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Optimal Doubling, Reifenberg Flatness and Operators of p -Laplace type

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Abstract

In this paper we consider operators of p -Laplace type of the form $\nabla \cdot A(x, \nabla u) = 0$. Concerning A we assume, for $p \in (1, \infty)$ fixed, an appropriate ellipticity type condition, Hölder continuity in x and that $A(x, \eta) = |\eta|^{p-1} A(x, \eta/|\eta|)$ whenever $x \in \mathbb{R}^n$ and $\eta \in \mathbb{R}^n \setminus \{0\}$. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, let D be a compact subset of Ω . We say that $\hat{u} = \hat{u}_{p,D,\Omega}$ is the A -capacitary function for D in Ω if $\hat{u} \equiv 1$ on D , $\hat{u} \equiv 0$ on $\partial\Omega$ in the sense of $W_0^{1,p}(\Omega)$ and $\nabla \cdot A(x, \nabla \hat{u}) = 0$ in $\Omega \setminus D$ in the weak sense. We extend $\hat{u}$ to $\mathbb{R}^n \setminus \Omega$ by putting $\hat{u} \equiv 0$ on $\mathbb{R}^n \setminus \Omega$. Then there exists a unique finite positive Borel measure $\hat{\mu}$ on $\mathbb{R}^n$, with support in $\partial\Omega$, such that

$$\int \langle A(x, \nabla \hat{u}), \nabla \phi \rangle dx = - \int \phi d\hat{\mu} \quad \text{whenever} \quad \phi \in C_0^\infty(\mathbb{R}^n \setminus D).$$

In this paper we prove that if Ω is Reifenberg flat with vanishing constant, then

$$\lim_{r \rightarrow 0} \inf_{w \in \partial\Omega} \frac{\hat{\mu}(B(w, \tau r))}{\hat{\mu}(B(w, r))} = \lim_{r \rightarrow 0} \sup_{w \in \partial\Omega} \frac{\hat{\mu}(B(w, \tau r))}{\hat{\mu}(B(w, r))} = \tau^{n-1},$$

for every τ , $0 < \tau \leq 1$. In particular, we prove that $\hat{\mu}$ is an asymptotically optimal doubling measure on $\partial\Omega$.

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1 Introduction

In the following we let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Recall that the solution to the Dirichlet problem for the Laplace operator in Ω is the unique smooth function $u \in C^\infty(\Omega)$ which is harmonic in Ω and equals $u = f$ on $\partial\Omega$ where $f \in C(\partial\Omega)$, the latter being the space of continuous functions defined on $\partial\Omega$. If the boundary $\partial\Omega$ is regular then u is continuous on the closure of Ω and u achieves the boundary data at each boundary point. The maximum principle and the Riesz representation theorem yield the representation formula

$$u(x) = \int_{\partial\Omega} f(y) d\omega^x(y) \quad \text{whenever} \quad x \in \Omega,$$

where $\omega = \omega^x$ is referred to as the harmonic measure at x associated to the Laplace operator. Let also $g = g(\cdot) = g(\cdot, x)$ be the Green function for Ω with pole at $x \in \Omega$ and extend g to $\mathbb{R}^n \setminus \Omega$ by putting $g \equiv 0$ on $\mathbb{R}^n \setminus \Omega$. We note that ω is the Riesz measure associated to g in the sense that

$$\int \langle \nabla g(y, x), \nabla \phi(y) \rangle dy = - \int \phi(y) d\omega(y) \quad \text{whenever} \quad \phi \in C_0^\infty(\mathbb{R}^n \setminus \{x\}).$$

Let $B(w, r) = \{y \in \mathbb{R}^n : |w - y| < r\}$ whenever $w \in \mathbb{R}^n$, $r > 0$, and where $|w - y| = \langle w - y, w - y \rangle^{1/2}$ is the Euclidean norm of $w - y$. To proceed we note that it is well-known that if Ω is a NTA-domain, with parameters M and r_0 , in the sense of [JK], see also Definition 2.1 stated below, then ω is a doubling measure on $\partial\Omega$ in the sense that there exists a constant $c = c(n, M)$, $1 \leq c < \infty$, such that if $0 < r < r_0$, and $w \in \partial\Omega$, then

$$\omega(B(w, 2r)) \leq c\omega(B(w, r)).$$

Furthermore, recently Kenig and Toro, see [KT], [KT1] and below, have established more refined doubling properties for harmonic measure in Reifenberg flat domains and they have, in particular, studied questions concerning measures which are asymptotically optimal doubling. Recall from [K] that ω is said to be asymptotically optimal doubling if ω is a doubling measure, and if, for every τ , $0 < \tau \leq 1$, one has

$$\lim_{r \rightarrow 0} \inf_{w \in \partial\Omega} \frac{\omega(B(w, \tau r))}{\omega(B(w, r))} = \lim_{r \rightarrow 0} \sup_{w \in \partial\Omega} \frac{\omega(B(w, \tau r))}{\omega(B(w, r))} = \tau^{n-1}. \quad (1.1)$$

To formulate some of the results of Kenig and Toro we first have to introduce some notation and notions. In particular, in the following we let $\bar{E}, \partial E, \text{diam } E$, be the closure, boundary, diameter, of the set $E \subset \mathbb{R}^n$ and we let $d(y, E)$ equal the distance from $y \in \mathbb{R}^n$ to E . Moreover, we let $h(E, F) = \max(\sup\{d(y, E) : y \in F\}, \sup\{d(y, F) : y \in E\})$ be the Hausdorff distance between the sets $E, F \subset \mathbb{R}^n$. By dx we denote the Lebesgue n -measure on $\mathbb{R}^n$.

Definition 1.1 *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Then $\partial\Omega$ is said to be uniformly (δ, r_0) -approximable by hyperplanes, provided there exists, whenever $w \in \partial\Omega$ and $0 < r < r_0$, a hyperplane Λ containing w such that*

$$h(\partial\Omega \cap B(w, r), \Lambda \cap B(w, r)) \leq \delta r. \quad (1.2)$$

We let $\mathcal{F}(\delta, r_0)$ denote the class of all domains Ω which satisfy Definition 1.1. Let $\Omega \in \mathcal{F}(\delta, r_0)$, $w \in \partial\Omega$, $0 < r < r_0$, and let Λ be as in Definition 1.1. We say that $\partial\Omega$ separates $B(w, r)$, if

$$\{x \in \Omega \cap B(w, r) : d(x, \partial\Omega) \geq 2\delta r\} \subset \text{one component of } \mathbb{R}^n \setminus \Lambda. \quad (1.3)$$

Definition 1.2 *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Then Ω and $\partial\Omega$ are said to be (δ, r_0) -Reifenberg flat provided $\Omega \in \mathcal{F}(\delta, r_0)$ and provided (1.3) holds whenever $0 < r < r_0, w \in \partial\Omega$.*

For short we say that Ω and $\partial\Omega$ are δ -Reifenberg flat whenever Ω and $\partial\Omega$ are (δ, r_0) -Reifenberg flat for some $r_0 > 0$. We note that an equivalent definition of a Reifenberg flat domain is given in [KT]. As in [KT] one can show that there exists $\hat{\delta} = \hat{\delta}(n)$, $0 < \hat{\delta} < 1$, such that if Ω is a δ -Reifenberg flat domain for some $0 < \delta < \hat{\delta}$, then Ω is a NTA-domain with constant $M = M(n)$.

Definition 1.3 *Let $\Omega \subset \mathbb{R}^n$ be a (δ, r_0) -Reifenberg flat domain for some $0 < \delta < \hat{\delta}$, $r_0 > 0$, and let $w \in \partial\Omega$, $0 < r < r_0$. We say that $\partial\Omega \cap B(w, r)$ is Reifenberg flat with vanishing constant, if, for each $\epsilon > 0$, there exists $\tilde{r} = \tilde{r}(\epsilon) > 0$ with the following property. If $x \in \partial\Omega \cap B(w, r)$ and $0 < \rho < \tilde{r}$, then there is a plane $\Lambda' = \Lambda'(x, \rho)$ containing x such that*

$$h(\partial\Omega \cap B(x, \rho), \Lambda' \cap B(x, \rho)) \leq \epsilon\rho.$$

For short we say that Ω and $\partial\Omega$ are Reifenberg flat with vanishing constant if $\partial\Omega \cap B(w, r)$ is Reifenberg flat with vanishing constant whenever $w \in \partial\Omega$, $0 < r < r_0$. The following two theorems are proved in [KT] and [KT1], respectively.

Theorem 1.4 *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain which is Reifenberg flat with vanishing constant. Then ω is asymptotically optimal doubling in the sense of (1.1).*

Theorem 1.5 *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and let μ be a doubling Radon measure, with support on $\partial\Omega$ and assume that μ is asymptotically optimal doubling. Then the following is true. If $n = 2, 3$, then Ω is Reifenberg flat with vanishing constant. If $n \geq 4$, and if Ω is δ -Reifenberg flat for some $\delta < 1/8$, then Ω is Reifenberg flat with vanishing constant.*

Note that in Theorem 1.5 no assumptions are initially made concerning the geometry of Ω . In the case $n \geq 4$, the assumption that Ω is δ -Reifenberg flat for some $\delta < 1/8$ is imposed to rule out the existence of the so called Preiss cone, see page 9 in [CKL]. The proof of this result uses strongly the remarkable result of Kowalski and Preiss, see [KP]. Furthermore, note that Theorem 1.4 and Theorem 1.5 give, if we initially assume that Ω is δ -Reifenberg flat for some $\delta < 1/8$, a characterization of the subset of such domains which are Reifenberg flat with vanishing constant in terms of the property that the harmonic measure is asymptotically optimal doubling in the sense of (1.1). For an extension of Theorem 1.4 to elliptic measures associated to linear divergence form elliptic operators we refer to [MT].

The purpose of this paper is to prove that Theorem 1.4 remain valid if the harmonic measure is replaced by measures being associated to solutions of p -Laplace type equations. As such we establish an analogue of Theorem 1.4 in the non-linear setting of p -Laplace type operators. To describe our result we first need to introduce some more notation. If $O \subset \mathbb{R}^n$ is open and $1 \leq q \leq \infty$, then by $W^{1,q}(O)$ we denote the space of equivalence classes of functions f with distributional gradient

$\nabla f = (f_{x_1}, \dots, f_{x_n})$, both of which are in $L^q(O)$. Let $\|f\|_{1,q} = \|f\|_q + \|\nabla f\|_q$ be the norm in $W^{1,q}(O)$ where $\|\cdot\|_q$ denotes the usual Lebesgue q -norm in O . Next let $C_0^\infty(O)$ be the set of infinitely differentiable functions with compact support in O and let $W_0^{1,q}(O)$ be the closure of $C_0^\infty(O)$ in the norm of $W^{1,q}(O)$. By $\nabla \cdot$ we denote the divergence operator.

Definition 1.6 Let $p, \beta, \alpha \in (1, \infty)$ and $\gamma \in (0, 1)$. Let $A = (A_1, \dots, A_n) : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$, assume that $A = A(x, \eta)$ is continuous in $\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$ and that $A(x, \eta)$, for fixed $x \in \mathbb{R}^n$, is continuously differentiable in η_k , for every $k \in \{1, \dots, n\}$, whenever $\eta \in \mathbb{R}^n \setminus \{0\}$. We say that the function A belongs to the class $M_p(\alpha, \beta, \gamma)$ if the following conditions are satisfied whenever $x, y, \xi \in \mathbb{R}^n$ and $\eta \in \mathbb{R}^n \setminus \{0\}$:

$$\begin{aligned} (i) \quad & \alpha^{-1}|\eta|^{p-2}|\xi|^2 \leq \sum_{i,j=1}^n \frac{\partial A_i}{\partial \eta_j}(x, \eta) \xi_i \xi_j \leq \alpha|\eta|^{p-2}|\xi|^2, \\ (ii) \quad & |A(x, \eta) - A(y, \eta)| \leq \beta|x - y|^\gamma|\eta|^{p-1}, \\ (iii) \quad & A(x, \eta) = |\eta|^{p-1}A(x, \eta/|\eta|). \end{aligned}$$

For short, we write $M_p(\alpha)$ for the class $M_p(\alpha, 0, \gamma)$.

Definition 1.7 Let $p \in (1, \infty)$ and let $A \in M_p(\alpha, \beta, \gamma)$ for some (α, β, γ) . Given a bounded domain G we say that u is A -harmonic in G provided $u \in W^{1,p}(G)$ and

$$\int \langle A(x, \nabla u(x)), \nabla \theta(x) \rangle dx = 0, \quad (1.4)$$

whenever $\theta \in W_0^{1,p}(G)$. If $A(x, \eta) = |\eta|^{p-2}(\eta_1, \dots, \eta_n)$, then u is said to be p -harmonic in G . As a short notation for (1.4) we write $\nabla \cdot A(x, \nabla u) = 0$ in G .

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, let D be a compact subset of Ω , let $p \in (1, \infty)$ and let $A \in M_p(\alpha, \beta, \gamma)$ for some (α, β, γ) . We say that $\hat{u} = \hat{u}_{p,D,\Omega}$ is the A -capacitary function for D in Ω if $\hat{u} \equiv 1$ on D and $\hat{u} \equiv 0$ on $\partial\Omega$ in the sense of $W_0^{1,p}(\Omega)$ and if $\hat{u}$ is A -harmonic in $\Omega \setminus D$. It is well known that a unique such $\hat{u}$ exists, see [HKM], and that

$$\int_{\Omega} \langle A(x, \nabla \hat{u}), \nabla \hat{u} \rangle dx = \inf \left\{ \int_{\Omega} \langle A(x, \nabla \theta), \nabla \theta \rangle dx \right\}, \quad (1.5)$$

where the infimum is taken over all $\theta \in C_0^\infty(\Omega)$ with $\theta \equiv 1$ on D . In particular, $\hat{u}$ solves the minimization problem on the right hand side in (1.5). We extend $\hat{u}$ to $\mathbb{R}^n \setminus \Omega$ by putting $\hat{u} \equiv 0$ on $\mathbb{R}^n \setminus \Omega$. Then there exists, see Lemma 2.6, a unique finite positive Borel measure $\hat{\mu}$ on $\mathbb{R}^n$, with support in $\partial\Omega$, such that

$$\int \langle A(x, \nabla \hat{u}), \nabla \phi \rangle dx = - \int \phi d\hat{\mu} \quad \text{whenever} \quad \phi \in C_0^\infty(\mathbb{R}^n \setminus D). \quad (1.6)$$

In the following we say that the measure $\hat{\mu}$ is asymptotically optimal doubling if $\hat{\mu}$ is a doubling measure and if (1.1) hold with ω replaced by $\hat{\mu}$. In this paper we prove the following theorem.

Theorem 1.8 *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, let D be a compact subset of Ω , let $p \in (1, \infty)$ and let $A \in M_p(\alpha, \beta, \gamma)$ for some (α, β, γ) . Let $\hat{u} = \hat{u}_{p,D,\Omega}$ be the A -capacitary function for D in Ω and let $\hat{\mu}$ be the measure, with support in $\partial\Omega$, associated to $\hat{u}$ as in (1.6). Then, if $\Omega \subset \mathbb{R}^n$ is a Reifenberg flat domain with vanishing constant, then $\hat{\mu}$ is asymptotically optimal doubling.*

Concerning the proof of Theorem 1.8 we here outline the proof of Theorem 1.8 in a special case. In particular, we let $p \in (1, \infty)$ and we consider the special case when $\Omega = \{y = (y', y_n) \in \mathbb{R}^n : y_n > 0\} \cap B(0, 1)$, $D = \overline{B}(e_n/2, 1/4)$, $A \in M_p(\alpha)$ for some α . Moreover, we let $\hat{u} = \hat{u}_{p,D,\Omega}$ be the A -capacitary function for D in Ω . Furthermore, we let $\hat{\mu}$ be the measure, with support in $\partial\Omega$, associated to $\hat{u}$ as in (1.6) and in the following we assume that $w = 0$ and $0 < r < 1/8$. We then first note, using the main result in [Li], that $\nabla \hat{u}$ is continuous on the closure of $\Omega \cap \{y = (y', y_n) \in \mathbb{R}^n : |y'| < 1/2, 0 < y_n < 1/8\}$ and from Lemma 2.8 together with Lemma 3.4 in [GT] we see that $\langle \nabla \hat{u}(y', 0), e_n \rangle > 0$ whenever $|y'| < 1/2$. Therefore, by repeating the approximation arguments in Lemma 2.4 [LLuN], we can construct a sequence of functions $\{\hat{u}_\epsilon\}$, $\hat{u}_\epsilon \in C^{1,\sigma}(\bar{\Omega} \cap B(0, 1/8))$, such that $\{\hat{u}_\epsilon\}, \{\nabla \hat{u}_\epsilon\}$ converge uniformly to $\hat{u}, \nabla \hat{u}$ in $\bar{\Omega} \cap B(0, 1/8)$. Using this fact it follows that

$$\int_{\Omega \cap B(0, 1/8)} \langle A(\nabla \hat{u}(y)), \nabla \phi(y) \rangle dy = - \int_{\partial\Omega \cap B(0, 1/8)} \phi(y', 0) \langle A(\nabla \hat{u}(y', 0)), e_n \rangle dy' \quad (1.7)$$

for all $\phi \in W_0^{1,p}(B(0, 1/8))$. Let $\hat{\mu}$ be the restriction of the measure in (1.6), associated to $\hat{u}$, to the set $\{y = (y', 0) \in \mathbb{R}^n : |y'| < 1/8\}$. Then using (1.7) we see that

$$\begin{aligned} d\hat{\mu}(y', 0) &= \langle A(\nabla \hat{u}(y', 0)), e_n \rangle dy' \\ &= A_n(\langle \nabla \hat{u}(y', 0), e_n \rangle e_n) dy' = (\langle \nabla \hat{u}(y', 0), e_n \rangle)^{p-1} A_n(e_n) dy', \end{aligned} \quad (1.8)$$

whenever $|y'| < 1/8$. Furthermore, using Theorem 1 in [LLuN], or Lemma 2.10 stated below, see also Lemma 2.8 in [LLuN], we deduce that

$$|\log(\partial_{x_n} \hat{u}(y'_1, 0)) - \log(\partial_{x_n} \hat{u}(y'_2, 0))| \leq c|y'_1 - y'_2|^\sigma, \quad (1.9)$$

whenever $|y'_1| < 1/32, |y'_2| < 1/32$. Here $c = c(p, n, \alpha)$, $\sigma = \sigma(p, n, \alpha) \in (0, 1)$. Using (1.8) and (1.9) we hence see that

$$\frac{e^{-c(p-1)(\tau r)^\sigma} (\tau r)^{n-1}}{e^{c(p-1)r^\sigma} r^{n-1}} \leq \frac{\hat{\mu}(\{y = (y', 0) : |y'| < \tau r\})}{\hat{\mu}(\{y = (y', 0) : |y'| < r\})} \leq \frac{e^{c(p-1)(\tau r)^\sigma} (\tau r)^{n-1}}{e^{-c(p-1)r^\sigma} r^{n-1}}. \quad (1.10)$$

Hence,

$$\lim_{r \rightarrow 0} \frac{\hat{\mu}(\{y = (y', 0) : |y'| < \tau r\})}{\hat{\mu}(\{y = (y', 0) : |y'| < r\})} = \tau^{n-1}.$$

Remark 1.8 *Note that in the above proof we considered the p -capacitary function defined relative $\Omega = \{y = (y', y_n) \in \mathbb{R}^n : y_n > 0\} \cap B(0, 1)$ and D . However the conclusion in (1.10) remains valid for any positive function which is p -harmonic inside a cylinder $Q = \{(y', y_n) \in \mathbb{R}^n : |y'| < 1, 0 < y_n < 1\}$ and vanishes on $y_n = 0$.*

While we at the end of the proof of Theorem 1.8 will return to the special case outlined above, the general proof is more involved and we refer the reader to the bulk of the paper for details.

The rest of the paper is organized as follows. In section 2 we state a number of estimates for non-negative A -harmonic functions in a bounded domain $\Omega \subset \mathbb{R}^n$, assuming that Ω is a NTA-domain and, at instances, also assuming that Ω is (δ, r_0) -Reifenberg flat. All of these results can be found in [LLuN] where the second and third author, together with John Lewis, began the generalization of the results in [LN, LN1, LN2] to a class of p -Laplace type operators allowing, in particular, for operators with variable coefficients. In particular, in [LLuN] new results concerning boundary Harnack inequalities and the Martin boundary problem, in Reifenberg flat domains, for operators of p -Laplace type of the form $\nabla \cdot A(x, \nabla u) = 0$ were established. Section 3 is devoted to the proof of Theorem 1.8.

2 Estimates for A -harmonic functions

In this section we state a number of estimates for non-negative A -harmonic functions in a bounded domain $\Omega \subset \mathbb{R}^n$ assuming that Ω is a NTA-domain and, at instances, also assuming that Ω is (δ, r_0) -Reifenberg flat. Moreover, for the proofs of Lemma 2.2-Lemma 2.10 stated below we refer to [LLuN] unless otherwise stated. We first recall the following definition of NTA-domains, see Definition 1.3 in [LN2].

Definition 2.1 *A bounded domain Ω is called non-tangentially accessible (NTA) if there exist $M \geq 2$ and r_0 such that the following are fulfilled:*

- (i) *corkscrew condition: for any $w \in \partial\Omega$, $0 < r < r_0$, there exists $a_r(w) \in \Omega$ satisfying $M^{-1}r < |a_r(w) - w| < r$, $d(a_r(w), \partial\Omega) > M^{-1}r$,*
- (ii) *$\mathbb{R}^n \setminus \bar{\Omega}$ satisfies the corkscrew condition,*
- (iii) *uniform condition: if $w \in \partial\Omega$, $0 < r < r_0$, and $w_1, w_2 \in B(w, r) \cap \Omega$, then there exists a rectifiable curve $\gamma : [0, 1] \rightarrow \Omega$ with $\gamma(0) = w_1$, $\gamma(1) = w_2$, and such that*
 - (a) $H^1(\gamma) \leq M |w_1 - w_2|$,
 - (b) $\min\{H^1(\gamma([0, t])), H^1(\gamma([t, 1]))\} \leq M d(\gamma(t), \partial\Omega)$.

In Definition 2.1, H^1 denotes length or the one-dimensional Hausdorff measure. We note that (iii) is different but equivalent to the usual Harnack chain condition given in [JK] (see [BL], Lemma 2.5). Moreover, using Theorem 3.1 in [KT] we see that there exists $\hat{\delta} = \hat{\delta}(n) > 0$ such that if $\Omega \subset \mathbb{R}^n$ is a (δ, r_0) -Reifenberg flat domain and if $0 < \delta \leq \hat{\delta}$, then Ω is a NTA-domain in the sense described above with constant $M = M(n)$. In the following we assume $0 < \delta \leq \hat{\delta}$ and we refer to M as the NTA-constant of Ω . Furthermore, we will always assume $r_0 < 1$.

Throughout this section we will assume that $A \in M_p(\alpha, \beta, \gamma)$ or $A \in M_p(\alpha)$ for some (α, β, γ) and that $1 < p < \infty$. Also in this paper, unless otherwise stated, c will denote a positive constant ≥ 1 , not necessarily the same at each occurrence, depending only on $p, n, M, \alpha, \beta, \gamma$ where M denotes the NTA-constant for $\Omega \subset \mathbb{R}^n$. In general, $c(a_1, \dots, a_m)$ denotes a positive constant ≥ 1 , which may depend only on $p, n, M, \alpha, \beta, \gamma$ and $a_1, \dots, a_m$, not necessarily the same at each occurrence. If $A \approx B$ then A/B is bounded from above and below by constants which, unless otherwise stated, only depend on $p, n, M, \alpha, \beta, \gamma$. Moreover, we let $\max_E u$, $\min_E u$ be the essential supremum and infimum of u on E whenever $E \subset \mathbb{R}^n$ and whenever u is defined on E . We put $\Delta(w, r) = \partial\Omega \cap B(w, r)$ whenever

$w \in \partial\Omega$, $0 < r$. Finally, e_i , $1 \leq i \leq n$, denotes the point in $\mathbb{R}^n$ with one in the i th coordinate position and zeroes elsewhere.

Lemma 2.2 *Let u be a positive A -harmonic function in $B(w, 4r)$. Then*

$$(i) \quad r^{p-n} \int_{B(w, r/2)} |\nabla u|^p dx \leq c \left(\max_{B(w, r)} u \right)^p,$$

$$(ii) \quad \max_{B(w, r)} u \leq c \min_{B(w, r)} u.$$

Furthermore, there exists $\sigma = \sigma(p, n, \alpha, \beta, \gamma) \in (0, 1)$ such that if $x, y \in B(w, r)$, then

$$(iii) \quad |u(x) - u(y)| \leq c \left(\frac{|x-y|}{r} \right)^\sigma \max_{B(w, 2r)} u.$$

Lemma 2.3 *Let $\Omega \subset \mathbb{R}^n$ be a bounded NTA-domain. Let $w \in \partial\Omega$, $0 < r < r_0$, and suppose that u is a non-negative continuous A -harmonic function in $\bar{\Omega} \cap B(w, 4r)$ and that $u = 0$ on $\Delta(w, 4r)$. Then*

$$(i) \quad r^{p-n} \int_{\Omega \cap B(w, r/2)} |\nabla u|^p dx \leq c \left(\max_{\Omega \cap B(w, r)} u \right)^p.$$

Furthermore, there exists $\sigma = \sigma(p, n, M, \alpha, \beta, \gamma) \in (0, 1)$ such that if $x, y \in \Omega \cap B(w, r)$, then

$$(ii) \quad |u(x) - u(y)| \leq c \left(\frac{|x-y|}{r} \right)^\sigma \max_{\Omega \cap B(w, 2r)} u.$$

Lemma 2.4 *Let $\Omega \subset \mathbb{R}^n$ be a bounded NTA-domain. Let $w \in \partial\Omega$, $0 < r < r_0$, and suppose that u is a non-negative continuous A -harmonic function in $\bar{\Omega} \cap B(w, 4r)$ and that $u = 0$ on $\Delta(w, 4r)$. There exists $c = c(p, n, M, \alpha, \beta, \gamma)$, $1 \leq c < \infty$, such that if $\tilde{r} = r/c$, then*

$$\max_{\Omega \cap B(w, \tilde{r})} u \leq c u(a_{\tilde{r}}(w)).$$

Lemma 2.5 *Let $\Omega \subset \mathbb{R}^n$ be a bounded NTA-domain. Let $w \in \partial\Omega$, $0 < r < r_0$, and suppose that u is a non-negative continuous A -harmonic function in $\bar{\Omega} \cap B(w, 4r)$ and that $u = 0$ on $\Delta(w, 4r)$. Extend u to $B(w, 4r)$ by defining $u \equiv 0$ on $B(w, 4r) \setminus \Omega$. Then u has a representative in $W^{1,p}(B(w, 4r))$ with Hölder continuous partial derivatives of first order in $\Omega \cap B(w, 4r)$. In particular, there exists $\hat{\sigma} \in (0, 1]$, depending only on $p, n, \alpha, \beta, \gamma$ such that if $x, y \in B(\hat{w}, \hat{r}/2)$, $B(\hat{w}, 4\hat{r}) \subset \Omega \cap B(w, 4r)$, then*

$$c^{-1} |\nabla u(x) - \nabla u(y)| \leq (|x-y|/\hat{r})^{\hat{\sigma}} \max_{B(\hat{w}, \hat{r})} |\nabla u| \leq c \hat{r}^{-1} (|x-y|/\hat{r})^{\hat{\sigma}} \max_{B(\hat{w}, 2\hat{r})} u.$$

Lemma 2.6 *Let $\Omega \subset \mathbb{R}^n$ be a bounded NTA-domain. Let $w \in \partial\Omega$, $0 < r < r_0$, and suppose that u is a non-negative continuous A -harmonic function in $\bar{\Omega} \cap B(w, 4r)$ and that $u = 0$ on $\partial\Omega \cap B(w, 4r)$. Extend u to $B(w, 4r)$ by defining $u \equiv 0$ on $B(w, 4r) \setminus \Omega$. Then there exists a unique finite positive Borel measure μ on $\mathbb{R}^n$, with support in $\Delta(w, 4r)$, such that whenever $\phi \in C_0^\infty(B(w, 4r))$, then*

$$\int \langle A(x, \nabla u), \nabla \phi \rangle dx = - \int \phi d\mu.$$

Moreover, there exists $c = c(p, n, M, \alpha, \beta, \gamma)$, $1 \leq c < \infty$, such that if $\tilde{r} = r/c$, then

$$c^{-1} r^{p-n} \mu(B(w, \tilde{r})) \leq (u(a_{\tilde{r}}(w)))^{p-1} \leq c r^{p-n} \mu(B(w, \tilde{r})). \quad (2.1)$$

Proof. The existence of the measure μ can be found in [HKM] (Theorem 21.2). To prove (2.1) we first note, using Lemma 2.4, Lemma 2.3 (ii) and Harnack's inequality, that there exists a constant $c(p, n, M, \alpha, \beta, \gamma)$ such that if $\tilde{r} = r/c$ then

$$\max_{\Omega \cap B(w, 2\tilde{r})} u \approx u(a_{\tilde{r}}(w)). \quad (2.2)$$

To prove the left hand side inequality in (2.1) we let $\theta \in C_0^\infty(\Omega)$ be such that $\theta = 1$ in $B(w, \tilde{r})$, $\theta = 0$ outside $B(w, 2\tilde{r})$, and $|\nabla \theta| \leq c\tilde{r}^{-1}$. From the definition of the class $M_p(\alpha, \beta, \gamma)$ together with Lemma 2.4, Lemma 2.3, Hölders inequality and (2.2), we see that

$$\begin{aligned} \mu(B(w, \tilde{r})) &\leq \int \theta d\mu \leq \int |\nabla \theta| |A(x, \nabla u)| dx \leq c\tilde{r}^{-1} \int_{\Omega \cap B(w, 2\tilde{r})} |\nabla u|^{p-1} \\ &\leq c \tilde{r}^{n-n(p-1)/p-1} \left(\int_{\Omega \cap B(w, 2\tilde{r})} |\nabla u|^p dx \right)^{(p-1)/p} \leq cr^{n-p} (u(a_{\tilde{r}}(w)))^{p-1}. \end{aligned}$$

This proves the left hand side inequality in (2.1). To prove the right hand side inequality one can use Lemma 3.1 in [KZ]. Let $M(\tilde{r}) = \max_{B(w, \tilde{r})} u$. Lemma 3.1 in [KZ] states that if

$$M(t\tilde{r}) \leq \theta M(\tilde{r}), \text{ for some } 0 < t < 1 \text{ and } \theta \text{ given by the statement of the lemma,} \quad (2.3)$$

then

$$M(\tilde{r}) \leq c \left(r^p \frac{\mu(B(w, 2\tilde{r}))}{|B(w, 2\tilde{r})|} \right)^{1/(p-1)}. \quad (2.4)$$

Once (2.4) is established the right hand side inequality in (2.1) follows from a simple rearrangement of terms, (2.2) and Harnack's inequality. Hence we only have to verify (2.3). However, using Lemma 2.3 (ii) we see that

$$u(x) \leq \tilde{c} t^\sigma M(r)$$

whenever $x \in B(w, t\tilde{r})$, $0 < t < 1/2$ and where $\sigma, \tilde{c}$ are as in Lemma 2.3. In particular, choosing $t = (\theta/\tilde{c})^{1/\sigma}$ completes the proof of (2.3). $\square$

Remark 2.6 Note that assumption (2.4) in [KZ] differs from the assumption made in Definition 1.6 (iii). However, examining the proof of Lemma 3.1 in [KZ] one realizes that the lemma remains unchanged.

Lemma 2.7 Let $G \subset \mathbb{R}^n$ be an open set, suppose that p , $1 < p < \infty$, is given and let $A \in M_p(\alpha, \beta, \gamma)$ for some (α, β, γ) . Let $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be the composition of a translation, a rotation and a dilation $z \rightarrow rz$, $r \in (0, 1]$. Suppose that u is A -harmonic in G and define $\hat{u}(z) = u(F(z))$ whenever $F(z) \in G$. Then $\hat{u}$ is $\hat{A}$ -harmonic in $F^{-1}(G)$ and $\hat{A} \in M_p(\alpha, \beta, \gamma)$.

Lemma 2.8 Let $\Omega \subset \mathbb{R}^n$ be a bounded (δ, r_0) -Reifenberg flat domain, $w \in \partial\Omega$, and $0 < r < r_0$. Suppose that u is a positive A -harmonic function in $\Omega \cap B(w, 4r)$, that u is continuous in $\bar{\Omega} \cap B(w, 4r)$, and that $u = 0$ on $\Delta(w, 4r)$. There exist $\delta_1 = \delta_1(p, n, \alpha, \beta, \gamma)$, $c = c(p, n, \alpha, \beta, \gamma)$ such that if $0 < \delta \leq \delta_1$, then

$$c^{-1} \frac{u(y)}{d(y, \partial\Omega)} \leq |\nabla u(y)| \leq c \frac{u(y)}{d(y, \partial\Omega)} \quad \text{whenever} \quad y \in \Omega \cap B(w, r/c).$$

Lemma 2.9 *Let $\Omega \subset \mathbb{R}^n$ be a bounded (δ, r_0) -Reifenberg flat domain. Let $w \in \partial\Omega$, $0 < r < r_0$ and suppose that u is a positive A -harmonic function in $\Omega \cap B(w, 4r)$, u is continuous in $\bar{\Omega} \cap B(w, 4r)$, and $u = 0$ on $\Delta(w, 4r)$. Then there exist, for $\epsilon > 0$ given, $\delta_2 = \delta_2(p, n, \alpha, \beta, \gamma, \epsilon) > 0$ and $c = c(p, n, \alpha, \beta, \gamma, \epsilon)$, $1 \leq c < \infty$, such that*

$$c^{-1} \left(\frac{\hat{r}}{r} \right)^{1+\epsilon} \leq \frac{u(a_{\hat{r}}(w))}{u(a_r(w))} \leq c \left(\frac{\hat{r}}{r} \right)^{1-\epsilon},$$

whenever $0 < \delta \leq \delta_2$ and $0 < \hat{r} < r/4$.

Lemma 2.10 *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain which is Reifenberg flat with vanishing constant. Let $w \in \partial\Omega$, $0 < r < r_0$. Assume that u, v are positive A -harmonic functions in $\Omega \cap B(w, 4r)$, u, v are continuous in $\bar{\Omega} \cap B(w, 4r)$ and $u = 0 = v$ on $\Delta(w, 4r)$. There exist $r_1 = r_1(p, n, \alpha, \beta, \gamma) < r$, $c = c(p, n, \alpha, \beta, \gamma) \geq 1$ and $\sigma = \sigma(p, n, \alpha, \beta, \gamma) > 0$ such that if $w' \in \partial\Omega \cap B(w, r)$ and $0 < r' < r_1$, then*

$$\left| \log \frac{u(y_1)}{v(y_1)} - \log \frac{u(y_2)}{v(y_2)} \right| \leq c \left(\frac{|y_1 - y_2|}{r'} \right)^\sigma,$$

whenever $y_1, y_2 \in \Omega \cap B(w', r')$.

3 Proof of Theorem 1.8

The purpose of the section is to prove Theorem 1.8. Note that we can without loss of generality assume that $\Omega \subset \mathbb{R}^n$ is a (δ, r_0) -Reifenberg flat domain with $0 < \delta \leq \hat{\delta}$ and for some r_0 , $0 < r_0 < 1$. Let $p \in (1, \infty)$ and let $A \in M_p(\alpha, \beta, \gamma)$ for some (α, β, γ) . Let $\hat{u} = \hat{u}_{p,D,\Omega}$ be the A -capacitary function for D in Ω . Extend $\hat{u}$ to $\mathbb{R}^n \setminus \Omega$ by putting $\hat{u} \equiv 0$ on $\mathbb{R}^n \setminus \Omega$ and let $\hat{\mu}$ be a measure on $\mathbb{R}^n$, with support in $\partial\Omega$, as in (1.6). Using Lemma 2.6 and the Harnack inequality we can conclude that $\hat{\mu}$ is a doubling measure. To prove Theorem 1.8 we have to prove, for τ , $0 < \tau \leq 1$, fixed, that

$$\lim_{r \rightarrow 0} \inf_{w \in \partial\Omega} \frac{\hat{\mu}(B(w, \tau r))}{\hat{\mu}(B(w, r))} = \lim_{r \rightarrow 0} \sup_{w \in \partial\Omega} \frac{\hat{\mu}(B(w, \tau r))}{\hat{\mu}(B(w, r))} = \tau^{n-1}.$$

As such we intend to develop estimates, for τ , $0 < \tau \leq 1$, fixed, of $\hat{\mu}(B(w, \tau r))/\hat{\mu}(B(w, r))$ whenever $w \in \partial\Omega$, and r is small enough.

3.1 Preliminary reductions

In the following we let

$$\lambda = \lambda(p, n, \alpha, \beta, \gamma) \gg 1 \text{ be a degree of freedom fixed in (3.9) below.}$$

Given $\lambda \gg 1$ we let $\tilde{r}_1$ be such that $B(w, 16\lambda\tilde{r}_1) \cap D = \emptyset$ whenever $w \in \partial\Omega$. Furthermore, we let $\tilde{r}_2 = \tilde{r}_2(p, n, \alpha, \beta, \gamma)$ be such that Lemma 2.9 holds with $\epsilon = \sigma/(2(p-2))$, σ as in Lemma 2.3, as well as with $\epsilon = 1/p$ whenever $w \in \partial\Omega$ and $r < \tilde{r}_2$. Finally, for $\hat{\tau}$, $0 < \hat{\tau} \leq 1$, fixed we let, using that $\Omega \subset \mathbb{R}^n$ is a Reifenberg flat domain with vanishing constant, $\tilde{r}_3 = \tilde{r}_3(p, n, \alpha, \beta, \gamma, \hat{\tau})$ be such that if $w \in \partial\Omega$ and $r < \tilde{r}_3$, then (1.2) and (1.3) hold with δ such that

$$32\lambda\delta < \hat{\tau}/100. \tag{3.1}$$

We now consider

$$0 < r < \min\{\tilde{r}_3, \tilde{r}_2, \tilde{r}_1, r_0\}. \quad (3.2)$$

With r fixed satisfying (3.2) we see, using Definition 1.1, that there exists a hyperplane Λ , containing w , such that

$$h(\partial\Omega \cap B(w, 8\lambda r), \Lambda \cap B(w, 8\lambda r)) \leq 8\lambda\delta r \quad (3.3)$$

with λ, δ satisfying (3.1). Moreover, using (1.3) we deduce that

$$\{y \in \Omega \cap B(w, 8\lambda\delta r) : d(y, \partial\Omega) \geq 16\lambda\delta r\} \subset \text{one component of } \mathbb{R}^n \setminus \Lambda. \quad (3.4)$$

Furthermore, given $\tilde{\epsilon}$ small we can always choose $\tilde{r} = \tilde{r}(\tilde{\epsilon}) < \min\{\tilde{r}_3, \tilde{r}_2, \tilde{r}_1, r_0\}$ such that if $0 < r < \tilde{r}$ then (3.3), (3.4) hold with δ replaced by $\tilde{\epsilon}$. In the following we assume (3.3) and (3.4). Furthermore, it is easily seen, using Lemma 2.7, that in our context we can assume, without loss of generality, that $w = 0$ and that $\Lambda = \{(y', y_n) : y' \in \mathbb{R}^{n-1}, y_n = 0\}$, i.e., we assume in the following that

$$\begin{aligned} \{y \in \Omega \cap B(0, 8\lambda r) : d(y, \partial\Omega) \geq 16\lambda\delta r\} &\subset \{y \in \mathbb{R}^n : y_n > 0\}, \\ \{y \in (\mathbb{R}^n \setminus \bar{\Omega}) \cap B(0, 8\lambda r) : d(y, \partial\Omega) \geq 16\lambda\delta r\} &\subset \{y \in \mathbb{R}^n : y_n < 0\}. \end{aligned} \quad (3.5)$$

Given the reduction in (3.5) we introduce, for $a, b \in \mathbb{R}^+$ and $z = (z_1, \dots, z_n) \in \mathbb{R}^n$, the cylinders,

$$\begin{aligned} Q_a(z) &= \{y = (y', y_n) : |y' - z'| < a, |y_n - z_n| < a\}, \\ Q_{a,b}^+(z) &= \{y = (y', y_n) : |y' - z'| < a, 0 < y_n - z_n < b\}, \end{aligned}$$

Furthermore, we let $Q_a^+(z) = Q_{a,a}^+(z)$ and if $z = 0$ then we let $Q_a = Q_a(0)$ and $Q_a^+ = Q_a^+(0)$. Using this notation we first note that $Q_{4\lambda r} \subset B(0, 8\lambda r)$ and in the following we restrict our attention to the cube $Q_{4\lambda r}$. Furthermore, we also introduce the notation

$$\tilde{Q}_{k\lambda r}^+ := Q_{k\lambda r, k\lambda r + 40\lambda\delta r}^+(-40\lambda\delta r e_n) \text{ for } k \in \{2, 4\}.$$

We now focus on $\hat{u}$ and we set, given $A \in M_p(\alpha, \beta, \gamma)$, $A_2(y, \eta) = A(y, \eta)$, $A_1(\eta) = A(0, \eta)$. Then $A_1, A_2 \in M_p(\alpha, \beta, \gamma)$. To make estimates we next introduce two auxiliary functions, u_1 and u_2 . In particular, we define u_2 to be the A_2 -harmonic function in $\tilde{Q}_{4\lambda r}^+$ which satisfies $u_2 = \hat{u}$ on $\partial\tilde{Q}_{4\lambda r}^+$. Then, by the maximum principle for A -harmonic functions we see that

$$\hat{u}(y) \leq u_2(y) \quad \text{whenever } y \in \Omega \cap Q_{4\lambda r}. \quad (3.6)$$

Furthermore, we let u_1 be the A_1 -harmonic function in $\tilde{Q}_{2\lambda r}^+$ which has continuous boundary data on $\partial\tilde{Q}_{2\lambda r}^+$ given by $u_1 = u_2$. Note that A_1 does not depend on x , and hence we are in the same setting as in Remark 1.8 when dealing with u_1 and A_1 . Next, using Lemma 2.8 we see that we can choose $\lambda = \lambda(p, n, \alpha, \beta, \gamma) \gg 1$ so that

$$c^{-1} \frac{\hat{u}(y)}{d(y, \partial\Omega)} \leq |\nabla \hat{u}(y)| \leq c \frac{\hat{u}(y)}{d(y, \partial\Omega)} \quad \text{whenever } y \in \Omega \cap Q_{4r}, \quad (3.7)$$

and such that

$$c^{-1} \frac{u_2(y)}{y_n + 40\lambda\delta r} \leq |\nabla u_2(y)| \leq c \frac{u_2(y)}{y_n + 40\lambda\delta r}, \quad (3.8)$$

for some $c = c(p, n, \alpha, \beta, \gamma)$, whenever $y \in \tilde{Q}_{2\lambda r}^+ \cap Q_{4r}$. In particular, in the following we fix

$$\lambda = \lambda(p, n, \alpha, \beta, \gamma) \gg 1 \text{ so that (3.7) and (3.8) hold.} \quad (3.9)$$

Furthermore, using (3.2) and Lemma 2.9 we see that there exists, with $\epsilon = \sigma/(2(p-1))$ or $\epsilon = 1/p$ given, a constant $\hat{c} = \hat{c}(p, n, \alpha, \beta, \gamma)$, $1 \leq \hat{c} < \infty$, such that

$$\hat{c}^{-1} \left(\frac{d(y, \partial\Omega)}{r} \right)^{1+\epsilon} \leq \frac{\hat{u}(y)}{\hat{u}(4re_n)} \leq \hat{c} \left(\frac{d(y, \partial\Omega)}{r} \right)^{1-\epsilon} \quad \text{whenever } y \in \Omega \cap Q_{4r}$$

and such that

$$\hat{c}^{-1} \left(\frac{y_n + 40\lambda\delta r}{r} \right)^{1+\epsilon} \leq \frac{u_2(y)}{u_2(4re_n)} \leq \hat{c} \left(\frac{y_n + 40\lambda\delta r}{r} \right)^{1-\epsilon}, \quad (3.10)$$

whenever $y \in \tilde{Q}_{2\lambda r}^+ \cap Q_{4r}$. To proceed we extend u_1, u_2 to $Q_{2\lambda r} \setminus \tilde{Q}_{2\lambda r}^+$ by defining $u_1 \equiv u_2 \equiv 0$ on $Q_{2\lambda r} \setminus \tilde{Q}_{2\lambda r}^+$. Furthermore, we let μ_1 and μ_2 be the measures associated to u_1 and u_2 , respectively, in the sense of Lemma 2.6, with support in $\partial\tilde{Q}_{2\lambda r}^+ \cap Q_{2\lambda r}$. Note that the support is a subset of the hyperplane $y_n = -40\lambda\delta r$. To prove Theorem 1.8 we first prove the following lemmas.

Lemma 3.1 *Let $\hat{\mu}$ and μ_2 be as above and let $\hat{\tau}$, $0 < \hat{\tau} \leq 1$, be fixed. Consider τ , $\hat{\tau} \leq \tau \leq 1$. Then there exist a constant $c_1 = c_1(p, n, \alpha, \beta, \gamma, \hat{\tau})$, $1 \leq c_1 < \infty$, such that*

$$(1 - c_1 \delta^{\sigma/(8p)}) \leq \frac{\hat{\mu}(Q_{\tau r})}{\mu_2(Q_{\tau r})} \leq (1 + c_1 \delta^{\sigma/(8p)}),$$

where $\sigma \in (0, 1)$ is the Hölder exponent from Lemma 2.3.

Lemma 3.2 *Let μ_1 and μ_2 be as above and let $\hat{\tau}$, $0 < \hat{\tau} \leq 1$, be fixed. Consider τ , $\hat{\tau} \leq \tau \leq 1$. Then there exists constants $c_2 = c_2(p, n, \alpha, \beta, \gamma, \hat{\tau})$, $1 \leq c_2 < \infty$, $\bar{\sigma} = \bar{\sigma}(p, n, \alpha, \beta, \gamma)$, $\bar{\sigma} \in (0, 1)$, such that*

$$(1 - c_2 r^{\bar{\sigma}}) \leq \frac{\mu_1(Q_{\tau r})}{\mu_2(Q_{\tau r})} \leq (1 + c_2 r^{\bar{\sigma}}).$$

3.2 Proof of Lemma 3.1 and Lemma 3.2

Proof of Lemma 3.1. To start the proof we first observe, for $x, \xi_1 \in \mathbb{R}^n \setminus \{0\}, \xi_2 \in \mathbb{R}^n \setminus \{0\}$, that

$$\begin{aligned} A_i(x, \xi_1) - A_i(x, \xi_2) &= \int_0^1 \frac{d}{dt} A_i(x, t\xi_1 + (1-t)\xi_2) dt \\ &= \sum_{j=1}^n ((\xi_1)_j - (\xi_2)_j) \int_0^1 \frac{\partial A_i}{\partial \eta_j}(x, t\xi_1 + (1-t)\xi_2) dt, \end{aligned} \quad (3.11)$$

for $i \in \{1, \dots, n\}$. Notice that (3.11) holds with $\xi_2 = 0$ if we remove the term $A_i(x, 0)$. Hence, using Definition 1.6 and simple estimates we deduce that

$$|A_i(x, \xi_1) - A_i(x, \xi_2)| \leq c(n, \alpha) |\xi_1 - \xi_2| (|\xi_1| + |\xi_2|)^{p-2}.$$

Let in the following $\sigma \in (0, 1)$ be the Hölder exponent from Lemma 2.3. Let $\phi \in C_0^\infty(\mathbb{R}^n)$, $0 \leq \phi \leq 1$, $\phi \equiv 1$ on $Q_{\tau r}$, $\phi \equiv 0$ on $\mathbb{R}^n \setminus Q_{(1+\delta^{\sigma/8})\tau r}$ and such that $|\nabla \phi| \leq c\delta^{-\sigma/8}(\tau r)^{-1}$. Using Lemma 2.6 we see that

$$|\hat{\mu}(Q_{\tau r}) - \mu_2(Q_{\tau r})| \leq \left| \int \langle A(y, \nabla \hat{u}) - A(y, \nabla u_2), \nabla \phi \rangle dy \right| + [\hat{\mu} + \mu_2](Q_{(1+\delta^{\sigma/8})\tau r} \setminus Q_{\tau r}),$$

and hence, using (3.11) together with Definition 1.6 (i), we see that

$$|\hat{\mu}(Q_{\tau r}) - \mu_2(Q_{\tau r})| \leq c\delta^{-\sigma/8}\tau^{-1}(I_1 + I_2 + I_3) + [\hat{\mu} + \mu_2](Q_{(1+\delta^{\sigma/8})\tau r} \setminus Q_{\tau r}),$$

where

$$\begin{aligned} I_1 &= r^{-1} \int_{Q_{4\lambda r}^+(80\lambda\delta re_n) \cap Q_{2r}} |\nabla(\hat{u} - u_2)|(|\nabla \hat{u}| + |\nabla u_2|)^{p-2} dy, \\ I_2 &= r^{-1} \int_{(Q_{4\lambda r}^+(-40\lambda\delta re_n) \setminus Q_{4\lambda r}^+(80\lambda\delta re_n)) \cap Q_{2r} \cap \Omega} (|\nabla \hat{u}|^{p-1} + |\nabla u_2|^{p-1}) dy, \\ I_3 &= r^{-1} \int_{(Q_{4\lambda r}^+(-40\lambda\delta re_n) \cap Q_{2r}) \setminus \Omega} |\nabla u_2|^{p-1} dy. \end{aligned}$$

We claim that there exist constants $\tilde{c} = \tilde{c}(p, n, \alpha, \beta, \gamma)$, $1 \leq \tilde{c} < \infty$, and $\hat{c} = \hat{c}(p, n, \alpha, \beta, \gamma, \hat{\tau})$, $1 \leq \hat{c} < \infty$, such that

$$I_1 + I_2 + I_3 \leq \tilde{c}(\delta^{1-\sigma/2} + \delta^{\sigma/2})\mu_2(Q_r) \quad (3.12)$$

and

$$[\hat{\mu} + \mu_2](Q_{(1+\delta^{\sigma/8})\tau r} \setminus Q_{\tau r}) \leq \hat{c}\delta^{\sigma/(8p)}\mu_2(Q_r). \quad (3.13)$$

In particular, assuming (3.12) and (3.13), we see that there exists $c = c(p, n, \alpha, \beta, \gamma, \hat{\tau})$, $1 \leq c < \infty$, such that

$$|\hat{\mu}(Q_{\tau r}) - \mu_2(Q_{\tau r})| \leq c(\delta^{-\sigma/8}(\delta^{\sigma/2} + \delta^{1-\sigma/2}) + \delta^{\sigma/(8p)})\mu_2(Q_r) \leq c\delta^{\sigma/(8p)}\mu_2(Q_r) \quad (3.14)$$

since $\sigma \in (0, 1)$. Finally, using (3.10), Lemma 2.6 and the Harnack inequality we see that we can replace $\mu_2(Q_r)$ on the right-hand side in (3.14), at the expense of changing c to a new constant having the same dependence as c , by $\mu_2(Q_{\tau r})$. This observation and (3.14) completes the proof of Lemma 3.1 assuming (3.12) and (3.13). The rest of the proof is therefore devoted to proving (3.12) and (3.13). In the following c will be a constant, not necessarily the same at all instances, such that $1 \leq c < \infty$ and depending at most on $p, n, \alpha, \beta, \gamma, \hat{\tau}$. We start by proving (3.12). To estimate I_3 we can use (3.8), (3.10) with $\epsilon = \sigma/(2(p-1))$, Lemma 2.6 and the Harnack inequality for A -harmonic functions in a straightforward calculation to conclude that

$$I_3 \leq cr^{n-p}\delta^{1-\epsilon(p-1)}(u_2(4re_n))^{p-1} \leq c\delta^{1-\sigma/2}\mu_2(Q_r). \quad (3.15)$$

Similarly, using (3.6)-(3.10), Lemma 2.6 and the Harnack inequality for A -harmonic functions we also see that

$$\begin{aligned}
I_2 &\leq cr^{n-p}\delta^{1-\epsilon(p-1)}(\hat{u}(4re_n))^{p-1} + cr^{n-p}\delta^{1-\epsilon(p-1)}(u_2(4re_n))^{p-1} \\
&\leq cr^{n-p}\delta^{1-\epsilon(p-1)}(u_2(4re_n))^{p-1} \leq c\delta^{1-\sigma/2}\mu_2(Q_r).
\end{aligned} \tag{3.16}$$

To estimate I_1 we first use the Schwartz inequality to conclude that

$$\begin{aligned}
I_1 &\leq r^{-1} \left(\int_{Q_{4\lambda r}^+(80\lambda\delta re_n) \cap Q_{2r}} |\nabla(\hat{u} - u_2)|^2 dy \right)^{1/2} \times \\
&\quad \left(\int_{Q_{4\lambda r}^+(80\lambda\delta re_n) \cap Q_{2r}} (|\nabla\hat{u}| + |\nabla u_2|)^{2(p-2)} dy \right)^{1/2}.
\end{aligned} \tag{3.17}$$

Let $y \in Q_{4\lambda r}^+(80\lambda\delta re_n) \cap Q_{2r}$ and assume that $\rho > 0$ is such that $B(y, 2\rho) \subset Q_{4\lambda r}^+(80\lambda\delta re_n)$. Then, using (3.11) and the A_2 -harmonicity of $\hat{u}, u_2$, we deduce that $e = \hat{u} - u_2$ is a weak solution to $L\zeta = 0$ in $B(y, 2\rho)$, where

$$\begin{aligned}
L\zeta(x) &= \sum_{i,j=1}^n \frac{\partial}{\partial x_i} (a_{ij}(x)\zeta_{x_j}), \\
a_{ij}(x) &= \int_0^1 \frac{\partial A_i}{\partial \eta_j}(x, t\nabla\hat{u}(x) + (1-t)\nabla u_2(x)) dt \quad \text{for } 1 \leq i, j \leq n.
\end{aligned}$$

Furthermore, by the structure assumptions on A , see Definition 1.6, we also see that

$$c^{-1}(|\nabla\hat{u}(x)| + |\nabla u_2(x)|)^{p-2} |\xi|^2 \leq \sum_{i,j=1}^n a_{ij}(x) \xi_i \xi_j \leq c(|\nabla\hat{u}(x)| + |\nabla u_2(x)|)^{p-2} |\xi|^2,$$

whenever $x \in B(y, 2\rho)$ and $\xi \in \mathbb{R}^n$. Moreover, using (3.7), (3.8) and the Harnack inequality for A -harmonic functions we can conclude that L is uniformly elliptic in $B(y, 2\rho)$. Hence we can, by standard arguments, conclude that the following Cacciopoli or energy type inequality holds,

$$\int_{B(y,\rho)} |\nabla e(x)|^2 dx \leq c\rho^{-2} \int_{B(y,2\rho)} |e(x)|^2 dx. \tag{3.18}$$

To continue we note, using Theorem 6.44 in [HKM], Lemma 2.3, Lemma 2.4 and Harnack's inequality for A -harmonic functions, that

$$0 < u_2(y) \leq c\delta^\sigma u_2(4re_n) \quad \text{whenever } y \in \partial\Omega \cap Q_{4\lambda r},$$

and hence, by the maximum principle

$$\hat{u}(y) \leq u_2(y) \leq c\delta^\sigma u_2(4re_n) + \hat{u}(y) \quad \text{whenever } y \in \Omega \cap Q_{4\lambda r}. \tag{3.19}$$

In the following we use the preliminaries and statements in (3.11)-(3.19) to estimate I_1 . In particular, we consider a Whitney decomposition of $Q_{4\lambda r}^+$ into cubes of side length $2^{-k}r$, $k \in \mathbb{Z}$, and we let Γ_k

be the set of all Whitney cubes Q with side length $2^{-k}r$, satisfying $Q \cap Q_{4\lambda r}^+(80\lambda\delta re_n) \cap Q_{2r} \neq \emptyset$ and $d(Q, \partial\Omega) \approx 2^{-k}r$. Let k_0 be the largest k such that $\Gamma_k \neq \emptyset$ and let k_1 be the smallest k such that $\Gamma_k \neq \emptyset$. Then $2^{-k_0} \approx \lambda\delta$ and $2^{-k_1} \approx 1$. Note also that if $\Gamma_k \neq \emptyset$ then the number of cubes in Γ_k is of the order $2^{k(n-1)}$. In the following let $p \geq 2$ and let, whenever $Q \in \Gamma_k$ for some k , x_Q and r_Q be, respectively, the center and the side length of the cube Q . Then using, (3.17), (3.18), (3.19), (3.7), (3.8), the fact that $\hat{u} \leq u_2$ in $Q_{4\lambda r}^+(80\lambda\delta re_n) \cap Q_{2r}$ and the Harnack inequality we first see that

$$\begin{aligned} I_1 &\leq cr^{-1} \sum_{k=-\infty}^{k_0} \sum_{Q \in \Gamma_k} \left(\int_Q |\nabla e|^2 dy \right)^{1/2} \left(\int_Q (|\nabla \hat{u}| + |\nabla u_2|)^{2(p-2)} dy \right)^{1/2} \\ &\leq cr^{-1} \sum_{k=-\infty}^{k_0} \sum_{Q \in \Gamma_k} r_Q^{n-1} \delta^\sigma u_2(4re_n) \left(\frac{u_2(x_Q)}{r_Q} \right)^{p-2}. \end{aligned} \quad (3.20)$$

Next, using (3.10), (3.20) and the Harnack inequality we can conclude that

$$I_1 \leq cr^{-1} \sum_{k=-\infty}^{k_0} \sum_{Q \in \Gamma_k} r_Q^{n-p+1} \delta^\sigma u_2(4re_n)^{p-1} \left(\frac{r_Q}{r} \right)^{(1-\epsilon)(p-2)}. \quad (3.21)$$

Next we let $\epsilon = \frac{\sigma}{2(p-2)}$ in (3.21), and use that there are about $2^{k(n-1)}$ cubes in Γ_k , each having side length $2^{-k}r$, together with Lemma 2.6 to conclude that

$$\begin{aligned} I_1 &\leq cr^{-1} \sum_{k=k_1}^{k_0} 2^{k(n-1)} (2^{-k}r)^{n-p+1} \delta^\sigma u_2(4re_n)^{p-1} 2^{-k(1-\epsilon)(p-2)} \\ &= c \sum_{k=k_1}^{k_0} r^{n-p} \delta^\sigma u_2(4re_n)^{p-1} 2^{\epsilon k(p-2)} \\ &\leq cr^{n-p} \delta^{\sigma-\epsilon(p-2)} u_2(4re_n)^{p-1} \leq c \delta^{\sigma-\epsilon(p-2)} \mu_2(Q_r) = c \delta^{\sigma/2} \mu_2(Q_r). \end{aligned} \quad (3.22)$$

A similar argument shows that (3.22) also holds for $1 < p < 2$. In particular, through (3.15), (3.16), (3.22) the proof of (3.12) is complete. We next prove (3.13), i.e., we estimate

$$[\hat{\mu} + \mu_2](Q_{(1+\delta^{\sigma/8})\tau r} \setminus Q_{\tau r}).$$

To proceed we first cover the ring $(Q_{(1+\delta^{\sigma/8})\tau r} \setminus Q_{\tau r}) \cap \partial\Omega$ with k_δ balls, $B(\omega_j, \delta^{\sigma/8}\tau r)$ with center $\omega_j \in (Q_{(1+\delta^{\sigma/8})\tau r} \setminus Q_{\tau r}) \cap \partial\Omega$ for all $j = 1, 2, \dots, k_\delta$. Note that due to the flatness assumption on $\partial\Omega$ we only need to use $k_\delta \approx (\delta^{\sigma/8})^{2-n}$ balls for such a covering and this is in contrast to the general case where one is expected to need the number $\approx \delta^{1-n}$ balls. Using the covering it follows from Lemma 2.6 that

$$\hat{\mu}(Q_{(1+\delta^{\sigma/8})\tau r} \setminus Q_{\tau r}) \leq \sum_{j=0}^{k_\delta} \hat{\mu}(B(\omega_j, \delta^{\sigma/8}\tau r)) \leq c \sum_{j=0}^{k_\delta} (\delta^{\sigma/8}\tau r)^{n-p} \hat{u}(a_{\delta^{\sigma/8}\tau r}(\omega_j))^{p-1}.$$

Using Lemma 2.9 with $\epsilon = 1/p$, the fact that $\hat{u} < u_2$, and the Harnack inequality, we get that

$$\begin{aligned}\hat{\mu}(Q_{(1+\delta^{\sigma/8})\tau r} \setminus Q_{\tau r}) &\leq c \sum_{j=0}^{k_\delta} (\delta^{\sigma/8}\tau r)^{n-p} (\delta^{(\sigma/8)(1-\epsilon)(p-1)}) u_2(a_r(0))^{p-1} \\ &\leq c (\delta^{\sigma/8})^{2-n} (\delta^{\sigma/8}r)^{n-p} (\delta^{(\sigma/8)(1-\epsilon)(p-1)}) u_2(a_r(0))^{p-1} \\ &\leq c \delta^{\frac{\sigma}{8p}} \mu_2(Q_r).\end{aligned}\tag{3.23}$$

We can proceed in a similar fashion to get that

$$\mu_2(Q_{(1+\delta^{\sigma/8})\tau r} \setminus Q_{\tau r}) \leq c \delta^{\frac{\sigma}{8p}} \mu_2(Q_r).\tag{3.24}$$

Putting together (3.23) and (3.24) proves (3.13) and hence the proof of Lemma 3.1 is complete. $\square$

Proof of Lemma 3.2. We again let $\phi \in C_0^\infty(\mathbb{R}^n)$, $0 \leq \phi \leq 1$, $\phi \equiv 1$ on $Q_{\tau r}$, $\phi \equiv 0$ on $\mathbb{R}^n \setminus Q_{(1+(\tau r)^\kappa)\tau r}$ and such that $|\nabla \phi| \leq c(\tau r)^{-1-\kappa}$, where $0 < \kappa < 1$ is to be chosen. Now, using Lemma 2.6 and the definition of $M_p(\alpha, \beta, \gamma)$, we see that

$$\begin{aligned}|\mu_2(Q_{\tau r}) - \mu_1(Q_{\tau r})| &\leq \left| \int \langle A_2(y, \nabla u_2) - A_2(y, \nabla u_1), \nabla \phi \rangle dy \right| \\ &\quad + \left| \int \langle A_2(y, \nabla u_1) - A_1(\nabla u_1), \nabla \phi \rangle dy \right| \\ &\quad + [\mu_1 + \mu_2](Q_{(1+(\tau r)^\kappa)\tau r} \setminus Q_{\tau r}) \\ &\leq c \tau^{-\kappa-1} r^{-\kappa} (J_1 + J_2) + [\mu_1 + \mu_2](Q_{(1+(\tau r)^\kappa)\tau r} \setminus Q_{\tau r}),\end{aligned}\tag{3.25}$$

where

$$\begin{aligned}J_1 &= r^{-1} \int_{\tilde{Q}_{4\lambda r}^+ \cap Q_{2r}} |\nabla(u_2 - u_1)| (|\nabla u_2| + |\nabla u_1|)^{p-2} dy, \\ J_2 &= r^{\gamma-1} \int_{\tilde{Q}_{4\lambda r}^+ \cap Q_{2r}} |\nabla u_1|^{p-1} dy.\end{aligned}$$

We next estimate J_1 and J_2 . To start with, using the Hölder inequality we see that $J_1 \leq r^{-1} J_{11} J_{12}$ where

$$\begin{aligned}J_{11} &= \left(\int_{\tilde{Q}_{4\lambda r}^+ \cap Q_{2r}} |\nabla(u_2 - u_1)|^p dy \right)^{1/p}, \\ J_{12} &= \left(\int_{\tilde{Q}_{4\lambda r}^+ \cap Q_{2r}} (|\nabla u_1| + |\nabla u_2|)^{\frac{(p-2)p}{p-1}} dy \right)^{\frac{p-1}{p}}.\end{aligned}$$

To estimate J_{11} we first note, by construction, that

$$|A_1(y, \eta) - A_2(y, \eta)| \leq c r^\gamma |\eta|^{p-1} \quad \text{whenever} \quad y \in \tilde{Q}_{2\lambda r}^+.\tag{3.26}$$

Moreover, if $y \in \mathbb{R}^n$, $\xi_1 \in \mathbb{R}^n \setminus \{0\}$, $\xi_2 \in \mathbb{R}^n \setminus \{0\}$ and $A \in M_p(\alpha, \beta, \gamma)$, then using (3.11) and the definition of the class $M_p(\alpha, \beta, \gamma)$ we see that

$$c^{-1}(|\xi_1| + |\xi_2|)^{p-2}|\xi_1 - \xi_2|^2 \leq \langle A(y, \xi_1) - A(y, \xi_2), \xi_1 - \xi_2 \rangle \leq c(|\xi_1| + |\xi_2|)^{p-2}|\xi_1 - \xi_2|^2. \quad (3.27)$$

Now, using (3.27) we deduce that

$$J_{11}^p \leq c\hat{J}, \quad \hat{J} := \int_{\tilde{Q}_{2\lambda r}^+} \langle A_1(y, \nabla u_1(y)) - A_1(y, \nabla u_2(y)), \nabla u_1(y) - \nabla u_2(y) \rangle dy, \quad (3.28)$$

whenever $p \geq 2$. Furthermore, if $1 < p < 2$, then we have, using (3.27) and Hölder's inequality, that

$$J_{11}^p \leq c\hat{J}^{p/2} \left(\int_{\tilde{Q}_{2\lambda r}^+} (|\nabla u_1|^p + |\nabla u_2|^p) dy \right)^{1-p/2}, \quad (3.29)$$

where $\hat{J}$ is as in (3.28). As $\nabla \cdot (A_1(y, \nabla u_1(y))) = 0 = \nabla \cdot (A_2(y, \nabla u_2(y)))$ whenever $y \in \tilde{Q}_{2\lambda r}^+$ and as $\theta = u_2 - u_1 \in W_0^{1,p}(\tilde{Q}_{2\lambda r}^+)$ we see, using the definition of $\hat{J}$, that

$$\hat{J} = \int_{\tilde{Q}_{2\lambda r}^+} \langle A_2(y, \nabla u_2(y)) - A_1(y, \nabla u_2(y)), \nabla u_1(y) - \nabla u_2(y) \rangle dy. \quad (3.30)$$

Next, using (3.26), (3.28), (3.30), and Hölder's inequality we can conclude, for $p \geq 2$, that

$$J_{11}^p \leq cr^\gamma \int_{\tilde{Q}_{2\lambda r}^+} (|\nabla u_1|^p + |\nabla u_2|^p) dy.$$

Furthermore, for $1 < p < 2$ we can use (3.29) to similarly find that

$$J_{11}^p \leq cr^{\gamma p/2} \int_{\tilde{Q}_{2\lambda r}^+} (|\nabla u_1|^p + |\nabla u_2|^p) dy.$$

Now, using (3.11) with $\xi_2 = 0$, Young's inequality, and the definition of the class $M_p(\alpha, \beta, \gamma)$ we see that

$$\begin{aligned} \int_{\tilde{Q}_{2\lambda r}^+} |\nabla u_1|^p dy &\leq c \int_{\tilde{Q}_{2\lambda r}^+} \langle A_1(y, \nabla u_1(y)), \nabla u_2(y) \rangle dy \\ &\leq \frac{1}{2} \int_{\tilde{Q}_{2\lambda r}^+} |\nabla u_1|^p dy + c \int_{\tilde{Q}_{2\lambda r}^+} |\nabla u_2|^p dy. \end{aligned}$$

Thus,

$$\int_{\tilde{Q}_{2\lambda r}^+} |\nabla u_1|^p dy \leq c \int_{\tilde{Q}_{2\lambda r}^+} |\nabla u_2|^p dy. \quad (3.31)$$

Let $a = \min\{1, p/2\}$, then using the above calculations and the same arguments as in (3.16) we see that

$$J_{11} \leq cr^{\gamma a/p} \left(\int_{\tilde{Q}_{2\lambda r}^+} |\nabla u_2|^p dy \right)^{1/p} \leq cr^{\gamma a/p} r^{\frac{n}{p}-1} u_2(4re_n). \quad (3.32)$$

We next bound J_{12} . Using Hölder's inequality as well as the arguments above we see that

$$\begin{aligned} J_{12} &\leq cr^{n\left(\frac{p-1}{p}-\frac{p-2}{p}\right)} \left(\int_{\tilde{Q}_{2\lambda r}^+} (|\nabla u_1| + |\nabla u_2|)^p dy \right)^{\frac{p-2}{p}} \leq cr^{n/p} \left(\int_{\tilde{Q}_{2\lambda r}^+} |\nabla u_2|^p dy \right)^{\frac{p-2}{p}} \\ &\leq cr^{n/p + (\frac{n}{p}-1)(p-2)} (u_2(4re_n))^{p-2}. \end{aligned} \quad (3.33)$$

As $J_1 \leq r^{-1} J_{11} J_{12}$ we see, by combining (3.32), (3.33) and Lemma 2.6, that

$$J_1 \leq r^{-1} J_{11} J_{12} \leq cr^{\gamma a/p} r^{n-p} u_2(4re_n)^{p-1} \leq cr^{\gamma a/p} \mu_2(Q_r).$$

Furthermore, using Hölder's inequality, (3.31) and the same arguments as in (3.32) we can also conclude that

$$J_2 \leq cr^{\gamma} \mu_2(Q_r),$$

hence

$$r^{-\kappa} (J_1 + J_2) \leq c(r^{\gamma a/p} + r^{\gamma}) r^{-\kappa} \mu_2(Q_r). \quad (3.34)$$

Before we can complete the proof of the lemma we also need to estimate

$$[\mu_1 + \mu_2](Q_{(1+(\tau r)^{\kappa})\tau r} \setminus Q_{\tau r}).$$

To do this we first note from the maximum principle that $u_1(a_r(0)) \leq u_1(\tilde{y})$ for some $\tilde{y} \in \partial \tilde{Q}_{2\lambda r}^+$ secondly by construction, Lemma 6.44 [HKM] and Harnack's inequality we conclude that $u_1(\tilde{y}) = u_2(\tilde{y}) \leq cu_2(a_r(0))$. Hence, $u_1(a_r(0)) \leq cu_2(a_r(0))$ and we can proceed as in the proof of Lemma 3.1 to get

$$\mu_1(Q_{(1+(\tau r)^{\kappa})\tau r} \setminus Q_{\tau r}) \leq cr^{\kappa/p} \mu_2(Q_r), \quad (3.35)$$

and

$$\mu_2(Q_{(1+(\tau r)^{\kappa})\tau r} \setminus Q_{\tau r}) \leq cr^{\kappa/p} \mu_2(Q_r). \quad (3.36)$$

Hence, using (3.25), (3.34), (3.35) and (3.36) we can conclude that there exists $c = c(p, n, \alpha, \beta, \gamma, \hat{\tau})$, $1 \leq c < \infty$, such that

$$|\mu_1(Q_{\tau r}) - \mu_2(Q_{\tau r})| \leq c(r^{\gamma a/p - \kappa} + r^{\gamma - \kappa} + r^{\kappa/p}) \mu_2(Q_r).$$

If we now let $\kappa = \frac{\gamma}{4p}$ then

$$|\mu_1(Q_{\tau r}) - \mu_2(Q_{\tau r})| \leq cr^{\bar{\sigma}} \mu_2(Q_r), \quad (3.37)$$

where $\bar{\sigma} = \min\{\gamma a/p - \kappa, \gamma - \kappa, \kappa/p\}$, $\bar{\sigma} > 0$. As in the proof of Lemma 3.1 we see, using (3.10), Lemma 2.6 and the Harnack inequality, that we can replace $\mu_2(Q_r)$ on the right-hand side in (3.37), at the expense of changing c to a new constant having the same dependence as c , by $\mu_2(Q_{\tau r})$. This observation completes the proof of Lemma 3.2. $\square$

3.3 Proof of Theorem 1.8

We next use Lemma 3.1 and Lemma 3.2 to complete the proof of Theorem 1.8. In particular, we again let $\hat{\mu}$ be as above and we let $\hat{\tau}$, $0 < \hat{\tau} \leq 1$, be fixed. We consider τ , $\hat{\tau} \leq \tau \leq 1$. To proceed we first note, using (3.1) and Reifenberg flatness, that

$$\frac{\hat{\mu}(Q_{(1-c\delta)\tau r})}{\hat{\mu}(Q_r)} \leq \frac{\hat{\mu}(B(0, \tau r))}{\hat{\mu}(B(0, r))} \leq \frac{\hat{\mu}(Q_{\tau r})}{\hat{\mu}(Q_{(1-c\delta)r})}$$

for some constant $c > 0$. Furthermore,

$$\frac{\hat{\mu}(Q_{\tau r})}{\hat{\mu}(Q_{(1-c\delta)r})} = \frac{\hat{\mu}(Q_{\tau r})}{\mu_2(Q_{\tau r})} \frac{\mu_2(Q_{\tau r})}{\mu_1(Q_{\tau r})} \frac{\mu_1(Q_{\tau r})}{\mu_1(Q_{(1-c\delta)r})} \frac{\mu_1(Q_{(1-c\delta)r})}{\mu_2(Q_{(1-c\delta)r})} \frac{\mu_2(Q_{(1-c\delta)r})}{\hat{\mu}(Q_{(1-c\delta)r})},$$

and

$$\frac{\hat{\mu}(Q_{(1-c\delta)\tau r})}{\hat{\mu}(Q_r)} = \frac{\mu_2(Q_r)}{\hat{\mu}(Q_r)} \frac{\mu_1(Q_r)}{\mu_2(Q_r)} \frac{\mu_1(Q_{(1-c\delta)\tau r})}{\mu_1(Q_r)} \frac{\mu_2(Q_{(1-c\delta)\tau r})}{\mu_1(Q_{(1-c\delta)\tau r})} \frac{\hat{\mu}(Q_{(1-c\delta)\tau r})}{\mu_2(Q_{(1-c\delta)\tau r})}.$$

Hence, using Lemma 3.1 and Lemma 3.2 we see that

$$\frac{\hat{\mu}(Q_{\tau r})}{\hat{\mu}(Q_{(1-c\delta)r})} \leq (1 + c_1 \delta^{\sigma/(8p)})(1 - c_2 r^{\bar{\sigma}})^{-1}(1 + c_2 r^{\bar{\sigma}})(1 - c_1 \delta^{\sigma/(8p)})^{-1} \frac{\mu_1(Q_{\tau r})}{\mu_1(Q_{(1-c\delta)r})},$$

and

$$\frac{\hat{\mu}(Q_{(1-c\delta)\tau r})}{\hat{\mu}(Q_r)} \geq (1 - c_1 \delta^{\sigma/(8p)})^{-1}(1 - c_2 r^{\bar{\sigma}})(1 + c_2 r^{\bar{\sigma}})^{-1}(1 - c_1 \delta^{\sigma/(8p)}) \frac{\mu_1(Q_{(1-c\delta)\tau r})}{\mu_1(Q_r)},$$

where $c_1 = c_1(p, n, \alpha, \beta, \gamma, \hat{\tau})$, $1 \leq c_1 < \infty$, $c_2 = c_2(p, n, \alpha, \beta, \gamma, \hat{\tau})$, $1 \leq c_2 < \infty$, $\bar{\sigma} = \bar{\sigma}(p, n, \alpha, \beta, \gamma)$, $\bar{\sigma} \in (0, 1)$, and $\sigma = \sigma(p, n, \alpha, \beta, \gamma)$, $\sigma \in (0, 1)$. We let

$$\begin{aligned} F_1(r, \delta) &= (1 + c_1 \delta^{\sigma/(8p)})(1 - c_2 r^{\bar{\sigma}})^{-1}(1 + c_2 r^{\bar{\sigma}})(1 - c_1 \delta^{\sigma/(8p)})^{-1}, \\ F_2(r, \delta) &= (1 - c_1 \delta^{\sigma/(8p)})^{-1}(1 - c_2 r^{\bar{\sigma}})(1 + c_2 r^{\bar{\sigma}})^{-1}(1 - c_1 \delta^{\sigma/(8p)}). \end{aligned}$$

Using this notation and combining the estimates above we can conclude that

$$F_2(r, \delta) \frac{\mu_1(Q_{(1-c\delta)\tau r})}{\mu_1(Q_r)} \leq \frac{\hat{\mu}(B(0, \tau r))}{\hat{\mu}(B(0, r))} \leq F_1(r, \delta) \frac{\mu_1(Q_{\tau r})}{\mu_1(Q_{(1-c\delta)r})}.$$

Next, since we are in the same setting as in Remark 1.8, we can argue in the same way as in the introduction to obtain, for some $c = c(p, n, \alpha)$ and $\sigma = \sigma(p, n, \alpha) \in (0, 1)$, that

$$\frac{\mu_1(Q_{\tau r})}{\mu_1(Q_{(1-c\delta)r})} \leq \frac{e^{c(p-1)(\tau r)^\sigma} (\tau r)^{n-1}}{e^{-c(p-1)((1-c\delta)r)^\sigma} ((1-c\delta)r)^{n-1}}, \quad (3.38)$$

$$\frac{\mu_1(Q_{(1-c\delta)\tau r})}{\mu_1(Q_r)} \geq \frac{e^{-c(p-1)((1-c\delta)\tau r)^\sigma} ((1-c\delta)\tau r)^{n-1}}{e^{c(p-1)r^\sigma} r^{n-1}}. \quad (3.39)$$

Hence we can conclude, also using the remark following (3.4), that

$$F_2(0, \epsilon)((1 - c\epsilon)\tau)^{n-1} \leq \lim_{r \rightarrow 0} \frac{\hat{\mu}(B(0, \tau r))}{\hat{\mu}(B(0, r))} \leq F_1(0, \epsilon)(\tau/(1 - c\epsilon))^{n-1}, \quad (3.40)$$

for $\epsilon > 0$ arbitrary. In particular, letting $\epsilon \rightarrow 0$ in (3.40) concludes the proof of Theorem 1.8. $\square$

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