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Boundary Values of Plurisubharmonic Functions and Related Topics

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Till min familj

Abstract

This thesis consists of three papers concerning problems related to plurisubharmonic functions on bounded hyperconvex domains, in particular boundary values of such functions. The papers summarized in this thesis are:

Paper I Urban Cegrell and Berit Kemppe, *Monge-Ampère boundary measures,* Ann. Polon. Math. 96 (2009), 175–196.

Paper II Berit Kemppe, *An ordering of measures induced by plurisubharmonic functions,* manuscript (2009).

Paper III Berit Kemppe, *On boundary values of plurisubharmonic functions,* manuscript (2009).

In the first paper we study a procedure for sweeping out Monge-Ampère measures to the boundary of the domain. The boundary measures thus obtained generalize measures studied by Demailly. A number of properties of the boundary measures are proved, and we describe how boundary values of bounded plurisubharmonic functions can be associated to the boundary measures.

In the second paper, we study an ordering of measures induced by plurisubharmonic functions. This ordering arises naturally in connection with problems related to negative plurisubharmonic functions. We study maximality with respect to the ordering and a related notion of minimality for certain plurisubharmonic functions. The ordering is then applied to problems of weak*-convergence of measures, in particular Monge-Ampère measures.

In the third paper we continue the work on boundary values in a more general setting than in Paper I. We approximate measures living on the boundary with measures on the interior of the domain, and present conditions on the approximation which makes the procedure suitable for defining boundary values of certain plurisubharmonic functions.

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Key words: Plurisubharmonic functions, boundary measures, boundary values, complex Monge-Ampère operator, weak*-convergence.

Sammanfattning

Denna avhandling består av tre artiklar som alla behandlar problem relaterade till plurisubharmoniska funktioner på begränsade hyperkonvexa områden, särskilt randvärden för sådana funktioner.

I den första artikeln, *Monge-Ampère boundary measures*, studerar vi ett sätt att sopa ut Monge-Ampère-mått till randen av området. De randmått vi erhåller generaliserar mått som Demailly studerat. Vi bevisar ett antal egenskaper som gäller för randmåtten, och vi beskriver hur randvärden för begränsade plurisubharmoniska funktioner kan associeras till randmåtten.

I den andra artikeln, *An ordering of measures induced by plurisubharmonic functions*, studerar vi en plurisubharmonisk ordningsrelation mellan mått. Denna ordning uppkommer naturligt i samband med problem relaterade till negativa plurisubharmoniska funktioner. Vi studerar maximalitet med avseende på denna ordning samt ett relaterat minimalitetsbegrepp för vissa plurisubharmoniska funktioner. Ordningen tillämpas sedan på problem gällande svag*-konvergens av mått, särskilt Monge-Ampère-mått.

I den tredje artikeln, *On boundary values of plurisubharmonic functions*, fortsätter vi arbetet med randvärden, men med en mer allmän utgångspunkt än i första artikeln. Vi approximerar randmått med mått som lever på det inre av området, och presenterar villkor på approximationen som gör att den kan användas till att definiera randvärden för vissa plurisubharmoniska funktioner.

Tack

Först och främst vill jag tacka min handledare Urban Cegrell, som med brinnande intresse och engagemang, stort tålamod och skön humor har lotsat mig fram hit. Jag vill också tacka min biträdande handledare Anders Fällström, som var en av de som fick mig att fastna för komplex analys och har varit ett stort stöd hela vägen. Tack till Lisa Hed och Linus Carlsson som varit mina närmaste komplexa klasskamrater och gjort denna resa så mycket roligare. Tack till alla andra som korsat min komplexa doktorandväg och bjudit på sällskap vid seminarier och konferenser – särskilt alla deltagare i KAUS. Ett extra tack till Per Åhag för flitig genomläsning av mina artiklar.

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Contents

1. Introduction

1.1 Pluripotential theory - basic notions

Potential theory in $\mathbb{R}^N$, also called classical potential theory, can be described as the study of subharmonic functions, i.e. the functions that "hang under harmonic functions" in the following sense.

Definition. Let Ω be a domain in $\mathbb{R}^N$. An upper semicontinuous function $u : \Omega \to \mathbb{R} \cup \{-\infty\}$ is said to be *subharmonic* if for every relatively compact open subset U of Ω and every continuous function $h : \overline{U} \to \mathbb{R}$ that is harmonic on U, we have the implication

$$u \leq h \text{ on } \partial U \implies u \leq h \text{ on } U.$$

Potential theory in $\mathbb{R}^2$ has been of great importance in the study of holomorphic functions in $\mathbb{C}$. But in higher dimension (i.e. in $\mathbb{C}^n \simeq \mathbb{R}^{2n}$, $n > 1$), classical potential theory has not had the same impact. One reason is that the class of subharmonic functions is too large, in the sense that it is not invariant under biholomorphic change of coordinates. In pluripotential theory, one therefore looks at the subclass of subharmonic functions whose composition with holomorphic mappings are subharmonic – this is the class of plurisubharmonic functions, which can also be defined as below. Note that a *complex line* in $\mathbb{C}^n$ is a set of the form $\{a + b\zeta : \zeta \in \mathbb{C}\}$, where $a, b \in \mathbb{C}^n$ are fixed.

Definition. Let Ω be a domain in $\mathbb{C}^n$. An upper semicontinuous function $u : \Omega \to \mathbb{R} \cup \{-\infty\}$ is said to be *plurisubharmonic* if for each complex line $\{a + b\zeta\}$ in $\mathbb{C}^n$, the function

$$\zeta \mapsto u(a + b\zeta)$$

is subharmonic on the set $\{\zeta \in \mathbb{C} : a + b\zeta \in \Omega\}$. The set of plurisubharmonic functions on Ω is denoted by $PSH(\Omega)$, and the subset of nonpositive functions is denoted by $PSH^-(\Omega)$.

Note that the constant function $u = 0$ is included in $PSH^-(\Omega)$. However, by the maximum principle for subharmonic functions, all other functions in $PSH^-(\Omega)$ are strictly negative. Also note that if $n = 1$, then plurisubharmonic is the same as subharmonic.

Example. If f is a holomorphic function on Ω, then $\log |f|$ and $|f|^p$, $p > 0$, are plurisubharmonic functions.

As already noted, the class of plurisubharmonic functions is a natural counterpart of the class of subharmonic functions. Similarly, a natural substitution for the class of harmonic functions is the class of maximal plurisubharmonic functions, see the definition below. Note that harmonic functions are *maximal subharmonic* functions in the same sense.

Definition. Let Ω be a domain in $\mathbb{C}^n$. A plurisubharmonic function u on Ω is said to be *maximal* if for every relatively compact open subset U of Ω and every upper semicontinuous function $v : \overline{U} \to \mathbb{R} \cup \{-\infty\}$ that is plurisubharmonic on U, we have the implication

$$v \leq u \text{ on } \partial U \implies v \leq u \text{ on } U.$$

One of the main tools in pluripotential theory is the complex Monge-Ampère operator. Before we can define this operator, we need some notation.

Remark. Good references for pluripotential theory are for example Klimek's book [12], Kiselman's overview [11] and Kołodziej's book [13].

Notation We denote a point in $\mathbb{C}^n$ by $z = (z_1, \ldots, z_n)$ and use the standard notation

$$\begin{gathered} z_j = x_j + iy_j, \quad \bar{z}_j = x_j - iy_j, \\ dz_j = dx_j + idy_j, \quad d\bar{z}_j = dx_j - idy_j, \\ \frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right), \quad \frac{\partial}{\partial \bar{z}_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right). \end{gathered}$$

We denote by dV the standard Euclidean volume form on $\mathbb{C}^n$, i.e.

$$\begin{aligned} dV &= (dx_1 \wedge dy_1) \wedge \cdots \wedge (dx_n \wedge dy_n) \\ &= \left(\frac{i}{2}\right)^n (dz_1 \wedge d\bar{z}_1) \wedge \cdots \wedge (dz_n \wedge d\bar{z}_n). \end{aligned}$$

If Ω is a domain (i.e. an open connected set) in $\mathbb{C}^n$ and the function $f : \Omega \to \mathbb{C}$ is differentiable at a point in Ω, then the differential of f at that point, df, is an $\mathbb{R}$-linear mapping. Hence df can be split into a $\mathbb{C}$-linear part ∂f and an anti-$\mathbb{C}$-linear part $\bar{\partial} f$:

$$df = \partial f + \bar{\partial} f,$$

and we have that

$$
\begin{aligned}
df &= \sum_{j=1}^{n} \left(\frac{\partial f}{\partial x_j} dx_j + \frac{\partial f}{\partial y_j} dy_j \right), \\
\partial f &= \sum_{j=1}^{n} \frac{\partial f}{\partial z_j} dz_j, \\
\bar{\partial} f &= \sum_{j=1}^{n} \frac{\partial f}{\partial \bar{z}_j} d\bar{z}_j.
\end{aligned}
$$

If $f \in C^2(\Omega)$ (i.e. twice differentiable), then the *complex Hessian* of f is defined as the matrix

$$
\left[\frac{\partial^2 f}{\partial z_j \partial \bar{z}_k} \right]_{j,k=1}^{n}.
$$

It can be shown that a real-valued C^2-function v is subharmonic on $\Omega \subset \mathbb{C}^n$ if and only if the Laplacian of v, Δv, is nonnegative on Ω. If $n = 1$ then $\Delta v = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 4 \frac{\partial^2 v}{\partial \zeta \partial \bar{\zeta}}$, where $\zeta = x + iy \in \mathbb{C}$. Therefore, if u is a real-valued C^2-function on a domain $\Omega \subset \mathbb{C}^n$, then u is plurisubharmonic if and only if

$$
\Delta_\zeta u(a + b\zeta) = 4 \frac{\partial^2 u}{\partial \zeta \partial \bar{\zeta}} \bigg|_{a+b\zeta} = \sum_{j,k=1}^{n} \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k} \bigg|_{a+b\zeta} \cdot b_j \bar{b}_k \geq 0,
$$

for every $a, b \in \mathbb{C}^n$ and $\zeta \in \mathbb{C}$ such that $a + b\zeta \in \Omega$. Hence u is plurisubharmonic if and only if the complex Hessian of u is positive semidefinite at each point of Ω. Note that if this is the case, then the determinant of the Hessian is nonnegative.

Let the operator d^c be defined by

$$
d^c = i(\bar{\partial} - \partial).
$$

Using that $d^2 = 0$ and $d = \partial + \bar{\partial}$ (see above), it follows that $dd^c = 2i\partial\bar{\partial}$. We are now ready to define the complex Monge-Ampère operator.

The complex Monge-Ampère operator For a C^2-function u on a domain Ω in $\mathbb{C}^n$, the complex Monge-Ampère operator $(dd^c \cdot)^n$ is defined as

$$
(dd^c u)^n = (dd^c u) \wedge \cdots \wedge (dd^c u) = 4^n n! \det \left[\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k} \right] dV. \tag{1.1}
$$

Hence, if $u \in C^2(\Omega) \cap PSH(\Omega)$, then $(dd^c u)^n$ is a positive measure on Ω. Note that $(dd^c \cdot)^n$ is a nonlinear operator if $n > 1$.

If $n = 1$ then $(dd^c u) = 4 \frac{\partial^2 u}{\partial z \partial \bar{z}} dV = \Delta u \, dV$, so u is subharmonic if and only if $(dd^c u) \geq 0$ and harmonic if and only if $(dd^c u) = 0$. It is also true in higher dimension that a plurisubharmonic function $u \in C^2(\Omega)$ is maximal if and only if

$(dd^c u)^n = 0$. However, $(dd^c u)^n \geq 0$ might hold even if u is not plurisubharmonic, note for example that if $n = 2$ then $(dd^c u)^2 = (dd^c(-u))^2$.

By a result of Bedford and Taylor [1], $(dd^c u_1) \wedge \cdots \wedge (dd^c u_n)$ is a well defined positive measure if $u_1, \ldots, u_n$ are locally bounded plurisubharmonic functions on a domain in $\mathbb{C}^n$. This is highly nontrivial, since one cannot in general multiply distributions.

In the next section we will see that if we restrict ourselves to a certain type of domains, then the complex Monge-Ampère operator can be further extended. But first some additional useful concepts.

Pluripolar sets A subset E of $\mathbb{C}^n$ is said to be *pluripolar* if for each $z \in E$ there is a neighbourhood U of z and a function $u \in PSH(U)$, $u \not\equiv -\infty$, such that $u = -\infty$ on $E \cap U$. By a result of Josefson [10], the local definition of pluripolarity coincides with the global definition, i.e. if E is pluripolar then there is a function $u \in PSH(\mathbb{C}^n)$, $u \not\equiv -\infty$, such that $u = -\infty$ on E. Pluripolar sets are in some sense small, for example they have Lesbegue measure 0.

Weak*-convergence In the following, by a *measure* (or *positive measure*) we always mean a positive regular Borel measure.

Now, let $\{\mu_j\}_{j=1}^{\infty}$ and μ be measures on a domain Ω in $\mathbb{C}^n$. We say that μ_j is *weak*-convergent* (*tends weak**) to μ on Ω if

$$\lim_{j \to \infty} \int_\Omega \phi \, d\mu_j = \int_\Omega \phi \, d\mu, \quad \forall \, \phi \in C_0(\Omega),$$

where $C_0(\Omega)$ denotes the set of continuous real-valued functions on Ω whose support is compact. By standard measure theory, it is equivalent to require that the convergence holds for each $\phi \in C_0^\infty(\Omega) = C_0(\Omega) \cap C^\infty(\Omega)$. Under some additional assumption (on the domain and on the measures), we can also use certain plurisubharmonic functions as test functions, see the next section.

Note that when Ω is bounded and $\{\mu_j\}_{j=1}^{\infty}$ and μ are finite measures on $\overline{\Omega}$, then they can be viewed as measures on the whole of $\mathbb{C}^n$. Hence we may consider weak*-convergence on $\mathbb{C}^n$, and μ_j tends weak* to μ if and only if

$$\lim_{j \to \infty} \int_\Omega \phi \, d\mu_j = \int_\Omega \phi \, d\mu, \quad \forall \, \phi \in C(\overline{\Omega}),$$

since all the measures have their support contained in the compact set $\overline{\Omega}$. Moreover, weak*-convergence on $\mathbb{C}^n$ is, in this setting, equivalent to weak*-convergence on any domain containing $\overline{\Omega}$.

Remark. In Paper I and III we consider weak*-convergence on $\overline{\Omega}$, which means weak*-convergence on $\mathbb{C}^n$, whereas in Paper II it is weak*-convergence on Ω.

1.2 Important subclasses of $PSH^-(\Omega)$

We start by defining hyperconvexity.

Definition. A bounded domain $\Omega \subset \mathbb{C}^n$ is said to be *hyperconvex* if there exists a negative plurisubharmonic exhaustion function for Ω, i.e. a function $\phi \in PSH^-(\Omega)$ such that the set $\{z \in \Omega : \phi(z) < c\}$ is relatively compact in Ω for each $c < 0$.

From now on, Ω will be a bounded hyperconvex domain in $\mathbb{C}^n$. Balls $\mathbb{B}(\xi, R)$ and polydiscs $\mathbb{D}(\xi_1, r_1) \times \cdots \times \mathbb{D}(\xi_n, r_n)$ are examples of hyperconvex domains. If Ω is hyperconvex, then the exhaustion function ϕ can (see [8]) be chosen in $C^\infty(\Omega) \cap C(\overline{\Omega})$ and such that $\int_\Omega (dd^c\phi)^n < \infty$. This implies for example that the classes of plurisubharmonic functions defined below are nontrivial. Moreover, the function $\varphi = 1/(-\phi)$ is a continuous plurisubharmonic function such that the set $\{z \in \Omega : \varphi(z) < c\}$ is relatively compact in Ω for each $c \in \mathbb{R}$, so Ω is pseudoconvex. Conversely, if Ω is a bounded pseudoconvex domain with Lipschitz boundary, then Ω is hyperconvex [9].

Given a bounded hyperconvex domain $\Omega \subset \mathbb{C}^n$, consider the following classes, defined in [3] and [5]:

- $\mathcal{E}_0(\Omega)$ is the set of functions $u \in PSH^-(\Omega) \cap L^\infty(\Omega)$ such that $\int_\Omega (dd^c u)^n < \infty$ and $\lim_{z\to\xi} u(z) = 0$, for all $\xi \in \partial\Omega$.
- $\mathcal{F}(\Omega)$ is the set of functions $u \in PSH^-(\Omega)$ such that there is a sequence $\{u_j\}_{j=1}^\infty$ in $\mathcal{E}_0(\Omega)$ with $u_j \searrow u$ and $\sup_j \int_\Omega (dd^c u_j)^n < \infty$.
- $\mathcal{E}(\Omega)$ is the set of functions $u \in PSH^-(\Omega)$ such that for each $\omega \subset\subset \Omega$ there is function $u_\omega \in \mathcal{F}(\Omega)$ with $u_\omega \geq u$ on Ω and $u_\omega = u$ on ω.

It follows that $\mathcal{E}_0(\Omega) \subset \mathcal{F}(\Omega) \subset \mathcal{E}(\Omega)$ and $PSH^-(\Omega) \cap L^\infty_{\text{loc}}(\Omega) \subset \mathcal{E}(\Omega)$.

Lemma. *If $\phi \in C_0^\infty(\Omega)$, then there are $\phi_1, \phi_2 \in \mathcal{E}_0(\Omega) \cap C(\overline{\Omega})$ such that $\phi = \phi_1 - \phi_2$.*

This lemma (Lemma 3.1 in [5]) explains why the functions in $\mathcal{E}_0(\Omega)$ are sometimes called *test functions*. For example, if $\{\mu_j\}_{j=1}^\infty$ and μ are finite measures on Ω with uniformly bounded total mass (i.e. $\sup_j \int_\Omega d\mu_j < \infty$), then μ_j tends weak* to μ on Ω if and only if $\lim_{j\to\infty} \int_\Omega \phi\, d\mu_j = \int_\Omega \phi\, d\mu$ for each $\phi \in \mathcal{E}_0(\Omega) \cap C(\overline{\Omega})$.

Theorem. *Suppose that $u \in PSH^-(\Omega)$. Then there is a sequence $\{u_j\} \subset \mathcal{E}_0(\Omega) \cap C(\bar{\Omega})$ such that $u_j \searrow u$ on Ω and supp $(dd^c u_j)^n \subset\subset \Omega$ for each j.*

Theorem. *For $k = 1, \ldots, n$, let $u_k \in \mathcal{E}(\Omega)$ and $\{g_{kj}\}_{j=1}^\infty \subset \mathcal{E}_0(\Omega)$ be such that $g_{kj} \searrow u_k$ as $j \to \infty$. Then $dd^c g_{1j} \wedge \ldots \wedge dd^c g_{nj}$ is weak*-convergent and the limit measure is independent of the sequences $\{g_{kj}\}_{j=1}^\infty$.*

These theorems (Theorem 2.1 and Theorem 4.2 in [5]) show that the complex Monge-Ampère operator is well defined on the class $\mathcal{E}(\Omega)$; if $u_1, \ldots, u_n \in \mathcal{E}(\Omega)$, then $dd^c u_1 \wedge \ldots \wedge dd^c u_n$ is defined as the limit measure obtained by combining the two theorems. Moreover, a function $u \in \mathcal{E}(\Omega)$ is maximal if and only if $(dd^c u)^n = 0$ (see [2] and [7]).

Functions in $\mathcal{F}(\Omega)$ have finite total Monge-Ampère mass. The next formula for integration by parts (proof can be found in [5]) shows that they have in some sense boundary value zero.

Theorem. *Let $u_0, \ldots, u_n \in \mathcal{F}(\Omega)$. Then*

$$\int_\Omega u_0\, dd^c u_1 \wedge dd^c u_2 \wedge \ldots \wedge dd^c u_n = \int_\Omega u_1\, dd^c u_0 \wedge dd^c u_2 \wedge \ldots \wedge dd^c u_n.$$

The following lemma contains some useful basic properties of the classes.

Lemma. *Let $\mathcal{K} \in \{\mathcal{E}_0, \mathcal{F}, \mathcal{E}\}$, then the following holds.*

(i) If $u, v \in \mathcal{K}(\Omega)$ and $\alpha, \beta \geq 0$, then $\alpha u + \beta v \in \mathcal{K}(\Omega)$.

(ii) If $u \in \mathcal{K}(\Omega)$ and $v \in PSH^-(\Omega)$, then $\max\{u, v\} \in \mathcal{K}(\Omega)$. In particular, if $u \in \mathcal{K}(\Omega)$, $v \in PSH^-(\Omega)$ and $v \geq u$, then $v \in \mathcal{K}(\Omega)$.

Example. Let Ω be the unit ball $\mathbb{B} = \mathbb{B}(0,1)$ in $\mathbb{C}^n$ and define the functions

$$u_1(z) = \max\{\log|z|, c\}, c < 0, \quad u_2(z) = \log|z|, \quad u_3(z) \equiv -1.$$

Then $u_1 \in \mathcal{E}_0(\mathbb{B})$, $u_2 \in \mathcal{F}(\mathbb{B}) \setminus \mathcal{E}_0(\mathbb{B})$ and $u_3 \in \mathcal{E}(\mathbb{B}) \setminus \mathcal{F}(\mathbb{B})$. Moreover $(dd^c u_1)^n = (2\pi)^n \sigma_c$, $(dd^c u_2)^n = (2\pi)^n \delta_0$ and $(dd^c u_3)^n = 0$. Here σ_c is the normalized area measure on the sphere $\partial\mathbb{B}(0, e^c)$ and δ_0 is the Dirac measure at 0. If $n = 2$ then $u_4(z_1, z_2) = \log|z_2|$ is a function in $PSH^-(\mathbb{B}) \setminus \mathcal{E}(\mathbb{B})$ (see [4]).

1.3 Boundary values

If a plurisubharmonic function u on Ω is continuous on the closure $\overline{\Omega}$, then there is no doubt what the boundary value of u is, it is simply the restriction of u to $\partial\Omega$. If in addition u is maximal and $u|_{\partial\Omega} = 0$, then $u \equiv 0$.

For a general $u \in PSH(\Omega)$ it is not completely straightforward to define boundary values. Let for example $\mathbb{B}$ be the unit ball in $\mathbb{C}^n$, $n > 1$. In [15], Rashkovskii constructs a non-constant function $v \in PSH^-(\mathbb{B}) \cap C^\infty(\mathbb{B}) \cap L^\infty(\mathbb{B})$ which is maximal and has the property that $v_r \to 0$ in $L^p(\partial\mathbb{B})$ $(1 \leq p < \infty)$ as $r \to 1$, where $v_r(\xi) = v(r\xi)$ for $\xi \in \partial\mathbb{B}$ and $0 < r < 1$. This implies that $\limsup_{\Omega \ni z \to \xi} v(z) = 0$ for each $\xi \in \partial\mathbb{B}$. However, since v is a strictly negative maximal function, it would not be a good definition to say that v *has boundary value zero.*

In [6], boundary values are defined in terms of maximal functions as follows. Denote the set of maximal plurisubharmonic functions in $\mathcal{E}(\Omega)$ by $\mathcal{M}(\Omega)$. Let $\mathcal{N}(\Omega)$ be the set of functions in $\mathcal{E}(\Omega)$ with smallest maximal plurisubharmonic majorant identically zero. The class $\mathcal{F}(\Omega)$ is a subset of $\mathcal{N}(\Omega)$. Given $H \in \mathcal{M}(\Omega)$, define the class $\mathcal{N}(\Omega, H)$ as the set of functions $u \in \mathcal{E}(\Omega)$ such that $v + H \leq u \leq H$

for some $v \in \mathcal{N}(\Omega)$. A function $u \in \mathcal{N}(\Omega, H)$ is then said to have *boundary value* H. In particular, the functions in $\mathcal{N}(\Omega) = \mathcal{N}(\Omega, 0)$ are said to have boundary value zero. The subclass $\mathcal{F}(\Omega, H) \subset \mathcal{N}(\Omega, H)$ is also considered, this is the set of functions $u \in \mathcal{E}(\Omega)$ such that $v + H \leq u \leq H$ for some $v \in \mathcal{F}(\Omega)$. Note that with this approach, the boundary values are actually functions defined on the interior of the domain (although sometimes assumed to be continuous on the closure).

In Paper I and III we use a different approach. The idea there is to associate boundary values of plurisubharmonic functions to measures living on the boundary. We do this by approximating the boundary measures with measures on the interior in a certain way.

2. Summary of papers

Paper I: Monge-Ampère boundary measures

If u is a function in $\mathcal{F}(\Omega)$, where Ω is a bounded hyperconvex domain in $\mathbb{C}^n$, then $(dd^cu)^n$ is a finite measure on Ω. In this paper we show how $(dd^cu)^n$ can be swept out to a measure μ_u on the boundary. This is done by the following procedure.

Let $\{\Omega_j\}_{j=1}^{\infty}$ be a *fundamental sequence* for Ω, i.e. a sequence of strictly pseudoconvex domains with C^2-boundary such that $\Omega_j \subset\subset \Omega_{j+1} \subset\subset \Omega$ and $\cup_{j=1}^{\infty}\Omega_j = \Omega$. Given $u \in \mathcal{F}(\Omega)$ define

$$u^j = \sup\{\varphi \in PSH^-(\Omega) : \varphi|_{\Omega\setminus\Omega_j} \le u|_{\Omega\setminus\Omega_j}\}.$$

Then $u^j \le 0$ is plurisubharmonic and $u \le u^j \le u^{j+1} \le 0$, in particular each $u^j \in \mathcal{F}(\Omega)$. Moreover, $(dd^cu^j)^n$ is concentrated on $\Omega \setminus \Omega_k$ since u^j is maximal on Ω_j, and $\int_\Omega (dd^cu^j)^n = \int_\Omega (dd^cu)^n$ by Stokes' theorem (note that $u_j = u$ on $\Omega\setminus\Omega_j$). In Theorem 3.1 it is shown that $\{(dd^cu^j)^n\}_{j=1}^{\infty}$ is weak*-convergent to a measure μ_u on $\partial\Omega$ and that $\lim_{j\to\infty}\int_\Omega \varphi(dd^cu^j)^n$ exists for every $\varphi \in PSH(\Omega)\cap L^\infty(\Omega)$. Moreover, the measure μ_u does not depend on the sequence $\{\Omega_j\}_{j=1}^{\infty}$.

This procedure generalizes an out-sweeping of Monge-Ampère measures studied by Demailly in [9], and Theorem 3.2 generalizes the Lelong-Jensen formula considered there:

Theorem. *Assume that $u \in \mathcal{F}(\Omega)$, $h \in \mathcal{E}(\Omega)$, $\int_\Omega h\,(dd^cu)^n > -\infty$ and that $dd^ch \wedge (dd^cu)^{n-1}$ vanishes on pluripolar sets. Then*

$$\lim_{j\to\infty}\int_\Omega h\,(dd^cu^j)^n = \int_\Omega h\,(dd^cu)^n - \int_\Omega u\,dd^ch \wedge (dd^cu)^{n-1}.$$

It follows for example that if $u \in \mathcal{F}(\Omega)$ and $h \in PSH(\Omega)\cap C(\overline{\Omega})$, then $\int_\Omega h\,(dd^cu)^n = \int_\Omega u\,dd^ch\wedge(dd^cu)^{n-1} + \int_{\partial\Omega} h\,d\mu_u$ (see Remark 1). Similar formulas have been used in [14].

In Section 4, we present a number of properties of the boundary measures μ_u. For example

- if $u \le v$ then $\mu_u \ge \mu_v$ (Proposition 4.3),
- $\mu_u = \mu_{\max\{u,-1\}}$ (Corollary 4.4),

- μ_u vanishes on pluripolar sets (Proposition 4.7),
- all μ_u-measures have the same support (Corollary 4.10).

If Ω is B-regular (i.e. if each continuous function on $\partial\Omega$ can be extended continuously to a plurisubharmonic function on Ω, see e.g. [16]), then $\operatorname{supp}\mu_u = \partial\Omega$. This follows from Theorem 4.1, which shows that each finite measure on the boundary of a B-regular domain is in the weak*-closure of the μ_u-measures. On the other hand, if $\Omega = \Omega_1 \times \Omega_2$ is the product of two hyperconvex domains, then $\operatorname{supp}\mu_u \subset \partial\Omega_1 \times \partial\Omega_2$ (see page 186).

In Section 5, we define boundary values of bounded plurisubharmonic functions with respect to the measures μ_u. Here we most often assume that $u \in \mathcal{F}^a(\Omega)$, where $\mathcal{F}^a(\Omega)$ is the set of functions $u \in \mathcal{F}(\Omega)$ such that $(dd^c u)^n$ vanishes on pluripolar sets. Note that there is no loss of generality in doing this, since $\mu_u = \mu_{\max\{u,-1\}}$ where $\max\{u,-1\} \in \mathcal{F}^a(\Omega)$ (bounded functions cannot put mass on pluripolar sets, see e.g. [1]). It follows from Lemma 5.1 that if $u \in \mathcal{F}^a(\Omega)$ and $g \in PSH(\Omega) \cap L^\infty(\Omega)$, then $g(dd^c u^j)^n$ tends weak* to $g^u \mu_u$ where $g^u \in L^\infty(\partial\Omega, \mu_u)$. The function g^u is considered as boundary value of g with respect to μ_u, and it depends, at least formally, on both g and u. However, if g is plurisubharmonic and bounded on some strictly larger domain $W \supset \overline{\Omega}$, then $g^u = g|_{\partial\Omega}$ a.e. (μ_u) for each u, see Theorem 5.3. In Theorem 5.4, the g^u-boundary values are related to the notion of boundary values in terms of maximal plurisubharmonic functions:

Theorem. *Suppose that $H \in \mathcal{M}(\Omega) \cap L^\infty(\Omega)$. Then, for every $u \in \mathcal{F}^a(\Omega)$ and every $g \in \mathcal{F}(\Omega, H)$ such that $\int_\Omega g\,(dd^c u)^n > -\infty$, $g\,(dd^c u^j)^n$ is weak*-convergent to $H^u\,d\mu_u$.*

In Section 6, we study more general boundary measures on a more restricted class of hyperconvex domains (including for example polydiscs and strictly pseudoconvex domains). The idea is to start with a measure μ on $\partial\Omega$ with certain properties, and then approximate μ by a procedure similar to the one in Section 3, see Theorem 6.1 for details. By Example 6.3, we reach more boundary measures by the method in this section.

Paper II: An ordering of measures induced by plurisubharmonic functions

In this paper we study the following order-relation between measures on Ω.

Definition. Let μ and ν be measures on Ω. Then we define

$$\mu \preceq \nu \quad \Longleftrightarrow \quad \int_\Omega \varphi\,d\mu \geq \int_\Omega \varphi\,d\nu, \quad \forall\,\varphi \in \mathcal{E}_0(\Omega) \cap C(\overline{\Omega}),$$

and we say that μ is *plurisubharmonically less than* ν.

It follows that if $\mu \preceq \nu$, then $\int_\Omega \varphi\, d\mu \geq \int_\Omega \varphi\, d\nu$ holds for each $\varphi \in PSH^-(\Omega)$ (choose $\{\varphi_j\}_{j=1}^\infty \subset \mathcal{E}_0(\Omega) \cap C(\overline{\Omega})$ such that $\varphi_j \searrow \varphi$), in particular $\mu(\Omega) \leq \nu(\Omega)$. Moreover, if $\mu \leq \nu$ as measures, then $\mu \preceq \nu$. Two basic examples are the following.

Example. Fix $z \in \Omega$ and choose $r > 0$ such that $\mathbb{B}(z,r) \subset\subset \Omega$. Let σ_r be the normalized area measure on the sphere $\partial\mathbb{B}(z,r)$ and δ_z the Dirac measure at z. Then $\sigma_r \preceq \delta_z$ by the mean-value inequality for subharmonic functions. Similarly, $\sigma_r \preceq \sigma_s$ if $r \geq s$.

Example. If u and v are functions in $\mathcal{F}(\Omega)$ such that $u \geq v$ then $(dd^c u)^n \preceq (dd^c v)^n$, via integration by parts.

The first example shows that $\mu \preceq \nu \nRightarrow \mu \leq \nu$. The second example can be considered as the inspiration for the plurisubharmonic ordering. It follows for example that if $\{u_j\}_{j=1}^\infty$ is a decreasing sequence in $\mathcal{F}(\Omega)$, then $\{(dd^c u_j)^n\}_{j=1}^\infty$ is a plurisubharmonically increasing sequence of measures. Hence, the plurisubharmonic ordering provides a way of generalizing monotone sequences in $\mathcal{F}(\Omega)$. Note that by Example 3.4 in this paper, $(dd^c u)^n \preceq (dd^c v)^n$ does not imply that $u \geq v$.

In Section 4, we study maximality with respect to the $\preceq$-ordering, and a related notion of minimality for certain plurisubharmonic functions.

Definition. A finite measure μ on Ω is said to be *maximal* if for any measure ν on Ω such that $\nu(\Omega) = \mu(\Omega)$, the relation $\nu \succeq \mu$ implies that $\nu = \mu$.

We show for example that $\mu = \sum_{j=1}^N a_j \delta_{z_j}$, where δ_{z_j} is the Dirac measure at $z_j \in \Omega$ and $a_j > 0$, is a maximal measure (Example 4.5). In Theorem 4.7 it is proved that each finite measure on Ω with compact support is majorized (in the plurisubharmonic ordering) by a maximal measure with the same total mass.

Definition. A function $u \in \mathcal{F}(\Omega)$ is said to be *minimal* if for any function $v \in \mathcal{F}(\Omega)$ such that $\int_\Omega (dd^c v)^n = \int_\Omega (dd^c u)^n$, the relation $v \leq u$ implies that $v = u$.

If u is a function in $\mathcal{F}(\Omega)$ such that $(dd^c u)^n$ is a maximal measure, then u is a minimal function (Proposition 4.9). The converse holds if $n = 1$, i.e. if $u \in \mathcal{F}(\Omega)$ is a minimal function then $(dd^c u)^1$ is a maximal measure (Proposition 4.11). It is unknown to the author if this is also true in higher dimension.

If $u \in \mathcal{F}(\Omega)$ and $(dd^c u)^n$ is carried by a pluripolar set, then u is minimal (Proposition 4.12). However, there are functions in $\mathcal{F}(\Omega)$ whose Monge-Ampère measure is maximal and lives on a non-pluripolar set (Example 4.15). In Theorem 4.17 we prove that each function in $\mathcal{F}(\Omega)$ is minorized by a minimal function with the same total Monge-Ampère mass.

In Section 5, we apply the plurisubharmonic ordering to the problem of weak*-convergence of measures. The first result is that if $\{\mu_j\}_{j=1}^\infty$ is a plurisubharmonically increasing sequence of measures on Ω with uniformly bounded total mass, then μ_j tends weak* to a measure ν on Ω, and $\int_\Omega \varphi\, d\mu_j$ decreases to $\int_\Omega \varphi\, d\nu$ for each $\varphi \in PSH^-(\Omega)$ (Proposition 5.1). If we in addition have that the measures are given by $\mu_j = (dd^c u_j)$ where $u_j \in \mathcal{F}(\Omega)$ and u_j tends to $u \in \mathcal{F}(\Omega)$ in the sense of distributions, then we also know that $(dd^c u)^n \preceq \nu = \lim_{j\to\infty}(dd^c u_j)^n$ (Proposition 5.2). By Example 5.3, these assumptions are not sufficient for $(dd^c u)^n = \lim_{j\to\infty}(dd^c u_j)^n$ to hold true. However, if the functions are in the subclass $\mathcal{F}_1(\Omega) = \{u \in \mathcal{F}(\Omega) : \int_\Omega u\,(dd^c u)^n > -\infty\}$ and there is a minorant $u_0 \in \mathcal{F}_1(\Omega)$ for the sequence, then we have equality. This is proved in Theorem 5.5:

Theorem. *Assume that $\{u_j\}_{j=0}^\infty$ is a sequence in $\mathcal{F}_1(\Omega)$ such that*

- *u_j tends to $u \in \mathcal{F}_1(\Omega)$ in the sense of distributions,*
- *$\{(dd^c u_j)^n\}_{j=1}^\infty$ is plurisubharmonically increasing,*
- *$u_j \geq u_0$ for each j.*

Then $(dd^c u_j)^n$ tends weak to $(dd^c u)^n$ and $\int_\Omega \varphi\,(dd^c u_j) \searrow \int_\Omega \varphi\,(dd^c u)^n$ for each $\varphi \in PSH^-(\Omega)$.*

Finally in Theorem 5.6, we use the notion of maximal measure to prove weak*-convergence:

Theorem. *Suppose that $\{u_j\}_{j=1}^\infty$ is a sequence in $\mathcal{F}(\Omega)$ such that*

- *u_j tends to $u \in \mathcal{F}(\Omega)$ in the sense of distributions,*
- *$(dd^c u)^n$ is a maximal measure,*
- *$\lim_{j\to\infty} \int_\Omega (dd^c u_j)^n = \int_\Omega (dd^c u)^n$.*

Then $(dd^c u_j)^n$ tends weak to $(dd^c u)^n$.*

Paper III: On boundary values of plurisubharmonic functions

In this paper, we continue the work on boundary values of plurisubharmonic functions, but in a more general setting than in Paper I. The idea is to approximate a boundary measure μ with measures μ_j on the interior, and to do this in a way suitable for plurisubharmonic functions. The plurisubharmonic ordering defined in Paper II is a useful tool in order to have uniform control of the measures in the approximating sequence. Given a bounded hyperconvex domain $\Omega \subset \mathbb{C}^n$ and finite measures μ and $\{\mu_j\}_{j=0}^\infty$ on $\overline{\Omega}$, we consider the following conditions:

(1) μ is carried by $\partial\Omega$ and μ_j by Ω.
(2) $\mu_j \preceq \mu_0$ for each j.
(3) μ_0 vanishes on pluripolar sets.
(4) $\lim_{j\to\infty} \mu_j = \mu$ (weak*-limit as measures on $\mathbb{C}^n$).
(5) $\lim_{j\to\infty} \int_\Omega \varphi\, d\mu_j$ exists for each $\varphi \in PSH(\Omega) \cap L^\infty(\Omega)$.

In Section 3, we study measures satisfying some or all of the conditions above. Note that if $u \in \mathcal{F}^a(\Omega)$ and we take $\mu_0 = (dd^c u)^n$, $\mu_j = (dd^c u^j)^n$ for $j \geq 1$ and

$\mu = \mu_u$, with notation from Paper I, then (1) – (5) are satisfied. However, to assume (1) – (5) is a more general approach, since it allows for more boundary measures, see Example 3.1. We obtain the following results.

- If (1) – (4) are satisfied, then $\lim_{j\to\infty} \int_\Omega \varphi \, d\mu_j = \int_{\partial\Omega} \varphi \, d\mu$ for each $\varphi \in PSH(W) \cap L^\infty(W)$, where $W \supset \overline{\Omega}$ is hyperconvex (Theorem 3.2). Using a standard argument (Lemma 3.3), it follows that $\psi\mu_j$ tends weak* to $\psi\mu$, for each $\psi \in PSH(W) \cap L^\infty(W)$ (Corollary 3.4).
- If (1), (4) and (5) are satisfied, then for each $\psi \in PSH(\Omega) \cap L^\infty(\Omega)$ there is a function $\psi_\mu \in L^\infty(\partial\Omega, \mu)$ such that $\psi\mu_j$ tends weak* to $\psi_\mu\mu$ (Proposition 3.6).
- If (1) – (3) are satisfied (actually, the boundary measure μ is not needed here) and $\lim_{j\to\infty} \mu_j(K) = 0$ for each compact set $K \subset \Omega$, then $\lim_{j\to\infty} \int_\Omega u \, d\mu_j = 0$ for each $u \in \mathcal{N}(\Omega) \cap L^\infty(\Omega)$, in particular $u\mu_j$ tends weak* to 0 (Theorem 3.7).
- If (1) – (5) are satisfied and $H \in \mathcal{M}(\Omega) \cap L^\infty(\Omega)$ is given, then $v\mu_j$ tends weak* to $H_\mu\mu$ for each $v \in \mathcal{N}(\Omega, H) \cap L^\infty(\Omega)$ (Theorem 3.8).

In Section 4, we use measures tending to the boundary as above, to define boundary values of certain plurisubharmonic functions:

Definition. Let $\mathfrak{M}_0$ be the set of pairs $(\mu, \{\mu_j\}_{j=0}^\infty)$ such that μ and $\{\mu_j\}_{j=0}^\infty$ are finite measures on $\overline{\Omega}$ with properties (1) – (5). A function $\varphi \in PSH(\Omega) \cap L^\infty(\Omega)$ is said to have *boundary value* φ_b *with respect to* $\mathfrak{M}_0$ if $\varphi_b : \partial\Omega \to \mathbb{R}$ is a bounded Borel measurable function such that

$$\lim_{j\to\infty} \varphi\mu_j = \varphi_b\mu, \text{ for each pair } (\mu, \{\mu_j\}_{j=0}^\infty) \text{ in } \mathfrak{M}_0,$$

where the limit is in weak*-sense on $\mathbb{C}^n$. We denote the set of plurisubharmonic functions on Ω that have boundary values with respect to $\mathfrak{M}_0$ by $\mathcal{B}(\Omega, \mathfrak{M}_0)$.

Two basic properties of the class $\mathcal{B}(\Omega, \mathfrak{M}_0)$ are given in Proposition 4.3; if $\varphi, \psi \in \mathcal{B}(\Omega, \mathfrak{M}_0)$ then $\varphi + \psi \in \mathcal{B}(\Omega, \mathfrak{M}_0)$ with $(\varphi + \psi)_b = \varphi_b + \psi_b$, and if $\varphi \in \mathcal{B}(\Omega, \mathfrak{M}_0)$, $\varphi \geq 0$, then $\varphi^2 \in \mathcal{B}(\Omega, \mathfrak{M}_0)$ with $(\varphi^2)_b = (\varphi_b)^2$.

Because of the weak*-convergence, we immediately have that if φ is a function in $PSH(\Omega) \cap C(\overline{\Omega})$ then $\varphi \in \mathcal{B}(\Omega, \mathfrak{M}_0)$ with $\varphi_b = \varphi|_{\partial\Omega}$. Given a general function $\varphi \in PSH(\Omega) \cap L^\infty(\Omega)$ and a pair $(\mu, \{\mu_j\}_{j=0}^\infty) \in \mathfrak{M}_0$, we know (Proposition 3.6) that we have a boundary value $\varphi_\mu \in L^\infty(\partial\Omega, \mu)$ with respect to the chosen pair. But if we want to say that $\varphi \in \mathcal{B}(\Omega, \mathfrak{M}_0)$, we need to find a function φ_b such that $\varphi_b = \varphi_\mu$ a.e. (μ) for each $(\mu, \{\mu_j\}_{j=0}^\infty) \in \mathfrak{M}_0$. The results of Section 3 leads to Theorem 4.2, which presents three situations where we can find such a boundary value function:

Theorem.

- *If* $u \in \mathcal{N}(\Omega) \cap L^\infty(\Omega)$ *then* $u \in \mathcal{B}(\Omega, \mathfrak{M}_0)$ *with* $u_b = 0$.
- *If* $\psi \in PSH(W) \cap L^\infty(W)$ *where* $W \supset \overline{\Omega}$ *is bounded and hyperconvex, then* $\psi \in \mathcal{B}(\Omega, \mathfrak{M}_0)$ *with* $\psi_b = \psi|_{\partial\Omega}$.

- *If* $H \in \mathcal{M}(\Omega) \cap L^\infty(\Omega)$ *is in* $\mathcal{B}(\Omega, \mathfrak{M}_0)$ *and* $v \in \mathcal{N}(\Omega, H) \cap L^\infty(\Omega)$*, then* $v \in \mathcal{B}(\Omega, \mathfrak{M}_0)$ *with* $v_b = H_b$*.*

Finally we consider the class $\mathfrak{M}_1$, defined as the set of pairs $(\mu, \{\mu_j\}_{j=0}^\infty)$ satisfying (1) – (4). In Proposition 4.5 we show that if $\varphi \in PSH(\Omega) \cap L^\infty(\Omega)$ has the property that $\lim_{j\to\infty} \int_\Omega \varphi \, d\mu_j$ exists for all pairs $(\mu, \{\mu_j\}_{j=0}^\infty)$ in $\mathfrak{M}_1$, then the boundary value φ_μ depends only on φ and the boundary measure μ, not on the choice of approximating sequence $\{\mu_j\}_{j=0}^\infty$. Moreover, if we have two different pairs in $\mathfrak{M}_1$, then we can find a function which serves as boundary value of φ relative to both boundary measures.

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