament T on $[n]$ uniformly at random and considering the Erdős–Hajnal construction $C(T)$ yields a lower bound of $\frac{n}{4} - O(\sqrt{n \log n})$ via a standard Chernoff bound on the probability that a given pair has low codegree and a union bound over all pairs. One can do a little better, however, by exploiting results about the distribution of primes congruent to 3 (mod 4).

Indeed, suppose that $c > 0$ is such that for every n sufficiently large, there exists an integer m with $n < m \leq n + n^c$ such that there exists a skew Hadamard matrix of order m . Then, as shown in Proposition 4.4, there is a tournament T on $m - 1$ vertices such that $\delta_2(C(T)) = \frac{m}{4}$. Now delete a set of $(m - 1 - n)$ vertices of T chosen uniformly at random to obtain a new tournament T' on n vertices exactly. Taking Chernoff bounds for the hypergeometric distribution (see, for example, [16, Lemma 2]) and a union bound over all pairs yields that w.h.p.

$$\text{ex}_2(n, K_4^-) \geq \delta_2(C(T')) \geq \frac{n}{4} - O\left(\sqrt{(m - n) \log n}\right) = \frac{n}{4} - O\left(n^{c/2+o(1)}\right).$$

As stated in the introduction, Seberry's conjecture is known to hold for all values of m of the form $m = 2^t \prod_{i \in I} (q_i + 1)$, where $t \in \mathbb{Z}_{\geq 0}$ and q_i is a prime power congruent to 3 (mod 4) for all $i \in I$ (see [34, Theorem 4.1]). In particular it holds for all m such that $m - 1$ is a prime congruent to 3 (mod 4). Now it is known [2, Theorem 3(I)] that for $n \in \mathbb{N}$, there exists a prime $m - 1$ congruent to 3 (mod 4) in the interval $n \leq m - 1 \leq n + n^{0.55+o(1)}$. Together with the argument in the paragraph above, this implies

$$\text{ex}_2(n, K_4^-) \geq \frac{n}{4} - O\left(n^{0.275+o(1)}\right).$$

We believe, however, that the correct bound should be of the form $n/4 - O(1)$, and we propose the following problem, which can be viewed as a weakening of Seberry's conjecture.

Problem 6.1. Show that there exists a constant $C > 0$ such that for any $n \in \mathbb{N}$ there exists a tournament T on n vertices with $\delta_2(T) \geq \frac{n}{4} - C$.

6.2 | Codegree density and smooth Turán density

As mentioned in Section 1.3, the codegree density of K_4^- coincides with various 'smooth' or 'weakly quasirandom' versions of Turán density, and the near-extremal constructions are the same. What is more, the conjectured value of the still unknown codegree density of K_4 is the same as the values of several of its 'weakly quasirandom' Turán densities, as shown by Reiher, Rödl and Schacht [29, 31] (see also [30, 32]). Again the (random) extremal constructions are the same. It is natural to ask whether there is any relationship in general between the codegree density of a 3-graph and its

'smooth' Turán densities. The 3-graph $F_{3,2} = \{abc, abd, abe, cde\}$ has $\pi_2(F_{3,2}) = 1/3$, as proved in [11], but has trivial, zero density for all notions of smooth Turán density, so such a relationship would have to take the form of a one-sided upper bound.

To make this discussion more precise, let us consider the weakest notion of weak quasirandomness: a 3-graph G on n -vertices is said to be $K_{1,1,1}$ -weakly quasirandom (wqr) with parameters (ε, p) if for all subsets $X \subseteq V(G)$, $||G[X]| - p \binom{|X|}{3}|| \leq \varepsilon n^3$. A sequence of 3-graphs $(G_n)_{n \in \mathbb{N}}$ with $|V(G_n)| \rightarrow \infty$ as $n \rightarrow \infty$ is then said to be $K_{1,1,1}$ -wqr with density p if each G_n is $K_{1,1,1}$ -wqr with parameters (p, ε_n) and $\varepsilon_n \rightarrow 0$. The $K_{1,1,1}$ -wqr Turán density of a 3-graph F is the infimum over all $p \geq 0$ such that in every $K_{1,1,1}$ -wqr sequence $(G_n)_{n \in \mathbb{N}}$ with density at least p , all but finitely many of the G_n contain a copy of F as a subgraph. Stronger notions of weak quasirandomness and associated Turán densities (which are lower bounded by the $K_{1,1,1}$ -wqr Turán density) exist, see [31]. Our question above in its strongest form thus asks.

Question 6.2. Is the codegree density of a 3-graph F always an upper bound on its $K_{1,1,1}$ -wqr Turán density?

Shortly after the first draft of this paper was written, Falgas-Ravry and Lo [10] gave a positive answer to Question 6.2 when $K_{1,1,1}$ -wqr is replaced by a slightly stronger form of weak quasirandomness. One reason to believe the answer to Question 6.2 might be positive is that it would be enough to show that one can extract from any $K_{1,1,1}$ -wqr 3-graph with density p a 'large' subgraph in which 'most' of the pairs have codegree density p . This follows from an argument of [10]; we give here a simpler proof of this fact tailored to the present setting.

Proposition 6.3. *Let F be a 3-graph and $\varepsilon > 0$ be fixed. Then there exist $\eta > 0$ and $n_0 \in \mathbb{N}$ such that if G is a 3-graph on $n \geq n_0$ vertices in which all but ηn^2 pairs of vertices have codegree at least $(\pi_2(F) + \varepsilon)n$, then G must contain a copy of F .*

Proof. Let G be a 3-graph in which all but ηn^2 pairs of vertices have codegree at least $(\pi_2(F) + \varepsilon)n$. Add at most ηn^3 triples to G to obtain a new 3-graph G' on n vertices with $\delta_2(G') \geq (\pi_2(F) + \varepsilon)n$. Provided that n_0 is sufficiently large, $\delta_2(G') > \text{ex}_2(n, F) + \frac{\varepsilon}{2}n$.

As shown by Mubayi and Zhao in [24, Proposition 1.4], the codegree density function exhibits *supersaturation*, which in our case implies that G' contains $C_\varepsilon n^{v(F)}$ copies of F , where $C_\varepsilon > 0$ is a constant depending on ε . Now let us remove the triples from $G' \setminus G$. Each of these is adjacent to at most $v(F)! \cdot n^{v(F)-3}$ copies of F in G' . Thus there must remain at least $(C_\varepsilon - \eta v(F)!) \cdot n^{v(F)}$ copies of F in G . Taking $\eta = \eta(\varepsilon, F)$ sufficiently small, this is strictly positive, establishing our claim. \square

Proposition 6.3 says, in essence, that allowing for $o(n^2)$ pairs to have low codegree does not change the codegree threshold by more than $o(n)$. Finally, let us note that embarrassingly we do not know the answer to the following question.

Question 6.4. Let $(G_n)_{n \in \mathbb{N}}$ be a sequence of $K_{1,1,1}$ -wqr 3-graphs with density $p > 0$. Must there exist a sequence of subgraphs $(H_n)_{n \in \mathbb{N}}$ with $H_n \subseteq G_n$, $v(H_n) \rightarrow \infty$ and $\delta_2(H_n)/v(H_n)$ bounded away from zero?

In other words, does a smooth distribution of the edges in a 3-graph imply the existence of a reasonably large subgraph with high internal codegree density? Although the instinctive reaction of many researchers we have consulted was that the answer to Question 6.4 should be negative,

a counterexample seems hard to come by: the quasirandomness condition pulls in the direction of random constructions, where it is difficult to control the codegree of subgraphs on $O(\log n)$ vertices, while the necessity to have no large subgraph with large internal codegree density pushes us towards structured constructions, which fail to be sufficiently quasirandom. Whether positive or negative, we expect that an answer to Question 6.4 would thus be quite interesting.

We note that Falgas-Ravry and Lo [10] gave a positive answer to Question 6.4 albeit again only for a stronger notion of weak quasirandomness than $K_{1,1,1}$ -wqr. However, they also provided examples of $K_{1,1,1}$ -wqr graphs in which the order of a largest subgraph with non-zero minimum codegree only grew as the inverse of the error parameter ε , suggesting that the situation is not entirely obvious one way or the other.

6.3 | Turán problems with codegree conditions

The codegree-extremal construction for K_4^- is quite different from the recursive construction of Frankl and Füredi which is conjectured to be extremal for the Turán problem (indeed the later construction has pairs of vertices with codegree zero). A natural question is what happens if we interpolate between the two problems. Explicitly, for $c \in [0, \pi_2(F)]$, let $\text{ex}^{\delta_2 \geq c}(n, F)$ denote the maximum number of edges in an F -free 3-graph on n vertices with minimum codegree at least $c(n-2)$ (if such a 3-graph exists).

Problem 6.5. Determine the asymptotic behaviour of $\text{ex}^{\delta_2 \geq c}(n, K_4^-)$ for $c \in [0, 1/4]$.

Write $f(c)$ for the limit of $\text{ex}^{\delta_2 \geq c}(n, K_4^-)/\binom{n}{3}$ as $n \rightarrow \infty$ (it can be shown this limit exists). The function $f(c)$ is non-increasing in $[0, \pi_2(F)]$ and takes values in $[0, \pi(F)]$. We have shown $f(1/4) = 1/4$, and Mubayi conjectured in [22, Conjecture 2] $f(0) = 2/7$. We can give a lower bound for $f(c)$ in the range $(0, 1/4)$ as follows.

Let H_6 denote the 3-graph on $[6]$ with edges $\{123, 234, 345, 145, 125, 136, 356, 256, 246, 146\}$. This 3-graph was constructed by Frankl and Füredi [13] and has the property that the link graph of every vertex is a cycle on 5-vertices; in particular, H_6 , its blow-ups and its iterated blow-ups are all K_4^- -free. Given $c > 0$, let $t = \lfloor \frac{\log(1/4c)}{\log 6} \rfloor$. Add all triples from a balanced blow-up of H_6 on n vertices, then repeat this construction inside each of the 6 parts, iterating this procedure a total of t times. Finally, inside each of the 6^t parts of size $(1 + o(1))n/6^t$ that remain place a $C(T)$ construction. This gives a K_4^- -free 3-graph with minimum codegree $(1/4 + o(1))n/6^t \geq (c + o(1))n$ and $(2/7 - 6^{-2t}/28 + o(1))\binom{n}{3}$ edges.

Corollary 6.6. $f(c) \geq \frac{2}{7} - \frac{1}{28} 6^{-2\lfloor \log(1/4c)/\log 6 \rfloor}$.

Note that our lower bound is not a continuous function of c ; however, as it does not seem possible to shift in a continuous way from a $C(T)$ (codegree-extremal) construction to an iterated blow-up of H_6 (the conjectured extremal construction for Turán density), this is a behaviour we could plausibly encounter.

APPENDIX A: DESCRIPTION OF THE SUPPLEMENTARY COMPUTER PROGRAMS

We split the task of a formal verification of our proof of Lemma 2.8 into six steps (A.1–A.6), where each step will be accompanied with a small script written in SAGE [35] that performs the described

task. Note that we also have independent implementations in C of each of the tasks listed below, which do run faster, however, they are lengthier and verifying their correctness is somewhat more tedious.

Before we continue, let us briefly recall some notation from Section 2. For two 3-graphs H and G , $p(H, G)$ denotes the probability that a random $v(H)$ -vertex subgraph of G is isomorphic to H . Also recall that for a k -vertex type σ and a fixed σ -flag F , the value p_F^σ denotes the probability that a random injection from $[k]$ to F^\emptyset yields a σ -flag that is isomorphic to F . Finally, for two σ -flags F_1 and F_2 and a $(v(F_1) + v(F_2) - v(\sigma))$ -vertex σ -flag H , $p(F_1, F_2, H)$ denotes the probability that a random partition of the unlabelled vertices of H into two parts of respective sizes $v(F_1) - v(\sigma)$ and $v(F_2) - v(\sigma)$ yields σ -flags isomorphic to F_1 and F_2 .

To simplify the following presentation, let us define the so-called *flag pair density*. Given a pair of σ -flags F_1 and F_2 , and a 3-graph G with $v(F_1) + v(F_2) - v(\sigma)$ vertices, the flag pair density $\tilde{p}(F_1, F_2, G)$ is defined as

$$\sum_{\substack{F \in \mathcal{F}_{v(G)}^\sigma \\ F^\emptyset = G}} p_F^\sigma \cdot p(F_1, F_2, F).$$

For a 3-graph G satisfying $v(G) > v(F_1) + v(F_2) - v(\sigma)$, we generalise the above and define the flag pair density $\tilde{p}(F_1, F_2, G)$ as follows:

$$\tilde{p}(F_1, F_2, G) := \sum_{H \in \mathcal{F}_{v(F_1)+v(F_2)-v(\sigma)}} p(H, G) \cdot \tilde{p}(F_1, F_2, H).$$

A.1 | The list of 7-vertex K_4^- -free 3-graphs

We identify 3-graphs on the vertex set $[k]$ with the collection of their edges. We order the collection of such 3-graphs in the lexicographic order inherited from the order on $[k]$.

Given k , we generate the list of k -vertex K_4^- -free 3-graphs on the vertex set $[k]$ that are lexicographically minimal within their isomorphism class in the following way: we iteratively go over all the labelled k -vertex K_4^- -free 3-graphs G such that the vertex subset $[k-1]$ induces some lexicographically minimal $(k-1)$ -vertex K_4^- -free 3-graph, and include in the list exactly those 3-graphs G that are lexicographically minimal. We note that our SAGE script outputs the list in the same order and format as Flagmatic 1.5.1 does.

For $k = 7$, generating this list readily yields that $|\mathcal{F}_7| = 8157$. Moreover, the maximum number of edges of a 7-vertex K_4^- -free 3-graph is 15, and the number of 3-graphs in \mathcal{F}_7 with exactly 2, 3, 4, ..., 14 and 15 edges is equal to 3, 9, 32, 102, 304, 752, 1451, 2022, 1909, 1118, 374, 70, 8 and 1, respectively.

SAGE script: `gen7.sage`

Output file: `graph_lists/list7`

A.2 | The lists of 6-vertex ι_i -flags, $i \in [6]$, and the corresponding pair densities

Firstly, for every $F \in \mathcal{F}_6$ we consider all possible injections of $[5]$ into $V(F)$ and generate the appropriate lists $\mathcal{F}_6^{\iota_i}$. Then, for every $G \in \mathcal{F}_7$, we consider all the injections $m : [5] \rightarrow V(G)$ together with the partitions of $V(G) \setminus m([5])$ into two non-empty parts. This allows us to directly compute the coefficients $\tilde{p}(F_1, F_2, G)$ for every expression of the form

$$\llbracket F_1 \times F_2 \rrbracket_{\iota_i} = \sum_{F \in \mathcal{F}_7^{\iota_i}} p_F^{\iota_i} \cdot p(F_1, F_2, F) \cdot F^\emptyset = \sum_{G \in \mathcal{F}_7} \tilde{p}(F_1, F_2, G) \cdot G.$$

As in the previous subsection, we output this data in the same format as Flagmatic 1.5.1 does.

SAGE script: `gen_flags.sage`

Output files: `flagmatic_flags-pruned.rat` and `graph_lists/list6_iotaX` for $X \in [6]$

A.3 | The codegree expressions $D \in \mathcal{D}$ as linear combinations of \mathcal{F}_7

Similarly to the previous section, we first generate the list \mathcal{F}_6^τ by considering all the possible injections of $[2]$ into the elements of \mathcal{F}_6 . For each $F \in \mathcal{F}_6^\tau$ we then compute the coefficients $\tilde{p}(F, E^\tau, G)$ and $\tilde{p}(F, N^\tau, G)$ for the expressions

$$\llbracket F \times E^\tau \rrbracket_\tau = \sum_{G \in \mathcal{F}_7} \tilde{p}(F, E^\tau, G) \cdot G \quad \text{and} \quad \llbracket F \times N^\tau \rrbracket_\tau = \sum_{G \in \mathcal{F}_7} \tilde{p}(F, N^\tau, G) \cdot G.$$

As we have noted in Section 2.2, although the set \mathcal{F}_6^τ has size 1643, the symmetry of the type τ yields that $|D| = 905$.

SAGE script: `codeg.sage`

Output file: `ineq_codeg`

A.4 | The tight-path expressions $\mathcal{P}_0, \mathcal{P}_1$ and \mathcal{P}_2 as linear combinations of \mathcal{F}_7

For the 4-vertex types σ_i with $i \in \{0, 1, 2\}$ edges, we generate the lists $\mathcal{F}_5^{\sigma_i}$ by considering all the possible injections of $[4]$ into the elements of \mathcal{F}_5 , and then identify the corresponding elements of $\mathcal{P}_i \subseteq \mathcal{F}_5^{\sigma_i}$. Next, in a similar fashion as in the two previous sections, we compute for every $F_1, F_2 \in \mathcal{F}_5^{\sigma_i}$ and $G \in \mathcal{F}_7$ the coefficients $\tilde{p}(F_1, F_2, G)$ in

$$\llbracket F_1 \times F_2 \rrbracket_{\sigma_i} = \sum_{F \in \mathcal{F}_6} \tilde{p}(F_1, F_2, F) \cdot F = \sum_{F \in \mathcal{F}_6} \tilde{p}(F_1, F_2, F) \cdot \sum_{G \in \mathcal{F}_7} p(F, G) \cdot G = \sum_{G \in \mathcal{F}_7} \tilde{p}(F_1, F_2, G) \cdot G.$$

SAGE script: `tightpath.sage`

Output file: `ineq_tightpath`

A.5 | The size of the set $\mathcal{E}_T \cap \mathcal{F}_7$ is equal to 247

Firstly, we generate all the 456 non-isomorphic tournaments on 7-vertices. Next, for each such a tournament T , we construct the 3-graph $C(T)$, and check whether it is isomorphic to $C(T')$ for some other tournament T' . It turns out that $|\mathcal{E}_T \cap \mathcal{F}_7| = 247$. Observe that for the sole purpose of verifying (2.2), it is enough to compute the size of $\mathcal{E}_T \cap \mathcal{F}_7$. Indeed, any identity of the form (2.2) may have a positive slack on at most $|\mathcal{F}_7 \setminus \mathcal{E}_T| = 7910$ coordinates by complementary slackness, and the identity for our particular choice of Q_i, I_i, c_j and u_D , where $i \in [6]$, $j \in [3]$ and $D \in \mathcal{D}$, has a positive slack on exactly 7910 coordinates.

SAGE script: `ext7.sage`

A.6 | Verifying the identity (2.2) stated in Lemma 2.8

Given the input data for the entries of the matrices I_i and Q_i , where $i \in [6]$, the positive rationals c_0, c_1, c_2 and u_D for $D \in \mathcal{D}$, we firstly check the positive definiteness of every Q_i . Next, we compute the right-hand side of (2.2) and compare its 8157 coefficients one by one with the left-hand side of (2.2). Finally, we verify that there are exactly 7910 coordinates with a positive slack.

SAGE script: `lemma28.sage`

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