Incidence, a scoring positional game on graphs ☆

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ABSTRACT

Positional games have been introduced by Hales and Jewett in 1963 and have been extensively investigated in the literature since then. These games are played on a hypergraph where two players alternately select an unclaimed vertex of it. In the Maker-Breaker convention, if Maker manages to fully take a hyperedge, she wins, otherwise, Breaker is the winner. In the Maker-Maker convention, the first player to take a hyperedge wins, and if no one manages to do it, the game ends by a draw. In both cases, the game stops as soon as Maker has taken a hyperedge. By definition, this family of games does not handle scores and cannot represent games in which players want to maximize a quantity.

In this work, we introduce scoring positional games, that consist in playing on a hypergraph until all the vertices are claimed, and by defining the score as the number of hyperedges a player has fully taken. We focus here on INCIDENCE, a scoring positional game played on a 2-uniform hypergraph, i.e. an undirected graph. In this game, two players alternately claim the vertices of a graph and score the number of edges for which they own both end vertices. In the Maker-Breaker version, Maker aims at maximizing the number of edges she owns, while Breaker aims at minimizing it. In the Maker-Maker version, both players try to take more edges than their opponent.

We first give some general results on scoring positional games such that their membership in Milnor's universe and some general bounds on the score. We prove that, surprisingly, computing the score in the Maker-Breaker version of INCIDENCE is PSPACE-complete whereas in the Maker-Maker convention, the relative score can be obtained in polynomial time. In addition, for the Maker-Breaker convention, we give a formula for the score on paths by using some equivalences due to Milnor's universe. This result implies that the score on cycles can also be computed in polynomial time.

1. Introduction

1.1. Positional games

Positional games have been introduced by Hales and Jewett in 1963 [15] and popularized by Erdős and Selfridge in 1973 [10]. Interest in them has increased due to the large number of games they can handle.

In the standard definition of positional games, the board is a hypergraph on which two players alternately select an unclaimed vertex. In the Maker-Breaker convention, if Maker manages to claim all the vertices of a hyperedge, he wins, otherwise, Breaker is the winner. In the Maker-Maker convention, the first player, if any, who takes a hyperedge wins. If no player manages to claim all the vertices of a hyperedge, the game ends by a draw.

Maker-Maker games are often considered as harder than Maker-Breaker games, since the objective of trying to fill a hyperedge and controlling at the same time that the opponent does not win, is hard to meet.

Positional games are finite perfect information two-players games. As such, there exists a winning strategy for one of the players or both players can insure a draw. The main issue is then to compute, for a given hypergraph, which player has a winning strategy. This problem has been proven to be PSPACE-complete for both conventions, even if all the hyperedges have size at least 11 by Schaefer [26]. This result was recently improved to hypergraph with hyperedges of size at least 6 by Rahman and Watson [25]. On the other side, Galliot et al. proved that the winner can be computed in polynomial time on 3-uniform hypergraphs [12].

In practice, positional games are studied in specific hypergraphs. Historically, they are almost always derived from hypergraphs built from a grid or a complete graph (see for example the reference books [2,16]). More recently, some positional games played on hypergraphs derived from general graphs have been studied. For such games, Maker aims at building a structure in a given graph, and Breaker aims at preventing him to do so. The structure could be, for example, a copy of a graph *H* (H-GAME [17]) or a dominating set (MAKER-BREAKER DOMINATION GAME [8]).

1.2. Scoring games

In parallel to the study of positional games, scoring games have been introduced in the 1950 s by Milnor [21] and Hanner [13]. Their study was almost forgotten until the 2000 s, when different formalisms for such games have been introduced by Ettinger [11], Stewart [28], or Larsson, Nowakowski and Santos [20]. The survey paper [19] summarizes these different approaches.

In scoring games, two players, usually Left and Right, alternate moves with a score adjoined to the game. Each move of a player can modify this score, Left aims at maximizing the score at the end of the game, while Right tries to minimize it. Since scoring games are also finite perfect information games, if both players play optimally, the score at the end of the game is well-defined and only depends on who starts.

Despite the fact that scoring games were less studied, mainly due to the difficulty to build a general framework for them, particular scoring games on graphs have still been introduced recently. One can cite the game INFLUENCE introduced by Duchêne et al. in 2021 [7] which has been proven PSPACE-complete in 2022 [9], or the largest connected subgraph game, introduced by Bensmail et al., firstly as a scoring connection game [5], and then as a Maker-Breaker connection game [4]. In [19], there is a list of other particular scoring games on graphs that have been recently studied.

1.3. Scoring positional games and outline of the paper

In the current paper, we introduce a general scoring version of positional games. Left and Right alternately select vertices of a hypergraph until all the vertices are selected. Points are given when a hyperedge is fully selected by a player. In the Maker-Maker convention, both players get points and the score is the difference between the number of hyperedges taken by Left and Right. In the Maker-Breaker version, the score is only the number of hyperedges taken by Left.

Outline of the paper In Section 2, after giving a formal definition of these games, we provide some general results on them. In particular, we prove that they belong to Milnor's universe and that determining the score is PSPACE-complete in the two conventions. In the rest of the paper, we explore the game INCIDENCE that corresponds to the subcase of 2-uniform hypergraphs (or equivalently to graphs). In Section 3, we prove that, unlike for standard positional games, the Maker-Maker version of INCIDENCE is the easiest one since computing the score is linear in this case. Then we focus on the Maker-Breaker version of INCIDENCE. In Section 4, we give some general bounds on the score as well as some nice properties to deal with twin vertices. This allows us to calculate the exact value of the score for complete binary trees. The next section shows that computing the score in Maker-Breaker convention is PSPACE-complete but fixed-parameter-tractable when parameterized by the *neighborhood diversity* of the graph (introduced in [18]), which implies in particular that it is also FPT when parameterized by vertex cover. The last section is dedicated to the study of paths and cycles. We prove some equivalence relations between paths, which lead to a closed formula for paths and cycles. In particular, we can compute exactly the score for a path of length *n*, which is equal to $n/5 + c$ where *c* only depends on $n \bmod 5$.

2. General results on scoring positional games

2.1. Definitions

Scoring positional games are played on hypergraphs by two players, Left and Right, with the same rules as for standard positional games. The only difference lies in the winning convention. In a scoring positional game, the game ends when all vertices have been claimed. The score of a player is then defined as the number of hyperedges he manages to take. In the Maker-Maker convention, each player tries to maximize his score. In the Maker-Breaker convention, Maker (identified as Left) tries to maximize her score while Breaker (identified as Right) aims at minimizing the score of Maker.

More formally, as for any scoring game, two scores are defined depending on which player starts. Let $H = (V, E)$ be a hypergraph. We define the score of H as follows:

- in the Maker-Maker convention, $Ls(H)$ (resp. $Rs(H)$) as the difference between the scores of Left and Right when Left starts (resp. when Right starts) and both players play optimally.
- in the Maker-Breaker convention, $Ls(H)$ (resp. $Rs(H)$) as the score of Left when Left (resp. Right) starts and both players play optimally.

It is well-known in scoring game theory that these notions exist and are well-defined (by considering the game tree of all the possible moves). Note that in the Maker-Maker convention, by symmetry of the roles of both players, we have that $Ls(H) = -Rs(H)$, so computing $Ls(H)$ will be of sufficient interest. In the Maker-Breaker convention, we have that $Ls(H)$ and $Rs(H)$ are nonnegative values by definition.

In addition, it will be helpful to consider the scores obtained after some vertices have been claimed. A *position* of a scoring positional game is a triplet $P = (H, V_L, V_R)$ such that V_L and V_R are disjoint subsets of vertices. The set V_L corresponds to the vertices claimed by Left whereas V_R correspond to the vertices claimed by Right. The set of remaining vertices will be generally denoted by V_F. We have $V_F = V \setminus (V_L \cup V_R)$. For both conventions, we will denote by $Ls(P)$ (resp. $Rs(P)$) the score of H if Left has already claimed the vertices of V_L, and Right the vertices of V_R, when Left (resp. Right) starts. When $V_F \neq \emptyset$, the scores at a position P can be recursively defined as follows:

$$Ls(P) = \max_{x \in V_F} Rs(H, V_L \cup \{x\}, V_R)$$
$$Rs(P) = \min_{x \in V_F} Ls(H, V_L, V_R \cup \{x\}).$$

When $V_F = \emptyset$, the score depends on the convention. In Maker-Maker convention,

$$Ls(P) = Rs(P) = |\{e \in E | e \subseteq V_L\}| - |\{e \in E | e \subseteq V_R\}|$$

whereas in Maker-Breaker convention, we have

$$Ls(P) = Rs(P) = |\{e \in E | e \subseteq V_L\}|.$$

In the literature, there are few games that can be seen as scoring positional games. The famous Dots and Boxes game [3], that has recently be proven PSPACE-complete by Buchin et al. [6], could be an example, with the additional constraint that a player is forced to move again each time he gets points. By removing this constraint, we get a pure example of the above definition (in the Maker-Maker convention), and the game is known as Picarête [1]. More recently, the Constructor-Blocker game introduced by Patkos et al. [23] in 2022, in which Constructor aims at maximizing the number of copies of a graph H with a forbidden graph F, can be seen as a scoring positional game when F is empty.

Incidence   In most of this paper, we will mainly focus on an example of scoring positional game that is called INCIDENCE. It corresponds to the game played on a hypergraph where all hyperedges are of size two. In other terms, this game can be defined as follows on a simple graph $G = (V, E)$. Alternately, two players claim an unclaimed vertex of V. When all the vertices have been taken, the score of a player is defined as the number of edges in the subgraph of G induced by the vertices he claimed.

Hence, in both conventions, Left (that is always Maker) aims at collecting points by claiming the two extremities of an edge. The main difference concerns the role of Right, that aims at touching the maximum number of edges (hence prohibiting a maximum number of points for Left) in the Maker-Breaker convention. See Fig. 1 for an example of computations of the score at the end of a game.

2.2. Milnor's universe

In 1953 [21], Milnor introduced a universe of scoring games having nice properties. This universe is the one of *dicotic nonzugzwang* games:
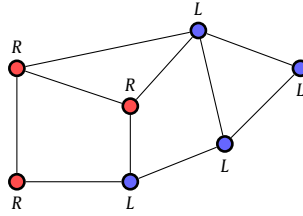
Fig. 1. An endgame of Incidence. In Maker-Maker convention the score of the position is 2 while it is 4 in Maker-Breaker convention.

- a game is *dicotic* if at any moment of the game, if a player can play, the other player can also play.
- a game is *nonzugzwang* if at any moment of the game, both players have no interest in skipping their turn. In the context of scoring positional games, it means that for a hypergraph H, we have $Ls(H, V_L, V_R) \geq Rs(H, V_L, V_R)$ for any sets of vertices V_L, V_R claimed by Left and Right during the game.

Being in Milnor's universe induces a couple of useful results concerning the sum operator and the equivalence of games. The *disjunctive sum operator* $+$ applied to scoring (positional) games G_1 and G_2 defines the game $G_1 + G_2$ as the game in which a move consists in moving either in G_1 or in G_2. The game ends when the moves in both components of the sum are exhausted. See [9] for the formal definition. Note that the sum of two scoring positional games, with the same convention, is still a scoring positional game with hypergraph the disjoint union of the two hypergraphs. As game sums appear in many games when playing, one could expect to simplify them by replacing large games by smaller ones. This leads to the notion of equivalence of games:

Definition 1 *(Milnor [21]).* Two scoring games G_1 and G_2 are *equivalent* (write $G_1 \equiv G_2$) if for any game G, we have $Ls(G + G_1) = Ls(G + G_2)$ and $Rs(G + G_1) = Rs(G + G_2)$.

In other terms, one can always exchange G_1 and G_2 in any sum of games if they are equivalent. In particular, games that are equivalent to the empty game can be removed from any sum of games.

Games belonging to Milnor's universe form an Abelian group with the sum operator [21]. In particular, this implies that every game G in Milnor's universe admits an inverse, i.e. a game G' such that $G + G' \equiv 0$ (where 0 is the empty game). More precisely, this inverse corresponds to the *negative* of G, i.e. the game where the roles of Left and Right are exchanged, together with their scores.

Moreover, proving equivalence in Milnor's universe is greatly simplified, thanks to the next lemma.

Lemma 2 *(Milnor [21]).* For any games G and H that are dicotic nonzugzwang, we have: $Ls(G - H) = Rs(G - H) = 0$ if and only if G and H are equivalent.

In addition, sums of games in Milnor's universe can be bounded as follows:

Lemma 3 *(Milnor [21]).* Let G and H be two dicotic nonzugzwang games, we have

$$Rs(G) + Rs(H) \leq Rs(G + H) \leq Ls(G) + Rs(H) \leq Ls(G + H) \leq Ls(G) + Ls(H).$$

In what follows, we will show that scoring positional games belong to Milnor's universe. Yet, the negative of a game cannot be defined in the Maker-Breaker convention, as the scores of Maker and Breaker can not be interchanged naturally, by asymmetry of the definition of the score. Therefore, we have decided to embed scoring positional games in a more general family that will be called PARTISAN SCORING POSITIONAL GAMES. The term partisan is derived from standard combinatorial games [3], meaning that Left and Right may have different moves (and also different ways of scoring points).

A partisan scoring positional game is played on a hypergraph H whose hyperedges are either colored blue, red or green. The two players, Left and Right, alternatively claim vertices of H. The score of Left corresponds to the blue and green hyperedges she claimed, whereas the score of Right corresponds to the red and green ones. As previously, the score of the game ($Ls(H)$ and $Rs(H)$, depending on who starts) is the difference between the score of Left and Right.

Partisan scoring positional games include both Maker-Maker and Maker-Breaker scoring positional games. Even more, the convention can be omitted, as it is deduced by the colors of the hypergraph. Indeed, if all the hyperedges are green, it means that both players can win any hyperedge, which corresponds to the Maker-Maker version. If all the hyperedges are blue, it corresponds to the Maker-Breaker convention, as only Left can get points. According to this definition, the negative of a partisan scoring positional game is well-defined, as it suffices to exchange the colors blue and red in the hyperedges, as well as the vertices already chosen by Left and Right (if any).

We will now give several general results about partisan scoring positional games. By inclusion, these results will also concern scoring positional games. First, we will prove that they belong to Milnor's universe and thus satisfy Lemma 2.

Lemma 4. *Partisan scoring positional games belong to Milnor's universe.*

Proof. Let $H = (V, E)$ be a hypergraph with hyperedges colored blue, red and green, and $V_L, V_R \subset V$ be vertices already claimed by Left and Right respectively such that $V_L \cap V_R = \emptyset$.

A partisan scoring positional game is dicotic: if $V_L \cup V_R = V$, then no moves are available, neither for Left nor for Right. Otherwise, let $v \in V \setminus \{V_L \cup V_R\}$. Both Left and Right are allowed to play v as it is an unclaimed vertex. Therefore, the game is dicotic.

A partisan scoring positional game is nonzugzwang: We need to prove that $Ls(H, V_L, V_R) \geq Rs(H, V_L, V_R)$. Let $k = Rs(H, V_L, V_R)$ with V_L, V_R vertices already claimed in H by Left and Right respectively. If $V_L \cup V_R = V$, we have $Ls(H, V_L, V_R) = Rs(H, V_L, V_R) = k$ as there is no move available in H. Otherwise, let $\mathcal{S}$ be an optimal strategy for Left when Right starts. We define a strategy $\mathcal{S}'$ for Left when she starts as follows:

- Left considers an arbitrary unclaimed vertex v_0 of the graph, and plays the vertex she would have played in $\mathcal{S}$ if Right plays v_0.
- Whenever, Right plays a vertex w in $V \setminus \{v_0\}$, she plays the vertex she would have played in $\mathcal{S}$ if Right has played w in $\mathcal{S}$ after having played v_0 on first move.
- If Right plays v_0, she considers an arbitrary unclaimed vertex v_1 in the graph, and continues this strategy by supposing that Right has played v_1 instead of v_0. More generally, when Right claims the vertex v_ℓ, she considers an unclaimed vertex $v_{\ell+1}$ and considers that Right has claimed $v_{\ell+1}$ instead.
- At the end, if she needs to consider that Right has played a vertex v_ℓ and no other vertex is available, she plays v_ℓ.

Following this strategy, all the vertices Left would have played in $\mathcal{S}$ if Right has played the vertices v_i s she has considered, have been played in $\mathcal{S}'$ by Left. Similarly, the vertices that Right have played in $\mathcal{S}'$ are a subset of the one he would have played in $\mathcal{S}$. Therefore, as $\mathcal{S}$ was an optimal strategy in H when Right starts, this strategy ensures that Left scores at least $k = Rs(H, V_L, V_R)$. Finally, we have $Ls(H, V_L, V_R) \geq k = Rs(H, V_L, V_R)$, and the game is nonzugzwang.

As the game is nonzugzwang and dicotic, it belongs to Milnor's universe. □

As a consequence, this result applies also to scoring positional games and, in particular, the game INCIDENCE. We will use this result in Section 6 to solve INCIDENCE on paths.

2.3. Algorithmic complexity

We now prove that computing the Left score of a scoring positional game is PSPACE-complete in both conventions. This result is a direct consequence of the PSPACE-complexity of (non-scoring) positional games.

MAKER-BREAKER POSITIONAL GAME
Instance: A hypergraph H, $P \in \{\text{Maker}, \text{Breaker}\}$.
Output: True if Maker wins the Maker-Breaker positional game played on H with first player P.

MAKER-BREAKER POSITIONAL GAME has been proved to be PSPACE-complete by Schaeffer [26] for 11-uniform hypergraphs (all the hyperedges have size 11). This result was recently improved to 6-uniform hypergraphs by Rahman and Watson [25].

Theorem 5 *([25]).* MAKER-BREAKER POSITIONAL GAME *is PSPACE-complete even restricted to* 6-*uniform hypergraphs.*

MAKER-BREAKER POSITIONAL GAME can easily be reduced to the two following problems on scoring positional games.

MAKER-BREAKER SCORING POSITIONAL GAME
Instance: A hypergraph H, an integer k, a first player $P \in \{Left, Right\}$.
Output: True if the P score in the scoring positional game played on H with Maker-Breaker convention is at least k, false otherwise.
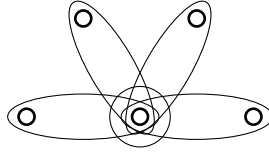
MAKER-MAKER SCORING POSITIONAL GAME
Instance: A hypergraph H, an integer k.
Output: True if the Left score in the scoring positional game played on H with Maker-Maker convention is at least k, false otherwise.

Corollary 6. MAKER-BREAKER SCORING POSITIONAL GAME *is PSPACE-complete even restricted to* 6-*uniform hypergraphs,* $P = Left$ *and* $k = 1$.

MAKER-MAKER SCORING POSITIONAL GAME *is PSPACE-complete even restricted to* 7-*uniform hypergraphs and* $k = 1$.

Proof. Since both games are played in $|V(H)|$ turns, they belong to PSPACE according to Section 6.1 in [14].

Fig. 2. A hypergraph satisfying $Ls(H) = \lfloor \frac{\Delta(H) + 1}{2} \rfloor$ in Maker-Maker convention.

Let H be a 6-uniform hypergraph and assume Left is the first player. We have $Ls(H) \geq 1$ in the Maker-Breaker convention if and only if Maker wins the Maker-Breaker positional game (without score) played on H with Maker as first player. Thus, by Theorem 5, Maker-Breaker Scoring Positional Game is PSPACE-complete even restricted to 6-uniform hypergraphs, $k = 1$ and $P = Left$.

Consider now H' the 7-uniform hypergraph obtained from H by adding a universal vertex v_0: each hyperedge of H is extended to contain v_0. There exists an optimal strategy in the Maker-Maker convention that starts by claiming v_0. Then the other player cannot score any point. Then, we have $Ls(H') \geq 1$ if and only if Maker wins playing second in the Maker-Breaker positional game (without score) played on H. Thus, by Theorem 5, Maker-Maker Scoring Positional Game is PSPACE-complete even restricted to 7-uniform hypergraphs and $k = 1$.   □

We will complete the results of Corollary 6 in next sections by proving that Maker-Breaker Scoring Positional Game is still PSPACE-complete for 2-uniform hypergraphs (Theorem 18). This will imply that Maker-Maker Scoring Positional Game is PSPACE-complete for 3-uniform hypergraphs. To complete the picture, we will give a linear algorithm to solve Maker-Maker Scoring Positional Game in 2-uniform hypergraphs (Theorem 11).

2.4. Bounds in Maker-Maker convention

In this subsection, we give an easy bound on the score in Maker-Maker convention, using the maximal degree of the hypergraph. Let H be a hypergraph. The *degree* of a vertex v of H is the number of hyperedges containing v. We denote by $\Delta(H)$ the maximal degree of H.

Lemma 7. *Let H be a hypergraph. In the Maker-Maker scoring positional game on H, we have $-\Delta(H) \leq Rs(H) \leq 0 \leq Ls(H) \leq \Delta(H)$.*

Proof. As noticed in Section 2.1, we have $Ls(H) = -Rs(H)$ in the Maker-Maker convention since players have symmetric roles. Since the game is nonzugzwang, we also have $Ls(H) \geq Rs(H)$ which implies that $Rs(H) \leq 0 \leq Ls(H)$.

To prove the upper bound with $\Delta(H)$, we just need to prove that $Ls(H) \leq \Delta(H)$. Let v_0 be the first vertex played in an optimal strategy. Consider the hypergraph H' obtained from H by removing v_0 and all the hyperedges containing it. If the second player applies the optimal strategy for H' during the rest of the game, he will score at least $Rs(H') \leq 0$ on it and the final score will be at most $|\{e | v_0 \in e\}| + Rs(H')$. Thus, we have $Ls(H) \leq deg(v_0) + Rs(H') \leq \Delta(H)$.   □

We do not think that the upper bound in Lemma 7 is tight if the hypergraph is simple (i.e. there are no two hyperedges that contain exactly the same vertices). Actually, the best example we know in this case is a hypergraph H having a universal vertex x, a hyperedge with x alone and $\Delta - 1$ hyperedges of size 2 containing x and another unique vertex, see Fig. 2. For this hypergraph, $Ls(H) = \lfloor \frac{\Delta(H)+1}{2} \rfloor$. Besides, we will prove that for 2-uniform hypergraphs (i.e. graphs), the score is at most $\Delta(H)/2$ (see Corollary 13. We believe that this bound remains true in any hypergraph.

Conjecture 8. *Let H be a simple hypergraph. In the Maker-Maker scoring positional game on H, we have $Ls(H) \leq \frac{\Delta(H)+1}{2}$.*

2.5. Bounds in Maker-Breaker

In Maker-Breaker convention, the bound from Lemma 7 is not valid anymore. Indeed, the score can actually be linear with the number of vertices of the hypergraph, even if the maximal degree is constant. Next, we derive a general tight bound, based on the same principle used to prove the Erdös-Selfridge criterion [10]. Some tight examples will be given in Section 4 for 2-uniform hypergraphs (see Corollary 14).

Theorem 9 (Erdős, Selfridge, 1973 [10]). *Let $H = (V, E)$ be a hypergraph. If $\sum_{e \in E} 2^{-|e|} < 1$, then Breaker wins on H when he starts. If $\sum_{e \in E} 2^{-|e|} < \frac{1}{2}$, then Breaker wins on H when Maker starts.*

The main idea to prove this theorem is that if the hyperedges are large enough, Breaker will have the time to play in all of them before Maker can fill one. A similar idea can be introduced when dealing with scores by computing how many

hyperedges Breaker can touch. The strategy used relies on a greedy strategy by introducing a potential function, as it was done by Erdős and Selfridge. Let H be a hypergraph. We denote by $\ell(H)$ the maximum number of hyperedges that contain a fixed pair of vertices. More formally, $\ell(H) = \max\limits_{x,y \in V^2} |\{e \in E | x, y \in e\}|$.

Theorem 10. *Let $H = (V, E)$ be a hypergraph. In the Maker-Breaker convention, we have $Ls(H) \geq \sum\limits_{e \in E} 2^{-|e|} - \frac{n\ell(H)}{8}$, and $Rs(H) \leq \sum\limits_{e \in E} 2^{-|e|}$.*

Proof. Let (H, V_L, V_R) be any position of a Maker-Breaker scoring positional game. We introduce the potential function:

$$P(H, V_L, V_R) = \sum_{e \in E, e \cap V_R = \emptyset} 2^{-|e \setminus V_L|}.$$

In this function, only hyperedges not played by Right are considered, and we only count the number of free vertices in the edge. Note that at the beginning of the game, $P(H, \emptyset, \emptyset) = \sum\limits_{e \in E} 2^{-|e|}$. At the end of the game, $V = V_L \cup V_R$ and $P(H, V_L, V_R) = |\{e \in E | e \cap V_R = \emptyset\}|$ is the final score. Furthermore, when a vertex v is played by Maker (respectively Breaker), the potential is increasing (resp. decreasing) by the quantity

$$\delta_P(H, V_L, V_R, v) = \sum_{e | e \cap V_R = \emptyset, v \in e} 2^{-|e \setminus V_L|}.$$

Let $\mathcal{S}$ be a strategy for Maker consisting in maximizing P at each move, i.e. Maker chooses the vertex v that maximizes $\delta_P(H, V_L, V_R, v)$. We prove that this strategy provides the desired bound. Suppose first that Maker starts. Suppose V_L and V_R have already been played by Maker and Breaker respectively. Let v_L the vertex played by Maker according to $\mathcal{S}$ and v_R the vertex played by Breaker after this move. As Maker has played v_L and not v_R, we have, before v_L was played, $\delta_P(H, V_L, V_R, v_L) \geq \delta_P(H, V_L, V_R, v_R)$.

However, $\delta_P(H, V_L \cup \{v_L\}, V_R, v_R)$ might be larger than $\delta_P(H, V_L, V_R, v_R)$ after v_L was played if there exist some hyperedges that contain both v_L and v_R. We actually have:

$$\delta_P(H, V_L \cup \{v_L\}, V_R, v_R) = \delta_P(H, V_L, V_R, v_R) + \sum_{e \cap V_R = \emptyset, v_L, v_R \in e} 2^{-|e \setminus V_L|}$$

$$\leq \delta_P(H, V_L, V_R, v_R) + \frac{\ell(H)}{4}.$$

Last inequality comes from the fact that $e \setminus V_L$ must contain v_L and v_R and thus has size at least 2. Therefore, we have

$$P(H, V_L \cup \{v_L\}, V_R \cup \{v_R\}) = P(H, V_L, V_R) + \delta_P(H, V_L, V_R, v_L) - \delta_P(H, V_L \cup \{v_L\}, V_R, v_R)$$

$$\geq P(H, V_L, V_R) - \frac{\ell(H)}{4}.$$

As there is n moves in the game by applying this step $\frac{n}{2}$ times for each pair of moves (recall that we consider here that Maker starts), we have at the end of the game $Ls(H) \geq P(H, V_L, V_R) \geq P(H, \emptyset, \emptyset) - \frac{n}{2} \times \frac{\ell(H)}{4}$, as required.

Suppose now that Breaker starts and considers this strategy for him (i.e. choosing the vertex v that maximizes $\delta_P(H, V_L, V_R, v)$). Suppose V_L and V_R have already been played by Maker and Breaker respectively. Let v_R be the vertex played by Breaker according to $\mathcal{S}$ and let v_L be the vertex answered by Maker. We have $\delta_P(H, V_L, V_R, v_R) \geq \delta_P(H, V_L, V_R, v_L)$. Note that here, $\delta_P(H, V_L, V_R \cup \{v_R\}, v_L)$ cannot increase after the move of Right, as it does not change the size of the hyperedges (it can only decrease if some edges containing v_L also contain v_R). Therefore, after these two moves, we obtain $P(G, V_L \cup \{v_L\}, V_R \cup \{v_R\}) \leq P(G, V_L, V_R)$. By applying this result from $V_L = V_R = \emptyset$ to the end of the game, we obtain $P(H, V_L, V_R) \leq P(H, \emptyset, \emptyset)$ for any sets V_L and V_R obtained after Right applies $\mathcal{S}$. In particular, when the game ends, this strategy ensures that $Rs(H) \leq P(H, \emptyset, \emptyset) = \sum\limits_{e \in H} 2^{-|e|}$. $\quad\square$

From now on and until the end of the paper, we will focus on the game INCIDENCE, i.e. the scoring positional game played on a graph.

3. MAKER-MAKER INCIDENCE is polynomial

In this section, we provide a linear time algorithm to compute the score of MAKER-MAKER INCIDENCE. A natural idea, while playing INCIDENCE, is that high degree vertices are interesting to play first, as they enable to score many points with

their multiple adjacent edges. Therefore, a simple strategy for both players would be to play greedily by always picking an available vertex of highest degree. We here prove that this strategy is optimal.

Later in Section 5, we will prove that MAKER-BREAKER INCIDENCE is PSPACE-complete, which induces that MAKER-MAKER SCORING POSITIONAL GAME is PSPACE-complete on 3-uniform hypergraphs.

Theorem 11. *Let G be a graph with n vertices. Let $d_1 \geq \ldots \geq d_n$ be the degree of the vertices in decreasing order. For the game* MAKER-MAKER INCIDENCE *played on G, we have*

$$Ls(G) = \frac{1}{2} \left(\sum_{i \text{ odd}} d_i - \sum_{i \text{ even}} d_i \right).$$

In particular, the score can be computed in linear time.

Proof. Let $G = (V, E)$ be a graph. Denote by $v_1, \ldots, v_n$ the vertices of G of degree $d_1, \ldots, d_n$ respectively, and arranged such that $d_1 \geq d_2 \geq \cdots \geq d_n$. Denote by $s = \frac{1}{2}(\sum_{i \text{ odd}} d_i - \sum_{i \text{ even}} d_i)$. We will prove that $Ls(G) = s$. Before proving the value of the score, we prove the following claim:

Claim. Denote by V_L the vertices claimed by Left, and by V_R the vertices claimed by Right at the end of a game played on G. The score obtained is $\frac{1}{2}(\sum_{v_l \in V_L} d_l - \sum_{v_r \in V_R} d_r)$.

Proof. Denote by e_L (resp. e_R) the number of edges where both endpoints were claimed by Left (resp. Right) and by e_0 the number of edges which have one extremity claimed by each player.

By definition, the score is $e_L - e_R$. Now, by a double counting argument, we have $\sum_{v_l \in V_L} d_l = 2e_L + e_0$, and $\sum_{v_r \in V_R} d_r = 2e_R + e_0$. Therefore, the score of the game is $e_L - e_R = \frac{1}{2}(\sum_{v_f \in V_L} d_l - \sum_{v_r \in V_R} d_r)$. ∎

Now we provide a strategy for Left that proves that $Ls(G) \geq s$. The same argument works for Right and leads to $Ls(G) \leq s$. Consider that Left claims at each turn the free vertex of highest degree. During her first turn, she claims a vertex of degree d_1, during the second turn, she claims either a vertex of degree d_2 or d_3, both having a value of at least d_3, ..., during here k-th turn, she will claim a vertex of degree $d_k, d_{k+1}, \ldots$ or d_{2k-1}, each of them have a value of at least d_{2k-1}. In the end, she will have played $\lceil \frac{n}{2} \rceil$ vertices, and the k-th of them will be of degree at least d_{2k-1}. Reciprocally, the highest degree played by Right has value at most d_2, the second highest has value at most d_4 and so on. Therefore, by using the result of the claim, the score obtained by this strategy is at least s.

The above score can be computed in linear time because it does not require to sort the list of the vertices, but only to know the number of vertices of any degree, which is bounded by $n - 1$. □

Corollary 12. *Let $n \in \mathbb{N}$. Denote by P_n the path of order n. In* MAKER-MAKER INCIDENCE, *we have $Ls(P_n) = -Rs(P_n) = 0$ if n is even and $Ls(P_n) = -Rs(P_n) = 1$ if n is odd.*

Proof. P_n has exactly $n - 2$ vertices of degree 2 and two vertices of degree 1. Therefore, if n is even, an optimal strategy gives $\frac{n}{2} - 1$ vertices of degree two and one vertex of degree one to each player, which provides a draw. If n is odd, Left has one more vertex of degree 2 to play, and her score is then 1. □

Corollary 13. *Let G be a graph of maximal degree Δ. In* MAKER-MAKER INCIDENCE, *we have $Ls(G) \leq \frac{\Delta}{2}$.*

Proof. Let G be a graph of maximal degree Δ. Up to adding an isolated vertex, suppose it has an even number of vertices. Denote by $d_1, d_2, \ldots, d_{2n}$ its degrees written in decreasing order. We have $Ls(G) = \frac{1}{2} \sum_{i=1}^{n} (d_{2i-1} - d_{2i}) = \frac{\Delta}{2} - \sum_{i=1}^{n} (d_{2i} - d_{2i+1})$, by setting $d_{2n+1} = 0$. For any $1 \leq i \leq n$, we have $d_{2i} \geq d_{2i+1}$. Hence, each term of the sum is nonnegative, and finally, we have $Ls(G) \leq \frac{\Delta}{2}$. □

4. General results on MAKER-BREAKER INCIDENCE

In the rest of the paper, we focus on the Maker-Breaker version of INCIDENCE. Contrary to the Maker-Maker version of this game, a greedy strategy is not always optimal. Thus, studying this game is much more challenging. In this section, we give some general results on this version. We start with a direct application of the bound given for general scoring positional games in Theorem 10.

Corollary 14. *Let G be a graph with n vertices and m edges. In the* Maker-Breaker Incidence *game, $Ls(G) \geq \frac{m}{4} - \frac{n}{8}$, and $Rs(G) \leq \frac{m}{4}$. These bounds are tight.*

Proof. This is a direct application of Theorem 10. Since the hypergraph is 2-uniform and simple, for each pair of vertices, there is at most one edge containing the two vertices. Thus we have $\ell(G) = 1$. Furthermore, each edge has size 2, thus $\sum_{e \in G} 2^{-|e|} = \frac{m}{4}$.

For tightness, consider first a graph G that is a complete graph of order $8k$, with $k \in \mathbb{N}$. The lower bound gives $Ls(G) \geq \frac{\binom{8k}{2}}{4} - k = \binom{4k}{2}$. By playing randomly, Left takes $4k$ vertices and each pair of vertices scores one point. Thus $Ls(G) = \binom{4k}{2}$

Consider the graph H made by a disjoint union of $2k$ paths on three vertices. Left playing second can take k central vertices and one leaf for each central vertex he has taken. This strategy gives at most k points to Left which is equal to the upper bound $\frac{m}{4}$ given in the statement. □

While playing Incidence, some moves are equivalent: playing one or the other will not change the final score. This is in particular the case when two vertices have the same neighborhood (up to the vertices already played). An interesting fact in this case is that, in Maker-Breaker convention, we can assume that each player will take exactly one of the two vertices. More formally, let $G = (V, E)$ be a graph and $P = (G, V_L, V_R)$ some position of the game on G. Let v_1, v_2 be two free vertices. Vertices v_1, v_2 are said to be *equivalent* in P if and only if we have $N(v_1) \cap V_F \setminus \{v_2\} = N(v_2) \cap V_F \setminus \{v_1\}$ and $|N(v_1) \cap V_L| = |N(v_2) \cap V_L|$. Note that the first equality is a set equality, while the second one only is on cardinals.

Lemma 15. *Let $G = (V, E)$ be a graph and let $P = (G, V_L, V_R)$ be a position of the game. Let v_1, v_2 be equivalent vertices in P. In* Maker-Breaker Incidence*, we have $Ls(P) = Ls(G, V_L \cup \{v_1\}, V_R \cup \{v_2\})$ and $Rs(P) = Rs(G, V_L \cup \{v_1\}, V_R \cup \{v_2\})$.*

Proof. We prove both results by induction on $|V_F| = |V \setminus (V_L \cup V_R)|$, the number of free vertices. The result is clear if there are only two free vertices v_1 and v_2 as each player will claim one of them, and they will have the same number of neighbors in V_L at the end. Let $P = (G, V_L, V_R)$ be a position with $|V_F| \geq 3$, and let $v_1, v_2 \in V_F$ be equivalent vertices in P.

We first prove that $Ls(P) = Ls(G, V_L \cup \{v_1\}, V_R \cup \{v_2\})$. Let x be an optimal move for Left in P. If $x \in \{v_1, v_2\}$, we have $Ls(P) = Rs(G, V_L \cup \{v_1\}, V_R)$. Indeed, exchanging the roles of v_1 and v_2 is possible since they will score exactly the same number of points at the end. Using the recursive definition of the scores we have, $Rs(G, V_L \cup \{v_1\}, V_R) \leq Ls(G, V_L \cup \{v_1\}, V_R \cup \{v_2\})$. Otherwise, we have $Ls(P) = Rs(G, V_L \cup \{x\}, V_R)$. Vertices v_1 and v_2 are still equivalent in $(G, V_L \cup \{x\}, V_R)$. By induction, $Rs(G, V_L \cup \{x\}, V_R) = Rs(G, V_L \cup \{v_1, x\}, V_R \cup \{v_2\})$. According to the recursive definition of the score, $Ls(G, V_L \cup \{v_1\}, V_R \cup \{v_2\}) \geq Rs(G, V_L \cup \{v_1, x\}, V_R \cup \{v_2\})$. Finally, in both cases, $Ls(P) \leq Ls(G, V_L \cup \{v_1\}, V_R \cup \{v_2\})$.

We now prove the other inequality. Let x be an optimal move for Left in $(G, V_L \cup \{v_1\}, V_R \cup \{v_2\})$. We have $Ls(G, V_L \cup \{v_1\}, V_R \cup \{v_2\}) = Rs(G, V_L \cup \{v_1, x\}, V_R \cup \{v_2\})$. By induction, since v_1 and v_2 are still equivalent in $(G, V_L \cup \{x\}, V_R)$, we have $Rs(G, V_L \cup \{v_1, x\}, V_R \cup \{v_2\}) = Rs(G, V_L \cup \{x\}, V_R)$. Using the recursive definition of the score, $Ls(P) \geq Rs(G, V_L \cup \{x\}, V_R)$, which leads to $Ls(P) \geq Ls(G, V_L \cup \{v_1\}, V_R \cup \{v_2\})$. Finally, we have proved $Ls(P) = Ls(G, V_L \cup \{v_1\}, V_R \cup \{v_2\})$.

We now turn to the proof of $Rs(P) = Rs(G, V_L \cup \{v_1\}, V_R \cup \{v_2\})$. Let x be an optimal move for Right in P. If $x \in \{v_1, v_2\}$, we have $Rs(P) = Ls(G, V_L, V_R \cup \{v_2\})$. Indeed, exchanging the roles of v_1 and v_2 is possible since they will score exactly the same number of points at the end. Using the recursive definition of the scores, we have $Ls(G, V_L, V_R \cup \{v_2\}) \geq Rs(G, V_L \cup \{v_1\}, V_R \cup \{v_2\})$. Otherwise, we have $Rs(P) = Ls(G, V_L, V_R \cup \{x\})$. Vertices v_1 and v_2 are still equivalent in $(G, V_L, V_R \cup \{x\})$. By induction, $Ls(G, V_L, V_R \cup \{x\}) = Ls(G, V_L \cup \{v_1\}, V_R \cup \{v_2, x\})$. According to the recursive definition of the score, $Rs(G, V_L \cup \{v_1\}, V_R \cup \{v_2\}) \leq Ls(G, V_L \cup \{v_1\}, V_R \cup \{v_2, x\})$. Finally, in both cases, $Rs(P) \geq Rs(G, V_L \cup \{v_1\}, V_R \cup \{v_2\})$.

We now prove the other inequality. Let x be an optimal move for Right in $(G, V_L \cup \{v_1\}, V_R \cup \{v_2\})$. We have $Rs(G, V_L \cup \{v_1\}, V_R \cup \{v_2\}) = Ls(G, V_L \cup \{v_1\}, V_R \cup \{v_2, x\})$. By induction, since v_1 and v_2 are still equivalent in $(G, V_L, V_R \cup \{x\})$, we have $Ls(G, V_L \cup \{v_1\}, V_R \cup \{v_2, x\}) = Ls(G, V_L, V_R \cup \{x\})$. Using the recursive definition of the score, $Rs(P) \leq Ls(G, V_L, V_R \cup \{x\})$, which leads to $Rs(P) \leq Rs(G, V_L \cup \{v_1\}, V_R \cup \{v_2\})$.

Finally, we have proved $Rs(P) = Rs(G, V_L \cup \{v_1\}, V_R \cup \{v_2\})$. □

Note that, this result is only true for equivalent vertices. In general, a good move for Left is not necessarily a good move for Right. For instance, in Fig. 3, if Left starts by playing u, the score is 4, and if she starts by playing any other vertex, the score is at most 3, thus her only optimal move is u. If Right starts by playing v, the score is 2, but if he starts by playing any other vertex, the score is at least 3. Hence, his only optimal move is v.

Lemma 15 is actually very useful to deal with similar vertices. We illustrate its power by computing the score for complete binary trees. A complete binary tree of depth k is a rooted tree such that each vertex at depth $j < k$ has exactly two children (and by definition of the depth, each vertex at depth k is a leaf).

Corollary 16. *Let T_k be a complete binary tree of depth $k \geq 1$. The scores in Maker-Breaker* Incidence *are $Ls(T_k) = 2^{k-1}$ and $Rs(T_k) = 2^{k-1} - 1$.*
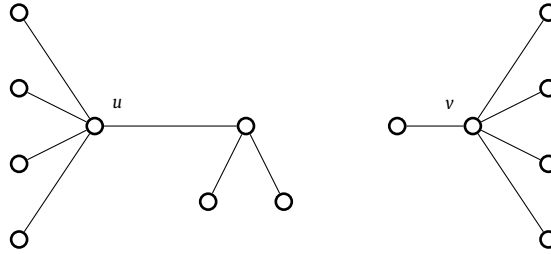
Fig. 3. A graph G for which $Ls(G) = 4$ with unique optimal move u and $Rs(G) = 2$ with unique optimal move v.

Proof. Let T_k be a complete binary tree of depth k. Its leaves are pairwise equivalent. By Lemma 15, we can assume that for any two leaves connected to a same vertex, one leaf can be given to Right, and the second one to Left. Then, the parents of the leaves have only one unclaimed neighbor and one neighbor claimed by Left. Therefore, any two parents of leaves with the same unclaimed neighbor have the same type. Thus, we can again apply Lemma 15 and assign one vertex of each pair to each player. By iterating this process from the leaves to the root, for any pair of vertices having the same parent, Maker and Breaker both get one of them. The game is then equivalent to the game where only the root is unclaimed, and thus the first player claims it. Finally, the number of edges taken by Maker satisfies $Ls(T_k) = Ls(T_{k-1}) + Rs(T_{k-1}) + 1$ (when Maker starts), and $Rs(T_k) = Ls(T_{k-1}) + Rs(T_{k-1})$ (when Breaker starts). Since $Ls(T_0) = Rs(T_0) = 0$, we get the result by induction. □

5. Complexity of Maker-Breaker Incidence

In this section, we first prove that Maker-Breaker Incidence is PSPACE-complete. Then, we consider the parameterized complexity of Maker-Breaker Incidence and prove that it is fixed parameter tractable when parameterized by the neighborhood diversity.

5.1. Maker-Breaker Incidence *is PSPACE-complete*

Reductions in (positional games) are often made from POS CNF (see for example [26,24,27,29]). In our cases, we need to deal with scores and not only a structure. To handle this problem, we use a quantified version of Max-2-SAT that we proved to be PSPACE-complete using 3-QBF.

Q-Max-2-SAT
Instance: A quantified boolean formula on the form $\varphi = Q_1 x_1, \ldots, Q_n x_n, \psi(x_1, x_2, \ldots x_n)$, with $Q_i \in \{\forall, \exists\}$ and ψ a 2-CNF formula on $x_1, \ldots, x_n$, an integer k
Output: True if at least k clauses of the formula are satisfied. False otherwise.

3-QBF
Instance: A quantified boolean formula $\Phi = Q_1 x_1, \ldots, Q_n x_n, \psi(x_1, x_2, \ldots x_n)$, with $Q_i \in \{\forall, \exists\}$ and ψ a 3-CNF formula on $x_1, \ldots, x_n$
Output: True iff Φ is true.

Theorem 17. Q-Max-2-SAT *is PSPACE-complete.*

Proof. The proof of PSPACE-completeness of Q-Max-2-SAT is similar to the proof of NP-completeness of Max-2-SAT from Papadimitriou [22].

First, Q-Max-2-SAT is in PSPACE, as any valuation can be computed in polynomial space. Therefore, by a min-max argument, it is possible to compute the number of satisfied clauses in polynomial space.

We provide a reduction from 3-QBF. Let $\phi = Q_1 x_1, \ldots, Q_n x_n \quad \psi(x_1, x_2, \ldots x_n)$ be a 3-QBF formula on m clauses. For each clause $c_i = l_1^i \vee l_2^i \vee l_3^i$ of ψ, we introduce a new variable d_i and construct a set $\mathcal{C}_i$ of 10 clauses $C_i^1, \ldots, C_i^{10}$ of at most 2:

$$\mathcal{C}_i = \{(l_1), (l_2), (l_3), (d_i), (\neg l_1 \vee \neg l_2), (\neg l_1 \vee \neg l_3), (\neg l_2 \vee \neg l_3), (\neg d_1 \vee l_1), (\neg d_1 \vee l_2), (\neg d_1 \vee l_3)\}$$

Claim. Given any valuation of the literals l_i's, if c_i is satisfied, then there exists a valuation of d_i such that exactly seven clauses in $\mathcal{C}_i$ are satisfied. Otherwise, at most six clauses of $\mathcal{C}_i$ are satisfied for any valuation of d_i.

Proof. The proof of the claim is a case analysis depending on the number of literals l_i that are true in c_i (since the literals play a symmetric role). The following tabular gives the number N_C of clauses in $\mathcal{C}_i$ that are satisfied depending on the number N_L of literals l_i that are true and the valuation of d_i.

N_L	0	0	1	1	2	2	3	3
d_i	F	T	F	T	F	T	F	T
N_C	6	4	7	6	7	7	6	7

■

Let $\varphi = Q_1 x_1, \ldots, Q_n x_n, \exists d_1, \ldots, \exists d_n, \bigwedge_{i=1}^{m} \bigwedge_{j=1}^{10} C_i^j$ and let $k = 7m$.

If ϕ is true, then, for any valuation obtained by the Q_i's that makes ψ true, there exists a valuation for each d_j such that there are exactly seven clauses satisfied in each set C_j. Thus, by taking this valuation for each d_j, we have that $k = 7m$ clauses satisfied in φ.

Reciprocally, if ϕ is false, then for any valuation provided by the Q_i s, there exists a clause C_j that is not satisfied. Therefore, at most six clauses in C_j are satisfied. For the other clauses, at most seven of them are satisfied. Thus the total number of satisfied clauses in φ is at most $7m - 1 = k - 1$.

Finally, the formula φ of Q-Max-2-SAT has at least $7m$ clauses satisfied if and only if ϕ is True.

Up to adding a variable in all the clauses of size 1 and quantifying it with a $\forall$, we can suppose that all the clauses of φ have size 2. $\square$

We now turn to the main proof of this section - that is the proof of the complexity of Maker-Breaker Incidence.

Maker-Breaker Incidence
Instance: A graph G, an integer k, a player $P \in \{\text{Left}, \text{Right}\}$.
Output: True iff the P score of G is at least k.

Theorem 18. Maker-Breaker Incidence *is PSPACE-complete.*

The construction provided in the proof will require some tools to order the moves of both player. Let $P = (G, V_L, V_R)$ be a game position of Incidence. Let u and v be free vertices. We say that v *dominates* u in P and write $v \geq_P u$ if in any position obtained from P, it is always more interesting to play v than u. More formally, $v \geq_P u$ if for any V_L', V_R' such that $V_L \subset V_L'$ and $V_R \subset V_R'$, $V_L' \cap V_R' = \emptyset$ and $u, v \notin V_L' \cup V_R'$, we have $Rs(G, V_L' \cup \{u\}, V_R') \geq Rs(G, V_L' \cup \{v\}, V_R')$ and $Ls(G, V_L' \cup \{u\}, V_R') \leq Ls(G, V_L' \cup \{v\}, V_R')$.

Lemma 19. *Let $G = (V, E)$ be a graph and $P = (G, V_L, V_R)$ a position of* Maker-Breaker Incidence*. Let u, v be two free vertices such that $|N(v) \cap V_L| \geq |N(u) \cap V_L| + |N(u) \setminus N(v) \cap V_F|$. Then $v \geq_P u$.*

Proof. Let S be a strategy in (G, V_L, V_R) that plays u before v. We define a strategy S' that plays v before u as follows:

- While S wants to claim a vertex $w \neq u$, claim w.
- If S wants to claim u while v is unclaimed, claim v instead, and still consider that u is claimed in S.
- When S wants to claim v, if it is already claimed, claim u instead. If the opponent has claimed u, consider that he has claimed v, and continue to follow S.

Following this strategy, according to the moves of the opponent, all the vertices claimed by S are claimed by S', with only a difference on u and v if they are not claimed by the same player.

If S was a strategy for Left, by following S', each edge that does not contain u nor v that was claimed by S is claimed by S', and reciprocally. Concerning the edges containing u or v, Left has scored at most $|N(u) \cap V_L| + |N(u) \cap V_F|$ points on them with S and $|N(v) \cap V_L| + |N(v) \cap V_F|$ by following S'. Therefore, as $|N(v) \cap V_L| \geq |N(u) \cap V_L| + |N(u) \setminus N(v) \cap V_F|$, Left has score at least the same number of edges following S'.

The same argument shows that Right will have more edges with a vertex claimed by him by playing v instead of u. $\square$

Proof of Theorem 18. First, Maker-Breaker Incidence is in PSPACE as the game last at most $|V|$ moves and the score is at most $|E|$. Thus, it can be computed in polynomial space, according to Section 6.1 in [14].

We prove that Maker-Breaker Incidence is PSPACE-complete by a reduction from Q-Max-2-SAT. In this proof, we consider a quantified formula as a two-player game. We first assume that the formula has the form $\exists x_{2n} \forall x_{2n-1} \exists x_{2n-2}, \ldots \forall x_1 \ \psi$, i.e. that the quantifiers $\exists, \forall$ are alternating and starting with a quantifier $\exists$. This can be done for any quantified formulas by adding some vertices with the desired quantifier that are put in no clause, and thus that does not change the number of clauses that are satisfied. The first player, Satisfier, tries to satisfy the formula by choosing the values of the even variables x_{2k} (i.e. that are quantified by an $\exists$-quantifier) while the second player, Falsifier, tries to spoil the formula and turn it to False by choosing the values of the odd variables x_{2k-1} (i.e. that are quantified by a $\forall$-quantifier). This classical technique to transform a quantified formula into a game has been used for instance by Rahman and Watson [25] to show the PSPACE-completeness of Maker-Breaker positional games.
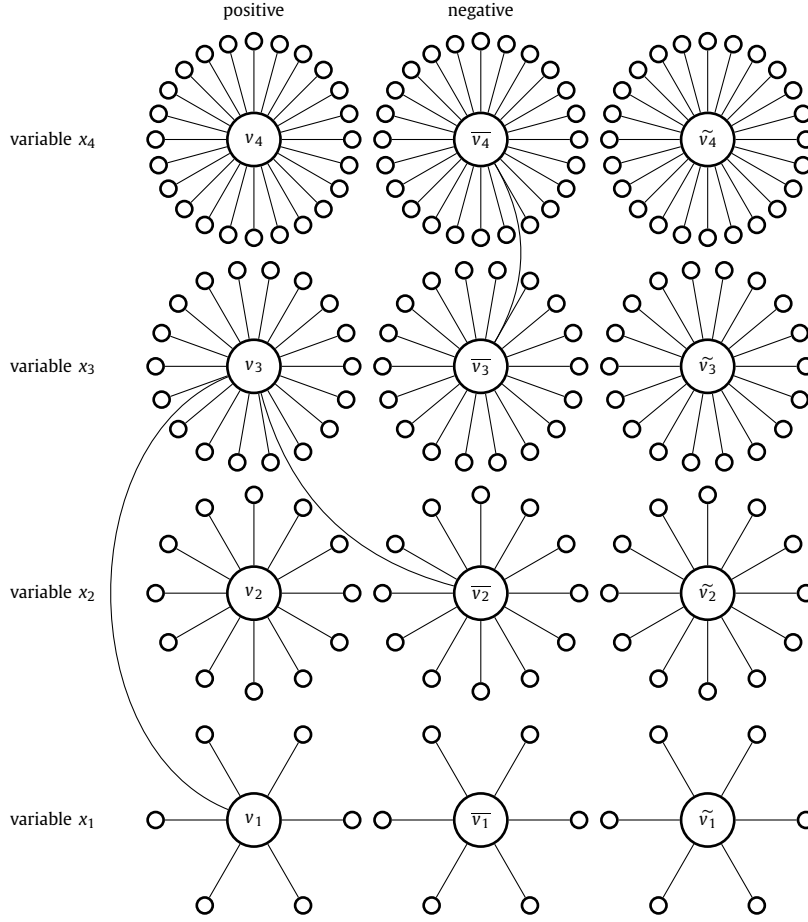
Fig. 4. Reduction of $\exists x_4 \forall x_3 \exists x_2 \forall x_1 (\neg x_2 \vee x_3) \wedge (x_1 \vee x_3) \wedge (\neg x_3 \vee \neg x_4)$.

Denote $\psi = \bigwedge_{j=1}^{m} (l_1^j \vee l_2^j)$ for l_1^j, l_2^j some literals. We build a graph $G = (V, E)$ as follows (see Fig. 4):

- For each variable x_i, we create $6mi + 3$ vertices. These vertices induce three stars of center v_i, $\overline{v_i}$ and $\widetilde{v_i}$, and with $2mi$ leaves each. We will denote by V_i the set $\{v_i, \overline{v_i}, \widetilde{v_i}\}$.
- We consider a function f defined by $f(x_i) = v_i$ and $f(\neg x_i) = \overline{v_i}$. For each clause $C_j = l_1^j \vee l_2^j$, we add an edge $e_j = (f(l_1^j), f(l_2^j))$.

The number of vertices outside sets V_i (i.e. the number of leaves) is $N = \sum_{i=1}^{2n} 6mi = 6mn(2n+1)$. Thus the total number of vertices in G is $N + 6n$ and the total number of edges is $N + m$, which is polynomial in the size of φ. An example of reduction is provided in Fig. 4 with $m = 3$ and $n = 2$.

Consider a game of MAKER-BREAKER INCIDENCE on G with Right starting. Using Lemma 15, for every $1 \leq i \leq 2n$, the leaves connected to vertices v_i, $\overline{v_i}$ and $\widetilde{v_i}$ respectively, are equivalent. Thus, half of them can be given to Left and the other half to Right. Since there are an even number of leaves for each star, the only free vertices after this operation are the $6n$ vertices in sets V_i for $1 \leq i \leq n$. Let $P^0 = (G, V_L^0, V_R^0)$ be this position, and denote by V_F^0 the set of free vertices in this position. By Lemma 15, we have $Rs(G) = Rs(P_0)$.

Now, if $1 \leq j < i \leq 2n$, for any $v_i^* \in V_i$ and $v_j^* \in V_j$, we have $|N(v_i^*) \cap V_L^0| = mi$, $|N(v_j^*) \cap V_L^0| = mj$ and $|N(v_j^*) \cap V_F^0| \leq m$. Therefore, by Lemma 19 we have $v_i^* \geq_{P_0} v_j^*$. Moreover, as $N(\widetilde{v_i}) \cap V_F^0 = \emptyset$, we also have $v_i \geq_{P_0} \widetilde{v_i}$ and $\overline{v_i} \geq_{P_0} \widetilde{v_i}$.

Hence, in any optimal strategy in P_0 with Right starting, the vertices are played in n rounds, from round $\ell = n$ to $\ell = 1$, with the following six steps in each round:

1. One vertex chosen by Right among $\{v_{2\ell}, \overline{v_{2\ell}}\}$
2. The other vertex among $\{v_{2\ell}, \overline{v_{2\ell}}\}$ is taken by Left.
3. The vertex $\widetilde{v_{2\ell}}$ is taken by Right.

4. One vertex among $\{v_{2\ell-1}, \overline{v_{2\ell-1}}\}$ is taken by Left.
5. The second vertex in $\{v_{2\ell-1}, \overline{v_{2\ell-1}}\}$ is taken by Right.
6. The vertex $\widetilde{v_{2\ell-1}}$ is taken by Left.

This way, Left will obtain exactly $N' = \sum_{\ell=1}^{n}(2\ell m + 2(2\ell-1)m) = 3mn(n+1) - 2mn$ edges in the stars and maybe some other edges in the clause edges. Let $k' = N' + m - k + 1$.

We will prove that $Rs(G) \geq k'$ at MAKER-BREAKER INCIDENCE if and only if Falsifier wins at Q-MAX-2-SAT on (φ, k).

Claim. If Satisfier has a strategy to satisfy k clauses in φ, then $Rs(G) < k'$.

Proof. We suppose that Satisfier has a winning strategy $\mathcal{S}$ in (φ, k). We consider that both Right and Left play optimally in G and thus we can assume that the game is played in P_0 and respects the previous order.

Consider the following strategy for Right. At each round ℓ from $\ell = n$ to $\ell = 1$, Right takes a decision only at Step 1. If Satisfier would turn x_{2i} to True in the game played on φ, then Right plays v_{2i}, otherwise, he plays $\overline{v_{2i}}$. Then, Steps 2 and 3 are determined. At Step 4, if Left plays v_{2i-1} then Right considers that Falsifier has turned x_{2i-1} to False, otherwise he considers she has turned it to True. Then again, Steps 5 and 6 are determined. By following this strategy, the underlying value obtained for φ is exactly the value that Satisfier would obtain by playing according to $\mathcal{S}$. Thus, at least k clauses are satisfied in φ.

Note that for a literal l^j, the vertex $f(l^j)$ is taken by Right if and only if l^j is True in the game of Q-MAX-2-SAT. Let $C_j = l_1^j \vee l_2^j$ be a clause. If Left has claimed the two extremities of e_j, it means that Left has played $f(l_1^j)$ and $f(l_2^j)$. Therefore, the underlying values of l_1^j and of l_2^j are both False, and C_j is not satisfied in ψ. Hence, Left claims at most $m - k$ edges e_j. Finally, Left claimed at most $k' - 1$ edges and we have $Rs(G) < k'$. ∎

Claim. If Falsifier has a strategy such that at most $k - 1$ clauses are satisfied in ϕ, then $Rs(G) \geq k'$.

Proof We now suppose that Falsifier has a winning strategy $\mathcal{S}$ in (φ, k). We consider that both Right and Left play optimally in G and thus we can assume that the game is played in P_0 and respects the previous order. Consider the following strategy for Left. At each round ℓ from $\ell = n$ to $\ell = 1$, Left takes a decision only at Step 4. At Step 1, if Right plays $v_{2\ell}$ then Left considers that Satisfier has turned $x_{2\ell}$ to True, otherwise she considers he has turned it to False. Then, Steps 2 and 3 are determined. At Step 4, if Falsifier would turn x_{2i-1} to False in the game played on φ, then Left plays v_{2i-1}, otherwise, she plays $\overline{v_{2i-1}}$. Then again, Steps 5 and 6 are determined.

By following this strategy, the underlying value obtained for φ is exactly the value that Falsifier would obtain by playing according to $\mathcal{S}$. Thus, it would satisfy at most $k - 1$ clauses in φ. As before, if a clause $l_1^j \vee l_2^j$ is not satisfied in φ it means that both vertices $f(l_1^j)$ and $f(l_2^j)$ are taken by Left and thus Left got the edge. Thus Left claims at least $N' + m - k + 1$ edges in the game G and $Rs(G) \geq k'$. ∎ □

Remark 20. Note that, up to adding a useless variable in φ, φ could start by a ∀-quantifier, implying that MAKER-BREAKER INCIDENCE is PSPACE-complete even if Left starts.

Corollary 21. 3-*uniform* MAKER-MAKER SCORING POSITIONAL GAME *is PSPACE-complete.*

Proof. The proof is similar to the second part of the proof of Corollary 6. From a graph $G = (V, E)$ of MAKER-BREAKER INCIDENCE, we consider the instance of 3-uniform MAKER-MAKER SCORING POSITIONAL GAME obtained by adding a universal vertex v_0. Consider the hypergraph $H = (V \cup \{v_0\}, \{e \cup \{v_0\} | e \in E\})$. When Left starts, any optimal strategy starts by playing v_0, otherwise Right plays it and the score will be at most 0. Then we are left to a Maker-Breaker position as Right cannot score any point, but starts. Finally the Left score of H in Maker-Maker convention is equal to the Right score of G in Maker-Breaker convention, which is PSPACE-complete to compute. □

5.2. Complexity parameterized by the neighborhood diversity

Neighborhood diversity is a graph parameter introduced by Lampis [18] to generalize FPT algorithms parameterized by vertex cover to larger classes of graphs. Let G be a graph. We say that two vertices u and v have the same *type* if $N(v) \setminus \{u\} = N(u) \setminus \{v\}$. The graph G has *neighborhood diversity* at most w if there exists a partition of V into at most w sets such that the vertices in each set have all the same type. Note that each set must induce a clique or an independent set. Furthermore, if a graph has bounded vertex cover, then it has bounded neighborhood diversity.

A decision problem has a *kernel* for a parameter w, if for any parameterized instance (P, w) of the problem, there exists an instance (P', w') and a computable function f, such that P reduces to P' in polynomial time in $(|P|, w)$ and such that $|P'|, |w'| \leq f(w)$. If $f(w) = O(w^3)$, the kernel is said to be cubic. If $f(w) = O(w \log(w))$, the kernel is said to be quasilinear. Having a kernel implies that the problem is fixed-parameter tractable for this parameter.

Theorem 22. Maker-Breaker Incidence *parameterized by the neighborhood diversity w has a cubic kernel.*

Proof. In this proof, we will consider as instances of Maker-Breaker Incidence triplets $(P, k, Left)$ where P is a position of Maker-Breaker Incidence played on G (i.e. some vertices are already played). Note that this does not change the complexity of the problem. Indeed, from any position $P = (G, V_L, V_R)$ one can obtain a graph G' with no vertices played for which the games are equivalent. First remove all the vertices in V_R of the graph. Then, duplicate each vertex in V_L by creating a twin vertex having the same neighborhood and free the vertices in V_L. By Lemma 15, one can assume that both players will take one vertex in each pair of twins.

Let $G = (V, E)$ be a graph of neighborhood diversity w. Consider a partition $(V_1, \ldots, V_w)$ of V such that the vertices in each part are all of the same type. We provide the following kernelization algorithm. Let $I = ((G, \emptyset, \emptyset), k, P)$ where $P \in \{Left, Right\}$ be an instance of Maker-Breaker Incidence. An example of the different steps is provided in Fig. 5.

Step 1: While there exists a part V_i, $1 \le i \le w$ such that there are at least two free vertices $u, v \in V_i$, add u to V_L and v to V_R. By Lemma 15, this transformation does not change the outcome of the game. At the end of Step 1, there are at most w free vertices in G. In Fig. 5(b), it consists in distributing vertices of same type between Left and Right.

Step 2: Remove all the edges included in V_L and set $k \leftarrow k - |e \subset V_L|$. Then remove from G all the vertices in V_R that cannot count for any point. This transformation does not change the outcome of I. At this moment, G only contains free vertices or vertices claimed by Left, and any edge has at least one free extremity. In Fig. 5(c), it consists in removing the 16 edges on which the two endpoints are claimed by Left, and to remove the red vertices and their incident edges. Therefore, k is decreased from 30 to 14.

Step 3: Let r the number of free vertices in P, we have $r \le w$. Let $v_1, \ldots, v_r$ be these vertices. For $1 \le i \le r$, let $p_i = |N(v_i) \cap V_L|$ and order the vertices such that $p_1 \ge p_2 \ge \cdots \ge p_r$. While there exists an integer i such that $p_i > p_{i+1} + r$ (with $p_{r+1} = 0$), by Lemma 19, there exists an optimal strategy in which the vertices $v_1, \ldots, v_i$ are played before the vertices $v_{i+1}, \ldots, v_r$. On these vertices, Left will score at least p_i at each Left move. Therefore, we can do the following transformation. Let $s = p_i - p_{i+1} - r$ for any $1 \le j \le i$, set $p_j \leftarrow p_j - s$ and set $k \leftarrow k - s \left\lceil \frac{i}{2} \right\rceil$. Repeat Step 3 until we have $p_i \le p_{i+1} + r$ for all $1 \le i \le r$. In particular, we have after these operations $p_1 \le r^2$. In Fig. 5(d), it happens only once, as $p_1 = 8$, $p_2 = 3$ and $w' = 4$. Therefore, we set $p_1 = 7$ and k is decreased from 14 to 13.

Step 4: Let $U = \{u_1, \ldots, u_{p_1}\}$ be p_1 new vertices and transform $(G, V_L, \emptyset)$ into $((G \setminus V_L) \cup U, U, \emptyset)$, and, for $1 \le i \le r$, connect the vertex v_i to any p_i vertices in U. This transformation does not change the outcome of the game, since only the number of neighbors in V_L matters when a vertex is played. In Fig. 5(e), we have $p_1 = 7$. Thus, U contains seven vertices and each remaining uncolored vertex v_i is connected to p_i of these seven vertices.

Finally, if $k \ge r^3$, as there are at most r^3 edges in the final graph, we can just transform P into a trivial False instance like the empty graph with $k = 1$. Thus, we can assume that $k \le r^3$.

The instance obtained has $p_1 + r \le r^2 + r \le w^2 + w$ vertices, at most $r * p_1 \le r^3 \le w^3$ edges, $k \le r^3 \le w^3$ and the same outcome as the input. Finally, this new instance has cubic size in w and thus Maker-Breaker Incidence has a cubic kernel. □

Corollary 23. *Let G be a graph of order n and neighborhood diversity w. In* Maker-Breaker Incidence $Ls(G)$ *and* $Rs(G)$ *can be computed in time $O(w^2 w! + n^2)$*

Proof. We can compute the kernel in time n^2, and then try all the possible games by testing all the moves in time $w^2 w!$. □

Note that the cubic size of the kernel is mostly due to the w^2 vertices that are already claimed by Left. As these vertices cannot be played any longer, by giving weight to the vertices, it is possible to have a quasilinear kernel by storing only the number of neighbors of each vertex that are already claimed by Left instead of vertices themselves.
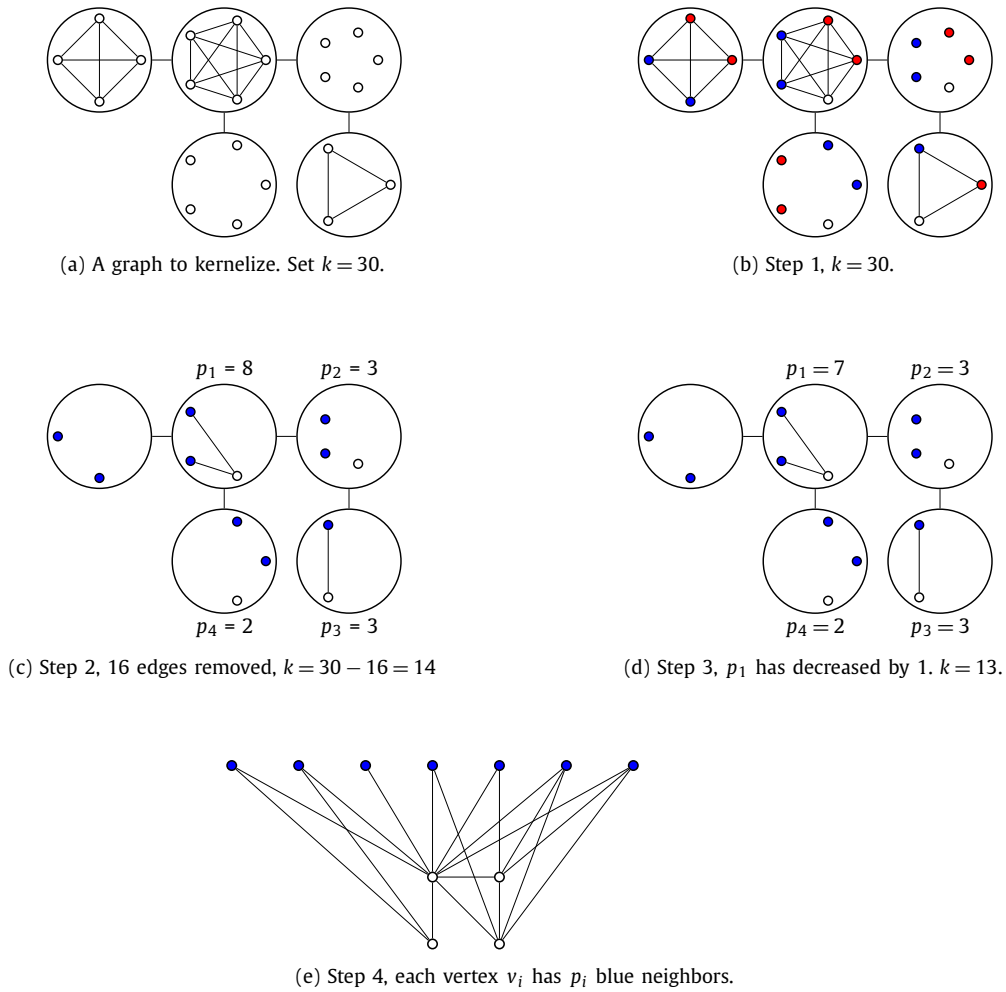
6. Paths and cycles

We here give the exact values of the score for Maker-Breaker Incidence played on paths and cycles. For that purpose, we will consider the equivalence properties of Milnor's universe detailed in Section 2. In particular, the notion of negative will be required, implying to consider the partisan version of Incidence. More precisely, in this section, instances of Maker-Breaker Incidence will correspond to paths or cycles where the edges are either colored all blue (i.e. only Left can get points) or all red (i.e. only Right can get points). The notations are defined as follows:

- P_n^L: path of order n where all the edges are colored blue. We denote the vertices of P_n^L by $\{v_0, \ldots, v_{n-1}\}$
- P_n^R: path of order n where all the edges are colored red. We denote the vertices of P_n^R by $\{v_0', \ldots, v_{n-1}'\}$

By definition, we have that $P_n^L = -P_n^R$.

(a) A graph to kernelize. Set $k = 30$.

(b) Step 1, $k = 30$.

(c) Step 2, 16 edges removed, $k = 30 - 16 = 14$

(d) Step 3, p_1 has decreased by 1. $k = 13$.

(e) Step 4, each vertex v_i has p_i blue neighbors.

Fig. 5. Example of a kernelization. Vertices in the same circle have same type. An edge between two circles means that all the edges between the vertices of the two circles are in the graph. Blue and red vertices are given to Left and Right respectively. We start with $n = 22$ and after Step 1 $r = 4$.

6.1. Equivalences of paths

We first give the main result about the equivalence between paths modulo 5. To present it, we introduce a usual notation in scoring game theory: for $k \in \mathbb{Z}$, we define by k the game with no option and where Left has a score of k points. Thus, in Maker-Breaker Incidence, the game 1 is equivalent to P_2^L in which Left has claimed the two vertices and -1 is equivalent to P_2^R in which Right has claimed the two vertices. Note that for any game G and any integer k, we have $G \equiv k$ if and only if $Ls(G) = Rs(G) = k$. The main theorem of this section states that paths of order at least 6 are equivalent to paths having five vertices less, with a difference of one in the score. This result remains true if an extremity of the path is already colored.

Theorem 24. Let $n \geq 1$ be an integer. We have $P_{n+5}^L \equiv P_n^L + 1$ and $P_{n+5}^R \equiv P_n^R - 1$.
Let $n \geq 2$ be an integer. We have $(P_{n+5}^L, \{v_0\}, \emptyset) \equiv (P_n^L, \{v_0\}, \emptyset) + 1$ and $(P_{n+5}^R, \emptyset, \{v_0'\}) \equiv (P_n^R, \emptyset, \{v_0'\}) - 1$.

The rest of this subsection will be dedicated to the proof of this theorem.

6.1.1. Strategy for Left when Right starts

Lemma 25. Let $n \geq 1$ be an integer. In Maker-Breaker Incidence, we have $Rs(P_{n+5}^L + P_n^R) \geq 1$.
Let $n \geq 2$ be an integer. In Maker-Breaker Incidence, we have $Rs(P_{n+5}^L + P_n^R, \{v_0\}, \{v_0'\}) \geq 1$.

This proof will be done by induction. Therefore, to handle the small cases, the scores of first paths will be required. They are recorded in Fig. 6 and Fig. 7 and can be easily checked by hand.

n	1	2	3	4	5	6	7	8	9	10
$Ls(P_n^L)$	0	0	1	1	1	1	1	2	2	2
$Rs(P_n^L)$	0	0	0	0	0	1	1	1	1	1

Fig. 6. First scores in short paths.

n	1	2	3	4	5	6	7	8	9	10	11
$Ls((P_n^L, \{v_0\}, \emptyset))$	0	1	1	1	1	2	2	2	2	2	3
$Rs((P_n^L, \{v_0\}, \emptyset))$	0	0	0	0	1	1	1	1	1	2	2

Fig. 7. First scores in short paths with an extremity claimed by Left.

Proof. In order to prove that $Rs(P_{n+5}^L + P_n^R) \geq 1$ ($Rs(P_{n+5}^L + P_n^R, \{v_0\}, \{v_0'\}) \geq 1$ resp.), we provide a strategy for Left by induction. If $1 \leq n \leq 5$ ($2 \leq n \leq 6$ resp.), a computation can verify that the result is true.

If $n \geq 6$ ($n \geq 7$ resp.), we consider the first move of Right:

- If Right plays a vertex v_i' for $0 \leq i \leq n-1$ ($1 \leq i \leq n-1$ resp.), Left answers by playing the vertex v_i. The resulting position is $(P_{n+5}^L + P_n^R, \{v_i\}, \{v_i'\})$ ($(P_{n+5}^L + P_n^R, \{v_0, v_i\}, \{v_0', v_i'\})$ resp.), which is equivalent to $(P_{i+1}^L + P_{i+1}^R, \{v_i\}, \{v_i'\}) + (P_{n+5-i}^L + P_{n-i}^R, \{v_0\}, \{v_0'\})$ ($(P_{i+1}^L + P_{i+1}^R, \{v_0, v_i\}, \{v_0', v_i'\}) + (P_{n+5-i}^L + P_{n-i}^R, \{v_0\}, \{v_0'\})$ resp.). As we have $(P_{i+1}^L + P_{i+1}^R, \{v_i\}, \{v_i'\}) \equiv 0$ ($(P_{i+1}^L + P_{i+1}^R, \{v_0, v_i\}, \{v_0', v_i'\}) \equiv 0$ resp.) and $(P_{n+5-i}^L + P_{n-i}^R, \{v_0\}, \{v_0'\})$ satisfies the induction hypothesis, and therefore the score is at least one.
- If Right plays a vertex v_i for $0 \leq i \leq n-1$ ($1 \leq i \leq n-1$ resp.), Left answers by playing the vertex v_i'. The resulting position is $(P_{n+5}^L + P_n^R, \{v_i'\}, \{v_i\})$ ($(P_{n+5}^L + P_n^R, \{v_0, v_i'\}, \{v_0', v_i\})$ resp.), which is equivalent to $(P_i^L + P_i^R) + (P_{n+5-(i+1)}^L + P_{n-(i+1)}^R)$ ($(P_i^L + P_i^R, \{v_0\}, \{v_0'\}) + (P_{n+5-(i+1)}^L + P_{n-(i+1)}^R)$ resp.). As we have $(P_i^L + P_i^R) \equiv 0$ ($(P_i^L + P_i^R, \{v_0\}, \{v_0'\}) \equiv 0$ resp.) and $(P_{n+5-(i+1)}^L + P_{n-(i+1)}^R)$ satisfies the induction hypothesis, the score is at least one.
- If Right plays a vertex v_i for $n \leq i \leq n+4$. Left answers by playing v_{i-5}', which exists as $n \geq 6$ ($n \geq 7$ resp.). The resulting position is $(P_{n+5}^L + P_n^R, \{v_{i-5}'\}, \{v_i\})$ ($(P_{n+5}^L + P_n^R, \{v_0, v_{i-5}'\}, \{v_0', v_i\})$ resp.), which is equivalent to $(P_i^L + P_{i-5}^R) + (P_{n-1-i}^L + P_{n-1-i}^R)$ ($(P_i^L + P_{i-5}^R, \{v_0\}, \{v_0'\}) + (P_{n-1-i}^L + P_{n-1-i}^R)$). Here, we have $(P_{n-1-i}^L + P_{n-1-i}^R) \equiv 0$ and $(P_i^L + P_{i-5}^R)$ ($(P_i^L + P_{i-5}^R, \{v_0\}, \{v_0'\})$ resp.) satisfies the induction hypothesis as $n+4 \geq i \geq n \geq 6$ ($i \geq n \geq 7$ resp.) and therefore the score is at least one.

This strategy ensures that $Rs(P_{n+5}^L + P_n^R) \geq 1$ ($Rs(P_{n+5}^L + P_n^R, \{v_0\}, \{v_0'\}) \geq 1$ resp.). $\quad\square$

6.1.2. Strategy for Right when Left starts

When Left starts, the induction made in the previous proof cannot be applied. Indeed, from the position $(P_{n+5}^L + P_n^R, \{v_0\}, \{v_0'\})$, Left can in one move make the position be $(P_{n+5}^L + P_n^R, \{v_0, v_{n+3}\}, \{v_0'\})$ and no move of right can transform it into a position handled by the induction hypothesis. Therefore, another strategy is required. We will consider a strategy for Right that consists, for the leftmost vertices of both paths, in mimicking any move of Left on the other path, and that ensures some minimal properties on the moves played on the rightmost vertices. We introduce the following lemma to handle the rightmost vertices.

Lemma 26. Consider the graph $G = P_6^L + \{v_0'\}$. Let v_0 be an extremity of P_6^L. In MAKER-BREAKER INCIDENCE, Right has a strategy, going second, such that Left claims either v_0 and v_0' without any point, or at most one of $\{v_0, v_0'\}$ and she scores at most one point on G.

Proof. Let $G = P_6^L + \{v_0'\}$. Recall that $v_0, \ldots, v_5$ are the vertices of P_6^L. We will describe a strategy for Right playing second such that Left scores no point or such that she does not claim both v_0 and v_0' with at most one point.

- If Left plays v_0, Right answers v_1,
  - if Left plays v_0', Right plays v_3 and pairs v_4 and v_5. Left cannot score a point.
  - If Left plays v_2 (v_5 resp.), Right plays v_3 (v_4 resp.) and pairs (v_4, v_5) ((v_2, v_3) resp.). This way, Left cannot score a point.
  - If Left plays in v_3 (v_4 resp.), Right plays v_4 (v_3 resp.) and pairs v_2 (v_5 resp.) with v_0'. Either Left scores a point or claims both v_0 and v_0'.
- If Left plays v_1, Right answers v_0. He has claimed one of (v_0, v_0'). He then pairs (v_2, v_3) and (v_4, v_5). With this pairing, Left can score at most one point.
- If Left plays v_2, Right answers v_3, He then pairs $(v_0, v_1$ and $v_4, v_5)$. The only one edge outside the pairing (and therefore that can be claimed by Left) is v_1, v_2 but with this pairing, Right then plays v_0 and claim one of v_0, v_0'. Otherwise, Left scores no point.

- If Left plays v_3 (v_5 resp.), Right answers v_4, he then pairs (v_0, v'_0) and (v_1, v_2). This way, Left scores at most one point on the edge (v_2, v_3) or (v_0, v_1) but she cannot take both. And Right will be able to take one of v_0 or v'_0.
- If Left plays v_4, Right answers v_3
  - If Left plays v_0, Right plays v_1 and pairs v'_0 with v_5. Either Left claims v'_0, and then by claiming v_5, Right ensures that Left scores no point, or Left claims v_5 and scores one point, but Right claims $v'_0 \in \{v_0, v'_0\}$.
  - If Left plays v_1 (v_2 resp.), Right plays v_0 and pairs v_2 (v_1 resp.) and v_5. By claiming one of them, Left scores one point but Right claims the second one, and therefore, Right ensures that Left scores only one point and does not claim both v_0 and v'_0.
  - If Left plays v_5 (v'_0 resp.), Right plays v_0 and pairs (v_1, v_2). Then, Left cannot score a second point (can score at most one point by playing v_5 resp.), and Right has already claimed one of v_0, v'_0.
- If Left plays v'_0, Right answers v_0. He has already claimed one of v_0, v'_0, and the remaining graph is equivalent to P^L_5 for which we already know that Left gets at most 1 when she starts. □

Lemma 27. *Let $n \geq 1$ be an integer. In* MAKER-BREAKER INCIDENCE*, we have $Ls(P^L_{n+5} + P^R_n) \leq 1$.*

Let $n \geq 2$ be an integer. In MAKER-BREAKER INCIDENCE*, we have $Ls(P^L_{n+5} + P^R_n, \{v_0\}, \{v'_0\}) \leq 1$.*

Proof. The proof below holds for the two cases, i.e. if the vertices v_0 and v'_0 are already colored or not.

Recall that $v_0, \ldots, v_{n+4}$ are the vertices of P^L_{n+5} and $v'_0, \ldots, v'_{n-1}$ are the vertices of P^R_n. We provide here a strategy for Right to ensure that the score is at most 1 as follows:

- If Left plays a vertex in a pair (v_i, v'_i) with $0 \leq i \leq n-2$, Right answers the second vertex of this pair.
- If Left plays another vertex, Right follows the strategy of Lemma 26 with $P^L_6 = \{v_0 = v_{n-1}, \ldots, v_{n+4}\}$ and $v'_0 = v'_{n-1}$.

According to this strategy, Right ensures that Left scores the same number of points as him on the subgraph induced by the vertices v_i, v'_i with $0 \leq i \leq n-2$. On the rest of the graph, from Lemma 26, either Left takes the two vertices v_0, v'_0 and gets no point, which can yield her overall at most one point with the edge (v_{n-2}, v_{n-1}) of P^L_{n+5}. Otherwise, she takes v'_0 or the extremity v_0 of the P^L_6 and scores one point. In this case, if this extremity corresponds to v'_{n-1} of P^R_n she does not score a second point, and if this extremity is v_{n-1}, she can score a point if she also takes v_{n-2}. But in this case, Right has claimed both v'_{n-2} by the pairing strategy and v'_{n-1} as he has also claimed the other extremity. Thus, Right also scores one point. Finally, Right ensures that the score is at most 1 with this strategy, and we have $Ls(P^L_{n+5} + P^R_n) \leq 1$. □

6.1.3. Proof of Theorem 24 and score on paths

Now we can prove Theorem 24.

Proof. By symmetry, as $P^L_n = -P^R_n$ for any n, we only need to prove the result for P^L_n.

As our game is in Milnor's universe, according to Lemma 2, it is sufficient to prove that $P^L_{n+5} - P^L_n - 1 \equiv 0$ (($P^L_{n+5} - P^L_n, \{v_0\}, \{v'_0\}) - 1 \equiv 0$ resp.), i.e. $Ls(P^L_{n+5} + P^R_n) = Rs(P^L_{n+5} + P^R_n) = 1$ ($Ls(P^L_{n+5} + P^R_n, \{v_0\}, \{v'_0\}) = Rs(P^L_{n+5} + P^R_n, \{v_0\}, \{v'_0\}) = 1$ resp.).

As the game is nonzugzwang, and according to Lemma 25 and Lemma 27, we have proven $1 \geq Ls(P^L_{n+5} + P^R_n) \geq Rs(P^L_{n+5} + P^R_n) \geq 1$ ($1 \geq Ls(P^L_{n+5} + P^R_n, \{v_0\}, \{v'_0\}) \geq Rs(P^L_{n+5} + P^R_n, \{v_0\}, \{v'_0\}) \geq 1$ resp.), which corresponds to the desired result. □

From Theorem 24, and since the score on small paths is provided by Fig. 6, the score of any path can be computed as follows:

Corollary 28. *Let $n \geq 1$ be an integer. Denote by $n = 5q + r$ with q and r the quotient and the rest of n divided by 5. In* MAKER-BREAKER INCIDENCE*, on the one hand, we have $Ls(P^L_n) = -Rs(P^R_n) = q$ if $0 \leq r \leq 2$, and $Ls(P^L_n) = -Rs(P^R_n) = q + 1$ if $3 \leq r \leq 4$. On the other hand, we have $Rs(P^L_n) = -Ls(P^R_n) = q - 1$ if $r = 0$, $Rs(P^L_n) = -Ls(P^R_n) = q$ if $1 \leq r \leq 4$.*

6.2. Union of paths and cycles

We will denote cycles as follows:

- C^L_n: cycle of length n where all the edges are colored blue.
- C^R_n: cycle of length n where all the edges are colored red.

Now that the equivalences of paths are known, union of paths can easily be reduced to union of paths of order at most 5. Yet, to deal with such unions, it is not sufficient in general to compute the score on them. The problem can be solved by considering new equivalences between small paths.

Lemma 29. *In* Maker-Breaker Incidence*, we have the following equivalences:*

$$P_1^L \equiv P_2^L \equiv 0 \tag{1}$$

$$2P_3^L \equiv 1 \tag{2}$$

$$P_4^L \equiv P_3^L \tag{3}$$

$$2P_5^L + P_3^L \equiv 2 \tag{4}$$

Proof. Recall that given a graph G and an integer k, in order to prove that $G \equiv k$, it is sufficient to prove $k \geq Ls(G)$ and $Rs(G) \geq k$.

1. We have $Ls(P_1^L) = Rs(P_1^L) = 0$ and $Ls(P_2^L) = Rs(P_2^L) = 0$ as in both games no edges are taken by a player. This proves, by Lemma 2, that $P_1^L = P_2^L = 0$

2. We prove $Ls(2P_3^L) = Rs(2P_3^L) = 1$. To do that, we just need to prove $Ls(2P_3^L) \leq 1$ and $Rs(2P_3^L) \geq 1$.
   - Suppose Left starts. If she plays in one path P_3^L, Right claims the middle vertex of the other path and then plays at least one vertex in the P_3^L where Left started. This way, Left scores at most one.
   - Suppose Right starts. He plays in one path P_3^L. By going first in the second path, Left can score one by playing the middle vertex and after that at least one of its two neighbors.

3. As $-P_3^L = P_3^R$, we will prove $P_4^L + P_3^R = 0$. Denote by (v_0, v_1, v_2, v_3) the vertices of P_4^L and by (v_0', v_1', v_2') the vertices of P_3^R
   - Suppose Left starts. If she plays in P_3^R, Right plays v_1 and pairs (v_2, v_3) to ensure that Left cannot score an edge. If Left plays v_0 or v_1 (v_2 or v_3 resp.), Right plays v_2 (v_1 resp.) and pairs (v_0', v_2') and v_1' with the available vertex in $\{v_0, v_1\}$ (in $\{v_2, v_3\}$ resp.). This way, Left and Right scores the same number of edges and this proves $Ls(P_3^R + P_4^L) \leq 0$
   - Suppose Right starts. Left considers the pairing (v_0, v_0'), (v_1, v_1'), (v_2, v_2'). This way, any point scored by Right is scored by Left. Therefore $Rs(P_4^L + P_3^R) \geq 0$.

4. Let $G = 2P_5^L + P_3^L$. Denote by $v_0, \ldots, v_4$ and $v_0', \ldots, v_4'$ the vertices of the two copies of P_5^L and by (u_0, u_1, u_2) the vertices of P_3^L. Let first prove $Rs(G) \geq 2$. Up to consider only 3 vertices of one copy of P_5^L, we can suppose that the first move of Right is in a P_5^L and we will prove that Left scores 2 on $P_5^L + P_3^L$. Suppose Right has played a vertex v_i' with $0 \leq i \leq 4$. Left plays v_2 and continues as follows:
   - If Right plays v_0 or v_1 (v_3 or v_4 resp.), Left plays v_3 (v_1 resp.) and pairs (v_4, u_1) ((v_0, u_1) resp.) and (u_0, u_2).
   - If Right plays u_0, u_1 or u_2, Left plays v_1 and pairs (v_0, v_3).
   In both cases, Left scores at least two points. Now we prove $Ls(G) \leq 2$. After the first move of Left, at least one of the two copies of P_5^L has its 5 vertices available. Suppose it is $v_0', \ldots, v_4'$. Right plays v_2' and pairs (v_0', v_1') and (v_3', v_4'), ensuring Left won't score any point on this copy of P_5^L. Left plays a second move:
   - If v_2 has not been played yet, Right plays v_2. Left plays a third move. If the three moves of Left are in P_3^L, Right pairs (v_0, v_1) and (v_3, v_4), ensuring Left does not score any other point. If at least one of them is not in P_3, Right plays any vertex of P_3, and know that at least one vertex of $(v_0, v_1, v_3, v_4, u_0, u_1, u_2)$ will be available for his next move. Thus, Left cannot score more than two points on the rest of them.
   - If Left has played v_2, at least one of v_1 or v_3 is available. Right plays it. By symmetry, suppose it is v_1. After the next move of Left, at least one of v_3, v_4, u_1 will be available. Right plays it, ensuring again that Left cannot score more than 2. □

We can now state the equivalence theorem for union of paths.

Corollary 30. *Let $P_1, \ldots, P_N$ be paths of lengths $n_1, \ldots, n_N$.*
Let $q_1, \ldots, q_N$ be positive integers and $1 \leq r_1, \ldots, r_N \leq 5$ be integers such that for any $1 \leq i \leq N$, we have $n_i = 5q_i + r_i$.
Denote for $1 \leq i \leq 5$ by N_i the number of r_j equal to i. In Maker-Breaker Incidence*, we have:*

$$\sum_{i=1}^{N} P_{5q_i+r_i}^L \equiv \sum_{i=1}^{N} q_i + \left\lfloor \frac{N_3 + N_4}{2} \right\rfloor + 3 \left\lfloor \frac{N_5}{4} \right\rfloor + (N_3 + N_4 \mod 2)P_3 + (N_5 \mod 4)P_5$$

Therefore, $Ls(\sum_{i=1}^{N} P_i)$ and $Rs(\sum_{i=1}^{N} P_i)$ are computable in linear time.

Proof. By Theorem 24, any path $P_{5q_i+r_i}^L$ is equivalent to $q_i + P_{r_i}$. Then, by Lemma 29, we have $P_3 \equiv P_4$, $2P_3 \equiv 1$, and $2(2P_5 + P_3) \equiv 4P_5 + 2P_3 \equiv 4P_5 + 1 \equiv 4$. Thus $4P_5 \equiv 3$. Note that these computations are possible thanks to Milnor's universe. □

Note that we consider $1 \le r_i \le 5$ and not $0 \le r_i \le 4$, so q_i and r_i are not exactly the quotient and the rest of the size of the path by 5.

Corollary 31. *Let $n \ge 1$. In* Maker-Breaker Incidence*, there exists a linear time algorithm to compute $Ls(C_n^L)$ and $Rs(C_n^L)$.*

Proof. First, note that $Rs(C_n^L) = Ls(P_{n-1}^L)$.

To compute $Ls(C_n^L)$, note that all the vertices are symmetric. Therefore, we can suppose that Left first plays any of them. The next move of Right will make the graph equivalent to $(P_k^L, \{v_0\}, \emptyset) + (P_{k'}^L, \{v_0'\}, \emptyset)$ with v_0, v_0' extremities of P_k^L and $P_{k'}^L$ and with $k + k' = n$. The score on these graphs can be computed in linear time by using Corollary 30, and therefore, $Ls(C_n^L)$ too as, by Theorem 24, at most 5 values are to be considered for the pair (k, k') according to the equivalences. $\square$

7. Perspectives

In this paper, we introduced positional scoring games in a general framework and then focused on Incidence, which corresponds to the case of 2-uniform hypergraphs. To conclude this paper, we list some relevant open problems.

- We have solved Maker-Breaker Incidence on union of paths using game equivalences. Next step would be to study trees.
- What is the complexity of Maker-Breaker Incidence when restricted to the class of cographs? Equivalent vertices have an important role and can be easily simplified. This could be a starting point for the study of cographs.
- We proved that Maker-Breaker Incidence is fixed-parameter tractable using the neighborhood diversity. It would be interesting to find other parameters for which the problem is FPT. For example, is it FPT parameterized by the score?
- The same question applies when considering general hypergraphs. The answer is negative for 6-uniform hypergraphs as it is PSPACE even for $k = 1$. What about 3-uniform hypergraphs? Since Maker-Breaker Positional Game is polynomial for 3-uniform hypergraphs [12], the question makes sense.
- We have proved that Maker-Maker Scoring Positional Game is PSPACE-complete even for 3-uniform hypergraphs but provided a linear algorithm for 2-uniform hypergraphs. It might be interesting to look at particular 3-uniform hypergraphs. For example, is it possible to compute the score in the scoring version of the Triangle Game (where players choose edges of a graph and try to construct triangles)? The hypergraph of this game has the particularity to be linear (hyperedges cross on at most one vertex). A more general question would be to find the complexity of Maker-Maker Incidence on linear 3-uniform hypergraphs.
- In Section 2.2, we have introduced *partisan scoring positional games* to include the two conventions of scoring positional games in a more general definition. Maker-Maker convention corresponds to games with only green hyperedges whereas Maker-Breaker convention corresponds to games with only blue edges. It would be interesting to consider games with both red and blue edges but no green edge.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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