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Small Youden Rectangles, Near Youden Rectangles, and Their Connections to Other Row-Column Designs

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In this paper we first study $k \times n$ Youden rectangles of small orders. We have enumerated all Youden rectangles for a range of small parameter values, excluding the almost square cases where $k = n - 1$, in a large scale computer search. In particular, we verify the previous counts for $(n, k) = (7, 3), (7, 4)$, and extend this to the cases $(11, 5), (11, 6), (13, 4)$ and $(21, 5)$.

For small parameter values where no Youden rectangles exist, we also enumerate rectangles where the number of symbols common to two columns is always one of two possible values, differing by 1, which we call *near Youden rectangles*.

For all the designs we generate, we calculate the order of the autotopism group and investigate to which degree a certain transformation can yield other row-column designs, namely double arrays, triple arrays and sesqui arrays.

Finally, we also investigate certain Latin rectangles with three possible pairwise intersection sizes for the columns and demonstrate that these can give rise to triple and sesqui arrays which cannot be obtained from Youden rectangles, using the transformation mentioned above.

Keywords: Youden squares, block designs, row-column designs

1 Introduction

An (n, k, λ) Youden rectangle (sometimes referred to as a Youden square) where $n \geq k$ is a $k \times n$ array on n symbols that satisfies the following two conditions:

1. There is no repeated symbol in any row or column, which we will call the *Latin condition*.
2. The number of shared symbols between any two columns is always λ, which we will call the *balance condition*.

Youden rectangles can be represented in different ways. In particular, by switching the roles of rows and symbols, one gets a representation in the form of a square matrix, typically with some empty cells. In previous literature, the term 'Youden square' has sometimes been used for the rectangular format as well,

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but we shall use the term ‘Youden rectangle’ for the rectangular format, reserving the term ‘square’ for the actual square format.

As indicated by the choice of terminology in the first part of the definition, a Youden rectangle can be viewed as a special case of a $k \times n$ Latin rectangle, which in this setting can be defined as a $k \times n$ array on n symbols, satisfying the Latin condition. In the present paper, we exclude the square, and almost square cases $k = n$, $k = n - 1$ as well as $k = 1$ for Youden rectangles, since for these parameter choices, all Latin rectangles trivially also satisfy the second condition.

Clearly, each row will contain each symbol exactly once, and so the array will also be *equireplicate*, that is, each symbol appears the same number of times, namely k. As is well known, divisibility and double counting considerations easily give that in order for a Youden rectangle to exist, $\lambda = \frac{k(k-1)}{n-1}$ must be an integer.

The reason for the use of the term ‘balance’, is that when treating the columns of a Youden rectangle as sets of symbols, these sets form the blocks of a *symmetric balanced incomplete block design* (SBIBD). Conversely, it was proven by Smith and Hartley [30] that the elements in the blocks of any SBIBD can be ordered to give a Youden rectangle. In fact, many different orderings are possible, so a single SBIBD will give rise to many different Youden rectangles. We have not employed this connection between SBIBDs and Youden rectangles in our computational work.

Alternatively, and equivalently, a Youden rectangle may be defined with more of an SBIBD approach as a $k \times n$ array on n symbols, where no symbol is repeated in any row, and when viewing the columns as sets of symbols, each pair of symbols occurs the same number of times, namely λ. The property that all pairs of blocks in an SBIBD intersect in the same number of elements is sometimes expressed by saying that the design is *linked*, and more recently rather by saying that the design has *constant block intersections*.

Already in the original paper [33] Youden points out that from a statistical point of view Youden rectangles suffer from the restricted set of feasible parameters. As one way around this problem we here introduce the class of *near Youden rectangles*. For given values of n and k a near Youden rectangle is a Latin rectangle with two allowed block intersection sizes, differing by 1, rather than one single intersection size. This relaxation significantly increases the set of allowed parameters while in a sense still keeping the design as balanced as possible. In Section 2.2 we discuss the theoretical properties of these designs in greater detail, and discuss their connections to existing design classes.

The early history of the study of Youden rectangles was chronicled by Preece [25], and a good starting point for further reading is the Youden chapter in the Handbook of Combinatorial Designs [10].

Little has been done on complete enumeration of these objects, though in [24] Youden rectangles with $n \leq 7$ were classified by Preece, and in [14] we performed a full enumeration of mutually orthogonal (in the Latin rectangle sense) triples of Youden rectangles for $n \leq 7$. Note that orthogonal Youden rectangles should not be confused with *multi-layered Youden rectangles*, as studied in [26]. In the present paper, our main aim has been to perform a complete enumeration of Youden rectangles for as large parameters as possible. The current state of knowledge on the number of Youden rectangles is tabulated in [10], which goes up to $n = 7$.

The rest of the paper is structured as follows. In Section 2 we give some further basic notation and formal definitions. In Section 3 we state the questions guiding our investigation, and describe briefly the method and algorithms used together with some practical information regarding the computer calculations. In Section 4 we present the data our computer search resulted in, in particular the number of different Youden rectangles of some small orders. In Section 5, we analyze the constructed objects with regards to other types of row-column designs. Section 6 concludes.

2 Preliminaries

2.1 Notions of Equivalence

We will use $\{0, 1, \ldots, n-1\}$ as the symbol set. We call a Youden rectangle *normalized* if it satisfies the following conditions:

(S1) (Ordering among columns) The first row is the identity permutation.

(S2) (Ordering among rows) The first column is $0, 1, 2, \ldots, k-1$.

Two Youden rectangles Y_A and Y_B are said to be *isotopic* if there exists a permutation π_s of the symbols, a permutation π_r of the rows and a permutation π_c of the columns such that when applying all three permutations to Y_A, we get Y_B. The equivalence concept isotopism is perhaps the most natural one when studying Youden rectangles, and isotopism classes are also known as *transformation sets* in this context.

Two normalized Youden rectangles Y_A and Y_B can be isotopic to each other, so grouping Youden rectangles according to which normalized rectangle they yield when renaming the symbols in the first column $0, 1, \ldots k-1$ in this order, and permuting the columns to satisfy **S1** gives a weaker notion of equivalence, by saying that Y_A and Y_B are equivalent if they yield the same normalized Youden rectangle in this way.

Other concepts of equivalence are also possible, and allowing for exchanging the roles of symbols and columns leads to the notion of *species* (also known as *main classes*). In the present paper, we will not be employing the last mentioned notion of equivalence, and we comment on this choice below. Taking transposes (that is, exchanging the roles of columns and rows), or exchanging the roles of symbols and rows, however, does not map $k \times n$ Youden rectangles to $k \times n$ Youden rectangles, and so we do not consider these transformations here.

Making this more formal, the group $G_{n,k} = S_k \times S_n \times S_n$ of isotopisms acts on the set of $k \times n$ Youden rectangles, where S_k corresponds to a permutation of the rows, the first S_n corresponds to a permutation of the columns, and the last S_n corresponds to a permutation of the symbols. Two rectangles Y_A and Y_B of size $k \times n$ are isotopic, and we say that they belong to the same *isotopism class* if there exists a $g \in G_{n,k}$ such that $g(Y_A) = Y_B$. The *autotopism group* of a Youden rectangle Y is defined as $\mathrm{Aut}(Y) := \{g \in G_{n,k} \mid g(Y) = Y\}$. When presenting examples, we use normalized representatives of isotopism classes. For a recent survey on the concept of isotopism in algebra and designs, see [11].

2.2 Near Youden Rectangles

For parameters where λ as calculated by $\lambda = \frac{k(k-1)}{n-1}$ is not an integer, no Youden rectangle exists. This divisibility is quite restrictive and from a statistical design theory perspective it is desirable to include more parameter choices here. One natural relaxation is to allow two different column intersection sizes, leading us to the following definition:

Definition 2.1. *A* near Youden rectangle *(NYR) is a $k \times n$ Latin rectangle where every column-column intersection has size either $\lambda_1 = \lfloor \lambda \rfloor$, or $\lambda_2 = \lceil \lambda \rceil$, where $\lambda = \frac{k(k-1)}{n-1}$.*

An example of a 4×6 NYR with column intersection sizes $\lambda_1 = 2$ and $\lambda_2 = 3$ is given in Figure 1. For example, the first column intersects the second, third and fourth columns in 2 symbols, and the remaining columns in 3 symbols.

0	1	2	3	4	5
1	0	5	4	3	2
2	4	0	5	1	3
3	5	4	0	2	1

Fig. 1: A 4×6 near Youden rectangle.

If λ_1 is zero, the resulting designs (when interpreting columns as blocks) may be *disconnected*, that is, the symbol set can be partitioned in two parts S_1 and S_2 such that the set of columns where the symbols in S_1 appear is disjoint from the set of columns where the symbols in S_2 appear. For example, two 7×3 Youden rectangles on disjoint symbol sets may be juxtaposed to form a 14×3 near Youden rectangle. Disconnectedness is undesirable from a statistical design point, but when $\lambda_1 \geq 1$, all near Youden rectangles are connected.

If we disregard the order of the elements in the columns of an NYR we get an equireplicate block design with the same intersection property as the NYR, i.e. pairs of blocks intersect in either λ_1 or λ_2 elements. However, in the study of block designs it has been more common to define designs in terms of covering numbers for pairs of symbols, i.e. the number of blocks which contain the pair of symbols, rather than intersection numbers. However, following Fisher's original proof of Fishers inequality in [12], rather than the now more common linear algebraic version, one can easily connect intersection numbers and covering numbers. The idea behind Fisher's proof is to calculate the variance of the intersection numbers in terms of the covering numbers, and as a corollary he also gets the result that in an SBIBD the intersection number is constant. This argument can also be done in the other direction, describing the variance of the covering numbers in terms of the intersection numbers. Instead of doing this from scratch we will use an identity given by Tsuji in [31], though we note that similar identities were used earlier in [8]. We here state the identity in a less general form, adapted to our current situation.

Theorem 2.2 (Lemma 1 in [31])**.** *Let $l_{p,q}$ denote the number of columns which contain the pair $\{p, q\}$ and $m_{i,j}$ the size of the intersection of the i:th and j:th columns. With λ as already defined, we then have*

$$\sum_{p,q}(l_{p,q}-\lambda)^2 = \sum_{i,j}\left(m_{i,j}^2 - \left(1+2\frac{(k-1)^2}{n-2}\right)m_{i,j} + \frac{k(k-1)}{n-1}\left(1-2\frac{k-1}{n-2}+\frac{nk(k-1)}{(n-1)(n-2)}\right)\right)$$

where the first sum is over 2-subsets of symbols and the second is over 2-subsets of columns.

Next we note that we can determine the number of column pairs with a given intersection size in a NYR, and that these intersection sizes are nicely distributed.

Proposition 2.3. *Let A be a $k \times n$ near Youden rectangle with column intersection sizes $\lambda_1 = \left\lfloor \frac{k(k-1)}{n-1} \right\rfloor$ and $\lambda_2 = \left\lceil \frac{k(k-1)}{n-1} \right\rceil$. Then any column c intersects $n_1 = \lambda_2(n-1) - k(k-1)$ other columns in λ_1 symbols and $n_2 = -\lambda_1(n-1) + k(k-1)$ other columns in λ_2 symbols.*

Proof: We fix an arbitrary column c and count the sum total S of the sizes of the intersections between c and all the other columns. Suppose c intersects n_1 columns in λ_1 symbols and intersects n_2 columns in

λ_2 symbols. Counting by columns, we then get $S = \lambda_1 n_1 + \lambda_2 n_2$.

Counting by symbols present in column c, we get $S = k(k-1)$, since c contains k symbols and A is equireplicate with replication number k, that is, each of the k symbols in c appears $k-1$ times outside of c.

Equating the different counts, and using that $n_1+n_2 = n-1$ and $\lambda_1+1 = \lambda_2$, we get a linear equation $\lambda_1 n_1 + \lambda_2(n-1-n_1) = k(k-1)$ in the variable n_1, with solution $n_1 = \lambda_2(n-1) - k(k-1)$. It follows that $n_2 = -\lambda_1(n-1) + k(k-1)$, and since the choice of c was not used, these values are equal for all columns c. □

Theorem 2.2 together with Proposition 2.3 gives the following:

Theorem 2.4. *If D is the block design obtained from a $k \times n$ NYR, then any pair of symbols is covered by either λ_1 or λ_2 blocks in D.*

Proof: Let us first note that the left hand side of the identity in Theorem 2.2 is a multiple of the variance of the covering numbers. The average covering number is λ, which is not an integer. Hence the smallest possible variance would be achieved if all covering numbers are one of λ_1 and λ_2. Since the variance is a convex function this minimum is also unique.

Using the values of n_1 and n_2 from Proposition 2.3 we can compute the right hand side of the identity in Theorem 2.2. Using λ_i and n_i for the covering numbers and their multiplicity in the left hand side produces the same value.

Hence the unique way to achieve the identity in Theorem 2.2 is to have all covering numbers equal to one of λ_1 and λ_2, with the stated frequencies for those two numbers. □

Thus the block design coming from a NYR has both intersection numbers and covering numbers taking only two possible values, doing so in the way that minimises the variance of those numbers. The block designs appearing here are in fact members of a known class, introduced by John and Mitchell in 1977 [15] called *regular graph designs*. The name comes from a property of their concurrence matrices which is in fact the dual of our Proposition 2.3. The class of regular graph designs, which includes non-symmetric designs, was later generalised to cases where equal replication is not possible, by Cheng and Wu in [9]. As far as we know, Theorem 2.4 has however not been noticed in the literature on regular graph designs. Analogous to how Smith and Hartley [30] connect SBIBDs to Youden rectangles we can obtain near Youden rectangles from regular graph designs by ordering the blocks and their elements. In fact, as observed e.g. by Bailey in Chapter 11.10 of [4], it is possible to order the elements of the blocks to become the columns of a row-column design for any equireplicate incomplete-block design with the same number of symbols as blocks.

Here we may also note that other types of designs where one allows the covering numbers, or intersection numbers to be non-constant have been studied. Bose and Nair [6] introduced and studied *partially balanced incomplete block designs* (PBIBD). The particular case where there are just two different values for the number of repetitions of pairs in a PBIBD was studied for example by Bose and Shimamoto in [7], and *symmetric* PBIBDs have also been studied, e.g., by Lawless and Stanton in [16]. Looking instead at sizes of intersections between blocks, the subclass of *balanced incomplete block designs* (BIBDs) where the block intersections only have two different sizes has been studied under the name *quasi-symmetric* designs, for example by Shrikande and Sane in [29]. Duals (exchanging the role of blocks and symbols) of PBIBDs have been studied under the name of *linked block* designs (LB), with different relaxations, see, e.g., [13, 20, 28].

3 Generating Data

In this section, we describe our computational work in general terms.

3.1 Guiding Questions

Our approach is complete enumeration by computer for as large parameter values as possible, and unless otherwise stated, we save all generated data. In particular, we not only record the *number* of Youden rectangles found, but we save the objects themselves.

With some exceptions due to size restrictions, the data generated is available for download at [2] and [1]. Further details about the organization of the data are given there.

The following questions serve as guides for what data to generate.

(Q1) How many isotopism classes of $k \times n$ Youden rectangles are there?

(Q2) What is the order of the autotopism group of each $k \times n$ Youden rectangle?

(Q3) If some condition is relaxed, how many objects satisfying the relaxed conditions are there?

3.2 Feasible parameter combinations

A necessary (but not sufficient) condition for the existence of a Youden rectangle is that $\lambda = \frac{k(k-1)}{n-1}$ is an integer. We exclude $k = 1, k = n-1$ and $k = n$, as being trivial, since all Latin rectangles for those values are Youden rectangles. We call non-trivial parameter values satisfying the divisibility condition *feasible*. The smallest feasible parameter combinations for nontrivial Youden rectangles are given in Table 1. Note that if (k, n) are feasible parameters for a Youden rectangle, then so are $(n-k, n)$.

$n\backslash k$	3	4	5	6	7	8	9	10	11	12	16
7	E	E									
11			E	E							
13		E					X				
15					X	X					
16				E?				X			
19							X	X			
21			E								X
23									X	X	

Tab. 1: All feasible parameter combinations for Youden rectangles with $7 \leq n \leq 23$. An E indicates full enumeration in the present paper, and an X indicates feasible parameters but no complete enumeration.

We have attempted to generate the Youden rectangles for all parameter combinations in Table 1, but in the remaining cases the number of partial objects was too large and the computation had to be stopped due to lack of storage space. The fact that we could handle one case for $n = 21$ illustrates the fact that growing n is not the only challenge for complete enumeration, but rather an interplay between n and k.

For parameter sets where there do not exist Youden rectangles, we have enumerated $k \times n$ near Youden rectangles, where the intersection sizes between symbol sets in columns (as noted above) are either $\lambda_1 = \left\lfloor \frac{k(k-1)}{n-1} \right\rfloor$ or $\lambda_2 = \left\lceil \frac{k(k-1)}{n-1} \right\rceil$.

For near Youden rectangles there are no simple divisibility conditions which have to be satisfied, like the ones for SBIBDs, and as we shall see we find numerous examples for all small parameters. However, a theorem of Brown [8] implies that for $n = 17, k = 6$, near Youden rectangles do not exist. So, while near Youden rectangles are much less restricted than Youden rectangles, the existence question is still non-trivial.

3.3 Implementation and Execution

We generated all non-isotopic rectangles by consecutively adding all possible columns, while observing that none of the conditions were violated. At suitable points, we reduced our list of partial objects by isotopism. Also, at selected stages, the list of partial objects was culled by running checks on whether they were at all extendible to a full Youden rectangle. We note here that using the definition in terms of constant sized column intersections, rather than the definition in terms of each symbol pair appearing a constant number of times, makes it possible to reduce the list of partial objects much more effectively. We also observe that for partial objects, it is not possible (at least not straightforwardly) to reduce the list of partial objects with respect to species (main classes). At this stage, therefore, it is natural to employ the equivalence notion of isotopism.

The algorithms used were implemented in C++ and run in a parallelized version on the Kebnekaise supercomputer at High Performance Computing Centre North (HPC2N).

The algorithm is divided into two parts. The first extends a given partial rectangle with k rows and t columns by one column, such that the new rectangle satisfies both the Latin condition and the balance condition. More specifically, we first add a column with k different symbols. We then check that in the extended $k \times (t + 1)$ rectangle, no symbols appear more than once in any row. We also check that the number of shared symbols between the added column and the t first columns is λ. In the case of generating near Youden rectangles, we instead check that all intersection sizes with the new column fall into one of the two allowed values. By checking all possible added columns, we find all extensions of the given $k \times t$ rectangle.

The second part of the algorithm checks whether a received $k \times (t + 1)$ rectangle could be chosen as a normalized representative of an isotopism class.

When a full Youden rectangle has been received, we check the order of the autotopism group. The group of possible autotopism actions on a $k \times n$ Youden rectangle is $S_k \times S_n \times S_n$, so potentially, the number of actions we need to check is $k! \cdot n! \cdot n!$.

Since we consider normalized rectangles this number can be reduced to $n \cdot k! \cdot (n - k)!$, since once we have chosen the first column (n options) and row permutation π_r ($k!$ options) we fix k symbols in the symbol permutation π_s (so $(n - k)!$ options remain).

The running time grows quickly as the rectangle parameters grow. We completely enumerated Youden rectangles of sizes 3×7, 4×7, 5×11 and 4×13 in a few minutes on a standard desktop computer. On the other hand, computation on sizes 6×11 and 5×21 required high performance computers and significantly more time. Using a parallelized version of the algorithms, enumerating 6×11 Youden rectangles took about 6000 core hours, which is a bit less than 1 year. The 5×21 case required several hundred core years.

Our methods and code can be applied to larger parameter values as well, but the number of partial rectangles, which is far larger than those for complete rectangles, become unmanageable. The running time per partial object has not been the bottleneck for our program, so there has been no reason to employ

(n,k,λ)		(7,3,1)	(7,4,2)
#YR		1	6
|Aut|	1	0	2
	3	0	3
	21	1	1

Tab. 2: The number of Youden rectangles with $n = 7$ sorted by autotopism group order.

(n,k,λ)		(11,5,2)	(11,6,3)
#YR		79 416	995 467 440
|Aut|	1	77 694	995 421 832
	2	1 423	40 831
	3	199	4 454
	4	45	124
	5	4	121
	6	38	62
	10	3	3
	12	7	10
	55	1	1
	60	2	2

Tab. 3: The number of Youden rectangles with $n = 11$ sorted by autotopism group order.

more sophisticated generation methods or equivalence checks. Instead the number of partial objects for large parameters became so large that disc space became the limiting factor.

4 Basic Computational Results

We now turn to the results and analysis of our computational work.

4.1 The Number of Youden Rectangles

Our first result is an enumeration of Youden rectangles. In Tables 2 to 5, we present data on the number of non-isotopic Youden rectangles, sorted by the order of the autotopism groups.

It is relevant to compare these numbers with the number of Latin rectangles. When no reduction at all is applied, there are $782\,137\,036\,800$ Latin rectangles of size 4×7, and only 512 Youden rectangles of the same size (note that this number is not given in any of the tables in the present paper). In [18], the numbers of *reduced* $n \times k$ Latin rectangles are given for $k \leq n$, $1 \leq n \leq 11$, that is, the number of Latin rectangles whose first row is the identity permutation and the first column is $0, 1, \ldots, k-1$, and there are $1\,293\,216$ reduced Latin rectangles of size 4×7. Finally, there are 1398 4×7 non-isotopic Latin rectangles [17], to be compared with only 6 non-isotopic Youden rectangles of the same size. As we can see, the proportion of Latin rectangles that additionally satisfy the balance condition is small.

We note again that the 3×7 and 4×7 Youden rectangles were completely classified by Preece [24], and that our enumerative results are in accordance with his classification.

(n,k,λ)		(13,4,1)
#YR		20
|Aut|	1	12
	3	7
	39	1

Tab. 4: The number of Youden rectangles with $n = 13$, $k = 4$ sorted by autotopism group order.

(n,k,λ)		(21,5,1)
#YR		3 454 435 044
|Aut|	1	3 454 384 100
	2	37 394
	3	13 349
	5	14
	6	109
	7	4
	9	55
	14	6
	18	7
	21	1
	42	3
	63	1
	126	1

Tab. 5: The number of Youden rectangles with $n = 21$, $k = 5$ sorted by autotopism group order.

0	1	2	3	4	5	6	7	8	9	10	11	12
1	4	5	6	2	9	10	11	0	7	8	12	3
2	5	7	8	9	11	0	3	4	12	1	6	10
3	6	8	9	10	0	7	4	12	1	11	2	5

Fig. 2: The 4×13 Youden rectangle Y with $|\mathrm{Aut}(Y)| = 39$.

The most common autotopism group order for Latin rectangles is 1 (see [18]). From the tables, we see that clearly the most common autotopism group order for Youden rectangles is also 1, but that there are also rare examples of rather symmetric Youden rectangles. One such example, a Youden rectangle of size 4×13, whose autotopism group order is 39 is presented in Figure 2. The autotopism group acts transitively on the columns of this Youden rectangle, that is, for any pair C_1 and C_2 of columns, there is an autotopism that takes C_1 to C_2.

As is well known, taking a (n, k, λ) difference set as first column and producing the remaining columns by developing this first column, that is, consecutively adding 1 to each entry, will produce a Youden rectangle. The autotopism group of the resulting Youden rectangle will then act transitively on the set of columns. We conclude that for $n = 7, 11, 13$, the very symmetric Youden rectangles we found, where the order of the autotopism group is divisible by the number of columns, correspond to those Youden rectangles generated from difference sets. The situation for $n = 21$ seems to be a bit more involved, since we see autotopism groups of orders 21, 42 (in fact, three such), 63 and even 126. A complete analysis of these Youden rectangles is beyond the scope of this paper, and we leave this as an open question.

For larger parameters, that is, where there exist more than one corresponding SBIBD, it would also have been interesting to group Youden rectangles according to which SBIBD they give if the ordering in the columns is ignored.

4.2 Near Youden Rectangles

In Tables 6 to 10, we list complete data for the number of isotopism classes of near Youden rectangles (NYR) from $n = 5$ to $n = 9$ for sets of parameters where there are no Youden rectangles, sorted by the order of the autotopism groups. We also display the number of NYRs which are self-conjugate as Latin rectangles, i.e., if we interchange the roles of columns and symbols we get a NYR in the same isotopism class.

We have excluded the cases $k = 1$, $k = n - 1$ and $k = n$, since as observed above, for these cases all Latin rectangles are Youden rectangles as well. We also excluded the case $k = 7, n = 9$, for which the number of partial rectangles was deemed too large for a straight-forward run of our program.

In Tables 11 to 14, we list data for the number of isotopism classes of near Youden rectangles from $n = 10$ to $n = 13$ for as large k as was feasible, with the same restrictions on parameter values as for $n = 5, \ldots, 9$.

Observation 4.1. *There exist NYRs for all parameters with* $n \leq 10$.

This follows from our enumeration together with the observation that if a $k \times n$ NYR is completed to an $n \times n$ Latin square then the new $n - k$ rows also form an $(n - k) \times n$ NYR.

We note that for $k = 2$ any NYR may be interpreted as a 2-regular graph. Such graphs can be easily enumerated by hand, and our data for this case is verified by such a manual count.

$(n, k, \lambda_1, \lambda_2)$		(5,2,0,1)	(5,3,1,2)
# NYR		1	2
#self-conjugate		1	2
$\lvert\mathrm{Aut}\rvert$	2	0	1
	10	1	1

Tab. 6: The number of near Youden rectangles with $n = 5$ sorted by autotopism group order.

$(n, k, \lambda_1, \lambda_2)$		(6,2,0,1)	(6,3,1,2)	(6,4,2,3)
#NYR		2	2	34
#self-conjugate		2	2	29
$\lvert\mathrm{Aut}\rvert$	1	0	0	9
	2	0	0	11
	4	0	0	5
	6	0	2	3
	12	1	0	4
	18	0	0	1
	36	1	0	1

Tab. 7: The number of near Youden rectangles with $n = 6$ sorted by autotopism group order.

$(n, k, \lambda_1, \lambda_2)$		(7,2,0,1)	(7,5,3,4)
# NYR		2	5 205
# self-conjugate		2	2 778
$\lvert\mathrm{Aut}\rvert$	1	0	4 889
	2	0	307
	4	0	8
	14	1	1
	24	1	0

Tab. 8: The number of near Youden rectangles with $n = 7$ sorted by autotopism group order.

$(n,k,\lambda_1,\lambda_2)$		(8,2,0,1)	(8,3,0,1)	(8,4,1,2)	(8,5,2,3)	(8,6,4,5)
# NYR		3	4	285	6 688	21 956 009
# self-conjugate		3	3	212	3 608	11 000 012
|Aut|	1	0	0	173	6 204	21 905 896
	2	0	0	78	381	48 865
	3	0	0	0	37	0
	4	0	0	15	29	1 208
	5	0	0	0	0	24
	6	0	2	0	18	0
	8	0	0	11	6	144
	10	0	0	0	0	6
	12	0	0	0	5	0
	16	1	1	4	5	36
	24	0	0	0	2	0
	30	1	0	0	0	0
	32	0	0	4	0	6
	48	0	1	0	1	0
	64	1	0	0	0	4

Tab. 9: The number of near Youden rectangles with $n = 8$ sorted by autotopism group order.

$(n,k,\lambda_1,\lambda_2)$		(9,2,0,1)	(9,3,0,1)	(9,4,1,2)	(9,5,2,3)	(9,6,3,4)
# NYR		4	11	5 342	2 757 904	731 801 066
# self-conjugate		4	11	2 955	1 388 084	98 054 401
|Aut|	1	0	3	4 881	2 750 174	731 727 683
	2	0	1	355	7 148	69 733
	3	0	1	20	290	3 079
	4	0	0	54	177	312
	6	0	4	15	86	213
	8	0	0	3	7	0
	9	0	1	3	6	16
	12	0	0	8	6	18
	18	1	0	2	8	5
	36	1	0	0	1	4
	40	1	0	0	0	0
	54	0	1	0	0	1
	72	0	0	1	1	0
	108	0	0	0	0	2
	324	1	0	0	0	0

Tab. 10: The number of near Youden rectangles with $n = 9$ sorted by autotopism group order.

$(n,k,\lambda_1,\lambda_2)$		(10,2,0,1)	(10,3,0,1)	(10,4,1,2)	(10,5,2,3)
# NYR		5	80	9 722	1 913 816
# self-conjugate		5	59	5 388	962 300
|Aut|	1	0	48	9 288	1 907 844
	2	0	23	331	5 952
	3	0	4	72	0
	4	0	0	9	0
	5	0	0	2	4
	6	0	2	2	0
	10	0	3	9	16
	12	0	0	9	0
	20	1	0	0	0
	42	1	0	0	0
	48	1	0	0	0
	100	1	0	0	0
	144	1	0	0	0

Tab. 11: The number of near Youden rectangles with $n = 10$ sorted by autotopism group order.

$(n,k,\lambda_1,\lambda_2)$		(11,2,0,1)	(11,3,0,1)	(11,4,1,2)
# NYR		6	852	1 598
# self-conjugate		6	501	865
|Aut|	1	0	759	1 597
	2	0	75	0
	3	0	12	0
	6	0	5	0
	11	0	1	1
	22	1	0	0
	48	1	0	0
	56	1	0	0
	60	1	0	0
	180	1	0	0
	192	1	0	0

Tab. 12: The number of near Youden rectangles with $n = 11$ sorted by autotopism group order.

$(n,k,\lambda_1,\lambda_2)$		(12,2,0,1)	(12,3,0,1)	(12,4,1,2)
# NYR		9	11 598	262
# self-conjugate		9	6 183	167
$\lvert$Aut$\rvert$	1	0	11 174	182
	2	0	333	46
	3	0	35	16
	4	0	13	4
	6	0	27	10
	8	0	2	0
	12	0	5	4
	18	0	3	0
	24	1	4	0
	54	1	0	0
	64	1	0	0
	70	1	0	0
	72	0	2	0
	120	1	0	0
	144	1	0	0
	216	1	0	0
	768	1	0	0
	388	1	0	0

Tab. 13: The number of near Youden rectangles with $n = 12$ sorted by autotopism group order.

$(n, k, \lambda_1, \lambda_2)$		(13,2,0,1)	(13,3,0,1)
# NYR		10	169 262
# self-conjugate		10	86 362
|Aut|	1	0	167 541
	2	0	1 626
	3	0	69
	6	0	24
	13	0	1
	26	1	0
	39	0	1
	60	1	0
	72	1	0
	80	1	0
	84	1	0
	144	1	0
	252	1	0
	300	1	0
	320	1	0
	1296	1	0

Tab. 14: The number of near Youden rectangles with $n = 13$ sorted by autotopism group order.

We see that for fixed n and growing k, at least for $n = 7$, $n = 11$ and $n = 13$, the number of near Youden rectangles grows faster than the number of Youden rectangles. The same holds for fixed k and growing n. As with Youden rectangles, most of the small near Youden rectangles have trivial autotopism groups.

We also note that for small n we always find self-conjugate near Youden rectangles, even though their number is typically smaller than the number of all near Youden rectangles.

Question 4.2. *Assume that a near Youden rectangle exists for given n and k. Does there always exist a self-conjugate near Youden rectangle for the same parameter combination?*

5 Relations to Triple Arrays and Related Row-Column Designs

In this section, we present data and give some new theoretical results on the connection between Youden rectangles and double, triple and sesqui arrays.

5.1 Theoretical background

A $(v, e, \lambda_{rr}, \lambda_{cc}, \lambda_{rc} : r \times c)$ *triple array* is an $r \times c$ array on v symbols satisfying the following conditions:

(TA1) No symbol is repeated in any row or column.

(TA2) Each symbol occurs e times (the array is *equireplicate*).

(TA3) Any two distinct rows contain λ_{rr} common symbols.

(TA4) Any two distinct columns contain λ_{cc} common symbols.

(TA5) Any row and column contain λ_{rc} common symbols.

If we relax condition (TA5), which is sometimes called *adjusted orthogonality*, the array is called a *double array*, and if condition (TA5) is expressly forbidden to hold, but all other conditions hold, we have a *proper* double array. If we relax condition (TA4), the array is called a *sesqui array*, and an array satisfying every condition except (TA4) we call a *proper* sesqui array. Our use of the term *proper* in this context should not be confused with how it is sometimes used to stress that the blocks of a block design all have the same size. Triple arrays were introduced by Agrawal [3], though examples were known previously, and a good general introduction to triple and double arrays is given in [19]. Sesqui array were introduced in [5].

In discussing these designs we will find a new class of Latin rectangles useful.

Definition 5.1. *A Latin rectangle with integer parameters (n, k, λ), with $\lambda = \frac{k(k-1)}{n-1}$ calculated from n and k as for a Youden rectangle, where the column intersections have sizes $\lambda - 1$, λ and $\lambda + 1$ is called a* triple-intersection *Latin rectangle.*

Note that these objects are defined only for such (n, k, λ) that allow Youden rectangles with these parameters, and that we require the intersection sizes to actually take on all these three values.

In [27] it was suggested that triple arrays could be constructed by taking an *arbitrary* Youden rectangle, removing one column and all symbols present in that column, and then exchanging the roles of columns and symbols. The argument employed used distinct representatives. However, in [32], the method was observed to be flawed, as the distinct representatives argument did not work, and an explicit counterexample was given. For ease of reference, we phrase the construction as follows.

Construction 5.2. *For a given Youden rectangle Y and a column C_0, let A be the array received from Y by first removing column C_0 and all occurrences in Y of symbols present in C_0, and then exchanging the roles of columns and symbols.*

We say that a Youden rectangle Y is compatible *with an array A if Y gives A via this construction for some suitable choice of column, and we say that Y* yields *A.*

Construction 5.2 was further investigated in [22], yielding among other the following results, reformulated to suit the terminology employed in the present paper:

Theorem 5.3 (Proposition 2 in [22])**.** *Using Construction 5.2, any Youden rectangle always yields an array that satisfies conditions (TA1), (TA2) and (TA4), regardless of the choice of column.*

In particular, when applied to a (n, k, λ) Youden rectangle, Construction 5.2 yields an equireplicate $r \times c = k \times (n - k)$ array on $v = n - 1$ symbols, with replication number $e = k - \lambda$ and column intersection size $\lambda_{cc} = \lambda$. We see then that Construction 5.2 may never (by definition of a proper sesqui array) yield a proper sesqui array, but it is possible that we would get the *transpose* of a proper $(n-k) \times k$ sesqui array.

Theorem 5.4 (Theorem 3 in [22])**.** *Using Construction 5.2, any Youden rectangle with $\lambda = 1$ always yields a proper double array for any choice of column.*

Theorem 5.5 (Theorem 7 in [22])**.** *For any triple array T with $v = r + c - 1$ and $\lambda_{cc} = 2$, there exists a Youden rectangle (with $k = r$, $n = v + 1$, $\lambda = 2$) that yields T using Construction 5.2.*

It was also conjectured in [22] that Theorem 5.5 would hold for triple arrays with λ_{cc} larger than 2.

When applying Construction 5.2 to near Youden rectangles or triple-intersection Latin rectangles, removing a column together with all the symbols present in that column will leave a $k \times (n-1)$ equireplicate array with some empty cells. For a near Youden rectangle, the empty cells are distributed so that the number of empty cells in a column is either λ_1 or λ_2. For a triple intersection Latin rectangle, the corresponding numbers of empty cells are $\lambda - 1$, λ or $\lambda + 1$. If more than one value occurs for the number of empty cells in a column, the array will not be equireplicate after exchanging the roles of columns and symbols, since the number of appearances of a symbol in the resulting array will be the number of non-empty cells in the corresponding column.

For near Youden rectangles Proposition 2.3 implies that the resulting array will never be equireplicate. However, the following theorem follows rather easily from results in [22].

Theorem 5.6. *For any $(v, e, \lambda_{rr}, 1, \lambda_{rc} : r \times c)$ triple array T with $v = r + c - 1$, there is a compatible $r \times (v - c)$ triple-intersection Latin rectangle Y with column intersection sizes* 0*,* 1 *and* 2*.*

The proof of this theorem uses the following result, where the RL-*form* R of a triple array T mentioned in the cited source is the array that results from exchanging the roles of columns and symbols in T.

Theorem 5.7 (Corollary 1 in [22])**.** *In the RL-form R of a triple array T with $v = r + c - 1$, for any two columns C_1 and C_2, the sum of the number of common non-empty rows and the number of common symbols of C_1 and C_2 is constant, namely e, the replication number.*

Proof Proof of Theorem 5.6: Since the parameters of T are not all independent of each other (in particular, when $v = r + c - 1$, it holds that $\lambda_{cc} = r - e$, see [19]), we may also observe that when exchanging the roles of symbols and columns in a T, there will be $r - e = \lambda_{cc}$ empty cells in each column in R (the number of rows in T in which the corresponding symbol does not appear). Reasoning similarly, there will be $r - \lambda_{cc}$ empty cells in each row of R (the number of columns in T where the corresponding symbol does not appear).

For $\lambda_{cc} = 1$, Theorem 5.7 then implies that in R, each pair of columns shares 0 symbols (when their empty cells lie in the same row) or 1 symbol (when their empty cells lie in different rows).

With this information, given a triple array T, we can construct a Youden rectangle Y compatible with T by first exchanging the roles of columns and symbols in T, yielding the array R, and then adding a new column C_0 with a set S of r new symbols, $s_1, s_2, \ldots s_r$ in this order. To fill the empty cells in row i in R, we then use the $r - 1$ symbols $S \setminus \{s_i\}$, in any order. This is the right number of symbols, since there are $r - 1$ empty cells in every row of R, and there will be no repeated symbol in any row or column.

The intersections between columns in Y may now have three different sizes. As observed above, pairs of columns in R shared either 0 symbols or 1 symbol, and after adding symbols to form Y, these numbers may have gone up by at most 1, since only one new symbol was added in each column. □

An example of the construction in the above proof is given in Figure 3. Since Theorem 5.6 shows that the same transformation that we applied to Youden rectangles could yield interesting row-column designs when applied to a triple-intersection Latin rectangle, we have also included this in our computational studies.

5.2 Computational Results for Youden Rectangles

In this section, we report on how many Youden rectangles yielded triple arrays, proper double arrays, or transposes of proper sesqui arrays, for all parameters (n, k, λ) for which we have complete data, except

0	2	1	4	5	6	8	7	10
11	3	8	5	6	7	9	1	2
5	7	4	9	3	11	0	10	8
1	0	3	2	10	4	6	9	11

(a) A 4×9 triple array T.

0	2	1		3	4	5	7	6		8	
	7	8	1		3	4	5	2	6		0
6			4	2	0		1	8	3	7	5
1	0	3	2	5		6			7	4	8

(b) The corresponding array R with the roles of symbols and rows in T interchanged

9	0	2	1	10	3	4	5	7	6	11	8	12
10	9	7	8	1	11	3	4	5	2	6	12	0
11	6	9	10	4	2	0	12	1	8	3	7	5
12	1	0	3	2	5	9	6	10	11	7	4	8

(c) A triple-intersection Latin rectangle compatible with T.

Fig. 3: Example of the construction in the proof of Theorem 5.6.

for $(21, 5, 1)$ Youden rectangles, where the computing time required was too great.

We ran checks even for properties guaranteed by Theorems 5.3, 5.4 and 5.5. Computational results were compatible with those of these theorems, which can be taken as an independent indication of the correctness of the computations.

5.2.1 Triple Arrays

Among the possible parameters for Youden rectangles for which we have complete data, there are just two sets of parameters where there is a chance of producing triple arrays, namely $(11, 5, 2)$ and $(11, 6, 3)$. All Youden rectangles with $\lambda = 1$ are excluded by Theorem 5.4, and $(7, 4, 2)$ would give a 4×3 triple array, the existence of which was excluded in [19].

In Table 15 for triple arrays and Table 17 for proper double arrays we give the following information:

1. The number of Youden rectangles that give a triple or double array via Construction 5.2 for *at least one* of its columns.
2. The total number of columns for which the construction yields a triple or double array (that is, Youden rectangles counted with 'multiplicities').
3. The number of non-isotopic triple or proper double arrays we observe appearing as a result of this operation.

The 5×6 triple arrays (and by taking transposes, also the 6×5 triple arrays) were completely classified into 7 isotopism classes in [23]. As predicted by Theorem 5.5, all 7 triple arrays appear in Table 15.

(n, k, λ)	# compatible YR	# compatible columns	# TA
(11,5,2)	52	52	7
(11,6,3)	826	826	7

Tab. 15: The number of Youden rectangles giving triple arrays.

Observation 5.8. *Each of the Youden rectangles with $n = 11$ that yields a triple array does so using a unique column.*

The 7 different triple arrays do not appear equally often. With classes numbered as in [23], the triple arrays appear with the frequencies given in Table 16. The orders of the autotopism groups of the triple arrays (in the row labelled TA |Aut|) are taken from [23]. It seems that it is easier to produce those triple arrays that have smaller autotopism groups.

TA class	1	2	3	4	5	6	7
TA $\lvert$Aut$\rvert$	60	12	12	6	4	3	3
# 5×6 YR	3	5	5	8	11	10	10
# 6×5 YR	23	62	62	115	168	198	198

Tab. 16: The number of Youden rectangles giving each of the 7 classes of 5×6 triple arrays.

We investigated the autotopism group orders of the Youden rectangles that produced triple arrays, but we observed no obvious patterns.

5.2.2 Proper Double Arrays

We also checked which Youden rectangles produced proper double arrays, and the results are given in Table 17. As predicted by Theorem 5.4, we see that all Youden rectangles with $\lambda = 1$ produced proper double arrays, for each column. For other values of λ, there is some indication that the proportion of compatible Youden rectangles decreases with growing λ, and that the most common case is that even in a compatible Youden rectangle, only one column is compatible.

(n, k, λ)	# compatible YR	# compatible columns	# DA
(7,3,1)	1	7	1
(7,4,2)	6	18	2
(11,5,2)	44 012	64 949	17 642
(11,6,3)	31 782 790	32 335 774	24 663
(13,4,1)	20	260	192

Tab. 17: The number of Youden rectangles giving proper double arrays.

We note also that for parameter pairs (n, k, λ_1), $(k, n-k, \lambda_2)$, the double arrays produced by the first have dimensions $k \times (n-k)$ and taking transposes yields an $(n-k) \times k$ double array, and vice versa. Despite this, we see different numbers of double arrays appearing through the construction both for the pair $(7,3,1)$, $(7,4,2)$ and the pair $(11,5,2)$, $(11,6,3)$. This would seem to indicate that there are double arrays that cannot be constructed using Construction 5.2.

We note that on the basis of these data, we can answer in the negative a question posed in [22], namely whether every Youden rectangle gives a double array using Construction 5.2 for *some* column. We phrase this as an observation. For examples for $v = 11$, see Figure 4.

Observation 5.9. *There are Youden rectangles that cannot be used to produce double arrays by removing a column and all the symbols in that column, and then interchanging the roles of symbols and columns.*

0	1	2	3	4	5	6	7	8	9	10
1	0	5	6	7	10	4	9	3	8	2
2	5	0	9	8	3	10	4	6	1	7
3	6	8	10	0	1	2	5	7	4	9
4	7	9	0	10	8	5	3	2	6	1

(a) An 11×5 Youden rectangle which does not give a double array for any column

0	1	2	3	4	5	6	7	8	9	10
1	0	9	4	7	8	10	5	3	6	2
2	3	4	7	5	6	1	8	10	0	9
3	6	7	1	8	4	5	9	2	10	0
4	7	0	9	10	1	2	3	6	5	8
5	8	6	10	0	9	7	2	4	3	1

(b) An 11×6 Youden rectangle which does not give a double array for any column

Fig. 4: Examples for Observation 5.9.

5.2.3 Transposes of Proper Sesqui Arrays

Using Construction 5.2, we checked for transposes of proper sesqui arrays, and the results are presented in Table 18.

(n,k,λ)	# compatible YR	# compatible columns	# SA^T
(7,3,1)	0	0	0
(7,4,2)	1	3	1
(11,5,2)	0	0	0
(11,6,3)	8 234	8 234	34
(13,4,1)	0	0	0

Tab. 18: The number of Youden rectangles giving transposes of proper sesqui arrays.

We observe that transposes of sesqui arrays are relatively rare, and that the compatible $(11,6,3)$ Youden rectangles are only compatible for one single column each. The one compatible $(7,4,2)$ Youden rectangle

0	1	2	3	4	5	6
1	2	3	4	6	0	5
2	4	5	6	0	3	1
3	5	6	1	2	4	0
	S	S			D	S

(a) The Youden rectangle.

0	1	4
1	4	2
2	3	5
3	5	0

(b) The transposed sesqui array.

Fig. 5: The unique 4×7 Youden rectangle compatible with the transpose of a sesqui array, with compatible columns marked by S, and a column compatible with a double array marked by D.

0	1	2	3	4	5	6	7	8	9	10
1	0	3	7	6	8	9	10	5	2	4
2	5	7	9	0	3	1	8	10	4	6
3	6	8	10	9	1	2	4	0	7	5
4	7	6	0	8	9	10	1	2	5	3
	T	D		D		D				D

Fig. 6: Example of a 5×11 Youden rectangle with maximum compatibility with respect to triple and proper double arrays. The column marked with T is compatible with a triple array, and the four columns marked with D are compatible with proper double arrays.

is given in Figure 5, together with the resulting transposed sesqui array.

5.2.4 Compatibility with Several Designs

In our data, we found some specimens of Youden rectangles exhibiting very good compatibility properties. To begin with, in Figure 5, we give a $(7,4,2)$ Youden rectangle which is compatible both with transposes of sesqui arrays, and with a proper double array.

Further, some of the Youden rectangles that gave triple arrays of dimensions 5×6 and 6×5 also gave proper double arrays for some other columns. Examples with maximum number of columns compatible with double arrays are given in Figures 6 and 7.

Even for Youden rectangles with $\lambda \neq 1$, we found Youden rectangles that for each column are compatible with some proper double array.

In Figure 8, we give the unique 4×7 Youden rectangle where each column is compatible with a double array. For any column, the resulting double array is isotopic to the one given in Figure 8(b). The Youden rectangle in Figure 8(a) has the largest autotopism group order, i.e., 21, and the autotopism group acts transitively on the columns. As observed above, this Youden rectangle can therefore be produced from a difference set. The double array has an autotopism group of order 3, which acts transitively on the columns.

For $n = 11$, the situation is a bit more complicated. In Figure 9, we give the two 5×11 examples we found, and in Figure 10, we give the unique 6×11 example.

The Youden rectangle in Figure 9(a) has an autotopism group of order 55, which acts transitively on the columns, and so comes from a difference set. All columns yield a double array isotopic to the one

0	1	2	3	4	5	6	7	8	9	10
1	0	4	10	5	7	8	2	3	6	9
2	3	6	9	7	8	0	10	1	5	4
3	6	9	5	10	0	4	1	2	8	7
4	7	3	6	8	9	2	5	10	1	0
5	8	7	0	3	2	10	6	9	4	1
			D						T	

Fig. 7: Example of a 6×11 Youden rectangle with maximum compatibility with respect to triple and proper double arrays. The column marked with T is compatible with a triple array, and the column marked with D is compatible with a proper double array.

0	1	2	3	4	5	6
1	2	4	5	3	6	0
2	4	3	6	5	0	1
3	5	6	1	0	2	4

(a) The Youden rectangle.

0	1	3
1	2	5
2	4	0
3	5	4

(b) The double array.

Fig. 8: The unique 4×7 Youden rectangle where each column is compatible with a double array.

0	1	2	3	4	5	6	7	8	9	10
1	2	5	6	7	3	8	9	4	10	0
2	5	3	8	9	6	4	10	7	0	1
3	6	8	7	0	4	9	1	10	2	5
4	7	9	0	5	10	1	3	2	6	8

(a)

0	1	2	3	4	5	6	7	8	9	10
1	0	5	6	7	3	4	2	9	10	8
2	5	0	8	9	4	10	6	1	3	7
3	6	8	0	10	7	2	9	4	5	1
4	7	9	10	0	8	5	3	6	1	2

(b)

Fig. 9: The only two 5×11 Youden rectangles where each column is compatible with a proper double array.

0	1	2	3	4	5	6	7	8	9	10
1	2	6	4	7	8	3	5	9	10	0
2	6	3	7	5	9	4	8	10	0	1
3	4	7	8	9	0	5	10	1	2	6
4	7	5	9	10	1	8	0	2	6	3
5	8	9	0	1	6	10	2	3	4	7

Fig. 10: The unique 6×11 Youden rectangle where each column is compatible with a proper double array.

0	1	2	3	4	6
1	2	5	6	7	8
2	5	3	8	9	4
3	6	8	7	0	9
4	7	9	0	5	1

Fig. 11: The double array produced from the 5×11 Youden rectangle with autotopism group order 55 given in Figure 9(a).

in Figure 11. The autotopism group order of this double array is 5, and it acts transitively on 5 of the columns, but keeps column 5 fixed.

The Youden rectangle in Figure 9(b) has an autotopism group of order 60, which acts transitively on two groups of columns, with 5 and 6 columns, respectively. All columns in the group with five columns yield the double array in Figure 12(a), and all columns in the group with six columns yield the double array in Figure 12(b). The autotopism group order of these double arrays are 12 and 10, respectively, and the group action for the first one is transitive on the columns, while the autotopism group for the second one acts transitively on all columns except the second column, which is fixed.

Finally, the Youden rectangle in Figure 10 has an autotopism group of order 55, which acts transitively on the columns, and so comes from a difference set. All columns yield the same double array, given in Figure 13, which has an autotopism group of order 5, which acts transitively on the columns.

It is interesting to note that the Youden rectangles in Figures 8–10 that produce a single double array (up to isotopism) for all columns have autotopism groups that act transitively on the columns. For an investigation of this topic, we refer the interested reader to [21].

0	1	2	5	6	9
1	5	4	0	3	7
2	3	6	4	0	8
3	6	8	7	9	1
4	7	5	9	8	2

0	1	2	3	7	8
1	0	3	4	9	5
2	5	6	9	8	0
3	6	5	7	2	4
4	7	8	6	1	9

Fig. 12: The double arrays produced from the 5×11 Youden rectangle with autotopism group order 60 given in Figure 9(b).

0	1	2	3	5
1	2	6	4	8
2	6	3	7	9
3	4	7	8	0
4	7	5	9	1
5	8	9	0	6

Fig. 13: The double array produced from the 6×11 Youden rectangle with autotopism group order 55 given in Figure 10.

(n, k)		(7,3)	(7,4)
# TILR		43	872
|Aut|	1	18	756
	2	21	101
	3	1	10
	4	0	3
	6	2	1
	14	1	1

Tab. 19: The number of triple-intersection Latin rectangles (TILR) with $n = 7$ sorted by autotopism group order.

5.3 Computational Results for triple-intersection Latin Rectangles

As we noted earlier, triple-intersection Latin rectangles both provide the missing source for the $\lambda = 1$ triple arrays and could potentially lead to additional row-column designs. In order to investigate this connection we have also generated all triple-intersection Latin rectangles with $n = 7$, but for larger n we deemed full enumeration infeasible. The number of such rectangles is given in Table 19, sorted by the order of the autotopism groups.

In Table 20 we give the number of such rectangles that are compatible with some proper double array. The maximum number of columns which are compatible with a double array is 2. Among the resulting non-isotopic double arrays for $(7, 3, 1)$ and $(7, 4, 2)$, we see three different double arrays, when taking transposes into account. The rectangles in Figure 14 are examples where the two compatible columns yield non-isotopic arrays, as indicated by subscripts.

(n, k, λ)	# compatible TILR	# compatible columns	# DA
(7,3,1)	6	8	2
(7,4,2)	97	104	2

Tab. 20: The number of triple-intersection Latin rectangles (TILR giving proper double arrays.

For triple-intersection Latin rectangles we have also found two examples which are compatible with proper sesqui arrays, as indicated in Table 21. We also found transposes of proper sesqui arrays in the case 4×7, as indicated in Table 22, and here the maximum number of compatible columns was three. We include all the resulting sesqui arrays here (in normalized form) in Figures 15 and 16, since such arrays

0	1	2	3	4	5	6
1	2	0	5	3	6	4
2	3	4	6	0	1	5
3	4	5	2	6	0	1
		D_2	D_1			

0	1	2	3	4	5	6
1	0	3	5	6	4	2
2	3	4	6	0	1	5
			D_1	D_3		

Fig. 14: Two examples of triple-intersection Latin rectangles with two columns that are compatible with non-isotopic proper double arrays. Subscripted D indicate the resulting non-isotopic double arrays, taking transposes into account.

are scarce in the literature. We note that we only find two non-isotopic sesqui arrays S_1 and S_2, when taking transposes into account, and that S_1 in fact recurs from Figure 5(b).

(n, k, λ)	# compatible TILR	# compatible columns	# SA
(7,3,1)	2	2	2
(7,4,2)	0	0	0

Tab. 21: The number of triple-intersection Latin rectangles (TILR) giving proper sesqui arrays.

(n, k, λ)	# compatible TILR	# compatible columns	# SA^T
(7,3,1)	0	0	0
(7,4,2)	73	78	2

Tab. 22: The number of triple-intersection Latin rectangles giving transposes of proper sesqui arrays.

6 Concluding remarks

With the computing time and storage available to us at present, we have exhausted the possibilities of complete enumeration of Youden rectangles. A further line of inquiry might be to enumerate some restricted class of Youden rectangles, satisfying some stronger conditions. Such conditions would have to go beyond the structure of the symbol intersections between columns, since by only employing the balance condition, we can only distinguish between non-isotopic SBIBDs.

The new class of objects which we have named near Youden rectangles (with only two column intersection sizes λ_1 and λ_2) shows some promise with regard to two desirable properties. First, they exist for far more parameter combinations than Youden rectangles. Second, they always have pairs of symbols covered either λ_1 or λ_2 times where $|\lambda_1 - \lambda_2| = 1$, so it may be expected that they perform reasonably well regarding statistical optimality. In a sense, they are as balanced as they can be. Investigating the statistical properties of these designs is beyond the scope of this paper.

In relation to near Youden rectangles, we would like to pose the following question:

Question 6.1. *For which combinations of k and n do near Youden rectangles exist?*

As we noted earlier a result by Brown [8] implies that for $n = 17, k = 6$ a near Youden rectangle does not exist.

0	1	2	3	4	5	6
1	2	0	4	5	6	3
2	3	4	5	6	1	0
						S_1

0	1	3	4
1	2	4	5
2	3	5	0

(a) S_1

0	1	2	3	4	5	6
1	2	4	0	5	6	3
2	3	1	4	6	0	5
			S_2			

0	1	3	4
1	2	4	5
2	0	5	3

(b) S_2

Fig. 15: The triple-intersection Latin rectangles of size 3×7 that give proper sesqui arrays, together with the corresponding sesqui arrays.

0	1	2	3	4	5	6
1	0	3	4	5	6	2
2	3	5	6	0	1	4
3	4	6	5	2	0	1
			S_1	S_1	S_2	

0	1	4
1	4	2
2	3	5
3	5	0

(b) S_1^T

0	1	4
1	4	0
2	3	5
3	5	2

(c) S_2^T

Fig. 16: Example of a triple-intersection Latin rectangle of size 4×7 that gives transposes of proper sesqui arrays for three compatible columns, together with the corresponding non-isotopic transposed sesqui arrays S_1^T and S_2^T.

In relation to triple, double and sesqui arrays, we would like to pose the following questions:

Question 6.2. *For a given set of parameters, how many double arrays are there that cannot be constructed from any Youden rectangle by removing a column and all the symbols in that column, and then exchanging the roles of symbols and columns?*

Question 6.3. *For a given set of parameters, can every double, triple, and (transpose of) sesqui array be obtained from a Youden rectangle or a triple-intersection Latin rectangle by Construction 5.2?*

Here one could of course extend the set of allowed intersection sizes in the Latin rectangle all the way up to k, so the focus is on whether a small span of intersection sizes suffices.

We hope to return to these questions in future work.

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