Estimation of Interregional Empty Rail Freight Car Flows

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Abstract: An important and to often neglected aspect in every freight transportation system is the presence of empty car flows. Empty car flows arise due to variations in the transportation demand, in the sense that in bound and out bound transportation flows differ with respect to volume or weight. This heterogeneity manifest variations in loaded as well as empty freight flows as a result of different branch structures and trade patterns between regions. Since a train or block generally is composed of both loaded and empty cars the empty ones contribute to a significant congestion effect in the freight transportation network. Thus it is motivated to explicitly consider empty freight car flows in tactical and strategic network flow models. This paper presents a method to estimate the empty car freight flows; the method has been tested on data from the Swedish freight railway system. We provide an estimation procedure for the static deterministic case, cast as a mathematical programming problem. Our approach enhances the modelling capabilities compared to existing models through the possibility to include expertise knowledge. The estimated empty car flow may serve as an input in a column generation model that solves a multimodal multicommodity network flow problem.

Keywords: Transportation Problems, Network Flows, Nonlinear Programming

Classification (JEL): C61, C63, R41
Preface

This report is the result of a research project conducted at the Center for Regional Science, CERUM, at Umeå University. A principal purpose of this research has been to propose a flexible and viable numerical method to estimate and update input data in the form of OD-matrices, which has been cast as a quadratic programming problem. These matrices in turn serve as input to a traffic assignment model. The proposed procedure allows incorporation of a priori data such as expertise knowledge, partial survey information, etc. In this study Robert Sörensson have both described and employed the proposed method for the purpose of estimating empty rail freight car flows. Finally, the financial support from VINNOVA under grant Dnr. 2001/03116 is hereby gratefully acknowledged.

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1. Introduction

An important but often neglected aspect of every freight transportation system is the presence of empty car flows. Empty car flows arise due to variations in the transportation demand, in the sense that inbound and outbound transportation flows, for a particular node differs with respect to volume or weight. Furthermore, the fact that different commodity groups require different car types which usually are not perfect substitutes, is a prime cause of empty car flows. This heterogeneity in car types induces a divergence imbalance at some nodes, and hence the empty cars need to be relocated when the car reaches their final destination. These manifested variations in loaded freight flows as well as empty freight flows are a reflection of different branch structures and trade patterns between regions. Since a train or block generally is composed of both loaded and empty cars the empty ones contribute to a significant congestion effects in the freight transportation network. Thus, it is motivated to explicitly consider empty freight car flows in tactical and strategic network flow models.

With a few exceptions, e.g. [CFL90] and [GlS85], most strategic network flow models has paid scant attention to empty car freight flows. For a review of the problem, the reader is refered to [DeC87]. This paper propose a method to estimate empty car freight flows. We provide an estimation procedure for the static deterministic case, cast as a mathematical programming problem. The proposed method amounts to give an integrated approach to the problem of simultaneously modelling loaded and empty freight vehicle movements. To this end the estimated empty car flow serves an input in a column generation method that solves a multimode multicommodity network flow problem.

The rest of this paper is organized as follows. In the next section we present a procedure to form an origin-destination matrix, which encompasses estimation of supply and demand, and the matching of those. Section three deal with a numerical method to solve the aforementioned problem as well a proof of convergence and rate of convergence for the method. In section four we apply the method from the previous section to a data set and perform some tests to indicate its ability. We conclude the paper with some final remarks.

2. Estimation procedure

In this section we present a procedure for the construction of an origin-destination matrix for empty freight car flows. We start by introducing some necessary no-
Consider a directed network $\mathcal{G} = (\mathcal{N}, \mathcal{A})$ where $\mathcal{N}$ is the set of nodes and $\mathcal{A}$ is the set of directed arcs. The nodes $n, n \in \mathcal{N}$, represent origins, destinations, centroids and intersections; the directed arcs $a, a \in \mathcal{A}$, represent the available transportation facilities in the network. The flows on link $a$ is denoted by $v_a$, if two or more commodities are associated with some link flow, then an additional super index $k$ will be used to denote the individual commodity flows, i.e. $v_a^k$. Flows between origin destination pairs $(i, j) \in \mathcal{C} \subseteq \mathcal{N} \times \mathcal{N}$, are denoted $g_{ij}$.

### 2.1. Supply and demand estimation

The supply and demand of empty freight cars is estimated as the difference between the number of outbound and inbound cars for each centroid (node). A positive (negative) divergence indicate a supply (demand) node. Depending on the information available, there are two possible ways to estimate supply and demand based on this approach.

In the first case the number of transported loaded cars or the transported volume is either known from observed freight flows or from a previous transportation assignment. Then the supply at the origin is given by

$$S_i = \max \left\{ 0, \sum_{k \in K} \sum_{a \in A_i^-} \frac{v_a^k}{w_k} - \sum_{k \in K} \sum_{a \in A_i^+} \frac{v_a^k}{w_k} \right\}, \quad \forall i \in \mathcal{I}, \tag{2.1}$$

and the demand at the destination is given by

$$D_j = \max \left\{ 0, \sum_{k \in K} \sum_{a \in A_j^-} \frac{v_a^k}{w_k} - \sum_{k \in K} \sum_{a \in A_j^+} \frac{v_a^k}{w_k} \right\}, \quad \forall j \in \mathcal{J}, \tag{2.2}$$

were $A^-$ and $A^+$ denote inbound and outbound connectors respectively and finally $w_k$ denote the loading factor for a freight car.

In the second case, supply and demand at the centroids is the only available information, were supply at the origin is given by

$$S_i = \max \left\{ 0, \sum_{k \in K} g_{ij}^k \frac{w_k}{w_k} - \sum_{k \in K} \sum_{i \in \mathcal{I}} g_{ij}^k \frac{w_k}{w_k} \right\}, \quad \forall i \in \mathcal{I}, \tag{2.3}$$

and the demand at the destination is given by

$$D_j = \max \left\{ 0, \sum_{k \in K} g_{ij}^k \frac{w_k}{w_k} - \sum_{k \in K} \sum_{j \in \mathcal{J}} g_{ij}^k \frac{w_k}{w_k} \right\}, \quad \forall j \in \mathcal{J}. \tag{2.4}$$
With this result at hand we are in a position to proceed to the next step which amounts to assign the outbound supplies and the inbound demands in order to get an origin-destination matrix.

2.1.1. On the selection of cost structure in the assignment

Since we want to match spatially distinct supply nodes and demand nodes two obvious candidates present themselves. First and probably the most obvious candidate is a distance dependent cost function. The virtue of such a choice is its simplicity, although it might be ambiguous as distance is but one of several components which influence the allocation decision. Yet an other possible candidate though a bit artificial is a cost function which is dependent on past observed flows, i.e. constructed as the reciprocal of past observed flows. A main advantage of such a cost function lies in its capability to perform forecasts of empty car flows, as it reflects a previous behavior in allocation decisions and since we in such a situation seeks an on average car distribution anyhow.

Obviously none of those candidates are unobjectionable, since one might expect a considerable amount of flows between spatially adjacent locations when a distance dependent cost function is used. On the other hand, a previous flow based cost function has a drawback in that it lacks the possibility to account for new OD relations in a forecast setting, which isn’t the situation at hand in the distance dependent case.

3. A Numerical Method for the Estimation problem

Although it’s perfectly clear that this kind of problem could be solved by a number of nonlinear programming algorithms, there exist however a number of more efficient special purpose algorithms for those problems. In the following subsections we review one such method tailored for this problem.

This section presents the separable strictly convex quadratic programming problem. More precisely, we consider the problem:

\[
\text{Minimize } \frac{1}{2} \left[ \sum_i \alpha_i \left( s_i - s_i^0 \right)^2 + \sum_{ij} \gamma_{ij} \left( x_{ij} - x_{ij}^0 \right)^2 + \sum_j \beta_j \left( d_j - d_j^0 \right)^2 \right], \tag{3.1}
\]

subject to
\[
\sum_{j} x_{ij} = s_i, \quad i = 1, ..., m, \quad (3.2)
\]
\[
\sum_{i} x_{ij} = d_j, \quad j = 1, ..., n; \quad (3.3)
\]

and to the inequality constraints
\[
0 \leq x_{ij} \leq u_{ij}, \quad \text{for } i = 1, ..., m; \quad j = 1, ..., n. \quad (3.4)
\]

The terms \(\alpha_i, \beta_j, \gamma_{ij}\) are > 0; \(u_{ij} > 0\); these terms, as well as \(s^0_i, d^0_j\), and \(x^0_{ij}\), are given for all \(i\) and \(j\).

Incorporating the equality constraints directly into the objective function and removing the constant terms, i.e. terms that don’t depend on \(x_{ij}\), the problem may be re-expressed as:

\[
\text{Minimize } \sum_{i} \left[ \frac{1}{2} \alpha_i \left( \sum_{j} x_{ij} \right)^2 - \alpha_i s^0_i \left( \sum_{j} x_{ij} \right) \right] + \sum_{ij} \left( \frac{1}{2} \gamma_{ij} x_{ij}^2 - \gamma_{ij} x^0_{ij} x_{ij} \right) + \\
\sum_{j} \left[ \frac{1}{2} \beta_j \left( \sum_{i} x_{ij} \right)^2 - \beta_j d^0_j \left( \sum_{i} x_{ij} \right) \right], \quad (3.5)
\]

subject to the constraints in (3.4). The optimality conditions can be stated, in turn, using equations (3.2) and (3.3) as:

For all \(i = 1, ..., m\) and \(j = 1, ..., n\);

\[
\alpha_i s_i - \alpha_i s^0_i + \gamma_{ij} x_{ij} - \gamma_{ij} x^0_{ij} \left\{ \begin{array}{lcl}
\leq -\beta_j d_j + \beta_j d^0_j, & \text{if } x_{ij} = u_{ij} \\
= \rho_{ij} \equiv -\beta_j d_j + \beta_j d^0_j, & \text{if } 0 < x_{ij} < u_{ij} \\
\geq -\beta_j d_j + \beta_j d^0_j, & \text{if } x_{ij} = 0.
\end{array} \right. \quad (3.6)
\]

In this particular problem, the row and column totals \(s_i\) and \(d_j\) are not known a priori and hence need to be estimated. Here, \(s^0_i\) denotes a given row total for row \(i\), and \(d^0_j\) denotes a given column total for column \(j\), and \(x^0_{ij}\) denotes a given matrix element in row \(i\) and column \(j\) of the \(m \times n\) matrix; and \(\alpha_i, \beta_j\) and \(\gamma_{ij}\), represents the respective weights. The aim is to estimate the row and column totals \(s_i\) and \(d_j\), and the matrix entries, \(x_{ij}\), so that the estimates is near in an euclidean sense, to the given values \(s^0_i, d^0_j, \) and \(x^0_{ij}\) respectively,
subject to constraints (3.2), (3.3), and (3.4). Inequalities in (3.4) permit the user to model the problem using upper bounds on the individual matrix entries. In the absence of upper bounds $u_{ij}$ imposed on the transaction estimates $x_{ij}$ the above model collapses to the exact demand equilibration procedure or the exact supply equilibration procedure presented in Dafermos and Nagurney [DaN89] and a constrained matrix problem with transportation type constraints in Nagurney and Eydeland [NaE92a]. Constraints (3.2) and (3.3) require that the estimated individual matrix elements sum to row and column totals.

Besides the possibility to impose upper bounds an other particular attractive feature is the flexible choice of weights in this formulation. For instance, if the weights are all equal to one, equation (3.1) results in a constrained least-squares problem, and if $\alpha_i = s_i^{0-1}$, $\beta_j = d_j^{0-1}$, and $\gamma_{ij} = x_{ij}^{0-1}$, for all $i$ and $j$, the objective function is the chi-square. Other possible weights include $\alpha_i = s_i^{0-1/2}$, $\beta_j = d_j^{0-1/2}$, and $\gamma_{ij} = x_{ij}^{0-1/2}$, which might be used if the initial estimates are known to be more likely to change; or $\alpha_i = s_i^{0-2}$, $\beta_j = d_j^{0-2}$, and $\gamma_{ij} = x_{ij}^{0-2}$, can on the other hand be used if we assume that the initial estimates of greater magnitude are less subject to change relative to those of smaller magnitude; or some suitable mixed weighting scheme might be applied.

3.1. A Gradient Projection Algorithm

In this subsection we present an algorithm proposed by Nagurney [Nag89] for computing a solution to the problem described in the previous section. The algorithm is of the relaxation type, that is, it solves the whole system by solving successively a restricted subproblem. At each step the restricted subproblem can be solved in closed form. We first present the outer loop of this algorithm and then the inner loop which is an exact procedure for the solution of each restricted subproblem.

The algorithm computes a sequence of feasible shipment patterns $[x^0_{ij}]$, $[x^1_{ij}]$, etc., which converges to a solution of (3.5) and which also satisfies conditions (3.6), via the following steps:

The Outer Loop

Step 0 Start with an arbitrary feasible shipment pattern $[x^0_{ij}]$, that is one that satisfies (3.4), i.e. $0 \leq x_{ij} \leq u_{ij}$.

Step $\tau$ ($\tau = 1, 2, \ldots$) Starting from the feasible shipment pattern $[x^{\tau-1}_{ij}]$ computed at step $\tau - 1$, and letting $l = \tau (\text{mod} n)$ (which is just a cyclic selection of the next
l), construct a new feasible shipment pattern $[x_{ij}^*]$, by modifying $x_{ij}^{r-1}$ in such a way that the equilibrium conditions (3.6) are satisfied for $j = l$.

To be more specific, let $[x_{ij}']$ denote the given feasible shipment pattern which has to be modified into a new feasible shipment pattern $[x_{ij}]$ so as to satisfy (3.6) for $j = l$. The new feasible shipment pattern will be obtained as follows. Fix all shipment patterns at their previous values except for the current $l$, i.e.,

$$x_{ij} = x_{ij}', \quad j \neq l; \quad i = 1, \ldots, m;$$

(3.7)

and then modify only $x_{il}$ for $i = 1, \ldots, m$ so as to satisfy equilibrium condition (3.6) for $j = l$. By utilizing constraints (3.2) and (3.3), condition (3.6) can be reexpressed as:

$$\alpha_i \left( x_{il} + \sum_{j \neq l} x_{ij}' \right) - \alpha_i s_i^0 + \gamma_i x_{il} - \gamma_i x_{il}' \begin{cases} \leq -\beta_i d_i + \beta_i d_i^0, & \text{if } x_{il} = u_{il} \\ = \rho_i \equiv -\beta_i d_i + \beta_i d_i^0, & \text{if } 0 < x_{il} < u_{il} \\ \geq -\beta_i d_i + \beta_i d_i^0, & \text{if } x_{il} = 0, \end{cases}$$

(3.8)

for $i = 1, \ldots, m$.

By use of the notation

$$g_i \equiv \alpha_i + \gamma_i,$$

(3.9)

and

$$h_{il} \equiv \alpha_i \sum_{j \neq l} x_{ij}' - \alpha_i s_i^0 - \gamma_i x_{il}'$$

(3.10)

we are in a position to rewrite condition (3.8) as:

$$g_1 x_1 + h_{il} \leq g_2 x_2 + h_{2l} \leq \ldots \leq g_t x_t + h_{tl} \leq g_{t+1} x_{t+1,l} + h_{t+1,l}$$

$$= \ldots = g_s x_{sl} + h_{sl} \leq g_{s+1} x_{s+1,l} + h_{s+1,l} \leq \ldots \leq g_m x_m + h_{ml}$$

(3.11)

where

$$\begin{align*}
x_{il} &= u_{il}, \quad i = 1, \ldots, t \\
0 < x_{il} < u_{il}, \quad i = t + 1, \ldots, s \\
x_{il} &= 0, \quad i = s + 1, \ldots, m.
\end{align*}$$

(3.12)
Note that condition (3.11) might be interpreted as a non decreasing ranking of supply market prices plus transshipment costs in quantity terms. Along the same line, we might analogously interpret the right hand side of (3.8) as the price at demand market l, in quantity terms.

We proceed to derive explicitly the value of \( \frac{1}{2} l \) defined in (3.8) in terms of known entities. We will then use the derived expression to construct an exact procedure for the solution of expression (3.11) and (3.12). From (3.3) and (3.12) one obtains

\[
\sum_{i=t+1}^{s} x_{id} = d_l - \sum_{i=1}^{t} u_{id}, \quad (3.13)
\]

and from (3.8), (3.9) and (3.10),

\[
\sum_{i=t+1}^{s} x_{id} = \sum_{i=t+1}^{s} \frac{(\rho_l - h_{id})}{g_i}. \quad (3.14)
\]

Equating the right hand side of equations (3.13) and (3.14), and solve for \( \rho_l \), utilizing the definition of \( \frac{1}{2} l \) in (3.8), yields

\[
\rho_l = -\beta_l d_l + \beta_l d_l^0 = \left( d_l - \sum_{i=1}^{t} u_{id} + \sum_{i=t+1}^{s} \frac{h_{id}}{g_i} \right) / \sum_{i=t+1}^{s} \frac{1}{g_i}. \quad (3.15)
\]

Solving equation (3.15) for \( d_l \), one obtains

\[
d_l = \left( -\sum_{i=t+1}^{s} \frac{h_{id}}{g_i} + \sum_{i=1}^{t} u_{id} + \beta_l d_l^0 \sum_{i=t+1}^{s} \frac{1}{g_i} \right) / \left( 1 + \beta_l \sum_{i=t+1}^{s} \frac{1}{g_i} \right). \quad (3.16)
\]

If we substitutes the expression for \( d_l \) in equation (3.16) into the left hand side of equation (3.15), it follows that

\[
\rho_l = \left( d_l^0 - \sum_{i=1}^{t} u_{id} + \sum_{i=t+1}^{s} \frac{h_{id}}{g_i} \right) / \left( \frac{1}{\beta_l} + \sum_{i=t+1}^{s} \frac{1}{g_i} \right). \quad (3.17)
\]

Provided that the critical values \( t \) and \( s \) are known then \( x_{id} \) can be calculated via

\[
\begin{align*}
x_{id} &= u_{id}, \quad i = 1, \ldots, t \\
x_{id} &= \frac{\rho_l - h_{id}}{g_i}, \quad i = t + 1, \ldots, s \\
x_{id} &= 0, \quad i = s + 1, \ldots, m.
\end{align*}
\]
The following procedure, which we call the inner loop, provide the critical values $t$ and $s$ for the exact solution of expressions (3.11) and (3.12) by means of (3.17) and (3.18). This procedure is applied at each step $\tau$ of the previously described outer loop. It might briefly be depicted as a sort, search and projection procedure, hence it’s algorithmic classification.

The Inner Loop

**Step 0** (Initialization) Sort the components $h_{il}$ (for $i = 1, ..., m$) in non-decreasing order and relabel the $h_{il}$ and $x_{il}$ accordingly. We assume henceforth that $h_{1l} \leq h_{2l} \leq ... \leq h_{ml}$. Let $h_{m+1,l} = \infty$, $\mathcal{M} = \{1, ..., m + 1\}$, and $\mathcal{H} = \emptyset$. Let $\mathcal{K} = \mathcal{M} \backslash \mathcal{H} = \{i_1, ..., i_{K+1}\}$, where $i_1 \leq ... \leq i_K \leq i_{K+1} = m + 1$. $\mathcal{H}$ denotes the set of elements consisting of indices of the $x_{il}$ variable which have been set to there upper bound values; $\mathcal{K}$ is the set of indices which have not been set to there upper bound. If $\beta_l d_l^0 < h_{1l}$, stop; all $x_{il} = 0$, $i = 1, ..., m$, else let $\bar{I} = 1$.

**Step 1** Define

$$
\rho_\tau \equiv \left( d_l^0 - \sum_{h \in \mathcal{H}} u_{hl} + \sum_{k=1}^{\bar{I}} h_{ikl} \right) / \left( \frac{1}{\beta_l} + \sum_{k=1}^{*} \frac{1}{g_{ik}} \right). \tag{3.19}
$$

**Step 2** If $\rho_\tau \in [h_{1\tau}, h_{\tau+1,l}]$ go to step 3. Otherwise, replace $\bar{I}$ by $\bar{I} + 1$, and go to step 1.

**Step 3** For $k = 1, ..., K$, let

$$
x_{il} = \begin{cases} 
\frac{\rho_l - h_{ikl}}{g_{ik}}, & k = 1, ..., \bar{I} \\
0, & k = \bar{I} + 1, ..., K.
\end{cases} \tag{3.20}
$$

**Step 4** For $k = 1, ..., \bar{I}$, if $x_{ikl} > u_{hl}$, redefine $x_{ikl}$ to be equal to $u_{hl}$, and transfer $i_k$ from $\mathcal{K}$ to $\mathcal{M}$. Let $\bar{I} = 1$ and return to step 1. Otherwise, stop.

### 3.2. Convergence of the Projection Algorithm

We now turn our attention to the issue of convergence and introduce the following assumptions on $F$ as well as an auxiliary proposition about contraction mappings which will be useful in the sequel. Some general results regarding contraction mappings, which we in due course will make use of, are relegated to the appendix.

**Assumption 3.1.**
There holds $F(x) \geq 0$ for all $x \in X$.

(b) \textit{(Lipschitz Continuity of $\nabla F$)} The function $F$ is continuously differentiable and there exist a constant $K$ such that

$$\| \nabla F(x) - \nabla F(y) \| \leq K \| x - y \|, \quad \forall x, y \in X. \tag{3.21}$$

The following result are based on a particular choice of norms, namely weighted quadratic norms. Consider a mapping $T : X \mapsto \mathbb{R}^n$ given by

$$T(x) = x - \gamma G^{-1} f(x), \quad \forall x \in X, \tag{3.22}$$

where $G$ is a symmetric positive definite matrix. The purpose of $G$ in (3.22) is to scale the direction in which $x$ is changed when the contraction mapping $T$ is applied. Thus, it makes sense to consider a norm that scales the components $x$ in a corresponding fashion. To this end we introduce the induced norm $\| \cdot \|_G$ defined by

$$\| x \| = (x' G x)^{1/2}$$

and determine under what conditions $T$ is a contraction with respect to $\| \cdot \|_G$. We make use of the weighted quadratic norm in the mapping (3.22) which yields

$$\| T(x) - T(y) \|_G^2 = \left( (x - y) - \gamma G^{-1} (f(x) - f(y)) \right)' G \left( (x - y) - \gamma G^{-1} (f(x) - f(y)) \right)$$

$$= \| x - y \|_G^2 + \gamma^2 (f(x) - f(y))' G^{-1} (f(x) - f(y)) - 2 \gamma (f(x) - f(y))' (x - y). \tag{3.23}$$

If $\gamma$ is chosen sufficiently small and the norm of $f(x) - f(y)$ is of the order of $\| x - y \|_G$, then the second last term involving $\gamma^2$ can be neglected. Hence, for $T$ to be a contraction, it is sufficient to assume that $\gamma$ is positive and small enough, and that

$$(f(x) - f(y))' (x - y) \geq \alpha \| x - y \|_G^2, \quad \forall x, y \in X, \tag{3.24}$$

where $\alpha$ is some positive constant. Inequality (3.24) is called a strong monotonicity condition and its significance will be evident once we are ready to determine the convergence and rate of convergence.

In the more general case $X$ is given as a Cartesian product set according to $X = \prod_{i=1}^m X_i \subset \prod_{i=1}^m \mathbb{R}^{n_i}$, and where $T$ is given by $T_i(x) = x_i - \gamma G_i^{-1} f_i(x)$ for each $i$. We assume that the matrix $G_i$ is symmetric and positive definite. For each $i$, we define the norm $\| \cdot \|_i$ on $\mathbb{R}^{n_i}$ by

$$\| x_i \|_i = (x_i' G_i x_i)^{1/2}.$$
This norm defines a block-maximum norm $\| \cdot \|$ given by $\| x \| = \max_i \| x_i \|_i$. The following proposition wraps up the preceding discussion in that it imposes a bound on $f(x) - f(y)$ and provides a monotonicity condition similar to (3.24).

**Proposition 3.1.** Suppose that each $G_i$ is symmetric and positive definite and let the norms $\| \cdot \|_i$ and $\| \cdot \|$ be as above. Furthermore, suppose that there exist positive constants $A_1, A_2, A_3$, with $A_3 < A_2$, such that for each $i$ and for each $x, y \in X$, we have

$$\| f_i(x) - f_i(y) \|_i \leq A_1 \| x - y \|,$$  \tag{3.25}

and

$$(f_i(x) - f_i(y))^\top (x_i - y_i) \geq A_2 \| x_i - y_i \|^2 - A_3 \| x - y \|^2$$  \tag{3.26}

Then, provided that $\gamma$ is positive and small enough, the mapping $T : X \mapsto \mathbb{R}^n$, defined by $T_i(x) = x_i - \gamma G_i^{-1} f_i(x)$, is a contraction with respect to the block-maximum norm $\| \cdot \|$.

**Proof.** Let $A_4$ be a positive constant such that $x_i' G_i^{-1} x_i \leq A_4 x_i' G_i x_i$ for every $x_i \in \mathbb{R}^n$. Assuming that $0 < \gamma < 1/2A_2$, we have

$$\| T_i(x) - T_i(y) \|^2 = \| x_i - y_i \|^2 + \gamma^2 (f_i(x) - f_i(y))^\top G_i^{-1} (f_i(x) - f_i(y))$$

$$-2\gamma (f_i(x) - f_i(y))^\top (x_i - y_i)$$

$$\leq \| x_i - y_i \|^2 + A_4^2 \gamma^2 \| x - y \|^2 - 2\gamma A_2 \| x_i - y_i \|^2 + 2\gamma A_3 \| x - y \|^2$$

$$\leq (1 - 2\gamma A_2 + A_4^2 \gamma^2 + 2\gamma A_3) \| x - y \|^2.$$  

If $\gamma$ is also smaller than $2(A_2 - A_3)/(A_4^2 A_4)$, which is possible because $A_2 > A_3$, the expression $1 - 2\gamma A_2 + A_4^2 A_4 \gamma^2 + 2\gamma A_3$ is smaller than 1, which proves the result. \[\square\]

We identify the last result as the by now familiar condition on the outer loop; start from a feasible shipment pattern $[x_{ij}^{-1}]$, and determining a new feasible shipment pattern $[x_{ij}^l]$ by modifying $x_{ij}^{-1}$ (for $i = 1, \ldots, m$) in such a manner that the optimality conditions (3.8) are satisfied for the particular $j = l$.

Finally, we need to verify that every sequence contain a convergent subsequence. In order to obviate the last hurdle, we make use of the upcoming proposition which subsequently carries over to convergence of the inner loop and the solution of our problem.

**Proposition 3.2.** (Geometric Convergence for Strongly Convex Problems) Suppose, in addition to Assumption 3.1, that there exist some $\alpha > 0$ such that

$$(\nabla F(x) - \nabla F(y))^\top (x - y) \geq \alpha \| x - y \|^2, \quad \forall x, y \in X.$$  \tag{3.27}
Then there exist a unique vector \( x^* \) that minimizes \( F \) over the set \( X \). Furthermore, provided that \( \gamma \) is chosen positive and small enough, the sequence \( \{x(\tau)\} \) generated by the gradient projection algorithm (3.22) converges to \( x^* \) geometrically.

**Proof.** Inequality (3.27) implies that the mapping \( T : X \mapsto \mathbb{R}^n \) defined by
\[
T(x) = x - \gamma \nabla F(x)
\]
is a contraction with respect to the Euclidean norm \( \| \cdot \| \), provided that \( \gamma \) is positive and sufficiently small. (The assurance of this particular fact is indeed established once we invoke Prop. 3.1.) In particular, the mapping \( T \) has a unique fixed point \( x^* \) and the sequence generated by the gradient projection algorithm \( x = T(x) \) converges to \( x^* \) geometrically. (This is a general result regarding contraction mappings, which is presented and verified in the appendix.) Such a fixed point satisfies \( \nabla F(x^*) = 0 \). Inequality (3.27) also implies that the function \( F \) is strictly convex, whence it follows, the unique vector \( x^* \) minimizes \( F \) over the set \( X \). (This is due to the fact that strong monotonicity is equivalent to strong convexity.)

Once we piece these parts together it’s apparent that the closed algorithmic map and convergent subsequence must find a minimum of the restricted strictly convex programming problem:

\[
\text{minimize } T([x^+]_{ij}) = \sum_i \left( \frac{1}{2} \gamma_i (x^+_i)^2 - \gamma_i a^0_i x^+_i + \sum_i \frac{1}{2} \alpha_i \left( x^+_i - \sum_{j \neq i} x^+_{ij} \right) \right)
\]
\[
+ \sum_i \left[ -\alpha_i a^0_i \left( x^+_i - \sum_{j \neq i} x^+_{ij} \right) \right] + \frac{1}{2} \beta_i \sum_i x^+_i \left( \sum_i x^+_i \right)^2 - \beta_i a^0_i \sum_i x^+_i \tag{3.28}
\]
subject to
\[
0 \leq x^+_i \leq u^+_i, \ i = 1, \ldots, m
\]
\[
x^+_{ij} = x^+_{ij} - 1, \ j \neq i, \ i = 1, \ldots, m. \tag{3.29}
\]
Or equivalently, to the solution of problem (3.1) subject to (3.2)-(3.4).

4. **A small application**

This section provide a small application which merely intend to illustrate the method from the previous section. Since the model require some prior or base year information on empty car flows we use observed flows, from the second half
of 1997 to the first half of 1998, between 23 Swedish counties. For a description of the data set see [Sor01b]. The expertise in the form of upper bounds is simply generated as a random uniform interval around the observed value. The interval range is plus minus ten percent whereupon the bound is rounded to the nearest integer. Initial supply and demand is obtained as in subsection 2.1.

In order to assess the merits of the proposed estimation method a twofold comparison is carried out. The first address the discrepancy between estimated and observed empty rail car freight flows. In this case we make use of two different discrepancy measures i.e., the Root Mean Square Error (RMSE) measure and Theil’s inequality coefficient (U), [The66]. The second comparison addresses the accuracy of the proposed estimation method compared to the frequently used Entropy maximizing procedure.

In the present context Theil’s inequality coefficient is defined as,

\[
U = \left[ \frac{\sum_{ij}(\hat{g}_{ij} - g_{ij})^2}{\sum_{ij}(g_{ij})^2} \right]^{1/2},
\]

which is expressed as the squared difference between an observed and an estimated empty car flow, weighted by it’s squared observed empty car flow. It’s a non-normalized measure, hence it ranges from zero to infinitely large values. Lower values indicate a greater degree of similarity between estimated and observed car flows. The results for the outbound supply of empty freight cars from the respective county is presented in table 4.1. In six of the cases the two methods estimates the car flows equally well, and in ten cases it out perform the Entropy maximizing method. The picture is slightly different regarding the inbound demand for empty cars, in four cases the two methods performs equally well. In fifteen cases the proposed method result in estimates with a lower variability, as can bee seen in table 4.2.

An alternative measure, all though a bit more sensitive to skew distributions is the RMSE,

\[
RMSE = \left[ \frac{1}{n} \sum_{ij} (\hat{g}_{ij} - g_{ij})^2 \right]^{1/2},
\]

which is given by the square root of the squared difference between an observed and an estimated empty freight car flow weighted by the number of observations. Just as in the former case the effective interval ranges from zero to an infinitely large number. The interpretation of this measure is the same as above; smaller values indicate a greater degree of similarity than greater ones. The results for
the outbound supply of empty freight cars from the respective county is given in table 4.3. In three of the cases the two methods estimates the outbound car flows

Table 4.1. U-Measure, Row sum.

<table>
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<th>Difference</th>
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Table 4.2. U-Measure, Col sum.

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<th>Difference</th>
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</table>

equally well, and in ten cases it once more perform better than the Entropy maximizing method. As depicted in table 4.4 regarding the inbound demand for empty cars, in just one case the two methods performs equally well, and in seventeen cases the proposed method result in estimates with a lower variability.

The fact that neither of the two utilized measures are normalized, makes the notion of closeness between observed and predicted values a bit vague. However, we might warrant their use since the comparison is made with respect to one and
the same population. Furthermore, each Euclidean distance measures has the property of not being rank preserving under certain transformations, whence a normalization might not be desirable anyhow.

<table>
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<td>881.4</td>
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Table 4.4. RMSE-Measure, Col sum.

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It should be noted that the comparison between estimated and observed empty car flows is given in terms of interregional origin destination path flows. Since the path flows are disjoint there exist a number of origin destination path flows compatible with a unique link flow. This is a consequence of the separable problem structure. Thus, any discrepancies should be interpreted with caution.

Finally, we shall briefly address the computational performance for the two numerical methods. We utilize the public domain Entropy maximizing subroutine by Persson [Per86] and the code by Nagurny [Nag89] except for the sorting routine.
which is replaced with a Quick Sort routine. Both numerical methods is coded in FORTRAN 77 and compiled using the Absoft 7.0 complier and was executed on a PC with a Pentium processor (333 MHz). The convergence limit $\epsilon$ were set to 0.4 i.e., the convergence condition (3.6) hold within $\epsilon$. As indicated in table 4.5 the projection method is seventeen times faster than the Lagrangean method and converges in only fractions of a second.

Table 4.5 Computational performance

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<tr>
<td>CPU-time (sec.)</td>
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5. Final remarks

In this paper we have presented an approach which enhance the accuracy and flexibility to estimate empty rail freight car flows. We have also given a proof of convergence and rate of convergence for the numerical method. The method has been applied to as small data set and the results indicate a better performance compared to an Entropy maximizing approach with respect to both accuracy and computational speed.
Acknowledgment
The financial support from VINNOVA under grant Dnr. 2001/03116 is gratefully acknowledged.

References


Appendix

In this appendix we collect some general results regarding contraction mappings for the readers convenience and ease of reference.

We consider contractions and the associated fixed point problem. A number of iterative algorithms can be written as

\[ x(\tau + 1) = T(x(\tau)), \quad \tau = 0, 1, \ldots, \]  

(A.1)

where \( T \) is a mapping from a subset \( X \) of \( \mathbb{R}^n \) into itself and has the property

\[ \| T(x) - T(y) \| \leq \alpha \| x - y \|, \quad \forall x, y \in X. \]  

(A.2)

Here \( \| \cdot \| \) is some norm, and \( \alpha \) is a constant belonging to the interval \([0, 1)\). Such a mapping is called a contraction mapping or a contraction, and iteration (A.1) is called a contracting iteration. The scalar \( \alpha \) is called the modulus of \( T \). A mapping \( T : X \mapsto Y \), where \( X, Y \subset \mathbb{R}^n \), that satisfies (A.2) will also be called a contraction mapping, even if \( X \neq Y \). Let there be given a mapping \( T : X \mapsto X \). Any vector \( x^* \in X \) satisfying \( T(x^*) = x^* \) is called a fixed point. The reason is that if the sequence \( \{x(\tau)\} \) converges to some \( x^* \in X \) and \( T \) is continuous at \( x^* \), then \( x^* \) is a fixed point of \( T \). We simply notice that contraction mappings are automatically continuous..

Remark 1. It’s important to keep in mind that a mapping \( T \) could be a contraction for some choice of the vector norm \( \| \cdot \| \) and yet, fail to be a contraction under a different choice of norm. Thus, the proper choice of a norm is crucial.

We close this appendix with the following basic result which shows that contraction mappings have a unique fixed point and the corresponding iteration \( x = T(x) \) converges to it.

Proposition 1. (Convergence of Contracting Iterations) Suppose that \( T : X \mapsto X \) is a contraction with modulus \( \alpha \in [0, 1) \) and that \( X \) is a closed subset of \( \mathbb{R}^n \).

(a) (Existence and Uniqueness of Fixed Points) Then the mapping \( T \) has a unique fixed point \( x^* \in X \).

(b) (Geometric Convergence) Then for every initial vector \( x(0) \in X \), the sequence \( \{x(\tau)\} \) generated by \( x(\tau + 1) = T(x(\tau)) \) converges to \( x^* \) geometrically. In particular,

\[ \| x(\tau) - x^* \| \leq \alpha^\tau \| x(0) - x^* \|, \quad \text{for all } \tau \geq 0. \]
Proof.
(a) Fix some $x(0) \in X$ and consider the sequence $\{x(\tau)\}$ generated by $x(\tau + 1) = T(x(\tau))$. We have, from inequality (A.2),

$$\|x(\tau + 1) - x(\tau)\| \leq \alpha \|x(\tau) - x(\tau - 1)\|,$$

for all $\tau \geq 1$, which implies

$$\|x(\tau + 1) - x(\tau)\| \leq \alpha^\tau \|x(1) - x(0)\|, \quad \text{for all } \tau \geq 0. $$

It follows that for every $\tau \geq 0$ and $m \geq 1$, we have

$$\|x(\tau + 1) - x(\tau)\| \leq \sum_{i=1}^{m} \|x(\tau + i) - x(\tau + i - 1)\| \leq \alpha^\tau (1 + \alpha + \ldots + \alpha^{m-1}) \|x(1) - x(0)\| \leq \frac{\alpha^\tau}{1 - \alpha} \|x(1) - x(0)\|.$$

Therefore, $\{x(\tau)\}$ is a Cauchy sequence and must converge to a limit $x^*$. Furthermore, since $X$ is closed, $x^*$ belongs to $X$. We have for all $\tau \geq 1$,

$$\|T(x^*) - x^*\| \leq \|T(x^*) - x(\tau)\| + \|x(\tau) - x^*\| \leq \alpha \|x^* - x(\tau - 1)\| + \|x(\tau) - x^*\|$$

and since $x(\tau)$ converges to $x^*$, we obtain $T(x^*) - x^*$. Therefore, the limit $x^*$ of $x(\tau)$ is a fixed point of $T$. It is a unique fixed point because if $y^*$ were another fixed point, we would have

$$\|x^* - y^*\| = \|T(x^*) - T(y^*)\| \leq \alpha \|x^* - y^*\|$$

which implies that $x^* = y^*$.

(b) We have

$$\|x(\tau') - x^*\| = \|T(x(\tau' - 1)) - T(x^*)\| \leq \alpha \|x(\tau' - 1) - x^*\|, $$

for all $\tau' \geq 1$, so by simply applying this relation successively for $\tau' = \tau, \tau - 1, \ldots, 1$, we obtain the desired result.
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