Modeling financial volatility
A functional approach with applications to Swedish limit order book data

Suad Elezović

Department of Statistics
Umeå University, Sweden
Abstract

This thesis is designed to offer an approach to modeling volatility in the Swedish limit order market. Realized quadratic variation is used as an estimator of the integrated variance, which is a measure of the variability of a stochastic process in continuous time. Moreover, a functional time series model for the realized quadratic variation is introduced. A two-step estimation procedure for such a model is then proposed. Some properties of the proposed two-step estimator are discussed and illustrated through an application to high-frequency financial data and simulated experiments.

In Paper I, the concept of realized quadratic variation, obtained from the bid and ask curves, is presented. In particular, an application to the Swedish limit order book data is performed using signature plots to determine an optimal sampling frequency for the computations. The paper is the first study that introduces realized quadratic variation in a functional context.

Paper II introduces functional time series models and apply them to the modeling of volatility in the Swedish limit order book. More precisely, a functional approach to the estimation of volatility dynamics of the spreads (differences between the bid and ask prices) is presented through a case study. For that purpose, a two-step procedure for the estimation of functional linear models is adapted to the estimation of a functional dynamic time series model.

Paper III studies a two-step estimation procedure for the functional models introduced in Paper II. For that purpose, data is simulated using the Heston stochastic volatility model, thereby obtaining time series of realized quadratic variations as functions of relative quantities of shares. In the first step, a dynamic time series model is fitted to each time series. This results in a set of inefficient raw estimates of the coefficient functions. In the second step, the raw estimates are smoothed. The second step improves on the first step since it yields both smooth and more efficient estimates. In this simulation, the smooth estimates are shown to perform better in terms of mean squared error.

Paper IV introduces an alternative to the two-step estimation procedure mentioned above. This is achieved by taking into account the correlation structure of the error terms obtained in the first step. The proposed estimator is based on seemingly unrelated regression representation. Then, a multivariate generalized least squares estimator is used in a first step and its smooth version in a second step. Some of the asymptotic properties of the resulting two-step procedure are discussed. The new procedure is illustrated
with functional high-frequency financial data.

Keywords: Realized quadratic variation; Swedish limit order book; two-step estimation procedure; functional time series; multivariate generalized least squares.

Acknowledgements

This thesis would never be finished without the opportunity to study and work at the Department of Statistics at Umeå University. I would like to give special thanks to all my colleagues and teachers at this inspiring place, particularly to those who encouraged me in any way.

First of all I would like to express my deepest gratitude to my supervisor Professor Xavier de Luna for his consistent and patient support during these five years of my doctoral studies. He has always been an attentive advisor, guiding and directing me thoughtfully when my mind was occupied with searching for a right direction. One of the most valuable things I learned from him is to be focused on details and simplicity in my research.

I would also like to thank my current co-supervisor Associate Professor OlegSeleznjev, from the Department of Mathematical Statistics at Umeå University, for discussions and comments on my works.

I am indebted to Dr. Maria Karlsson for constructive criticism on several of my papers. I am grateful to Dr. Anders Muszta for giving insightful comments on one of my papers. I am also thankful to Dr. Ingeborg Waernbaum for helping me with technical details during the preparation of this manuscript. I thank to Associate Professor Göran Björck, from the Department of Mathematical Statistics at Stockholm University, for helpful comments on one of my papers.

I would also like to thank Xavier de Luna and Kenny Bränberg for helping me with my future career. Very much thanks to all former and present PhD students for both academic and social discussions. In particular, I would like to thank the Board of the Department of Statistics at Umeå University for granting me a unique privilege to pursue the PhD studies.

At last but not least, I want to thank my beloved life partner Enisa for supporting me and tolerating my preoccupation with work.

Finally, thanks to anyone who finds this thesis worth reading.

Suad Elezović
Umeå, February 2009
## Contents

1. Introduction 1

2. The data 3
   2.1 Description of the Swedish limit order book 3
   2.2 Measurement issues: microstructure effects 6

3. Background and motivation 7
   3.1 Notes on theory of RQV 7
   3.2 Computation issues: choice of sampling frequency 9

4. Models for functional data 10
   4.1 Overview 10
   4.2 A functional linear model for longitudinal data 10
   4.3 A functional time series model 12

5. Summary of the papers 13
   5.1 Paper I: Estimating quadratic variation of prices and spreads from the Swedish limit order book 13
   5.2 Paper II: Functional modeling of volatility in the Swedish limit order book 13
   5.4 Paper IV: Functional autoregressive models: theory 14

6. Concluding remarks 15

Papers I–IV
List of papers

The thesis is based on the following papers:


Paper II: Copyright 2008 Elsevier, Computational Statistics & Data Analysis; reproduced with permission from the publisher.
1 Introduction

Modeling and forecasting volatility of asset prices (returns) has been a key issue in the financial markets during the last decades. In the late 1980s, it was found that financial market volatility is usually both time-varying and predictable (see, e.g., Bollerslev and Zhou, 2006). Consequently, a major part of the computations in financial engineering involves measuring the unobservable volatility. The accuracy of any appropriate measure (or model) for such a latent variable is extremely important in making decisions concerned with derivative pricing, risk management and asset allocation, as pointed out in Andersen and Bollerslev (1998). See also Poon and Granger (2003) for a more detailed discussion concerned with the importance of forecasting volatility in the financial markets.

Nowadays, data collection is computerized and financial data is collected in real time, either as transaction tick-by-tick data or as electronic order books. A common feature of this kind of data bases is the extremely large number of records sampled at very high-frequencies (frequencies in minutes or seconds). Quite naturally, the data analysts and researchers try to make use of this source of information which has resulted in a new field concerned with the analysis of ultimate high-frequency data.

An important research direction in this field is concerned with measures of volatility (variability)\(^1\) of asset prices (returns) from high-frequency data. However, certain measurements problems are likely to arise when the sampling time interval decreases (increasing frequency) due to a phenomenon commonly known as the market microstructure effects, discussed in Section 2.2. For more details about the market microstructure effects see, e.g., Smith (1994) and Gourieroux and Jasiak (2001, Ch. 14).

Most of the measures of volatility of prices from the ultimate high-frequency data, proposed to overcome the problem with microstructure effects, rely on a proper choice of the sampling frequency for a relevant estimator. A convenient approach makes use of the sums of squared inter-period returns (higher frequency) to estimate volatility over a single time period (lower frequency). The obtained estimator, commonly known as the realized quadratic variation (RQV), has been extensively studied in Barndorff-Nielsen and Shephard (2002.a, 2004.b) and Bollerslev and Zhou (2002, 2006), among others.

This thesis consists of four papers which may each be read independently.

\(^1\)Since variability and volatility are used interchangeably in the relevant literature, the term volatility is used throughout this work to refer to the variability of asset prices.
Even so, each of these papers is concerned with measures and models for volatility of share prices (returns) from the Swedish limit order book (LOB) data. The nature of this data allows for a continuous time modeling framework since the data records are sampled (observed) over fine time intervals. Another interesting feature of this data is the availability of the records on both prices and quantities of shares.

Consequently, the approach presented in this thesis uses the price-quantity relationship by creating a function, called the bid (ask) curve, which is basically the average price for a given quantity of shares that may be computed at any time record in the book. Then, the daily volatility may be measured by the RQV, which has been defined as the sum of squared intra-day returns, obtained from the above mentioned bid or ask curves.

As far as we know, all related studies create the mentioned volatility measures from the quoted (or simulated) stock prices for a quantity of one share. Hence, a novelty in this thesis is a functional approach where the volatility is modeled as a function of quantity (greater than one) of shares.

Moreover, the thesis adapts a two-step estimation procedure for functional linear models (Fan and Zhang, 2000) to a functional dynamic time series model. The first step is concerned with the estimation of parameters of a time series model for each given value of the argument (quantity) by a univariate ordinary least squares. The obtained estimates are called raw estimates since they are inefficient. The second step involves smoothing the obtained raw estimates by a nonparametric technique, such as local polynomial fitting (e.g. Fan and Gijbels, 1996). The main goal of this procedure is to obtain the estimates for the coefficient functions which are both smooth and more efficient than the raw estimates.

Furthermore, an evaluation of the proposed two-step estimation procedure is done by comparing the performance of fit of smooth estimates and the corresponding raw estimates. This evaluation is performed by a simulation study within a stochastic volatility (SV) continuous time modeling framework. The goal of this study is to motivate the use of the two-step estimation procedure for modeling the financial volatility of the Swedish limit order book data.

A theory for a functional autoregressive time series model is presented in the last paper in the thesis. A new two-step estimator for parameter functions is proposed. This estimator takes into account the contemporaneous correlation structure of the error terms by proposing a multivariate generalized least squares estimation followed by smoothing. Some aspects of inference related
to the proposed estimation procedure are discussed here.

In addition to this introductory part, the thesis is organized as follows. Section 2 gives a description of the data. In particular, some issues concerned with extracting the relevant information from the data are discussed here. In Section 3, a theoretical background for RQV, as a measure of volatility, is reviewed by giving a motivation for our approach. Section 4 describes models for functional data, focusing on a two-step procedure for a functional linear model and a functional time series model. Section 5 summarizes the contents of Papers I-IV. Finally, Section 6 gives some concluding remarks and suggestions for future research.

2 The data

2.1 Description of the Swedish limit order book

The data analyzed throughout this thesis comes from the Swedish electronic limit order book. Since 1 June 1990, most of the trading operations on the Stockholm Stock Exchange (SSE) have been carried out in an open electronic limit order book market. SSE was acquired by OMX (Nordic Exchange market) in 1998 which has then been part of the Nasdaq OMX group since the takeover in February 2008. Trading in this order-driven market is similar to other major markets around the world, such as Paris Bourse or Toronto Stock Exchange.

Any limit order book essentially consists of the complete records of un-executed limit orders. A limit order is an order to buy or sell a quantity of shares at a given limit price or better. These orders enter the computerized limit order book system where they are stored until execution or cancelation. An execution occurs when a matching market order arrives. A market order is an order to buy or sell a specified quantity of shares immediately, at the best available price. Priority of execution is given according to prices and times of submission.

The limit order trading strategy bears a risk of un-execution since the limit orders may be canceled if there is not enough liquidity in the market. The concept of liquidity, in terms of trading liquidity, is usually reflected by an asset’s ability to be transformed into another asset without any significant loss of value. In a more general way, market liquidity may be considered as a function of volume and trading activity in the market. Consequently, in a more liquid market, assets will be traded more frequently, at a smaller
bid/ask spread (the difference between bid and ask prices), and without much influence of the trading volumes on prices (price impact).

Limit order trading is crucial for understanding liquidity provision in the major stock exchange markets all around the world, as pointed out in Ahn et al. (2001). Many studies support the hypothesis that there exists a strong positive relationship between the spreads and price volatility in the limit order markets (e.g., Foucault et al., 2003). In a corresponding manner, Handa and Schwartz (1996) postulate that an increase in short-run volatility results in more incoming limit orders that supply liquidity to the market. In turn, an increase in limit order trading decreases short-run volatility. This suggests that an equilibrium level of limit order trading and short-run price volatility may exist.

In the Swedish limit order book market, information about the five best bids and offers is publicly available via computer screens to all market participants. At any time record (stamp), this information includes the bid and ask prices at five levels, as well as the bid and ask quantities of shares available at the respective prices; see, e.g., Sandås (2001) for more details about this market.

The larger is the quantity available at the best prices, the lower is the average price to be paid for a given quantity of shares, thereby lowering the transaction costs. This, in turn, implies more liquidity. Consequently, the average price per share for a given quantity of shares is likely to be more informative about the liquidity of an asset than the quoted price (for a volume of one share), as pointed out in Gourieroux and Jasiak (2001, p. 358). These average prices, commonly known as the bid (ask) curves, partly summarize the content of the limit order book.

Table 1: Hypothetical limit order book situation at time point \( t_1 \)

<table>
<thead>
<tr>
<th>Level</th>
<th>Bid price</th>
<th>Bid quant.</th>
<th>Ask price</th>
<th>Ask quant.</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>20</td>
<td>230</td>
<td>21</td>
<td>150</td>
</tr>
<tr>
<td>2</td>
<td>19</td>
<td>50</td>
<td>22</td>
<td>350</td>
</tr>
<tr>
<td>3</td>
<td>18</td>
<td>190</td>
<td>23</td>
<td>490</td>
</tr>
<tr>
<td>4</td>
<td>17</td>
<td>670</td>
<td>24</td>
<td>410</td>
</tr>
<tr>
<td>5</td>
<td>16</td>
<td>560</td>
<td>25</td>
<td>200</td>
</tr>
</tbody>
</table>

As an illustration, consider the bid and ask curves in Figure 1, obtained
from a hypothetical limit order book data situation, given in Table 1. For instance, an average price for buying a quantity of $q = 200$ shares is $ap = 21.25$ SEK per share. Buying a quantity of $q = 1600$ shares would increase the average price to $ap = 23.1$ SEK per share while selling a quantity $q = 250$ would cost $bp = 19.92$ SEK per share, on average. The price for a larger quantity, say, $q = 1750$ would lead to a lower average price $bp = 17.25$ SEK per share.

Figure 1: The bid, mid-quote and ask curves at a single time point (data from Table 1)
Hence, the bid curves are monotonically decreasing while the ask curves are monotonically increasing. An interesting feature of the bid (ask) curves is that the size of the spread between them determines the price volatility. Hence, the bid (ask) curves may be considered as measures incorporating both volatility and the liquidity aspect of a limit order book at any time point. Since they are functions, the bid (ask) curves will be used in a functional approach to measure volatility throughout this thesis.

2.2 Measurement issues: microstructure effects

The Swedish limit order book data is, so-called, very (ultimate, ultra) high-frequency financial time series, which means that observations are sampled (in real time) over fine time intervals. Since these intervals are rarely equidistant, the choice of an optimal frequency is of particular importance for an analyst. Furthermore, there may exist several records within a single second which, at this point in time, is the highest possible frequency in the order book. This usually causes problems with identification of the relevant observation at this time point. For a more general discussion of the issues related to very high-frequency data, see Brownless and Gallo (2006) or Gourieroux and Jasiak (2001).

Most of the proposed methods in the available literature treat the problem of sampling over irregular time intervals by aggregating over a fixed time interval. Then, some kind of interpolation is applied. Another common approach is to take the last observations at that particular interval (see, e.g., Barndorff-Nielsen and Shephard, 2005). These approaches also boost the observation errors by a potential loss of information that is included into non-taken (adjacent) observations.

Usually, an average between the bid and ask price is taken as a proxy for the efficient price at a given time point (e.g. at the end of a chosen sampling interval). In the financial literature, the concept of efficient price is associated with an efficient market, where the prices fully represent all available information; see, e.g., Hasbrouck (2002) for more discussion about efficient prices. Then, an appropriate volatility measure is computed using these observed proxies which inevitably introduces some errors. Part of these errors arises due to lack of accuracy of the created volatility measures. Another part emerges due to inaccuracy of the proposed proxies for efficient prices.

Under normal conditions, the efficient price is obscured by the effects of different trading activities, such as bid-ask bounce, irregular (non-synchronous)
trading, discreteness and non-uniqueness of prices etc. The bid-ask bounce effect arises as a result of fluctuations of trading prices between a slightly higher and a slightly lower value, depending on the type of entering orders (buy or sell) (Gourieroux and Jasiak, 2001, p. 359). The trading (and quoted) prices do not exist in continuous time since trading (and quoting) is irregular. Moreover, the prices are quoted in the discrete values which are integers given as multiples of a tick size. Furthermore, the prices are often not unique at a given time point, as pointed out above. All these effects are commonly known as the market microstructure effects; for further discussion about the microstructure effects, see Smith (1994) and Hasbrouck (1991), among others.

3 Background and motivation

3.1 Notes on theory of RQV

The horizon over which the volatility is measured, modeled or forecasted is indeed of crucial interest since different applications use different horizons. For instance, pricing an option requires knowledge about the volatility of the underlying asset over a horizon of time until the expiration date. An accurate assessment of risk of the assets included in a portfolio is another example of the need to measure volatility, since the financial risk and volatility of the assets are closely related to each other. As pointed out in Poon and Granger (2003), forecasting volatility of asset prices is an appropriate starting point in the risk assessment for any investment.

Volatility on a very high-frequency basis is usually considered as time-varying and sometimes predictable. However, on the longer horizons, it is less obvious how to make accurate forecasts of volatility (e.g. Christoffersen and Diebold, 2000). A common approach nowadays is to construct a measure of a lower frequency volatility from the so-called realized volatilities, obtained from the observed price changes over finer time intervals. This approach relies on the assumption that a price series represents the realization of a continuous time diffusion process, such as a stochastic volatility (SV) model.

A general form of an SV model for the efficient log-price process $X_t$ is given as a stochastic differential equation of the following form (see, e.g., Nielsen and Frederiksen, 2007)

$$dX(t) = \mu(t) \, dt + \sigma(t) \, dW(t), \quad (1)$$

where $\mu(t)$ represents the mean (drift) process and $\sigma(t)$ is the instantaneous
(spot) volatility of the process $X(t)$, while $W(t)$ stands for the standard Wiener process (sometimes called Brownian motion). A quantity of particular interest is the integrated variance (IV), given as

$$\sigma^2(T) = \int_0^T \sigma^2(t) \, dt, \quad (2)$$

which may further be divided into fractions over successive time periods

$$\sigma^2(T_1) = \int_0^{T_1} \sigma^2(t) \, dt, \quad \sigma^2(T_2) = \int_{T_1}^{T_2} \sigma^2(t) \, dt, \ldots, \quad (3)$$

obtaining the increments of IV, which are basically the quantities to be estimated. These increments are known as actual variances and they refer to the squared volatility of asset prices over a fixed interval of time (usually a day).

A very convenient way of estimating the cumulative squared volatility over a chosen time interval from, say, 0 to $T$ is to use the previously mentioned RQV, defined as the sum of squared incremental returns from the efficient prices $X_t$, as follows

$$[X, X]_T = \sum_{t_j} \left( X_{t_{j+1}} - X_{t_j} \right)^2, \quad (4)$$

where $t_j = j\Delta$ and $j = 0, 1, 2, \ldots, (n - 1)$; $\Delta$ represents the sampling time interval; $n = \lfloor \frac{T}{\Delta} \rfloor$ where $\lfloor x \rfloor$ stands for the largest integer smaller or equal to $x$. The return process $(X_{t_{j+1}} - X_{t_j})$ includes all available observations in $[0, T]$.

Barndorff-Nielsen and Shephard (2002.a) derive the asymptotic distribution of RQV, showing that RQV is a consistent estimator for the integrated variance (IV). This theory is based on the fact that IV is equal to the quadratic variation (QV) of the efficient price process, for all SV models. More specifically, as the number of sampling intervals $n$ increases (or $\Delta \rightarrow 0$), we will have

$$\langle X, X \rangle_T = \text{plim}_{n \rightarrow \infty} \sum_{t_j} \left( X_{t_{j+1}} - X_{t_j} \right)^2 = \int_0^T \sigma^2(t) \, dt, \quad (5)$$

where the mentioned QV, denoted as $\langle X, X \rangle_T$, represents the limit in probability. This quantity is central to the theory of stochastic calculus. Accordingly, the estimation error of RQV should decrease with increasing frequency,
thus implying that RQV from the highest possible frequency is to be considered as the best possible estimate of IV ($\int_0^T \sigma^2(t)$).

### 3.2 Computation issues: choice of sampling frequency

Unfortunately, the market microstructure effects create a mismatch between the continuous time asset pricing theory and the data sampled at very fine time intervals, as pointed out in Barndorff-Nielsen and Shephard (2004.b). Accordingly, the choice of an appropriate sampling frequency is crucial for accuracy of RQV as an estimator of IV (and QV). The sampling frequency is usually determined in an ad hoc procedure, ranging from 5 to 30 minutes in most studies in the literature.

An appropriate sampling frequency may be chosen by utilizing signature plots, introduced in Andersen et al. (2000) and extensively used in applied works (see, e.g., Barndorff-Nielsen and Shephard, 2002.b, 2005). The signature plots are created from averages of the (daily) RQV estimates for different inter-period (high-frequencies) over a sample of lower frequency periods (days). Then, a plausible frequency is chosen following the rule that the microstructure effects should not have any significant impact on RQV when its estimates are stable (do not show much variation) from one frequency to another. A very detailed classification of the estimators of IV that are aimed at dealing with the microstructure noise is given in Zhang et al. (2005).

The signature plots, as diagnostic tools for determining an optimal frequency for RQV, are used in this thesis (Paper I). Moreover, instead of using the observed (quoted) bid (ask) prices from the limit order book, the bid (ask) curves are used to compute RQV. In this way, the functional RQV time series are created. Since the RQV essentially measures volatility, which is considered as time-varying, a traditional dynamic time series method may be applied to model the RQV time series. In addition, the functional aspect of the RQV time series may imply internal correlations between RQV for different arguments (quantities). Hence, a two-step estimation procedure, first introduced by Fan and Zhang (2000), may be applied to improve the quality of the estimator. This approach is presented in Paper II and further studied in Papers III and IV.
4 Models for functional data

4.1 Overview

In general, functional data analysis is concerned with models and methods for analyzing curves and functions. In Ramsey and Silverman (2006), a collection of tools and techniques for extracting information from functional data is presented. Their approach is characterized by studying the variation of the curves using the rates of change or derivatives. For that purpose, the original discrete data is transformed into smooth curves (functions). These functions are usually represented by a set of basis functions of different kinds, such as the Fourier basis, the spline basis, the wavelet basis, etc. Within this framework, the authors introduce functional linear models to study the variability of a functional variable with respect to a set of covariates.

Some extensions of these methods are presented in Ferraty and Vieu (2006), where a non-parametric approach to functional data is studied together with some aspects of functional time series analysis. See also Müller and Stadtmüller (2005), where a generalized functional linear model is proposed and Guo (2002), who introduces functional models within a mixed-effect framework.

A slightly different approach to the analysis of functional data is introduced by Cai et al. (2000), who are interested in the estimation of functional-coefficient regression models for time series data. The authors apply a local linear regression technique to estimate the unknown coefficient functions. Such an approach is closely related to this thesis, since it involves an estimation of the coefficient functions and smoothing, which are the central concepts in our methodology. However, while Cai et al. (2000) treat the autoregressive coefficients as functions of covariates (random variables, usually lagged responses), our approach treats the coefficients as the unknown real valued functions of arguments (quantities of shares).

4.2 A functional linear model for longitudinal data

Almost any multivariate data may be put into a functional context. This is particularly convenient for data from repeated measurements and for longitudinal data. A functional linear model for longitudinal data is estimated by a two-step estimation procedure, first introduced in Fan and Zhang (1999) and adapted to functional data in Fan and Zhang (2000). Consider a longitudinal data set, with some observed response $y_{ij}$, a set of covariates $X_{ij}$ as
well as the times of measurements \( \{t_{ij}, j = 1, \ldots, T_i\} \), where \( i \) refers to the \( i \)th subject (individual). Then, the data is

\[
(t_{ij}, X_{ij}, y_{ij}), \quad j = 1, 2, \ldots, T_i, \quad i = 1, 2, \ldots, n. \tag{6}
\]

A model for studying the relation between \( X(t) \) and \( Y(t) \) is

\[
Y(t) = X(t)^\prime \beta(t) + \varepsilon(t), \tag{7}
\]

where \( \varepsilon(t) \) is a correlated process with zero-mean that accounts for the part of variation in \( Y(t) \) that cannot be explained by the covariates and \( \beta(t) \) is the coefficient vector.

Then, the repeated measurements data in (6) is considered as a random sample from model (7), which may result in the following representation

\[
Y_i(t_{ij}) = X_i(t_{ij})^\prime \beta(t_{ij}) + \varepsilon_i(t_{ij}), \tag{8}
\]

where \( Y_i(t_{ij}) = Y_{ij}, X_i(t_{ij}) = X_{ij} \) and \( \varepsilon_i(t) \) has zero mean and the covariance function \( \gamma(s, t) = \text{cov}(\varepsilon_i(s), \varepsilon_i(t)) \).

Model (7) belongs to a class of linear models for longitudinal data where the coefficients are allowed to vary over time. Within this framework, model (7) is usually estimated by smoothing spline or kernel methods, which have been shown to be difficult and computationally slow. To overcome the inflexibility of the existing methods, a two-step procedure is proposed. In the first step, a standard linear model is fitted with OLS using the data collected at each distinct \( t_j \) and obtaining the raw estimates:

\[
b(t_j) = (b_1(t_j), b_2(t_j), \ldots, b_d(t_j))^\prime
\]

for

\[
\beta(t_j) = (\beta_1(t_j), \beta_2(t_j), \ldots, \beta_d(t_j))^\prime,
\]

where \( d \) represents the number of covariates, including the intercept.

In the second step, the data \( \{t_j, b_r(t_j)\} \) is smoothed for each given component \( r = 1, \ldots, d \), over \( j = 1, 2, \ldots, T \), to obtain the smooth coefficient functions \( \hat{\beta}_r(t) \). A typical linear smoother is

\[
\hat{\beta}_r(q)(t) = \sum_{j=1}^{T} H_r(t_j, t)b_r(t_j),
\]

where the weights \( H \) are constructed by a nonparametric smoothing method.
The smoothing step improves the efficiency of the raw estimates since it allows for the imputation of the values of the coefficient curves at non-design points. Furthermore, the smooth estimates are likely to be more appropriate in a situation where a coefficient function is expected to be smooth. One more advantage of the procedure, as claimed by Fan and Zhang (2000), is its computational speed and its applicability with the existing software.

4.3 A functional time series model

The data in Fan and Zhang (2000) is actually not functional data, but the data from repeated measurements. In this thesis, the data is essentially functional. Moreover, a dynamic aspect of these functional observations is included, since we deal with time series of $RQV(q)$. A plausible model for these functional observations may be a dynamic time series model, such as an autoregressive model of a pre-specified order, as follows

$$y_t(q) = \theta_0(q) + \sum_{i=1}^{p} \theta_i(q)y_{t-i}(q) + \varepsilon_t(q),$$

where $y_t(q)$ is $RQV(q)$ at some time $t$, $\theta_i(\cdot)$, $i = 0, 1, \ldots, p$, are $p+1$ unknown real valued functions of $q \in \mathbb{R}$, $\varepsilon_t(q)$ is a stochastic process with mean zero and the covariance function $\gamma(q_j, q_k) = \text{cov}\{\varepsilon_t(q_j), \varepsilon_t(q_k)\}$.

Hence, at each $q_{j}, j = 1, 2, \ldots, K$, the data may be considered as generated by a classical linear autoregressive model of order $p$

$$y_t(q_j) = \theta_0(q_j) + \sum_{i=1}^{p} \theta_i(q_j)y_{t-i}(q_j) + \varepsilon_t(q_j).$$

(10)

Obviously, model (10) may be fitted by, e.g., an ordinary least squares procedure to obtain the raw estimates $\hat{\theta}(q_j)$

$$\hat{\theta}(q_j) = \left(\hat{\theta}_0(q_j), \hat{\theta}_1(q_j), \ldots, \hat{\theta}_p(w_j)\right)'$$

for the parameter vector

$$\theta(q_j) = (\theta_0(q_j), \theta_1(q_j), \ldots, \theta_p(w_j))'.$$

Following the methodology of the two-step procedure described in Section 4.2, the raw estimates are smoothed by an existing non-parametric technique,
such as local polynomial fitting (e.g. Fan and Gijbels, 1996), to obtain the smooth estimates of the coefficient vector.

The two-step estimator for model (9) is discussed in Papers III and IV.

5 Summary of the papers

5.1 Paper I: Estimating quadratic variation of prices and spreads from the Swedish limit order book

The realized quadratic variation is considered to be a suitable measure of volatility of high-frequency financial prices. In the approach presented here, the realized quadratic variation, as a function of quantity of shares, is used to measure the volatility of prices quoted in the Swedish limit order book. The main idea behind this approach is to reduce the microstructure effects, which affect data sampled at very high frequencies. Moreover, finding an optimal time interval for computing the squared returns, needed for computation of values of the realized quadratic variation, is considered to be a crucial step in creating more accurate measures of volatility. The main results, presented through a case study, confirm the empirical results from some other comparable studies of the microstructure effects: The bias of the proposed estimator indeed increases with the sampling frequency. The major reduction of microstructure effects is only obtained when the sampling frequency is fairly low.

5.2 Paper II: Functional modeling of volatility in the Swedish limit order book

The publicly available electronic limit order book at the Stockholm Stock Exchange consists of five levels of prices and quantities of a given stock with a bid and ask side. All changes in the book during one day can be recorded with a time quote. Studying the variation of the quoted price returns as a function of quantity is discussed. In particular, discovering and modeling dynamic behaviors in the volatility of prices and liquidity measures is considered. Applying a functional approach, estimation of the volatility dynamics of the spreads, created as differences between the ask and bid prices, is presented through a case study. For that purpose, a two-step estimation of functional linear models is used, adapting this method to a functional time series context.

A two-step procedure for volatility estimation is evaluated by a simulation study intended to mimic estimation from the Swedish limit order book. To simulate data with varying volatility, the Heston stochastic volatility model is used. From the simulated data, the time series of realized quadratic variation (RQV), for a given relative quantity of shares, are obtained. These time series are modeled in a functional time series context by fitting an autoregressive moving average model. This model may be estimated in two ways, either by obtaining the raw estimates of the coefficient functions (naive approach) or by smoothing the fitted coefficient functions (two-step approach). Our results show that the risk measures of the smooth coefficient functions are indeed smaller than the corresponding risk measures of the coefficient functions of raw estimates. Consequently, the two-step estimation procedure is considered to be more efficient than the naive approach within this framework.

5.4 Paper IV: Functional autoregressive models: theory

Consider situations where a real valued function is observed over time and has a dynamic dependence structure. Linear autoregressive models, which have proven useful for modeling dynamics of “pointwise” time series, can be generalized to such a functional time series situation. We call such models functional autoregressive models. Their parameters are functions of a real valued argument (the data) and we consider a two-step estimation procedure proposal for functional linear models. The latter proposal is based on a first step where the ordinary least squares method is used to estimate pointwise linear models for given values of the argument of the functions observed. The second step smooths the first-step estimates, regressing the latter on the above mentioned arguments. The second step does not only yield smooth estimates of the functional parameters but also provides less variable pointwise estimates at the price of a bias. We do not only contribute by presenting a functional autoregressive model but also by proposing a two-stage estimator where the first step takes into account the contemporaneous correlation structure through a multivariate generalized least squares estimator. Some of the properties of the resulting two-step procedure are given. Financial functional data is used as an illustration.
6 Concluding remarks

There has been a constant increase in the interest in improving the existing methods designed for modeling financial volatility during the last few decades. Electronic trading in financial markets has particularly boosted this interest, since the real time financial data has become more available to the public. More or less innovative statistical methods for the analysis of this kind of data have frequently been proposed in the literature. This dissertation makes a contribution to the existing methods for financial volatility modeling, where the original data may be treated as functions. This functional approach is central to the whole thesis, since such an approach seems to be neglected in the existing studies. In particular, the major focus is on very specific financial data that comes from the Swedish electronic limit order book.

The nature of the Swedish limit order book data allows for a functional approach, which is introduced in Paper I through a volatility measure created from prices as the function of quantities. This paper gives an incentive for the use of the proposed measure in a functional context, but also raises a question of the quality of such a measure. This issue is further studied in Paper II, where a dynamic functional time series model for the volatility measure is suggested. This model is estimated by a two-step estimation procedure, originally proposed for the functional linear models for longitudinal data in Fan and Zhang (2000). The main idea of such a procedure is to improve the efficiency of the proposed estimator for the coefficient functions. In this sense, this study illustrates how our approach may be considered to be an improvement in volatility estimation.

To test whether the two-step procedure may be considered as an improvement, a simulation study is performed in Paper III. This simulation is performed within a specific parametric modeling framework in continuous time, by utilizing an established theoretical relationship between parameters from the continuous time model and the traditional autoregressive moving average time series model. Worth noticing, though, is the fact that our approach is limited to the Heston stochastic volatility model, which has certain limitations in practical use. A natural question about whether this model is sufficiently reliable to accommodate for more general conclusions about the quality of the proposed estimator may be addressed. It would certainly be interesting to perform more comprehensive simulation studies using some other (stochastic volatility) models and comparing the results.
Finally, Paper IV presents a new two-stage estimator for functional autoregressive models. This estimator is based on seemingly unrelated regression representation. Consequently, a multivariate generalized least squares (GLS) estimator is used instead of the univariate ordinary least squares (OLS) estimator. When the parameter functions are assumed to be smooth, this estimator may be smooth. Some asymptotic properties of the GLS estimator and its smooth version are presented when a linear smoother is used. An application to high-frequency financial data is performed. The results indeed show that the estimated asymptotic variance of the GLS estimates is smaller than for the OLS estimates. However, it is not obvious whether smoothing GLS would lead to efficiency gains. Still, one may be willing to smooth the GLS estimates to obtain the smooth estimates of the parameter functions. Since this application is merely an illustration, further research is needed to study these issues.

References


