SHORT COMMUNICATION

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KEY IDEAS: WHAT ARE THEY AND HOW CAN THEY HELP US UNDERSTAND HOW PEOPLE VIEW PROOF?

ABSTRACT. This paper examines the views of proof held by university level mathematics students and teachers. A framework is developed for characterizing people's views of proof, based on a distinction between public and private aspects of proof and the key ideas which link these two domains.

KEY WORDS: proof, calculus, mathematical thinking, epistemology, mathematical ideas

1. OVERVIEW

The product of this research is a theoretical framework for characterizing people's views of proof. The framework, grounded in empirical data reported in Raman (2001), brings together two different but related ideas concerning the production and evaluation of mathematical proof. The first is a distinction between an essentially *public* and an essentially *private* aspect of proof (with connections, among others, to the notion of proof scheme suggested by Harel and Sowder, 1998). The second is the notion of *key idea* (with connections to, for example, Steiner's (1978) notion of explanatory proof).

PUBLIC AND PRIVATE ASPECTS OF PROOF

Research indicates that students at both high school and university level have difficulty, not only in producing proofs, but even in recognizing what a proof is (e.g. Chazan, 1993; Moore 1994). The difficulty in understanding the nature of proof has also been reported among prospective elementary school teachers (Simon, 1996) and experienced high school teachers (Knuth, 2002). Understanding the nature of proof, in addition to its theoretical interest, seems essential for thinking about how to teach students about proof, both at the university level and throughout the K-12 level, as is recommended by the new NCTM standards (NCTM, 2000).

While it is widely agreed that students have difficulty with the nature of proof, there is little agreement on what the nature of proof is. In fact, the nature of proof has been a matter of debate among mathematicians, philosophers, and historians for hundreds of years. One product of this debate has been various distinctions between different types of proof.

For instance as Steiner (1978) points out and Hanna (1990) emphasizes, mathematicians routinely distinguish between proofs that demonstrate and proofs that explain (though it remains an open question what it means for proofs to explain.)

Proof has also been described as a part of a discourse practice (Sfard 2000), with a distinction being made between discourse with oneself (in the case of trying to produce a proof) and discourse with others (in the case of trying to communicate a proof to someone else)[1]. This distinction is similar to that made by Harel and Sowder between the process of ascertaining ("removing one's own doubt"), and persuading ("removing other's doubt"). It also bears resemblance to Mason's (1985) pedagogical suggestion of how to create a proof: first convince yourself, then convince a friend, then convince an enemy.

In all these cases, people seem to distinguish between an essentially private and an essentially public aspect of mathematics; that is to say proof involves both public and private arguments. By "private argument" I mean, "an argument which engenders understanding", and by "public" I mean "an argument with sufficient rigor for a particular mathematical community". "Mathematical community" refers not only to a particular setting, but also the people involved along with their expectations for the kind of argument needed within that setting. Examples of mathematical communities could include formal settings like an exam, publication in a journal, or an informal setting like office hours or a conversation between two mathematicians in the same field.

Raman (2001) shows that university mathematicians and students think about the public and private aspects of mathematics in a subtly but fundamentally different way[2]. For the faculty, the public and private aspects are inextricably linked. Consider the way the following professor reasons about the task: Prove that the derivative of an even function is odd.

Prof A: Let's see, an even function. There is only one thing about it, and that is its graph is reflected across the axis. Yeah, and you can be quite convinced that it is true by looking at the picture. If you said enough words about the picture, you'd have a proof.

The argument that engenders understanding to this professor—that is the argument that convinces him personally that the claim is true—provides the underpinnings for a proof, an argument that he feels would communicate his private idea in rigorous language. However,

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[1] Note that it is not the act of communicating that distinguishes these two types of discourse. Sfard defines "discourse" as any type of communication and "thinking" as communication with oneself. The difference between the two types of discourse is the object of the communication.

[2] Note that I am not considering all professional mathematicians here (e.g. applied mathematicians working in industry). The focus is on university level students and teachers, with an eye towards improving communication between these two groups.

note that while the public and private expressions of his idea may be different, he sees the two as linked—they both represent the same idea.

That is not to say that all proofs are generated by mapping a privately held idea into the coin-of-the-realm. Another professor in the same study produced a proof of the same claim by using the formal definition of derivative. When asked if he had any pictures in mind when he generated that proof, he replied:

> Prof B: No, I didn't do it geometrically. If I had had trouble writing up the analytic solution, then I would have drawn myself a picture. But I looked at this and I thought, if this is going to be true it's got to come out of the definition of derivative.

Note, however, that even though he generated the proof without recourse to any informal sorts of understandings, he knows that he *could* have. His concept of proof, just like Prof. A's, is one in which the public and private aspects have an essential connection.

This view of proof appears in stark contrast to that held by many students, who see the public and private aspects of mathematics as essentially different. Consider the following student who (a) tried to generate a proof by looking at examples, (b) wrote down a formal definition of derivative and (c) got stuck. Later he was shown several proofs of the claim, and he remarked:

> Student A: You are creating something out of nowhere, when you prove. If you don't use the definition of derivative or you don't have the chain rule, I would say it is pretty impossible to go about proving this problem.

The crucial distinction between this student's view and those of the professors above is that he sees proving as "creating something out of nowhere". While he talks in other parts of the interview about differences between public and private aspects of proof (e.g. he is completely convinced that the claim has to be true based on examples that he has generated, however he doesn't think examples constitute proof.) This student, like many others both in this study and a similar one conducted at another site (Morrow, in prep), students appear limited in their ability to generate a proof not only by a lack of knowledge (that is to say, that they don't have the key idea of the proof like Prof A above), but also an insufficient epistemology. They do not see the essential connection between their privately held idea and what they expect to produce as a formal, public proof.

2. Analysis of underlying ideas

In order to further understand the differences between the student and faculty in this study, I suggest that there are three essentially different kinds of ideas involved in the production and evaluation of a proof (related to Hanna's and Steiner's distinction between proofs that demonstrate and proofs that explain discussed above). The data presented below come from interviews in which students, graduate students and faculty discussed the way they tried to prove the claim that the derivative of an even function is odd.

2.1.Heuristic idea

The first type of idea used in proof production is called a heuristic idea. This is an idea based on informal understandings, e.g. grounded in empirical data or represented by a picture, which maybe suggestive but does not necessarily lead directly to a formal proof. A heuristic idea gives a sense of understanding, but not conviction. It gives a sense that something *ought* to be true.

Here is an example of the expression of this idea by a student:

> Student A: So my understanding of derivative is that you subtract the power by one. Right, so if you have an even function, the power is even, so it always comes out to be odd. That's my… my intuitive understanding of the problem. And then… I don't know… I tried to get somewhere, but I really couldn't, so I just write down the formula for the… I guess the definition for what the derivative is. So, that's what I have. And I couldn't go anywhere from there.

Note that the student begins with a procedural view of derivative, closely linked to the kinds of computational activities expected of him in class. He also seems to limit the functions he considers to polynomials, again reflecting his class experience. His reasoning gives him a sense that the claim ought to be true, but he does not see any way to make this argument into a formal proof. In contrast, some of the faculty in the study also thought about the case of polynomials, but they would go on to argue that one could generalize to a Taylor series argument (if all functions were analytic). So we see that the student's approach here is not entirely naïve, but he does not have the resources to take his argument further.

2.2 Procedural idea

The second type of idea used in proof production is called a procedural idea. This is an idea based on logic and formal manipulations which leads to a formal proof without connection to informal understandings A procedural idea gives a sense of conviction, but not understanding. It demonstrates *that* something is true

Here is an example of the expression of this idea by a student:

> Grad A: So, I mean, my general approach to proofs like this is—it says to prove something. It's got a bunch of words in it. I know what they mean. I always write down, "let whatever it is be whatever I'm supposed to start with." Say what that means, definitionally. And then see if I could see an obvious way to take that and get the definition of the next thing.

Whereas Student A's approach might be seen as bottom-up, starting with concrete examples and trying to make a general proof, Grad A's approach can be seen as top-down. She has a general idea about how to go about doing proofs, at least the kind commonly found in textbooks, and tries to finesse a proof by following this procedure. By itself, the proof she generates, while correct, does not give her a sense of why the claim is true.

2.3 Key idea

Finally, the third type of idea that can lead to proof production is called a key idea. A key idea is an heuristic idea which one can map to a formal proof with appropriate sense of rigor. It links together the public and private domains, and in doing so gives a sense of understanding and conviction . Key ideas show *why* a particular claim is true.

This idea was expressed by a faculty member, Prof A, as already quoted above:

> Prof A: Let's see, an even function. There is only one thing about it, and that is its graph is reflected across the axis. Yeah, and you can be quite convinced that it is true by looking at the picture. If you said enough words about the picture, you'd have a proof.

What is it that distinguishes the professor's approach from those of the graduate student and undergraduate student? The professor does not proceed from concrete examples or a procedure for producing a proof argument, but rather from the key idea that makes the claim true. "There is only one thing" about an even function, its symmetry, which both provides an explanation for why the claim is true and can be translated into formulas which demonstrates readily that the claim is true.

3. CONCLUDING REMARKS

We now have two different ways of characterizing the differences between students and faculty in Raman (2001), one in terms of a distinction between public and private aspects of mathematics, and the other in terms of different kinds of ideas underlying the mathematical argumentation. How are these characterizations related? The connection is rather straightforward: the heuristic idea is essentially private, the procedural idea is essentially public, and the key idea provides the link between the two.

For mathematicians, proof is essentially about *key ideas*; for many students it is not. This is in part because students do not have the key idea (an issue of knowledge), but more interestingly because they do not realize that proof is about key ideas (an issue of epistemology). Further, it seems to be the case that even though mathematicians value key ideas in their own work, they do not tend to emphasize those key ideas in instruction, and more crucially, in assessment. Thinking about how to make key ideas a more central part of both the high school and college curriculum, then, seems to be an important step towards helping students develop a mature view of mathematical proof.

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