

A Theoretical Model of the Connection Between the Process of Reading and the Process of Solving Mathematical Tasks

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In this paper we suggest a theoretical model of the connection between the process of reading and the process of solving mathematical tasks. The model takes into consideration different types of previous research about the relationship between reading and solving mathematical tasks, including research about traits of mathematical tasks (a linguistic perspective), about the reading process (a psychological perspective), and about behavior and reasoning when solving tasks (a mathematics education perspective). In contrast to other models, our model is not linear but cyclic, and considers behavior such as re-reading the task.

Introduction

The relation between reading ability and mathematical ability is important to examine, especially since written tests are the predominant form of assessment in mathematics teaching. However, this relationship is also complex. On the one hand, tests which are intended to measure achievement in mathematics should not measure reading ability, which the framework for PISA highlights by noting that: “The wording of items should be as simple and direct as possible” (PISA, 2006, p. 108). On the other hand, a student has to be able to read and write in order to pass a paper and pencil test. Also, communication is one of the competences brought forward within frameworks describing school mathematics worldwide; see for example NCTM (2000) and PISA (2006). Reading (and writing) mathematics can then be seen as an important part of knowing mathematics. Thus, we cannot completely separate reading ability from mathematical ability.

If you, the typical reader of this article, read a mathematical task in order to solve it, when does the solving begin? It can be difficult to separate the reading process from the solving process (e.g. see the analysis by Österholm, 2007). In some cases, the process of reading seems to be clearly separated from the process of solving, and in other cases the two processes seem to be highly integrated. Sometimes you start to think of the solution before you are done reading. This might happen when you read the following task: “Seven girls had a backpack each, and in each backpack there were seven cats. Each cat had seven kittens, etc.” Sometimes you have solved the task at the same time as you are finished

reading. For example, this might happen when you read the following task: “What equals $2+3$?” Other times you have to read the task several times in order to solve it, but when you finally have figured out what the task’s wording actually meant, you already know the answer. This might happen when you read the following task: “If Anne was half Mary’s age when Mary was 14, then how old will Mary be when Anne is 50?” In fact, once you have decoded the text, the solution is right at hand. It seems unclear whether this process should be seen as an instance of utilizing a reading ability, a mathematical ability, or a mixture thereof. Through these examples we highlight, again, the complex relationship between reading ability and mathematical ability, and also the complex relationship between reading and solving a mathematical task. In addition, these simple examples highlight the need for a theoretical model of reading and solving mathematical tasks that takes these relationships into consideration.

The purpose of this paper is to present a theoretical model that describes aspects of both reading and solving mathematical tasks and that takes into account research from three perspectives; linguistics, psychology, and mathematics education. By examining existing research about relations between reading and solving mathematical tasks in each of these perspectives, we argue for the necessity of a model that combines them. More long-term goals, not to be fulfilled in this paper, are that the model should be possible to use to explain empirical data and as a guide for planning and designing empirical studies about the relationship between reading and solving mathematical tasks. The model is of a theoretical nature since it is not based on empirical data, and the purpose is not that it should act as a theory for analyzing all aspects of task solving or of reading comprehension. The purpose is to capture the dynamic relationship between reading and solving tasks, in order to be able to study this relationship in more detail.

Three perspectives on reading and solving

In the existing literature about reading and solving mathematical tasks, we find that articles tend to focus on properties of the task text (a linguistic perspective), on the reading process (a psychological perspective), *or* on behavior and reasoning when solving a task (a mathematics education perspective). The literature survey by Österholm (2007) shows that not much research exists that directly focuses on the relationship between reading and solving mathematical tasks, and in line with this result we have found that only in a few cases does a single study consider a combination of at least two of the mentioned perspectives.

In the following three sections we characterize each of the three perspectives, in particular regarding if and how they view connections between reading and solving mathematical tasks. We do not discuss many (empirical) studies in relation to reading and solving within these three large research areas, but the

purpose is now to localize and discuss frameworks and models that seem most useful for our purpose of creating a model that characterizes the connection between the process of reading and the process of solving mathematical tasks.

The linguistic perspective – the wording and grammar of tasks

Within linguistic research the object of analysis is often the wording or the grammatical properties of the text. There are some features of mathematical texts and tasks that might decrease the readability, according to linguistic research. Examples of such properties are technical vocabulary, multiple semiotic systems, and grammatical patterns (Schleppegrell, 2007). Linguistic research also considers in what ways readability formulas can predict the readers' difficulties understanding a text or a test item (Homan, Hewitt, & Linder, 1994). The underlying assumption is that texts with higher readability indices are more difficult to read than texts with lower indices. These studies often use predictors, for example the number of difficult words or word length, as basis for the calculation of an index (Homan et al., 1994). The implicit model of the reading comprehension of a mathematical task within linguistic research can therefore be seen as a simple function of two variables, the reader's prior knowledge (including different kinds of abilities) and the complexity of the text; that reading comprehension increases with better prior knowledge and with lower text complexity.

The psychological perspective – the process of reading

There exists plenty of research from a psychological perspective focusing on reading comprehension and the process of reading. In this kind of research, reading seems most often discussed through the characterization of mental processes and mental representations. Such characterizations are done in different ways in different theoretical frameworks, for example by relating to Bloom's taxonomy regarding mental processes (Graesser, León, & Otero, 2002) or by describing multiple levels of representation (van Oostendorp & Goldman, 1998).

There is no room here to cover the complexities of different types of characterizations of mental processes and mental representations in order to discuss different possibilities for describing the reading and solving of mathematical tasks as seen from a psychological perspective. However, there are some frameworks about reading, as seen from a psychological perspective, which have also been applied or related to the solving of mathematical tasks, which is an important aspect in our choice of framework. Kintsch's (1998) theory of *comprehension* will be used as the main framework describing aspects of reading in our model, a choice based on three reasons: (1) the work of Kintsch has had a great general influence on research about reading comprehension (Weaver, Mannes, & Fletcher, 1995), (2) the theory includes detailed models about both the mental process and the mental representation in reading comprehension, and

(3) the theory has been applied to the situation when students solve mathematics word problems.

Kintsch (1998) describes the mental representation of texts by distinguishing between three different levels, or components, of the mental representation; the surface component, the textbase, and the situation model. The surface component refers to when the words and phrases themselves, and not their meaning, are encoded in the mental representation. The textbase represents the meaning of the text, that is, the semantic structure of the text. The situation model is a construction that integrates the textbase and aspects of the reader's prior knowledge.

Besides the components of mental representation, Kintsch's theory also includes a model, the construction integration model, which describes how a mental representation is created in the comprehension process, in particular how the utilization of prior knowledge occurs through associative activation. This model describes a fundamental cognitive functioning when interpreting something 'external', which for example can refer to a given text and also a text you create yourself, such as when starting a calculation in the solving of a task.

When Kintsch (1998, chapter 10) applies his theory of comprehension to word problems, another component is added to the description of mental representations; the so called problem model, which is a mathematization of what is described in the text. The problem model consists of a schema from the reader's long-term memory that is activated based on some properties of the situation model. This reliance on schemas has been criticized by other researchers, for example since there are empirical results about word problems that do not seem compatible with a schema theory (Thevenot, Devidal, Barrouillet, & Fayol, 2007). Kintsch (1998, p. 354) also acknowledges that there are limitations to this part of the theory, since it assumes the existence of "full-fledged schemas [...] that need only to be applied correctly", while other studies he refers to show that this is usually not the case, and that acquiring such schemas "is a major facet of learning". Instead, the solving of word problems usually utilizes knowledge that is "less orderly, less abstract, and more situated" (p. 357).

It seems like a model of the reading and solving of mathematical tasks from a psychological perspective could be divided into two cases. First, we have a model that is based on the utilization of schemas. This model seems to be limited to tasks that are of a very familiar type for the reader, since it assumes the existence of "full-fledged schemas", which have been created through abstraction from plenty of experiences of similar types of tasks. Second, we have a model that turns "away from the abstract schema concept and toward the notion of situated cognition" (Kintsch, 1998, p. 355). In this model it seems unnecessary to introduce the notion of problem model, since this notion seems closely connected to abstract aspects and not situational aspects.

The mathematics education perspective – the process of solving

Much mathematics education research related to reading and solving mathematical tasks consider *word problems*. Although there is plenty of such research, a literature survey among journal articles showed that “not many studies exist that in a direct manner examines the relation between reading and problem solving among the 199 references about word problems” (Österholm, 2007, p. 141). Instead, most studies about word problems seem to focus on aspects of modeling and relationships between the task and the “real world” (e.g. see Verschaffel, Greer, Van Dooren, & Mukhopadhyay, 2009), where the *process of reading* is most often not problemized. In addition, studies about word problems that do focus on aspects of language and reading seem to see a separation between the process of reading and the process of solving, which is not congruent with our own view. For example, Roth (2009, p. 63) characterizes the problem an informant has as “related to understanding the text, not one of providing the sought-for problem” and that students could need assistance in “understanding first the text and then the task”. Therefore, instead of using perspectives on reading and solving from studies about one genre of mathematical tasks (word problems), which tend to focus on aspects not directly relevant for other types of tasks, we choose to direct our attention to frameworks that more generally describe mathematical task solving.

Within mathematics education research there are many theoretical frameworks that can be used to describe the *solving* of mathematical tasks, see for example Vinner (1997) and Lithner (2008). These authors both present two fundamentally different concepts that are relevant when describing how students solve a mathematical task; pseudo-analytic and analytic thinking (Vinner, 1997), and imitative and creative reasoning (Lithner, 2008). The first concept in each framework is connected to the student *knowing how* to solve a task without necessarily knowing why the method works, and the other to the student *knowing both how and why*. Lithner (2008) sees the wording of a task as a surface property that the solver sometimes can use in order to identify the type of solution strategy (cf. schema) appropriate to solve the task. Vinner (1997) similarly discusses how the solver can determine the appropriate solution procedure to a given question by using mental schemes to determine the *similarities* between the question and so called *typical questions*. He does not explicitly say so, but it is reasonable to assume that these similarities could for example be based on the wording or grammatical structures of the tasks. Schoenfeld (1985) considers the activity of reading as an important part of problem solving but as a behavior rather than as a mental process. Pòlya (1945) includes issues connected to reading in what he calls the first phase of problem solving; understanding the problem. He argues that the wording of the problem needs to be understandable

for the student, but he does not further examine what the connection is between reading and solving.

These and other similar frameworks do not focus on the reading *process*, but put more emphasis on reading as a type of activity or behavior when describing the solving of mathematical tasks. However, they do sometimes mention or consider the wording in the mathematical tasks when discussing the students' solution strategies.

Lithner's (2008) research framework will be used as the main framework describing the solving process in our model, since we see it as the most detailed framework, including more well-defined components, compared to similar frameworks, and since it is in many ways representative of the type of frameworks discussed above, for example Vinner (1997). Also, this framework neither focuses on nor excludes word problems. Lithner characterizes different types of reasoning and in this paper we apply the framework to the solving of mathematical tasks in particular. Therefore we see "reasoning" as any method that can be used to solve tasks (a standpoint that basically coincides with the definition of reasoning within the framework) whether the solution is correct or not. Lithner divides all types of reasoning into two major categories, imitative and creative reasoning. The basic types of imitative reasoning are memorized reasoning (i.e. remembering a whole answer) and algorithmic reasoning (i.e. remembering an algorithm and calculating an answer) and both are fundamentally different from creative reasoning.

The basic steps when performing imitative reasoning are *choosing a strategy* and *implementing the strategy* (Lithner, 2008). The choice is made based on surface properties of the task, for example the wording. The implementation consists of writing the answer down (memorized reasoning) or following the steps of the algorithm (algorithmic reasoning).

Creative reasoning is based on the intrinsic mathematical components of the task, and the reasoning is new (to the solver) and flexible. In order to solve a problem (a task that is basically new to the solver) it is necessary to use creative reasoning, since it is not possible to identify the task and choose a memorized method based on its surface properties.

A suggested model of reading and solving mathematical tasks

Criticism of existing models

Empirical studies related to Kintsch's (1998) theory of different components of mental representation in reading comprehension confirm the need to distinguish between these components. These studies also show an intricate relationship between text properties (in particular text coherence), the reader's prior knowledge, and the reader's reading comprehension. In particular, results show that "readers who know little about the domain of the text benefit from a

coherent text, whereas high-knowledge readers benefit from a minimally coherent text” (McNamara, Kintsch, Songer, & Kintsch, 1996, p. 1). These results from the psychological perspective highlight some shortcomings in the linear model of reading comprehension that is used, or implied, in linguistic research. We are however not dismissing linguistic research as a relevant influence on the process of creating our model. The plethora of research that statistically connect different linguistic properties of tasks to either reading comprehension or success rate in solving the tasks need to be considered in our model, which is not done in this paper, but planned for in future studies.

Even if psychological theories relate to, and make predictions of, students’ solving of mathematical tasks, they are in principle limited to the description of mental representations. However, the behavioral aspects of solving mathematical tasks cannot be reduced to something directly and unambiguously determined by mental representations of texts. As Kintsch (1998, p. 333) mentions: “What the student remembers and what the student does are related in informative ways and *mutually constrain each other*” (emphasis added). This relationship, in particular the mutual constraint, is not described within the psychological models of reading and solving mathematical tasks, since these models focus on describing mental representations (i.e. content and structure of memory). However, this relationship is not included in theories from mathematics education research either, since such theories try to explain phenomena through aspects of students’ explicit strategies and reasoning, thus focusing on behavior rather than mental representations. This type of research ignores the process of reading the task and the good predictions that can be made from psychological theories that focus on details of the reading process. Thus, an important starting point for our model is to analyze connections between these two perspectives; one that focuses on mental representations (the psychology perspective) and one that focuses on aspects of behavior (the mathematics education perspective).

A study of Hegarty, Mayer, and Monk (1995) focuses both on aspects of mental representation and on aspects of behavior when students are solving mathematical tasks, since it relates data on students’ reading behavior to the creation of different types of mental representations. However, there are some inconsistencies within this study, in particular regarding the relationship between the two steps of problem solving the authors refer to (p. 19); constructing a problem representation (how a student understands a problem) and solving a problem (including computational procedures and problem solving strategies). The eye fixation method used by the authors is described as a method “to gain insights into the nature of the comprehension processes” (p. 19), but the empirical results make it plausible that “the eye-fixation protocols cover both the comprehension and planning stages” (p. 25). Instead of forcing the descriptions of method and data into a model that presupposes a linear separation between

processes of comprehension and of solving (i.e. between aspects of mental representation and aspects of behavior), we argue for a model that can more easily describe the eye-fixation data, and that also includes central theoretical components from the perspectives of psychology and mathematics education.

The structure of our model

What is presented here is the first version of our model, which includes a basic structure of the relationship between aspects of reading and solving a mathematical task. This structure is a contrast to the common linear types of models, which are problematic since they, as described earlier, presuppose a separation between reading and solving. In addition, we use a structure that takes into account the mutual constraint between aspects of mental representation and aspects of behavior. Therefore, our model includes a cyclic component (see Figure 1) that allows for the behavioral component to affect the mental representation, and not only the other way around.

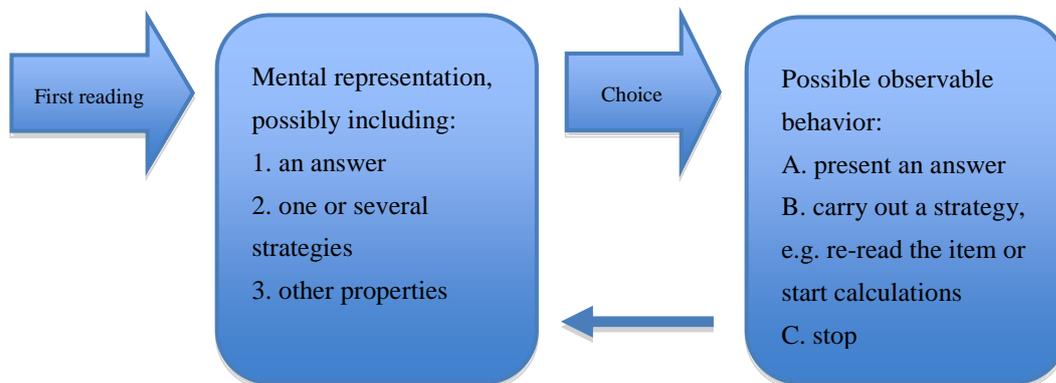


Figure 1: A visualization of the suggested model of reading and solving mathematical tasks.

The first step in the model is the *first reading* of the text, a reading that does not have to include the whole text from beginning to end. This reading creates a mental representation (MR) of the text through a process of comprehension (Kintsch, 1998). Through activation of prior knowledge (i.e. a creation of a situation model), this MR can include the explicit answer (1 in Figure 1) to the question asked in the task (e.g. when ‘5’ is activated in the reading of the task ‘What equals 2+3?’) or a strategy (2 in Figure 1) that can be used to get (closer) to an answer (e.g. when ‘derivation’ is activated when reading a question about maximum or minimum values). The word ‘strategy’ is here used in a wide sense; it could mean a mathematical strategy (e.g. an algorithm, cf. Lithner, 2008) or a more heuristic strategy, for example including the step ‘re-read the item’ or ‘ponder the question’.

These properties of the MR then guide the behavior, the choice, for example so that the solver presents the answer (A in Figure 1) if it is a part of the MR or so that he or she can carry out a strategy (B in Figure 1) that has been activated. However, it can also be other properties (3 in Figure 1) of the MR that guide the behavior (other than a direct activation of an answer or a strategy), for example that the MR is very fragmented, which could make the reader choose to re-read the text. Note that this strategy of re-reading could also have been activated in the MR, but it seems impossible to separate these possibilities in practice. Furthermore, even if the MR could activate a behavior that can be directly “applied”, a type of choice is always present, since the MR does not directly determine behavior. This is in line with Lithner (2008, p. 257) who sees the concept of *strategy choice* in this setting in a wide sense; “choose, recall, construct, discover, guess, etc.” The third option, to stop (C in Figure 1), includes giving up but also leaving the task for various other reasons (e.g. lack of time or becoming bored).

If you merely either present an answer (A) or stop (C), your behavior has little potential to affect the MR. However, every kind of *strategy implementation* (B) will affect the MR since what you do is directly related to the task and it thus becomes part of the comprehension of the task. For example, some types of activities might be seen as mostly adding something to the MR (e.g. carrying out calculations) while other types might be seen as mostly changing the existing MR (e.g. re-reading). These changes to the MR can, in the same way as it is done in the first reading of the text, activate parts of prior knowledge and can cause the new version of the MR to include new answers or strategies, thus starting a new turn in this cyclic process.

Other studies and theories in relation to our model

Let us now return to the study by Hegarty et al. (1995). Their data consist of patterns in students’ eye fixations as they read and re-read (parts of) the task text. Within our model, we have the first reading of the task as a first step, whereas the re-reading of the task is a behavioral aspect that has been performed based on some property of the MR created in the reading process. As mentioned, this property can be of different types, for example that the MR includes a specific strategy for creating the solution (e.g. to add two specific numbers given in the text) where the re-reading is performed in order to recall some specific information given in the text (e.g. the specific numbers to add), or it could be that the MR includes a more general strategy for how to handle mathematical tasks (perhaps of a specific kind). Since Hegarty et al. (1995) used relatively simple and familiar types of tasks it can be assumed that the students had knowledge of specific methods for solving these tasks. That is, the type of reasoning active for these students was of an imitative type (as labeled by Lithner, 2008). Thus, in our model, this type of reasoning can be described as the activation in the MR of a direct answer (corresponding to Lithner’s memorized reasoning) or of a strategy

(corresponding to Lithner's algorithmic reasoning). In the present version of our model it is these types of reasoning that can be described and modeled more specifically.

However, also aspects of creative reasoning could be located within our model. In particular, the constant “flow of information” in both directions between behavior and MR, which is present through the cyclic property of our model, can open up for a more dynamic and flexible pattern of thinking and behavior, which are typical criteria for creative reasoning (e.g. in Lithner, 2008). In addition, the process of comprehension (as defined by Kintsch, 1998) is active in many parts of the model, and this process includes the automatic associative activation of prior knowledge, which can include aspects of creative reasoning. For example, if you have never thought about X and Y at the same time before, but a task presents both these aspects, the activation in the process of comprehension can cause new connections to be made between X and Y in particular but also between associations to X and Y. This process can be seen as creative in a novelty aspect (a criterion given by Lithner, 2008).

Conclusions and further research

Our model is based on the assumption that we cannot separate the reading and solving of mathematical tasks. This assumption is argued for by reflecting on our own reading and solving (see the introduction) and by highlighting difficulties in the use of models that assume the opposite (see the description of our model). In addition, our model can be combined with more elaborate models related to the two main components of our model; Kintsch's (1998) theory in relation to mental representations and Lithner's (2008) theory in relation to behavioral aspects of task solving.

In the continued refinement of and argumentation around our model we will relate to more types of empirical results, in particular results from the linguistic perspective, and we will also refine the structure described in Figure 1, in particular regarding the meaning of the arrows used to denote some kind of transition and/or influence. Besides this type of continued research, we will also use this model as a basis for empirical studies in order to test the usefulness of the model and to make direct verifications or rejections of (parts of) the model.

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