Option Pricing in the Presence of Liquidity Risk

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Abstract

The main objective of this paper is to prove that liquidity costs do exist in option pricing theory. To achieve this goal, a martingale approach to option pricing theory is used and, from a model by Jarrow and Protter [JP], a sound theoretical model is derived to show that liquidity risk exists. This model, derived and tested in this extended theory, allows for liquidity costs to arise. The expression liquidity cost is used in this paper to measure liquidity risk relative to the option price.

The approach to show that there exist liquidity costs is carried out in two steps. The first step is to derive a model built on a theoretical model derived by Jarrow and Protter, and the second step is to use the model to attain the size of this cost, in this case relative to the option price.

The model uses a supply curve to account for the supply/demand in the market. The supply curve is integrated into the model and is also proven to exist, both in theory and in practice. To show how non horizontal supply curve affect option prices simulations are made using MATLAB. The impact of liquidity risk is presented as a percentage of the analytical option price.

The result show a real liquidity risk, although small, that can have a big impact in systems with a big turnover in options, i.e. financial institute, and also in a crisis when the supply curve is changed and has a bigger impact on the option price. In a "normal" case the liquidity costs is in the order of $10^{-2}$ to $10^{-3}$ percent of the option prices.

This paper uses the recent advances concerning the including liquidity risk of option pricing theory [JP]. Here new insights into the relevance of the classical techniques used in 'continuous time finance for practical risk management' are tested. The First and Second Fundamental Theorem are proven valid for this extension of the theory. Different supply curve models of liquidity issues in stock and option market trading are discussed.

*Keywords and phrases: liquidity risk, liquidity costs, option theory, supply curve, martingale approach in option pricing.*
Sammanfattning


Den här uppsatsen använder två steg för att bevisa att likviditetskostnader vid optionsprissättning existerar. Första steget är att ta fram en modell som bygger på en modell som Jarrow och Protter utvecklat och det andra steget är att använda modellen och ta fram en storlek på denna kostnad i förhållande till optionspriset.

Den framtagna modellen använder sig av utbudskurvan för att ta hänsyn till tillgång och efterfrågan på marknaden. Utbudskurvan är integrerad i modellen och det är bevisat att den existerar i teori och praktik. Simuleringar är utförda på den framtagna modellen för att visa hur likviditetskostnader påverkar optionspriset. Effekterna av likviditetsrisk presenteras som procentandel av det analytiska optionspriset.

Resultatet visar en verklig likviditetsrisk som är liten men kan ha en stor inverkan i system med en stor omsättning i optioner, till exempel finansiella institut, och även i en kris när utbudskurvan ändras och har en större inverkan på optionspriset. I ”normal” fallet ligger likviditetskostnaden storleksmässigt mellan $10^{-2}$ och $10^{-3}$ procent av optionspriset.

Denna uppsats använder de senaste framstegen inom likviditetsrisk i optionsvärderingsteorin. Här testas betydelsen av klassiska tekniker som används i praktisk riskhantering i kontinuerlig tid. Det Första och Andra Fundamentalta Teoremet inom optionsprissättning visar sig hålla i den här utökade teorin. Olika utbudskurvor diskuteras kring likviditetsfrågor vid handel på värdepappersmarknader.

Nyckelord och fraser: likviditetsrisk, likviditetskostnader, optionsteori, utbudskurva, martingal ansats vid optionsprissättning.
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1 Introduction

1.1 Background

Issues of liquidity have long been discussed between stock and option traders. If there is a given hedging strategy, the profit can be reduced by liquidity costs due to updating positions in the strategy. This liquidity costs should be included into the price one charges for the option. When calculating an option price, the price movement of the underlying asset to the maturity date has to be estimated.

Grossman and Miller [GM] in 1988 presented a supply and demand equilibrium approach to this problem. Perhaps liquidity in the stock market never could be precisely defined, but there should exist some kind of relation between price and size of order. This is the attitude of practitioners and academics alike. In the following a more modern approach of arbitrage free pricing is discussed, which has the benefit of allowing more explicit and detailed calculations.

In the approach presented here the existence of a supply curve is postulated as a new key feature of the option pricing theory. This theory is a relatively simple solution to issues of liquidity. It has been showed that a supply curve actually exists and is not just a line with slope zero. [CJP]

1.2 Purpose

Liquidity risk in option pricing is a fairly new area of research and is up to today mainly based on the experience of practitioners. Many of the results in this area are heuristic. Even though there has been some recent progress in the research concerning liquidity risk research there are still quite few mathematical approaches to liquidity risk modeling. This is partly due to quite sophisticated level of mathematics involved in this type of theory. The main purpose of this paper is to derive a sound option pricing model using the approach developed by Jarrow and Protter [JP] and to show the impact of liquidity risk by constructing a simulation tool. Simulation techniques have been successfully applied in financial mathematics but there does not exist a simulation tool for this specific case, when liquidity costs is taken into account in the option price.

The main focus is to work with existing theory in this field and derive a sound model, i.e. extending the already existing theory in option pricing, and base the model on the theory Jarrow and Protter have developed in this research area [JP]. This theory is then used to build a model of the extended economy. The model is then used to investigate if the liquidity risk has an impact on the option price set by analytical solutions, or approximated analytical solutions. The option has the famous Black Scholes model as underlying factor when deriving the price. The results presented in this paper are based on a theoretical sound, intuition regarding supply curves, i.e. liquidity costs, and the results of other researchers.
1.3 Limitations

The simulation tool will be developed to test three different options and the specific put and call types of these options, in total six different option types. The options used are chosen by how common they are in the option market and to some extent, my own interest. All options are of the type European. The Black Scholes pricing model is used when deriving the option price. This is used only to normalize the liquidity risk by present the numbers in percent of the option price. To calculate liquidity costs a supply curve needs to be determined.

1.4 Objectives

The main objective is to derive a model using the sound extended theory by Jarrow and Protter [JP], derived from the martingale approach to option theory and design a simulation tool to determine the impact liquidity costs have on the option price. This model is an extension of previously work done by Jarrow and Protter in liquidity risk theory using a martingale approach for arbitrage pricing of options. To analyze the outcome of this result a simulation tool is used to view the size of this liquidity risk in options. The result from simulations is then analyzed.

To sum up the objectives in this paper are:

1. Derive a model built on the work of Jarrow and Protter [JP] and show that liquidity risk exist when pricing options,

2. Build a simulation tool, i.e. a model of this extended theory, using the computer program MATLAB and apply input data,

3. Analyze the results from this simulation tool, and the input data.
1.5 Theoretical Introduction

The article of Jarrow and Protter concerning liquidity risk [JP] summarizes the recent advances on the inclusion of liquidity risk into option pricing theory. Classical asset pricing theory states that investors actions on the market has no impact on the price paid or received, traders act as price takers. Softening of the price due to this assumption and the impact to the actual realized returns in asset pricing models is called liquidity risk.

Liquidity risk has been extensively studied in the market microstructure literature but not as extensively in asset pricing literature. The market microstructure literature states that a big impact on prices can be due to asymmetric information or difference in risk tolerance [K]. Liquidity risk has been studied in an extreme case, market manipulation [CM]. The research done by Çetin [C] conclude that liquidity risk is related to transaction costs and thereby affect the price paid or received.

This article uses the classical option pricing theory with embedded liquidity risk. The investors act in this approach as price takers with respect to a $C^2$ supply curve on the underlying asset. [C]

Liquidity costs exist but are non-binding because no arbitrage opportunities can exist. Still with this extension appropriate generalizations of the First and Second Fundamental Theorem of asset pricing hold [CJP]. It can be stated that markets are arbitrage free if and only if there exists an equivalent martingale measure.

The supply curve must have discontinuity at 0 [C] or continuous trading strategies must be excluded to hold for the extended theory. Continuous trading strategies are not possible in practice, only approximations of simple strategies are doable [CJPW]. For simple strategies liquidity costs are binding which implies that the market is no longer complete and an upward sloping supply curve for options can exist.
1.6 Heuristic Explanation of Conceptions

Here are some words given some heuristic definitions however to get rigorous statements and concepts you are referred to Øksendal [O], Björk [B] and Hull [H]. This is soft descriptions of some concepts used in this paper to make it easier to read and understand without a deep knowledge in financial mathematics.

**Arbitrage** - A possibility to make a profit without taking any risk.

**Borel measure** - Any measure \( \mu \) defined on the \( \sigma \)-algebra of Borel sets is called a Borel measure.

**Brownian motion** - A continuous-time stochastic process with properties \( W_0 = 0 \), \( W_t \) is a.s. continuous function on \([0, T]\), has independent increments and \( W_t - W_s \sim \mathcal{N}(0, t - s) \) for any \( 0 \leq s < t \leq T \).

**FLVR.** - Free lunch with vanishing risk.

**Contingent claim** - Derivative (option with cash settlement).

**Derivative** - An agreement or contract that is not based on a real exchange, the real definition of a derivative is an agreement between two parties that is contingent on a future outcome of the underlying (option with cash settlement).

**Free lunch with vanishing risk** - Defined mathematically as a 'softer' no arbitrage definition.

**Liquidity cost** - The cost derived from liquidity risk, i.e. the premium you have to pay to free the money you have locked up in for example stocks.

**Liquidity risk** - Liquidity cost.

**Market-to-market value** - the value of an asset based on the current market price of the asset, or for similar assets, or based on another objectively assessed fair value.

**Martingale** - A stochastic process such that the conditional expected value of an observation at some time \( t \), given all the observations up to some earlier time \( s \), is equal to the observation at that earlier time \( s \).

**Maturity** - The final payment date of a loan or other financial instrument, at which point the principal (and all remaining interest) is due to be paid.

**Option** - A contract between a buyer and a seller that gives the buyer of the option the right, but not the obligation, to buy or to sell a specified asset (underlying) on or before the options expiration time, at an agreed price, the strike price.

**Replicating portfolio** - This is a fictitious portfolio used to evaluate the price of an option by buying and selling the underlying asset.

**Supply curve** - The curve explaining how the price depends on the size of a stock order.

**SFTS.** - Self financing trading strategy.

**Self financing trading strategy** - This is a trading strategy which generates no cash flows for all defined times and can be called a replicating portfolio. This portfolio borrows or invests in the money account to purchase or sell stocks.

**Semimartingale** - It can be decomposed as the sum of a local martingale and an adapted finite-variation process.

**Supermartingale** - A concave function of a martingale is a supermartingale, i.e. the expectation value decreases over time.


## 2 Liquidity and Options

Three different steps are used in this study. First a brief introduction in option theory is made. This is done to allow the reader to understand the option pricing theory and also see the problem in extending this to include liquidity risk. The next step is to derive a simple and sustainable model based on a theoretical model by Jarrow and Protter [JP] which the liquidity risk are included and also proven to exist theoretically. After these steps the model are simulated through a data program called MATLAB. This is done to ensure that the theoretical evidence also hold in practice.

The theory is presented by first defining liquidity risk and then describing what an option is and how it is priced. This is done for the reader so that a natural discussion can be made when deriving a sound theoretical framework. When liquidity risk and option pricing is defined in a suitable way the economy is discussed with a model of the economy. The model is defined and the market-to-market value and liquidity costs which arises in this model is discussed. The First Fundamental Theorem is tested in this economy and to test the Second Fundamental Theorem a brief introduction in the definition of a contingent claim (see the Heuristic Explanation of Conceptions for definition) is made. This definition is then tested in the assumption that the market is complete\(^1\) which leads to a formulation of the Second Fundamental Theorem is possible. Now examples of economies including liquidity risk can be constructed. The examples presented has the objective to illustrate the difficulty to define a theoretical sound economy with liquidity risk. A possible definition of supply curves are made and properties of such a supply curve is presented.

### 2.1 Liquidity Risk

There are two kinds of liquidity [P]: market liquidity, and funding liquidity. A security has good market liquidity if it is easy to trade, i.e. has a low bid ask spread, small price impact, high resilience and is easy to find in OTC markets. A bank or investor has good funding liquidity if it has enough available funding from its own capital or from loans.

With these definitions the meaning of liquidity risk can be described. Market liquidity risk is the risk that the market liquidity worsens when you need to trade. Funding liquidity risk is when a trader no longer has the funds to keep his position and is forced to reduce the position. An example is when a levered hedge fund loses the access to borrow from their bank and is forced to sell their holdings (securities) as a result. Another example is depositors to a bank may withdraw their funds and the bank may lose the ability to borrow from other banks or raise funds through debt issues.

Liquidity generally varies over time and across markets. The most extreme form of market liquidity risk is that dealers are shutting down (no bids). Funding liquidity risk occurs when banks are short on capital. When this is happening the banks are forced to scale back their trading that requires capital and the amount of capital they lend out, for example to other traders. If banks cannot fund themselves, they cannot fund their clients. The two forms

\(^1\)A market is complete with respect to a trading strategy if there exists a SFTS. such that at any time \(t\), the returns of the two strategies are equal.
of liquidity are linked and can reinforce each other in liquidity spirals where poor funding leads to less trading. This reduces market liquidity, increasing margins and tightening risk management to further worsening funding.

### 2.2 Basic Option Theory

Black and Scholes defined an option to be a security giving the right to buy/sell an asset under certain conditions within a specified period of time [BS]. An option is a contract between a buyer and a seller that gives the buyer of the option the right, but not the obligation to buy/sell a specified underlying asset on or before the options expiration date (maturity) at an agreed price (strike price).

There are lots of different types of options. A call option is an option that gives the right to buy an underlying asset and a put option gives the right to sell the underlying asset.

An option has a value to the person who holds this option. The buyer of an option has to pay a premium to the seller of the option and, consequently the right to buy or sell the underlying asset. When an options maturity is reached the owner may choose to exercise the right to buy or sell the asset. If in the case of a call option, the spot price is higher than the strike price, the option owner exercises the right to buy a predetermined amount of the underlying asset and sell it on the market at market price at profit. If the option chooses to exercise this right the seller of the option are obliged to sell the asset to the buyer (owner) of the option at the strike price. [H]

A call option with a strike price much lower than the asset price is likely to be exercised. The probability in this case, not to exercise the option is very small and the value of the option is close to the difference between the strike price and the spot price on the underlying asset. If

![Option Graph](image)

Figure 1: Illustrate the option value before maturity.
the strike price was much higher than the asset price the value of the option would be almost zero. The time remaining until maturity of an option also influence the value. If there is a long time until maturity the potential for the option to profitable for the buyer increases which also gives a higher value. If the underlying assets price does not change over time the value of the option drops as the maturity date comes closer which leads to a decrease in potential for a profitable option for the buyer. [BS]

The option value can be divided into two parts, the intrinsic value and the time value. These two values interact and makes up the total value of an option. The intrinsic value is a non-negative value and is non-zero if the option has a payoff.

A European option is the right to buy or sell the underlying asset only at the maturity date of the option. If these rights can be exercised at any time it is an American option. This value has different forms depending on which option you look at. Some different option types are European option, Digital option, Asian option, Lookback option, Barrier option, Spread option and Basket option. [H]

2.3 Basic Option Theory with the Martingale Approach

The theoretical approach in this paper begins with the martingale approach to arbitrage theory. This is the most general approach existing for arbitrage pricing and is also very efficient from a computational point of view.

A filtered probability space is needed when using the martingale approach to pricing options. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})$.

The market model consists of $N + 1$ a priori given asset price processes $S_0, S_1, ..., S_N$, in this case the Black Scholes model. Fundamental problems occur under these conditions; under what conditions is the market arbitrage free and complete. These problems have the famous solutions First and Second Fundamental Theorem of Mathematical Finance. These results are general and powerful but also quite deep so in this section only the main structural ideas of the proofs are presented and follows the approach made by Björk. [B]

Take a market model consisting of the asset price process $S_0, S_1, ..., S_N$ on the time interval $[0, T]$. The numeraire process $S_0$ is assumed to be strictly positive. Some simple assumptions are made. A T-claim $R$ is fixed as $R$. Derivative instruments are completely defined in terms of the underlying asset $S$ and can also be called contingent claims. The name, contingent claim, is used in this paper.

$\mathcal{F}_T^S$-adapted means that for each fixed time $T$ the process $R$ is a functional of the Wiener trajectory on the interval $[0, T]$. A T-claim $R$ can be replicated, alternatively reachable or

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2A priori knowledge is prior knowledge about a population, rather than that estimated by recent observation.

3A basic standard by which values are measured, in this case a normalized economy with $S_0$ as a numeraire, relative price process $S(t)/S_0(t)$.

4A contingent claim with date of maturity $T$, or T-claim, is any stochastic variable $R \in \mathcal{F}_T^S$. 

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hedgeable, if there exists a self financing trading strategy (SFTS.) $X$ such that

$$V^X(T) = R, \quad \text{in } \mathbb{P} \text{ a.s.}$$

This means that $X$ is a hedge, replicating portfolio or hedging portfolio against $R$.

To show that no arbitrage are possible and the market is complete, the First and Second Fundamental Theorem are stated. A normalized economy is used only to make the process easier. It still holds for non-normalized economies.

**Theorem 2.1 (First fundamental Theorem)** The market model is free of arbitrage if and only if there exists a martingale measure, a measure $Q \sim P$, such that the processes

$$S_0(t), S_1(t), \ldots, S_N(t)$$

are (local) martingales under $Q$. [B]

To exemplify this consider the case where the numeraire $S_0(t)$ is the money account

$$S_0(t) = e^{-\int_0^t r(u) du}$$

where $r$ is the (possible stochastic) short rate and if the assumption, that all process are Wiener driven, are made then a measure $Q \sim P$ is a martingale measure if and only if all assets $S_0, S_1, \ldots, S_N$ have the short rate as their local rates of return, i.e. if the $Q$-dynamics are of the form

$$dS_i(t) = S_i(t)r(t)dt + S_i(t)\sigma_i(t)dW^Q(t),$$

where $W^Q$ is a (multidimensional) $Q$-Brownian motion. The second result is used as conditions for market completeness. $\sigma$ is only a defined volatility.

**Theorem 2.2 (Second fundamental Theorem)** Assuming absence of arbitrage, the market model is complete if and only if the martingale measure $Q$ is unique. [B]

This summarizes the basic theory concerning pricing contingent claims. A consequence of this theorem is that in order to avoid arbitrage, $R$ must be priced according to the formula

$$\Gamma(t; R) = S_0(t)E^Q \left[ R \bigg| \mathcal{F}_t \right],$$

where $Q$ is a martingale measure for $[S_0, S_1, \ldots, S_N]$, with $S_0$ as the numeraire. If the bank account has $W(t)$ as the numeraire then $W(t)$ has the dynamics

$$dW(t) = r(t)W(t)dt,$$

where $r$ is as before the short rate process. In this case the pricing formula above reduces to

$$\Gamma(t; R) = S_0(t)E^Q \left[ e^{-\int_0^t r(u) du}R \bigg| \mathcal{F}_t \right].$$
Different $\mathbb{Q}$ could give different price processes for a fixed claim $R$, but if $R$ is attainable then all $\mathbb{Q}$ will produce the same price process and this price process is given by

$$\Gamma(t; R) = V(t; X),$$

where $X$ is the hedging portfolio. Different choices of hedging portfolios (if such exist) will produce the same price process. Especially, for every replicable claim $X$ it holds that

$$V(t; X) = S_0(t)E^\mathbb{Q}[e^{-\int_0^t r(u) du} R|\mathcal{F}_t].$$

Summing up the price of any derivative instrument in a complete market will be uniquely determined by the required no arbitrage. The price is unique because it can be replaced by the replicating portfolio. The price on the derivative does not depend on the risk the brokers or agents in the market are willing to take. This means that they can have any attitude towards risk as long as they prefer more (deterministic) money to less.

In an incomplete market the requirement of no arbitrage is no longer sufficient to determine a unique price for options. There are several martingale measures that can price derivatives in a no arbitrage economy. The price of a derivative using the martingale measure is chosen by the market and is determined by two major factors:

1. The derivative should be priced so that there are no arbitrage possibilities on the market. This requirement is derived from equation (2.1) where all derivatives use the same $\mathbb{Q}$.

2. When there is an incomplete market the price also depends on the supply/demand on the market. Supply and demand for a specific derivative depends on the aggregate risk aversion on the market, liquidity considerations and other factors. All these aspects are aggregated into a particular martingale measure used by the market. [B]
3 The Model

Let \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})\) be a filtered probability space satisfying usual conditions with \(T\) as a fixed time. Let's consider a security in a market and call it a stock with no dividends. There is also a money market account that uses the spot rate of interest as return. Assuming that the spot rate of interest is zero gives an initial value for the money at all times.

The supply curve is a curve for shares bought or sold of a stock within the trading interval where an arbitrage trader acts as a price taker. \(S(t, x, \omega)\) represents the stock price at time \(t \in [0, T]\) that the trader pays or receives for an order of the size \(x \in \mathcal{R}\) given the state \(\omega \in \Omega\). A positive order \((x > 0)\) represents a buy and a negative order represents a sale. The order zero \((x = 0)\) corresponds to the marginal trade. Now the trader has a supply curve that depends on the size of the order rather than a horizontal supply curve as in the classical theory. The supply curve is independent of the traders past actions which implies that the last action has no impact on the price process.

**Definition 3.1** The definition of the supply curve represents by [JP]

1. The stock price per share \(S(t, x, \cdot)\) is non-negative and \(\mathcal{F}_t\)-measurable.
2. \(x \mapsto S(t, x, \omega)\) is \(t\) almost everywhere and non-decreasing in \(x\) a.s. (if \(x \leq y\) then almost surely in \(\mathbb{P}\) almost everywhere in \(t\)).
3. \(S\) is \(C^2\) in the second argument, and \(\partial S(t, x) / \partial x\) and \(\partial^2 S(t, x) / \partial x^2\) is continuous in \(t\).
4. \(S(\cdot, 0)\) is a semi martingale (decomposed as the sum of a local martingale and an adapted finite-variation process \(X_t = M_t + A_t\)).
5. \(S(\cdot, 0)\) has continuous sample paths for all \(x\) including time 0.

The supply curve assumption states that the larger the purchase (or sale) the larger the price impact. This is usual in asset pricing markets where the impact is due to information effects or supply/demand imbalances [K].

A concrete example of a supply curve is if \(S(t, x) \equiv f(t, D_t, x)\) where \(D_t\) is an \(n\)-dimensional \(\mathcal{F}_t\)-measurable semi-martingale and \(f : \mathbb{R}^{n+2} \rightarrow \mathbb{R}^+\) is Borel measurable. This non-negative function \(f\) can translate to a simpler supply curve generated by a market equilibrium process in a complex and dynamic economy. \(D_t\) represents the uncertainty in the economy.

**Definition 3.2** The investors trading strategy is defined by a triplet \(((X_t, Y_t : t \in [0, T]), \tau)\) where \(X_t\) represents traders aggregate stock holding at time \(t\) and \(Y_t\) represents the traders aggregate money market account position at time \(t\). \(\tau\) represents the liquidation time of the stock position and has some restrictions

1. \(X_t\) and \(Y_t\) are predictable and optional processes with \(X_{0-} = Y_{0-} = 0\),
2. \(X_T = 0\). \(\tau\) is a predictable \(\mathcal{F}_t\)-measurable on \([0, T]\) stopping time with \(\tau \leq T\) and \(X = H1_{(0, \tau)}\) for some predictable process \(H(t, \omega)\).
The trading strategy which is interesting in this case is self financing, generates no cash flows for all times $t \in [0, T)$. This means that purchase or sales of stocks must be obtained by borrowing or investing in the money market account and leads to the possibility of determine $Y_t$. Here an arbitrary stock holding $(X_t, t)$ is used to define $Y_t$.

**Definition 3.3** A self financing trading system (SFTS.) is a strategy $((X_t, Y_t : t \in [0, T]), \tau)$ where

(a) $X_t$ is right continuous with left limits (càdlàg) if $\partial S(t, x)/\partial x \equiv 0$ for all $t$ and has finite quadratic variation $([X, X]_T < \infty)$ otherwise,

(b) $Y_0 = -X_0 S(0, X_0)$ for $t = 0$, and

$$Y_t = Y_0 + X_0 S(0, X_0) + \int_0^t X_u - dS(u, 0) - X_t S(t, 0) - \sum_{0 \leq u \leq t} \Delta X_u [S(u, \Delta X_u) - S(u, 0)] - \int_0^t \frac{\partial S}{\partial x}(u, 0) d[X, X]_u^c$$

for $0 < t \leq T$. (3.1)

The first condition sets restrictions on acceptable trading strategies. This gives a well defined equation (3.1) apart from the last two terms which may be negative infinity. The classical theory does not need these restrictions. An example is a discontinuous trading strategy with infinitely jumps give an undefined $Y_t$ [JP]. $Y_0 = -X_0 S(0, X_0)$ from the second condition implies that the strategy requires zero initial investment at time 0 but in complete markets this condition is removed. The last condition defines the self financing condition at time $t$. The money market account equals its initial value at time 0 plus the accumulated trading gains minus the cost of getting this position minus the price impact costs of discrete changes in share holdings minus the price impact cost of continuous changes in share holdings. This expression is an extension of the classical self financing condition when the supply curve is horizontal.

To show this the second conditions is put together forming

$$Y_t + X_t S(t, 0) = \int_0^t X_u - dS(u, 0) - \sum_{0 \leq u \leq t} \Delta X_u [S(u, \Delta X_u) - S(u, 0)] - \int_0^t \frac{\partial S}{\partial x}(u, 0) d[X, X]_u^c$$

for $0 \leq t \leq T$. (3.2)

The left side represent the classical value of the portfolio at time 0. The right side shows a decomposition of this. The first term gives the classical accumulated gains/losses to the portfolio value. The last two terms capture the impact of illiquidity (both with negative signs).

### 3.1 Market-to-Market Value and Liquidity Costs

The market-to-market value of a SFTS, and the liquidity cost which arises in this situation are discussed here. There exists no unique value of a trading strategy or portfolio at a time prior to liquidation. Any price on the supply curve is a plausible price to use in valuing the
portfolio. There exists at least three meaningful possibilities; the immediate liquidation value (if \( X_t > 0 \) then \( Y_t + X_tS(t, -X_t) \)), the accumulated cost of forming the portfolio \( (Y_t) \), and the portfolio evaluated at the marginal trade \( (Y_t + X_tS(t, 0)) \). The last possibility represents the value of the portfolio under the classical price taking condition and is defined as the market-to-market value of the self financing trading strategy \( (X, Y, \tau) \). These three valuations give the portfolio the same value at the liquidation time \( \tau \) due to \( X_\tau = 0 \).

The liquidity cost of trading strategies in a market-to-market value is defined as the difference between accumulated gains/losses to the portfolio, computed at marginal trade price \( (S(t, 0)) \), and the market-to-market value of the portfolio.

**Definition 3.4** The liquidity cost of a SFTS. \((X, Y, \tau)\) is

\[
L_t \equiv \int_0^t X_u - dS(u, 0) - [Y_t + X_tS(t, 0)]
\]

which comes from

\[
L_t = \sum_{0 \leq u \leq t} \Delta X_u [S(u, \Delta X_u) - S(u, 0)] + \int_0^t \frac{\partial S}{\partial x}(u, 0)d[X, X]_u^0 \geq 0
\]

where \( L_{0-} = 0 \), \( L_0 = X_0[S(0, X_0) - S(0, 0)] \) and \( L_t \) is non-decreasing in \( t \).

The second inequality follows from that \( S(u, x) \) is increasing in \( x \). This shows that the liquidity cost is non-negative and non-decreasing in \( t \). The two components is defined as discontinuous changes in the share holdings (first part) and continuous component (second part). Also \( \Delta L_0 = L_0 - L_{0-} = L_0 > 0 \) is due to \( X_{0-} = Y_{0-} = 0 \).

### 3.2 The First Fundamental Theorem

To evaluate a self financing trading strategy it is important to take into consideration the value after liquidation. The first fundamental theorem of asset pricing has hereby been generalized to an economy with liquidity risk and an arbitrage opportunity can be defined.

An arbitrage opportunity appears if a SFTS. \((X, Y, \tau)\) such that \( \mathbb{P}(Y_T \geq 0) = 1 \) and \( \mathbb{P}Y_T > 0 > 0 \). Define, for \( \alpha \geq 0 \), \( \Theta_\alpha \equiv SFTS. (X, Y, \tau) \mid \int_0^t X_u - dS(u, 0) \geq -\alpha \) for all \( t \) a.s. Then given an \( \alpha \geq 0 \) the SFTS. is \( \alpha \)-admissable if \( (X, Y, \tau) \in \Theta_\alpha \). The strategy is admissable if it is \( \alpha \)-admissible for some \( \alpha \).

\( Y_t + X_tS(t, 0) \) is a supermartingale if there exist a probability measure \( \mathbb{Q} \sim \mathbb{P} \) such that \( S \) is \( \mathbb{Q} \)-local martingale. If \( (X, Y, \tau) \in \Theta_\alpha \) for all \( \alpha \) then \( Y_t + X_tS(t, 0) \) is a \( \mathbb{Q} \)-supermartingale.

Because \( Y_t + X_tS(t, 0) = \int_0^t X_u - dS(u, 0) - L_t \) and, in \( \mathbb{Q}, \int_0^t X_u - dS(u, 0) \) is a \( \mathbb{Q} \)-local martingale then \( Y_t + X_tS(t, 0) \) is a supermartingale. \( L_t \) is non-negative and non-decreasing.

**Theorem 3.5** This theorem presents a condition for no arbitrage. If there exists a probability measure \( \mathbb{Q} \sim \mathbb{P} \) and \( S(\cdot, 0) \) is a \( \mathbb{Q} \)-local martingale then \( (X, Y, \tau) \in \Theta_\alpha \) have no arbitrage opportunities for any \( \alpha \). [JP]
From earlier \( Y_t + X_t S(t, 0) \) is a supermartingale and by definition of liquidation time element \( Y_\tau + X_\tau S(\tau, 0) = Y_\tau \). This gives for a SFTS. \( \mathbb{E}^Q[Y_\tau] = \mathbb{E}^Q[Y_\tau + X_\tau S(\tau, 0)] \leq 0 \) but by earlier definition an arbitrage opportunity arises when \( \mathbb{E}^Q[Y_\tau] > 0 \) which leads to the conclusion that there exists no arbitrage opportunity in this economy. The market-to-market portfolio is a hypothetical portfolio and has zero liquidity costs. If \( S(\cdot, 0) \) has equivalent martingale measure the hypothetical portfolio admit no arbitrage. The only difference between these portfolios and the actual portfolio is the subtraction of non-negative liquidity costs. This means that the actual portfolio cannot admit arbitrage either.

To get a good condition for an equivalent local martingale measure to exist, a free lunch with vanishing risk (FLVR.) is defined.

**Definition 3.6** A free lunch with vanishing risk can be an admissible SFTS. with an arbitrage opportunity or a sequence of \( \epsilon_n \)-admissible SFTS. \( (X^n, Y^n, \tau^n)_{n \geq 1} \) and a non-negative \( F_T \)-measurable random variable \( f_0 \). This variable is not identical 0 but \( Y^n_T \to f_0 \) when \( \epsilon_n \to 0 \) in probability. [JP]

### 3.3 The Fictitious Economy

A fictitious economy is introduced by using the previously used economy with \( S(t, x) \equiv S(t, 0) \) to state the first fundamental theorem. In this fictitious economy a SFTS. \( (X, Y^0, \tau) \) satisfies the classical condition with \( X_0 = 0 \) and \( Z^0_t \equiv \int_0^t X_u S(u, 0) - X_t S(t, 0) \) for all \( 0 \leq t \leq T \) (the value of the portfolio) where \( Y^0_t = \int_0^t X_u - dS(u, 0) \) and \( X \) is allowed to be a general \( S(\cdot, 0) \) integrable predictable process.

**Theorem 3.7** (First Fundamental Theorem [B]). If there are no arbitrage opportunities in the fictitious economy then there is no free lunch with vanishing risk if there exists a probability measure \( Q \sim \mathbb{P} \) such that \( S(\cdot, 0) \) is a \( Q \)-local martingale.

### 3.4 Contingent Claim

To study an economy with liquidity risk the second fundamental theorem of asset pricing can be extended by assuming that there exists a local martingale measure \( Q \) so that there are no arbitrage opportunities and (no free lunch with vanishing risk) NFLVR. Definition 3.3 of a SFTS. \( (X, Y, \tau) \) is generalized slightly to allow for non-zero investments at time 0, i.e. \( Y_0 + X_0 S(0, X_0) \neq 0 \).

To continue a definition need to be made. A definition of a space \( H^2_Q \) of semimartingales are made with respect to the equivalent local martingale measure \( Q \). Let \( Z \) be a special semimartingale with canonical decomposition \( Z = N + A \) where \( N \) is a local martingale under \( Q \) and \( A \) is a predictable finite variation process. The \( H^2 \) norm of \( Z \) is

\[
\|Z\|_{H^2} = \left\| [N, N]_\infty \right\|_{L^2} + \left\| \int_0^\infty |dA_s| \right\|_{L^2}
\]

where \( L^2 \)-norms are with respect to the equivalent local martingale measure \( Q \).

\[5\text{Due to the assumption } S(\cdot, 0) \in H^2_Q, \int XdS(u, 0) \text{ does not need to be uniformly bounded from below. [JP]}


A contingent claim is any $\mathcal{F}_t$-measurable random variable $C$ with $\mathbb{E}^\mathbb{Q}(C^2) < \infty$. This is considered at time $T$, prior to liquidation. If the payoff of a contingent claim depends on the stock price at time $T$ then the dependence must be made explicit or else the claim is not well-defined.

### 3.4.1 Example of Contingent Claim

To show an example of a well-defined contingent claim an European call option is considered. This option has a strike price $K$ and maturity $T_0 \leq T$. To use the modified boundary condition and incorporating the supply curve for the stock two cases must be considered, cash delivery and physical delivery.

If an option has a cash delivery, the long position receives cash at maturity if the option ends up in-the-money. To match the cash settlement, the synthetic option has to be liquidated prior to $T_0$. The underlying stock position is liquidated when the synthetic option is liquidated and the position in the stock at time $T_0$ is zero. To achieve this position by selling the stock at time $T_0$ the boundary condition is $C \equiv \max[S(T_0, -1) - K, 0]$, where $\Delta X_{T_0} = -1$. But the stock can be liquidated prior to time $T_0$ using a continuous and finite variation process. This can be used to try to avoid liquidity cost at time $T_0$. Here the boundary conditions is $C \equiv \max[S(T_0, 0) - K, 0]$ where $\Delta X_{T_0} = 0$. Because liquidation occurs right before $T_0$ the options payoff can only be approximately obtained.

If this option has physical delivery the synthetic option should match the underlying asset in the physical delivery. This means that the short position in this option contract has to deliver the stock shares. The stock position in the synthetic option is not sold but the model requires the stock position to be liquidated at time $T_0$ due to the construction. To approximate physical delivery the boundary condition is set to $C \equiv \max[S(T_0, 0) - K, 0]$ where $\Delta X_{T_0} = 0$. This gives theoretically no liquidity cost which would be the case at time $T_0$. When trading in options this case is an expansion of the economy which gives a possibility to avoid liquidity costs at time $T$.

### 3.5 Market Completeness

The market is complete if there exists a SFTS. (self financing trading strategy) $(X,Y,\tau)$ with $\mathbb{E}^\mathbb{Q}\left(\int_0^T X_u^2 d[S(u,0), S(u,0)]\right) < \infty$ for any contingent claim $C$ such that $Y_T = C$. A contingent claim $C$ is considered in $L^2(d\mathbb{Q})$ and has a SFTS. $(X,Y,\tau)$ such that $C = c + \int_0^T X_u dS(u,0)$ where $c \in \mathbb{R}$, $\mathbb{E}^\mathbb{Q}\left(\int_0^T X_u^2 d[S(u,0), S(u,0)]\right) < \infty$ and $(\mathbb{E}^\mathbb{Q}(C) = c)$ $^6$ In this case a long position in the contingent claim $C$ is redundant because there is no liquidity costs. $Y_0$ is chosen to $Y_0 + X_0 S(0,0) = c$ but liquidity costs in this trade is from Definition 3.4

$$L_t = \sum_{0 \leq u \leq t} \Delta X_u [S(u, \Delta X_u) - S(u, 0)] + \int_0^t \frac{\partial S}{\partial x}(u, 0) d[X, X]_u c \geq 0.$$  

$^6\int_0^0 X_0 dS(u,0) = X_0 \Delta S(0,0) = 0$ by the continuity of $S(\cdot, 0)$ at time 0.
Previous Definition 3.3 gives
\[ Y_T = Y_0 + X_0 S(0, X_0) + \int_0^T X_{u-} dS(u, 0) - X_T S(T, 0) - L_T + L_0, \]
and because \( \int_0^T X_{u-} dS(u, 0) = \int_0^T X_u dS(u, 0) \) [JP] the value of the money market account is
\[ Y_T = C - X_T S(T, 0) - L_T + L_0. \]
In this situation \( X_T = 0 \) because liquidation has occurred by time \( T \) which gives
\[ Y_T = C - (L_T - L_0) \leq C. \]
This trading strategy sub-replicates a long position (make profit when the underlying asset goes up in price) in this contingent claim payoffs. If \( -X \) is used to hedge a short position the payoff is
\[ Y_T = -C - (L_T - L_0) \leq -C \]
where \( Y \) is the value in the money market account and \( L \) is the liquidity cost to hedge a short position (gain profit when the underlying asset price goes down). The long and short trading strategies provide an upper and lower bound which can be used to acquire the contingent claims payoffs.

Two cases arises which are worth mention. The first case is when \( \frac{\partial S}{\partial x} (\cdot, 0) \equiv 0 \) then \( L = L_0 \) if \( X \) is a continuous trading strategy. Any contingent claim \( C \) with the same properties as before can be replicated because \( X \) is continuous. The second (general) case is when \( \frac{\partial S}{\partial x} (\cdot, 0) \geq 0 \). Then \( L = L_0 \) if \( X \) is a finite variation and continuous trading strategy. This leads to the conclusion that any contingent claims \( C \) with an existing SFTS. \((X, Y, \tau)\) such that \( C = c + \int_0^T X_u dS(u, 0) \) can be replicated.

These two properties makes it possible to approximate \( X \) using a finite variation and continuous trading strategy so that in a limited sense it is possible to avoid all liquidity costs when using the replication strategy.

**Theorem 3.8** To approximate continuous and finite variation SFTS. let \( C \in L^2(dQ) \). Then suppose that there exists a predictable \( X \) with \( \mathbb{E}^Q \left( \int_0^T X_u^2 d\lbrack S(u, 0), S(u, 0)\rbrack \right) < \infty \) such that \( C = c + \int_0^T X_u dS(u, 0) \) with \( c \in \mathbb{R} \). There also exists a self financing trading strategy (SFTS.) \((X^n, Y^n, \tau^n)_{n \geq 1} \) with \( X^n \) replicating portfolios that are bounded, continuous and of finite variation such that \( \mathbb{E}^Q \left( \int_0^T (X_u^n)^2 d\lbrack S(u, 0), S(u, 0)\rbrack \right) < \infty \), \( X_0^n = 0 \), \( X_T^n = 0 \), \( Y_0^n = \mathbb{E}^Q(C) \) and
\[
Y_T = Y_0 + X_0 S(0, X_0) + \int_0^T X_{u-} dS(u, 0) - X_T S(T, 0) - L_T + L_0 \rightarrow c + \int_0^T X_u dS(u, 0) = C
\] (3.3)
for all \( n \) in \( L^2(dQ) \).
To derive this it is noted that for a predictable $X$ that is integrable with respect to $S(\cdot,0)$, 
\[ \int_0^T X_u dS(u,0) = \int_0^T X_u 1_{[0,T]}(u) dS(u,0). \]
$X_0 = 0$ is assumed and given any $H \in \mathbb{L}$ (as cadlag set) with $H_0 = 0$ a statement could be made 
\[ H^n_t(\omega) = n \int_{t-1/n}^t H_u(\omega) du, \]
for all $t \geq 0$ and $H_u = 0$ for $u < 0$. $H$ is previously define in Definition 3.2b. This gives almost certain a pointwise limit of the sequence of adapted processes $H^n$ which are continuous and has finite variation. If $X$ with $X_0 = 0$ is predictable and $\mathbb{E}^Q \left( \int_0^T X^n_u^2 d[S(u,0), S(u,0)] \right) < \infty$ then there exists a sequence of continuous and bounded processes of finite variation $X^n_{n \geq 1}$ such that $\mathbb{E}^Q \left( \int_0^T (X^n_u)^2 d[S(u,0), S(u,0)] \right) < \infty$, $X^n_0 = 0$ and 
\[ \int_0^T X^n_u dS(u,0) \rightarrow \int_0^T X_u dS(u,0), \]
for all $n$ in $L^2(dQ)$ [JP].

It is now allowed to choose $X^n_T$ for all $n$. If $Y^n = \mathbb{E}^Q(C)$ for all $n$ and $Y^n_t$ is defined by equation 3.1 for $t > 0$, then by putting in $\tau^n = T$ for all $n$ the sequence $(X^n, Y^n, \tau^n)_{n \geq 1}$ satisfy equation 3.3.\(^7\)

### 3.6 The Second Fundamental Theorem

Theorem 3.8 makes it possible to motivate an extension of the second fundamental theorem of asset pricing. If given any contingent claim $C$ there exists a sequence of SFTS. $(X^n, Y^n, \tau^n)$ with $\mathbb{E}^Q \left( \int_0^T (X^n_u)^2 d[S(u,0), S(u,0)] \right) < \infty$ for all $n$ such that $Y^n_T \rightarrow C$ as $n \rightarrow \infty$ in $L^2(dQ)$ the market is approximately complete.

**Theorem 3.9 (Second Fundamental Theorem [B]).** If there exists a unique probability measure $Q \sim \mathbb{P}$ such that $S(\cdot,0)$ is a $Q$-local martingale then the market is approximately complete.

**Proof.** To show this the first step is to prove that the hypothesis guarantees that a fictitious economy with no liquidity costs is complete. The second step is used to show that this result implies that an economy with liquidity costs has approximate completeness.

A SFTS. $(X, Y^0, \tau)$ in the fictitious economy which was introduced in this paper but with $S(\cdot,x) \equiv S(\cdot,0)$ satisfies the classical condition $Y_t = Y_0 + X_0 S(0,0) + \int_0^t X_u \cdot dS(u,0) - X_t S(t,0)$. Because this holds the fictitious market is complete if and only if $Q$ is unique due to the classical second fundamental theorem [HP]. If $Q$ is unique the fictitious economy is complete and $S(\cdot,0)$ has martingale properties. Because of this there exists a predictable $X$ such that $C = c + \int_0^T X_u dS(u,0)$ with $\mathbb{E}^Q \left( \int_0^T (X^n_u)^2 d[S(u,0), S(u,0)] \right) < \infty$ and when applying theorem 3.8 the market is approximately complete. The assumption that the martingale measure is unique is stated. Any contingent claim $C$ then gives a sequence of SFTS.

\(^7\)Note that $L^n \equiv 0$ for all $n$ and $\int_0^T X^n_u dS(u,0) = \int_0^T X^n_u dS(u,0)$
\((X^n, Y^n, \tau^n)_{n \geq 1}\) such that \(\mathbb{E}^Q\left(\int_0^T (X^n)^2 d[S(u, 0), S(u, 0)]\right) < \infty\) for all \(n\). This leads to \(Y^n_T = Y^n_0 + X^n_0 S(0, X^n_0) + \int_0^T X^n_{u-} dS(u, 0) - L^n_T \to C\) and due to the previously stated SFTS, is called an approximating sequence for \(C\). [JP] ﬁnal

Now \(\Psi^C\) is the set of approximating sequence for \(C\) which is a contingent claim. The value of the contingent claim \(C\) at time 0 is

\[
\inf \left\{ \liminf_{n \to \infty} Y^n_0 + X^n_0 S(0, X^n_0) : (X^n, Y^n, \tau^n)_{n \geq 1} \in \Psi^C \right\}.
\]

If there exists a unique probability measure \(\mathbb{Q}\) such that \(S(\cdot, 0)\) is a \(\mathbb{Q}\)-local martingale then any contingent claim \(C\) is equal to \(\mathbb{E}^\mathbb{Q}(C)\) at time 0. Let \((X^n, Y^n, \tau^n)_{n \geq 1}\) be an approximating sequence for \(C\). This gives \(\mathbb{E}^\mathbb{Q}(Y^n_T - C) \to 0\). But since \(\mathbb{E}^\mathbb{Q}\left(\int_0^T (X^n)^2 d[S(u, 0), S(u, 0)]\right) < \infty\) for all \(n\) and \(\int_0^T X^n_{u-} dS(u, 0)\) is a \(\mathbb{Q}\)-martingale for all \(n\) then \(\mathbb{E}^\mathbb{Q}(Y^n_T) = Y^n_0 + X^n_0 S(0, X^n_0) - \mathbb{E}^\mathbb{Q}(L^n_T)\). \(L^n \geq 0\) for all \(n\) together with \(\mathbb{E}^\mathbb{Q}(Y^n_T - C) \to 0\) gives

\[
\liminf_{n \to \infty} Y^n_0 + X^n_0 S(0, X^n_0) \geq \mathbb{E}^\mathbb{Q}(C)
\]

for all approximating sequence. There also exists some approximating sequence \((\bar{X}^n, \bar{Y}^n, \bar{\tau}^n)_{n \geq 1}\) with \(\bar{T}^n = 0\) such that \(\liminf_{n \to \infty} \bar{Y}^n_0 + \bar{X}^n_0 S(0, X^n_0) = \mathbb{E}^\mathbb{Q}(C)\).

The value above is consistent with no arbitrage but if the contingent claim is sold at \(p > \mathbb{E}^\mathbb{Q}(C)\) one can short the contingent claim at \(p\) and construct a sequence of continuous and finite variation SFTS. \((X^n, Y^n, \tau^n)_{n \geq 1}\) with \(Y^n_0 = \mathbb{E}^\mathbb{Q}(C)\), \(X^n_0 = 0\) and \(\lim_{n \to \infty} Y^n_T = C\). So in probability there exists a FLVR. This is not allowed since \(\mathbb{Q}\) is an equivalent martingale measure for \(S(\cdot, 0)\). The same can be proved for \(p < \mathbb{E}^\mathbb{Q}(C)\).

The supply curve formulation, that at time 0 any contingent claim \(C\) is equal to \(\mathbb{E}^\mathbb{Q}(C)\), makes it possible to construct a continuous trading strategies of finite variation to both approximately replicate any contingent claim and avoid all liquidity cost. Continuous trading strategies in a continuous time setting are the reason for making this possible.

### 3.7 Example of an Economy with Liquidity Risk

An extension of the Black Scholes economy with liquidity risk is considered when an example of this theory is presented (see [CJPW] for more examples). Let

\[
S(t, x) = e^{\alpha x} S(t, 0) \quad \text{with} \quad \alpha > 0
\]  

\[
S(t, 0) \equiv S(0, 0)e^{(r-\frac{1}{2}\sigma^2)t+\sigma W_t}
\]

where \(r\) is the risk free rate, \(\sigma\) is the volatility and \(W\) is standard Brownian motion initialized at zero. \(r\) and \(\sigma\) are constants. The marginal stock price follows a geometric Brownian motion. These expressions characterize an extended Black Scholes economy. This supply curve satisfies the condition stated in the Definition 3.1. Under these conditions there exists a unique martingale measure for \(S(\cdot, 0)\) because the market hereby is arbitrage-free and approximately complete.
Consider a European call option with strike price $K$, maturity date $T$ and this stock has cash delivery. To avoid liquidity costs at time $T$, the payoff to the option at time $T$ is selected as $C_T = \max[S(T, 0) - K, 0]$.

The contingent claim valuation, earlier stated, gives the value of a long position in the option

$$C_0 = e^{-rT} \mathbb{E}^Q(\max[S(T, 0) - K, 0]).$$

The analytical value for the expectation value is (Black Scholes formula)

$$C_0 = S(0, 0) N(h(0)) - Ke^{-rT} N(h(0) - \sigma \sqrt{T})$$

where $N(\cdot)$ is the standard cumulative normal distribution function and

$$h(t) \equiv \frac{\log S(t, 0) - \log K + r(T - t)}{\sigma \sqrt{T - t}} + \frac{\sigma}{2} \sqrt{T - t}.$$ 

When Itô’s formula is applied the classical replicating strategy, $X = (X_t)_{t \in [0,T]}$ is given by

$$X_t = N(h(t)).$$

This hedging strategy is continuous but has not finite variation. In this economy $\frac{\partial S}{\partial x}(t, 0) = \alpha e^{0} S(t, 0) = \alpha S(t, 0)$ but the standard hedging strategy does not attain this value although the call value on the option is the Black Scholes formula. It can be shown that the classical Black Scholes hedging strategy give a case with non-zero liquidity costs

$$L_T = X_0 (S(0, X_0) - S(0, 0)) + \int_0^T \frac{\alpha (N'(h(u)))^2 S(u, 0)}{T - u} du.$$ (3.7)

An approximating hedging strategy that is continuous and of finite variation with zero liquidity costs actually is the sequence of SFTS. $(X^n, Y^n, \tau^n)_{n \geq 1}$ with

$$X_t^n = 1_{\left[\frac{1}{n}, T - \frac{1}{n}\right]}(t) n \int_{(t-\frac{1}{n})^+}^t N(h(u)) du,$$ if $0 \leq t \leq T - \frac{1}{n}$

$$X_t^n = (nTX^n_{(T-\frac{1}{n})} - ntX^n_{(T-\frac{1}{n})})$$, if $T - \frac{1}{n} \leq t \leq T$ (3.8)

and $Y_0^n = \mathbb{E}^Q(C_T)$. These expressions together with the use of a limited trading strategy gives the call options value in time $T$ to be $Y^n_T = Y^n_0 + X^n_0 S(0, X^n_0) + \int_0^T X^n_u dS(u, 0) - L^n_T \rightarrow C_T = \max[S(T, 0) - K, 0]$ in $L^2(dQ)$. 

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4 Economies with Supply Curves for Derivatives

The extensions of the First and Second Fundamental Theorem hold in the economy described in earlier chapters. If we use $C^2$ smooth supply curve for the stock and allowing continuous trading strategies there is a unique price for any option on the stock. This definition implies that the supply curve for trading an option is horizontal and therefore has no quantity impact on the price. If the supply curve is not horizontal arbitrage opportunities arises. The definition is not consistent with reality and a model that analyzes liquidity risk should imply supply curves for both stocks and options.

The described supply curve does exist for stocks, but not for options in the example of the economy with liquidity risk, i.e. the previous chapter. This is because continuous trading strategies of finite variation enable the investor to avoid all liquidity costs in the stock. Liquidity costs still exist but they are non-binding and a modified classical theory still applies. Liquidity costs are binding in practice and to make them binding in theory either the $C^2$ condition or continuous trading strategies must be removed. If continuous trading strategies are removed the model consists with practice because these strategies are impossible to utilize except as approximations thru simple trading strategies. But simple trading strategies have binding liquidity costs.

4.1 Discrete Trading Strategies

Now the previous theory is modified to consider only discrete trading strategies defined as any simple SFTS. $X_t$ where

$$X_t \in x_{\tau_0} 1_{\tau_0} + \sum_{j=1}^{N} x_{\tau_j} 1_{(\tau_{j-1}, \tau_j]} \left\{ \begin{array}{l}
\tau_j \text{ are } \mathcal{F} \text{ stopping times for each } j, \\
x_{\tau_j} \text{ is in } \mathcal{F}_{\tau_{j-1}} \text{ for each predictable } j \\
\tau_0 \equiv 0 \text{ and } \tau_j > \tau_{j-1} + \delta \text{ for a fixed } \delta > 0.
\end{array} \right.$$  

These trading strategies are discontinuous because between every trade that is executed there is a $\delta > 0$ big time unit gap.

An arbitrage-free environment is obtained by this restriction of trading strategies. By introducing the minimum distance $\delta$ between trades the market is not approximately complete. In an incomplete (not approximately complete) market the cost of replicating an option depends on the chosen trading strategy. The second fundamental theorem fails which implies that there can be a quantity impact on the price of an option and the supply curve does not need to be horizontal.

There is some constraint on the supply curve due to no arbitrage and this can be studied by the super-replication of options when using discrete trading strategies. The lower case values $x$ and $y$ denotes from now on discrete trading strategies. All discrete trading strategies has the liquidity cost

$$L_T = \sum_{j=0}^{N} \left[ x_{\tau_{j+1}} - x_{\tau_j} \right] \left[ S \left( \tau_j, x_{\tau_{j+1}} - x_{\tau_j} \right) - S \left( \tau_j, 0 \right) \right].$$  (4.1)
The hedging error for a discrete trading strategy with $x_T = 0$ is
\[ C_T - Y_T = C_T - \left[ y_0 + x_0 S(0, 0) + \sum_{j=0}^{N-1} x_{\tau_{j+1}} \left[ S(\tau_{j+1}, 0) - S(\tau_j, 0) \right] \right] + L_T. \]

There are two components to this hedging error, one of them is liquidity cost $L_T$ (equation 4.1) and the other, the remainder of the previous expression, is the error in replicating the options payoff $C_T$.

An upper bound on the price of a particular quantity of option is investigated. The cost of a single call option on the stock is obtained as follows. Defined $Z_t = X_t S(t, 0) + Y_t$ as the time $t$ market-to-market value of the replicating portfolio. The optimization problem is
\[ \min_{(X,Y)} Z_0 \text{ such that } Z_T \geq C_T = \max(S(T, 0) - Ke^{-rT}, 0) \]

where
\[ Z_T = y_0 + x_0 S(0, 0) + \sum_{j=0}^{N-1} x_{\tau_{j+1}} \left[ S(\tau_{j+1}, 0) - S(\tau_j, 0) \right] - L_T. \]

The solution to this problem requires a numerical approximation (see [CJPW] for a binomial approximation to solve the problem). Liquidity costs in the underlying stock are quantity dependent. This gives that the cost of super-replication also is quantity dependent. An upper bound on the entire supply curve is created for the option by the cost of super-replicating a number of shares of the option. There is a significant economically difference between the classical price and the super-replication cost. [CJPW]
4.2 Transactional Costs

Transactional costs can be seen as liquidity risk where the hypothesis of the $C^2$ supply curve is violated. Three different types of transaction costs are discussed. All costs are per share unless there is another statement. Mathematics of the supply curve is used to study transaction costs while ignoring liquidity issues. The supply curve is now known as the transaction curve and the goal is to see when continuous trading is possible. [C]

**Definition 4.1** There are three kinds of transaction costs

1. Fixed transaction costs are per share stock price and defined by

   \[ S(t, x) = S(t, 0) + \frac{a}{x} \]

2. Proportionate transaction costs depend proportionately on the money value of the trade

   \[ S(t, x) = S(t, 0)(1 + \beta \text{sign}(x)) \]

   where $\beta > 0$ is the proportionate transaction cost per unit value.

3. Combined fixed and proportionate transaction costs is defined different depending on the

   \[ a \]
   \[ S(t, x) = S(t, 0) + \beta x + \text{sign}(x) \gamma 1_{|x| > \delta} \]

   where $\beta$, $\gamma$ and $\delta$ are positive constants.

   \[ b \]
   \[ S(t, x) = S(t, 0) + \max(\alpha, |x| c) \]

   where $\alpha$ and $c$ are positive constants.

This implies that when the transaction costs are fixed, only piecewise constant trading strategies are needed to do a model.

4.2.1 Fixed Transaction Costs

Fixed transaction costs when having continuous trading gives infinite costs at infinite time. The definition of fixed transaction costs together with trading strategy $X = \sum_{i=0}^{n-1} X_i 1_{[T_i, T_{i+1})}$ for trading times $0 = T_0 \leq T_1 \leq ... \leq T_n = T$ gives the cumulative trading costs $\sum_{i=0}^{n-1} a 1_{(X_i \neq X_{i-1})}$.

If $X_i \neq X_{i-1}$ then this is equal to $na$. Now say that $X$ has continuous paths the trading costs $TC(X)$ with trading strategy $X$ is

\[
TC(X) = \limsup_{n \to \infty} \sum_{T_i \in \Pi_k} a 1_{(X_{T_i} \neq X_{T_{i+1}})} = \limsup_{n \to \infty} a N_{\Pi_n}(X)
\]

where $\Pi_n$ is a finite increasing sequence of stopping times covering the interval $[0, T]$ and the mesh of $\Pi_n$ tends to 0 as $n \to \infty$. $N_{\Pi_n}(X)$ is the number of times that $(X_i \neq X_{i-1})$. For the random stopping times of $\Pi_n$ $\limsup_{n \to \infty} N_{\Pi_n}(X) = \infty$, unless $X$ is almost certain piecewise constant. Continuous trading strategies have infinite transaction costs. If the trading strategy has both jumps and continuous parts the transaction costs will also be infinite.
4.2.2 Proportional Transaction Costs

With proportional transactional costs there is possible to trade continuously if you use a trading strategy with paths with finite variation which is not possible with for example the standard Black-Scholes hedge of a European call/put option. Proportional transaction costs with continuous trading are infinite if the trading strategy has paths of infinite variation. A strategy $X$ has paths of finite variation on $[0, T]$ for a subset $\Lambda$ of $\Omega$. The cumulative transaction costs are $b \int_0^T S(u, 0) |dX_u|$ almost surely on $\Lambda$ where $|dX_u|$ denotes the total variation (Stieltjes path by path integral) which are infinite on $\Lambda^c$. The constant $b$ is the proportionate transaction cost per unit value defined in definition 4.1, part 2.

If $\Pi_n$ is a sequence of random partitions tending to the identity on $[0, T]$ and $X$ is a continuous trading strategy, then the cumulative transaction costs for proportional costs can be written as

$$TC(X) = \limsup_{n \to \infty} \sum_{T^n_i \in \Pi_k} S(T^n_k, 0) |\Delta X^n_{n,k}| b$$

where $\Delta X^n_{n,k} = X^n_{T^n_k} - X^n_{T^n_{k-1}}$. When $X$ has paths of finite variation this expression converges to the path by path integral $b \int_0^T S(u, 0) |dX_u|$ (Stieltjes integral) and when $X$ has paths of infinite variation the expression diverges to $\infty$.

4.2.3 Combined Fixed and Proportional Transaction Costs

Combined fixed and proportional transaction costs with continuous trading create infinite costs in finite time. If a trading strategy $X$ has continuous paths then $TC(X)$ denote the transaction costs so that

$$TC(X) \geq \limsup_{n \to \infty} \sum_{T^n_i \in \Pi_k} \delta 1_{(X^n_{T^n_i} \neq X^n_{T^n_{i+1}})} = \limsup_{n \to \infty} \delta N_{\Pi_n}(X)$$

with some constant $\delta$, $\Pi_n$ is as before a sequence of random partitions tending to the identity and $N_{\Pi_n}(X)$ is the number of times $X^n_{T^n_i} \neq X^n_{T^n_{i+1}}$ for the random stopping times of $\Pi_n$. This leads to infinite costs.

4.3 Linear Supply Curves

The classical theory has unlimited liquidity and is embedded in the previously discussed structure. The standard price process $S_t = S(t, 0)$ can be deduced by reduce the supply curve $x \to S(t, x)$ to $x \to S(t, 0)$. This is a line with slope 0 and vertical axis intercept $S(t, 0)$. When the supply curve is linear it can be written on the form

$$x \to S(t, x) = M_t x + b_t$$  \hspace{1cm} (4.3)$$

where $M_t = 0$ if the classical theory should hold. Suppose this is taken as the null hypothesis Blais [Bl] has shown that this can be rejected at the 0.9999 significance level. From this the conclusion that the supply curve exists and is not trivial is derived.

By use linear regression it can be showed that the supply curve is linear for liquid stocks.
The slope and intercept is time varying and can be written as equation 4.3 where \( b_t = S(t, 0) \) and \((M_t)_{t \geq 0}\) is itself a stochastic process with continuous paths.

**Theorem 4.2** A liquid stock with linear supply curve

\[
x \to S(t, x) =: M_t x + b_t
\]

and a càdlàg (right continuous with left limits) trading strategy \( X \) with finite quadratic variation has for a SFTS. the value in the money market account

\[
Y_t = -X_t S(t, 0) + \int_0^t X_u dS(u, 0) - \int_0^t M_u d[X, X]_u
\]

The quadratic differential term can have jumps.

Non-liquid stocks raises a problem and the presented theory cannot hold in one particular point. The supply curve \( x \to S(t, x) \) is not \( C^2 \). Here the supply curve is jump linear and has one jump which can be described as the bid ask spread. The only place the \( C^2 \) hypothesis is used is in the derivation of the SFTS. which makes it possible to eliminate this hypothesis in the jump linear case. Because the supply curve is no longer continuous \( \gamma(t) = S(t, 0) - S(t, 0-) \) is defined as the bid ask spread where \( S(t, 0- \) is the marginal ask and \( S(t, 0) \) is the marginal bid. Now \( \Lambda = (u, \omega) : \Delta X_u(\omega) \) and the supply curve has a jump linear form like this

\[
S(t, x) = \begin{cases} 
\beta(t)x + S(t, 0) & (x \geq 0) \\
\alpha(t)x + S(t, 0-) & (x < 0)
\end{cases}
\]

**Theorem 4.3** For an illiquid stock with a jump linear supply curve like the equation above (4.4) and a càdlàg trading strategy \( X \) with finite quadratic variation the value in the money market account for a SFTS. is

\[
Y_t = -X_t S(t, 0) + \int_0^t X_u dS(u, 0) - \int_0^t \beta(u)1_{A^c}(u) + \alpha(u)1_A(u)d[X, X]_u - \int_0^t 1_A(u)d[\gamma, X]_u
\]

where \( \gamma(t) = S(t, 0) - S(t, 0-) \) is the bid ask spread.

Let’s start with the money market process \( Y \) to show this. \( Y \) should satisfy

\[
Y_t = Y_0 - \lim_{n \to \infty} \sum_{k \geq 1} (\Delta X_{n,k}) S(T_n^k, \Delta X_{n,k}) = \\
= X(0)S(0, X_0) - \lim_{n \to \infty} \sum_{k \geq 1} (\Delta X_{n,k}) [S(T_n^k, \Delta X_{n,k}) - S(T_n^k, 0)] - \lim_{n \to \infty} \sum_{k \geq 1} (\Delta X_{n,k}) S(T_n^k, 0)
\]
where $\Delta X_{n,k} = X_{T_n^k} - X_{T_n^{k-1}}$. From before we know that the last sum converges to $-X_0 S(0,0) - X_t S(t,0) + \int_0^t X_u - dS(u,0)$ and due to jump linear hypothesis the second sum

$$
\sum_{k \geq 1} (\Delta X_{n,k}) \left[ S(T_n^k, \Delta X_{n,k}) - S(T_n^k, 0) \right] = \\
= \sum_{k \geq 1} (\Delta X_{n,k}) 1(\Delta x_{n,k} \geq 0) [\beta(t) \Delta X_{n,k} + S(t, 0) + S(t, 0)] + \\
\sum_{k \geq 1} (\Delta X_{n,k}) 1(\Delta x_{n,k} < 0) [\alpha(t) \Delta X_{n,k} + S(t, 0-) - S(t, 0)]
$$

where $\Lambda = (u, \omega) : \Delta X_{u,\omega} < 0$. As the limit is applied, standard theorem from stochastic calculus get the expression converges uniformly on compact time sets in probability to

$$
- \int_0^t \beta(u) 1_{\Lambda^c}(u) + \alpha(u) 1_{\Lambda u} d[X,X]_u - \int_0^t 1_{\Lambda}(u) d[\gamma, X]_u
$$

This gives, compared to fixed transaction costs previously discussed, that bid ask spreads not necessarily generate infinite liquidity costs and due to this can use continuous trading strategies.
5 Method

Equation in Theorem 4.3 is used in the simulation. The second and third term is the actual liquidity costs. The first and second term is the calculated gain/loss and, in the end, the option price, i.e. the value this replicating portfolio has in the money account at time $T$ with no money invested in the underlying asset. The option price, in a perfect economy without liquidity risk, is calculated using the analytic solutions of the chosen assets. This is defined in the next section.

Liquidity costs depends on the replicating portfolio adjustments $X$ and the properties a supply curve. These properties are defined in the following sections. When all of this is defined the simulation can be done. I have taken the liquidity risk and presented this as percentage of the analytical price of the options which is the price a option would have in a perfect economy, i.e. a frictionless and competitive market.

5.1 Supply Curve

Liquid stock is defined in the sense that the order book is updated frequently and has many entries. A two piece jump linear and linear supply curve is tested and gives a heuristic measure of liquidity for a stock. In data set provided by Morgan Stanley less than 8 % would be considered liquid by this criterion. Almost all stocks in this set are considered as illiquid stocks. [BP]

Because only 8 % of stocks in the US stock market can be considered liquid the supply curve with jump linear form (equation (4.4)) is used in this simulation which represents illiquid stocks. This is the case with illiquid stocks and three parameters need to be defined to create the linear jump supply curve. Two slope parameters and one parameter which simulates the bid ask spread are created with this process. This involves a stochastic process and should emulate a real supply curve. [BP]

A stochastic mean-reverting process (Ornstein-Uhlenbeck equation) is used to derive the jump linear supply curve. [O]

$$dX_t = (m - X_t)dt + \sigma dW_t$$

where $W_t \in \mathbb{R}$ is a Brownian motion and $m$ is real.

The Ornstein-Uhlenbeck process is an example of a Gaussian process that has a bounded variance and admits a stationary probability distribution. The difference between this process and a Brownian motion is in the drift term. The Brownian motion has a constant drift term and the Ornstein-Uhlenbeck process has a drift term dependent on the current value of the process. This gives that if the current value of the process is less than the long term mean, the drift will be positive and if the current value of the process is greater than the long term mean, the drift will be negative. The mean acts as an equilibrium level for the process.

In this case $m$ is the value which the mean-reverting process uses as the equilibrium. Three processes are used to get samples of $\alpha$, $\beta$ and $\gamma$. $\alpha$, $\beta$ and $\gamma$ are also used as $m$ to evaluate
the stochastic supply curve with the expected value of $m$ because these are set as the expected value of the supply curve properties. $\sigma$ in the mean-reverting process is the term that determine the volatility and is defined as an arbitrary number with $X_t \geq 0$ as a boundary condition. This is done to ensure that the supply curve does not have a negative value, i.e. it is impossible to have a negative dependence between the underlying asset price and the order size. $B_t$ is a simple Brownian motion.

5.2 Derivatives

The derivatives which are used in this simulation are European, Digital and Asian options. Both put and call for each of the options are tested and all the options are of European style.

European options analytical solution for the price is derived from the Black Scholes pricing model [BS]. These solutions are used in comparing the liquidity cost and the price of the options.

\[
E_{C_t} = S_t N(h_1) - Ke^{-r(T-t)}N(h_2),
E_{P_t} = -S_t N(-h_1) + Ke^{-r(T-t)}N(-h_2)
\]

where $S_t$ is the stock price, $K$ is the strike price, $r$ is the risk free rate, $(T - t)$ is the time to maturity and $N(h)$ is the cumulative normal distribution of

\[
h_1 = \ln(S_t/K) + (r + \sigma^2/2)(T - t),
\]
\[
h_2 = h_1 - \sigma \sqrt{T - t}.
\]

In this case Digital Cash-or-Noting options are used in the simulations. The closed form analytical formula is given by [RR]

\[
D_{C_t} = e^{-r(T-t)}N(h_2),
D_{P_t} = e^{-r(T-t)}N(-h_2).
\]

The cash payout of this type of option is either 0 or 1 due to normalization. The price of this option can also be factorized with a predetermined cash amount.

The geometric averaging Asian options has closed analytical prices [KV]

\[
Ag_{C_t} = S_t N(g_1) - Ke^{-r(T-t)}N(g_2),
Ag_{P_t} = -S_t N(-g_1) + Ke^{-r(T-t)}N(-g_2)
\]

where

\[
g_1 = \ln(S_t/K) + (b + \sigma_g^2/2)(T - t),
\]
\[
g_2 = g_1 - \sigma_g \sqrt{T - t},
\]
\[
\sigma_g = \sigma/\sqrt{3},
\]
\[
b = (r - \sigma^2/6)/2.
\]
There are no closed form solutions for arithmetic average Asian options. In this study an analytical approximation is used. [TW]

\[
A_{AC_t} = S_t e^{(B-r(T-t))} N(a_1) - K_a e^{-r(T-t)} N(a_2),
A_{AP_t} = -S_t e^{(B-r(T-t))} N(-a_1) + K_a e^{-r(T-t)} N(-a_2)
\]

where

\[
\begin{align*}
a_1 &= \left[ \ln\left(\frac{S_t}{K_a}\right) + (B + \sigma_a^2/2)(T-t) \right] / \sigma_a \sqrt{T-t}, \\
\sigma_a &= \sqrt{\ln(\frac{M_2}{(T-t)^2} - 2B)}, \\
B &= \ln\left(\frac{M_1}{(T-t)}\right), \\
M_1 &= e^{r(T-t)} - 1, \\
M_2 &= \frac{2e^{(r+\sigma^2)(T-t)}}{(r + \sigma^2)(2r + \sigma^2)(T-t)^2} + \frac{2}{r(T-t)^2} \left( \frac{1}{2r + \sigma^2} - \frac{e^{r(T-t)}}{r + \sigma^2} \right), \\
K_a &= \frac{T}{T-t} K - \frac{t}{T-t} S_{avg},
\end{align*}
\]

and \( S_{avg} \) is the average asset price.

5.3 Liquidity Costs

To derive the liquidity cost the classical replicating strategy, \( X \) need to be derived for each of the options. This is easy to calculate using the analytical pricing formulas and in the case with Asian options, the approximated analytical pricing formulas.

When \( X \) is derived the equation stated in Theorem 4.3 is used in the simulations. The terms which adjust the value in the money market account to include liquidity risk is named \( L_1 \) and \( L_2 \) where \( L_1 \) depends on the slope of the supply curve and \( L_2 \) depends on the bid ask spread of the defined supply curve.

The liquidity cost \( L_1 \) and \( L_2 \) are discussed together and not separate. This gives a total impact on the option price. This simulation is tested with 10 000 trajectories and 29 timesteps. Also the variance of these costs is derived. The model has also been stress tested to analyze the stability and also get some results in extreme cases.

A simple null hypothesis test is done where \( \text{liquidity costs} = 0 \) are tested. The conclusion is that there definitely exist liquidity costs.

5.4 Empirical Liquidity Costs

A set of data is used from the Swedish stock Handelsbanken (SHB) to derive the real supply curve. The complete order book for SHB is obtained with tick data from 2005-05-03 to 2005-08-08 and are arbitrary chosen. This data set contains each posting of the top 5 bids and the top 5 asks, including both prices and shares available. The top 5 bids and asks in
this data set are aggregated. This data shows how many shares the market is willing to buy or sell at prices near the quoted price as well as tick data for trades. Unlike standard tick data, this gives information about supply and demand for a stock and also the actual points on the supply curve.

This order book gives the numerical values of the supply curve which are taken as input in the extended model and evaluated.
6 Result

To get liquidity costs in an option some input data need to be defined. The parts that need input data in the model are the Black Scholes model and the supply curve. These data can be selected to follow a specified stocks course but for now they are arbitrary decided.

6.1 Theoretical Supply Curve

The following simulations were done to show the size of the liquidity risk and also how different input data influence liquidity costs.

The basic input data are chosen as:
The spot price $S_0 = 100$,
The strike price $K = 100$,
The risk free rate $r = 0.05$ (5%),
The volatility $\sigma = 0.30$,
The slope of the supply curve $\alpha = \beta = 10^{-4}$,
The bid ask spread of the supply curve $\gamma = 10^{-4}$.
Time to maturity = 1 year,
Timesteps = 256 ($2^8$).
256 timesteps represent that the underlying asset is bought or sold 256 times in a year to replicate all options.

Now the replicating strategy, $X$ and the price of each option can be determined. Liquidity costs is presented as a percentage of the option price, analytically determined, to get a good view of the impact of liquidity costs on the option price and compare different types of options.
Figure 2: This is the total liquidity risk in a European call option over time up to maturity, in this case 1 year with the basic data defined above.

Figure 2 is normalized with the option price at time $T = 1$. This price is not aggregated. In this figure it is clear that liquidity risk increases over time.

6.1.1 Liquidity Cost Sensitivity

This simulation is done with the basic data specified above. The liquidity cost is separated in $L_1$ and $L_2$. $L_1$ is the liquidity cost depending on the impact of the supply curve slope. $L_2$ is the liquidity cost that the bid ask spread in the supply curve contributes to the total liquidity cost $L_1 + L_2$.

Liquidity cost $L_2$ is in the order $10^2$ larger than $L_1$ and contributes with the largest part of the total liquidity cost of the European put and call option. The Digital option has the largest liquidity cost in percent of the option price. $L_2$ in Asian options has a smaller contribution to the total liquidity cost compared to European and Digital options but the contribution is still $10^1$ larger than the contribution of $L_1$. The liquidity cost is in the order of $10^{-4}$ to $10^{-3}$ percent of the option price.
<table>
<thead>
<tr>
<th>Liquidity costs</th>
<th>L1</th>
<th>L2</th>
<th>L1+L2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Percent of european C</td>
<td>6.831144e-006</td>
<td>1.229315e-004</td>
<td>0.000128</td>
</tr>
<tr>
<td>Percent of european P</td>
<td>1.039274e-005</td>
<td>1.870250e-004</td>
<td>0.000195</td>
</tr>
<tr>
<td>Percent of digital C</td>
<td>3.844734e-005</td>
<td>1.037360e-003</td>
<td>0.001060</td>
</tr>
<tr>
<td>Percent of digital P</td>
<td>3.932158e-005</td>
<td>1.040257e-003</td>
<td>0.001071</td>
</tr>
<tr>
<td>Percent of asiang C</td>
<td>1.053989e-005</td>
<td>3.282329e-004</td>
<td>0.000335</td>
</tr>
<tr>
<td>Percent of asiang P</td>
<td>1.339956e-005</td>
<td>4.435834e-004</td>
<td>0.000452</td>
</tr>
<tr>
<td>Percent of asiana C</td>
<td>9.743524e-006</td>
<td>1.549660e-004</td>
<td>0.000164</td>
</tr>
<tr>
<td>Percent of asiana P</td>
<td>1.397039e-005</td>
<td>2.221923e-004</td>
<td>0.000235</td>
</tr>
</tbody>
</table>

Table 1: The liquidity cost are presented with L1 and L2, i.e. liquidity cost depending on the supply curve slope and the bid ask spread.

6.1.2 Risk Free Rate Sensitivity

The basic data are used in this simulation but the risk free rate is changed. This is done to analyze the sensitivity of this parameter in the model. The risk free rates are changed to extreme values to stresstest the model and get a ‘worst case scenario’.

The liquidity risk changes by a factor of two when the risk free rate changes from zero to five percent. Liquidity costs are bigger when the rate tends to zero. This simulation gives the same differences in liquidity costs between different options, i.e. Digital options have higher liquidity costs than other options. The non existing liquidity risk in arithmetic Asian options with rate zero depends on the approximate analytical solution. This solution does not work with risk free rate zero and also if the dividend is the same as the risk free rate [TW].

<table>
<thead>
<tr>
<th>Liquidity costs</th>
<th>r=0</th>
<th>r=5</th>
<th>r=10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Percent of european C</td>
<td>0.000381</td>
<td>0.000128</td>
<td>0.000067</td>
</tr>
<tr>
<td>Percent of european P</td>
<td>0.000381</td>
<td>0.000195</td>
<td>0.000155</td>
</tr>
<tr>
<td>Percent of digital C</td>
<td>0.001072</td>
<td>0.001060</td>
<td>0.000525</td>
</tr>
<tr>
<td>Percent of digital P</td>
<td>0.003942</td>
<td>0.001071</td>
<td>0.000400</td>
</tr>
<tr>
<td>Percent of asiang C</td>
<td>0.000952</td>
<td>0.000335</td>
<td>0.000156</td>
</tr>
<tr>
<td>Percent of asiang P</td>
<td>0.000860</td>
<td>0.000452</td>
<td>0.000324</td>
</tr>
<tr>
<td>Percent of asiana C</td>
<td>NaN</td>
<td>0.000164</td>
<td>0.000055</td>
</tr>
<tr>
<td>Percent of asiana P</td>
<td>NaN</td>
<td>0.000235</td>
<td>0.000113</td>
</tr>
</tbody>
</table>

Table 2: All the basic parameters are used except the risk free rate to show liquidity costs.

6.1.3 Volatility Sensitivity

In this simulation all basic parameters are used except the volatility parameter which changes to investigate the impact on liquidity costs. $\sigma$ describes the volatility of the underlying asset used to price the options.

The extreme case of volatility does not change liquidity costs significant (table 2 in Appendix). When $\sigma$ is small liquidity costs decreases and when $\sigma$ is large the liquidity risk
increases. This is related to the movement of the underlying asset. Larger movements in the stock leads to more adjustments in the replicating portfolio, hence larger liquidity costs.

6.1.4 Spot Price Sensitivity

To investigate the impact on liquidity costs depending on spot price of the underlying asset the spot price are changed and also the strike price. The strike price follows the spot price only to ensure that all simulations, i.e. trajectories could be used in the simulation. If a big gap between spot price and strike price is enforced there would be too few trajectories that could give a satisfying result. If this scenario is interesting a number of different Monte Carlo methods could be used like importance sampling [G].

As before the option that has most liquidity costs is Digital options (table 4 in Appendix). It has about $10^1$ larger liquidity costs than the other options. The spot prices 10, 100 and 1000 are tested and the liquidity risk changes in the order of the spot prices. Spot price 100 generates $10^{-1}$ smaller liquidity costs than spot price 10. This can be explained by the supply curve. If you have to buy/sell the underlying asset due to the replicating portfolio a cheaper stock demands bigger orders to get the same value of an expensive stock.

6.1.5 Strike Price Relative Spot Price Sensitivity

This simulation is done with a gap between the spot price and the strike price. The gap is 10 percent of the spot price. In this case all trajectories can be used to investigate liquidity costs. If the gap would be large the same problem arises as above [G].

This gap sensitivity show an increase in liquidity costs when the strike price is 10 percent larger than the spot price (table 5 in Appendix). This can be explained by the drive in the underlying asset which depends on the risk free rate. The small liquidity risks in the case when the spot price is 10 percent larger than the strike price are quite easy to intuitive understand. If there is almost no risk that the price drops under the strike price, there is no need to adjust the replicating portfolio as much as with the case when the strike price and the spot price are almost equal.

6.1.6 Supply Curve Slope Sensitivity

In this simulation the supply curve slope are changed but all the other input data are set as the basic data. The supply curve has two slope parameters but they are put as the same. This is only used when specified stocks are used and is not relevant in this specific simulation. The slopes used are $10^{-2}$, $10^{-4}$ and $10^{-6}$. $10^{-2}$ is an extreme case and illustrates a stock with very high liquidity risk. This does not regard stocks with high turnover, but in a financial crisis the supply curve has a steep slope.

As expected, liquidity costs increases when the curve has a steep slope (table 6 in Appendix). The change from $10^{-4}$ to $10^{-2}$ has about a $10^1$ percent impact to liquidity costs. The digital options have still the largest liquidity risk.
6.1.7 Supply Curve Bid Ask Spread Sensitivity

The bid ask spread of the supply curve are changed in this simulation but all other inputs have the basic inputs. The bid ask spread is the spread between the highest bid and the lowest ask price.

The bid ask spread has a larger impact on liquidity costs on all options than the supply curve slope (table 7 in Appendix). When the bid ask spread is $10^{-2}$ the liquidity risk is in the order of $10^{-1}$ percent in Digital options and $10^{-2}$ percent in the other options. The change in liquidity costs is of a ratio of one-to-one, a change in the bid ask spread of $10^1$ makes an impact of $10^1$ percent on liquidity costs.

6.1.8 Timestep Sensitivity

To investigate the impact of the frequency of adjusting the replicating portfolio the timestep input is changed. 256 timesteps are used in the basic data and can be described as adjusting the portfolio one time a day when the option expires after one year.

The impact of adjusting the replicating portfolio 64 times and 1024 times is not large on liquidity costs (table 8 in Appendix). It also decreases as the timesteps tends to infinity. Digital options have still a higher liquidity risk.

6.1.9 Time to Maturity Sensitivity

This simulation investigates the impact of the options lifespan. A half year, a year and two years are used as input data. All other parameters are set as the basic data defined above.

Liquidity costs depending on time to maturity does not change drastically (table 9 in Appendix). When the lifespan is changed from two years to a half year liquidity costs gets about 50 percent larger. The Digital options doubles in liquidity costs in this case. Digital options are more sensitive and liquidity in this case more than doubles.

6.1.10 Worst Case Scenario

A simulation is made, not with the basic data but with the worst thinkable cases in every parameter in a liquidity cost perspective. This is done to illustrate a 'worst case scenario'. The model can be used to simulate these scenarios but it is important to realize that the input data in the model is as significant as the result.
### Table 3

<table>
<thead>
<tr>
<th>Liquidity costs</th>
<th>Worst case</th>
</tr>
</thead>
<tbody>
<tr>
<td>Percent of european C</td>
<td>0.399261</td>
</tr>
<tr>
<td>Percent of european P</td>
<td>0.246262</td>
</tr>
<tr>
<td>Percent of digital C</td>
<td>1.229960</td>
</tr>
<tr>
<td>Percent of digital P</td>
<td>0.791695</td>
</tr>
<tr>
<td>Percent of asiang C</td>
<td>0.916483</td>
</tr>
<tr>
<td>Percent of asiang P</td>
<td>0.338761</td>
</tr>
<tr>
<td>Percent of asiana C</td>
<td>NaN</td>
</tr>
<tr>
<td>Percent of asiana P</td>
<td>NaN</td>
</tr>
</tbody>
</table>

Table 3: This simulations input data is $S_0 = 10$, $K = 11$, $r = 0$, $\sigma = 0.50$ $\alpha = \beta = 10^{-2}$, $\gamma = 10^{-2}$, Time to maturity = 1 year and timesteps = 64.

The parameter with the largest impact on liquidity costs is the bid ask spread. The liquidity risk is an add-on to the option. The analytic approximation of the arithmetic Asian options are not possible to calculate with $r=0$. Liquidity costs of these options should be a bit lower than for the geometric Asian options as the have throughout these results.

### 6.2 Empirical Supply Curve

The stock Handelsbanken AB has been used as a guideline of the real economy and has the properties $\alpha = 1.2 \times 10^{-7}$, $\beta = 1.4 \times 10^{-6}$ and $\gamma = 2.9 \times 10^{-3}$. This is a stock with significant movement in shares which gives an upper bound in the property parameters. The parameters in the Black Scholes model are defined as the basic parameters defined before but with the spot price 165. This is done to emulate the stock to some extent.

### Table 4

<table>
<thead>
<tr>
<th>Liquidity costs SHB</th>
<th>L1</th>
<th>L2</th>
<th>L1+L2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Percent of european C</td>
<td>5.222830e-008</td>
<td>0.002117</td>
<td>0.002121</td>
</tr>
<tr>
<td>Percent of european P</td>
<td>7.945890e-008</td>
<td>0.003220</td>
<td>0.003227</td>
</tr>
<tr>
<td>Percent of digital C</td>
<td>8.029939e-008</td>
<td>0.017954</td>
<td>0.018022</td>
</tr>
<tr>
<td>Percent of digital P</td>
<td>1.438868e-007</td>
<td>0.017821</td>
<td>0.017767</td>
</tr>
<tr>
<td>Percent of asiang C</td>
<td>7.406151e-008</td>
<td>0.005651</td>
<td>0.005660</td>
</tr>
<tr>
<td>Percent of asiang P</td>
<td>9.315360e-008</td>
<td>0.007639</td>
<td>0.007651</td>
</tr>
<tr>
<td>Percent of asiana C</td>
<td>7.116521e-008</td>
<td>0.002662</td>
<td>0.002664</td>
</tr>
<tr>
<td>Percent of asiana P</td>
<td>1.020376e-007</td>
<td>0.003817</td>
<td>0.003819</td>
</tr>
</tbody>
</table>

Table 4: This simulation presents the liquidity risk in an option with HSB as the underlying asset.

In this case the liquidity cost L2, which depend on the bid ask spread is the factor that contributes with almost all liquidity cost.
7 Conclusions

7.1 Theory

This paper extends the classical arbitrage pricing theory to include liquidity risk. This is accomplished by studying the underlying asset price depending on the trading size. The first and second fundamental theorems are extended and proved to hold. The economy is shown to be arbitrage free, like the first theorem, if the stochastic process for the price for a marginal trade has an equivalent martingale probability measure. The second theory also approximately holds. If the martingale measure is unique the markets will be approximately complete.

In an approximately complete market the derivative prices is equal to the classical arbitrage free price of the derivative security. This is a horizontal linear supply curve. To attain an upward sloping curve one solution is to preclude continuous trading strategies. This implies an incomplete market and minimal cost super-replicating trading strategies are discussed in this regard. A model is derived in the liquid and illiquid stock case and definitions are made regarding liquidity and illiquidity to attain a sound theoretical solution.

When a classical option price approach is done and no softening of the price is made, i.e. no impact on the price due to order size, the literature supports three general conclusions [J]. The first is that the classical option price is true on average even when liquidity risk is included. The second conclusion is that although the classical (theoretical) option hedge cannot be applied as theory prescribes, the discrete approximations often provide reasonable approximations [JT]. These discrete approximations are also consistent with upward sloping supply curves. The third conclusion is that risk management measures like Value-at-Risk (VaR) are biased low due to the exclusion of liquidity risk, but simple adjustments for liquidity risk to risk measures like VaR are available.

7.2 Result

Results in the theory are used in deriving a theoretical sound model that takes liquidity costs into consideration. This model has been tested in theory and practice.

The largest contribution to liquidity costs is the bid ask spread $\gamma$ on the supply curve. Liquidity costs with the basic data used are in the order $10^{-4}$ percent of the option price except for the Digital option with $10^{-3}$ percent of the option price. This is not a significant impact on the price. Only at large turnovers this could generate large costs.

The Digital options have a larger liquidity costs than all the other options because the replicating portfolio in this case has to be adjusted more frequently.

The arithmetic Asian options have a smaller impact on liquidity costs compared to the geometric Asian options. The geometric option does not exist and is only used to approximate the value or costs to the real arithmetic Asian option. The geometric option is in pricing models underpriced compared to the real arithmetic Asian option and this also affects the liquidity cost [KV]. The analytic approximation of the arithmetic Asian option cannot be defined when the risk free rate is zero but the geometric closed solution can which gives a
good approximation of liquidity costs.

The worst case scenario had liquidity costs as big as one percent. This does not seem large but if the financial institutions supporting the option market only take a commission of 0.1 to 1 percent this profit disappear.

Options using the stock HSB as the underlying asset has a liquidity risk in the order of $10^{-2}$ percent of the option prices. This is calculated with a stock that has a large turnover compared to other stocks in the Swedish stock market but it is still defined as an illiquid stock because of a real bid ask spread over the collected book data used.

Swedens stock market is small compared to the market in USA and almost all stocks in that market are considered illiquid with my definition. It is not strange that almost all stocks in the Swedish stock market are considered illiquid. A smaller market contributes with larger liquidity costs in options.

The liquidity is in this theory an add on and aggregates over time with linear properties (figure 2). The slope of the liquidity risk gets steeper in a financial crisis when the market is illiquid.

An interesting area to research further is liquidity costs in an economy in crisis, when the supply curve has a steep slope. This model of liquidity costs is a good tool in risk management and gives a good estimation of worst case scenario. Also exact liquidity costs can be calculated using the actions of the broker as input.

When brokers act perfect from a modeling perspective, they can arbitrarily approximate a perfect hedging strategy with continuous strategies with paths of finite variation. In practice continuous trading, both financially and physically are impossible, all trading is of necessity discrete. Despite of this the result shows that one can get arbitrarily small liquidity charges by trading with high frequency in small amounts. If the attention is directed to not only one option, but instead a portfolio of several options, liquidity issues do arise in an important way, using this model with linear supply curves. Liquidity risk cannot be hedged, it is a binding cost.

To investigate the sensitivity of the input in the extended theory would be interesting to research further. This is not only interesting when modeling a fictitious economy in a crisis, but also how different input affect how the brokers should act.

It would be interesting to test this model on a real financial institution and see the impact of this liquidity risk in actual money. The turnover in the option market supported by a large financial institute is substantial and a small percent in liquidity costs could be a large amount of money and the option price may need to be adjusted.
8 References


Appendix

Volatility Sensitivity

<table>
<thead>
<tr>
<th>Liquidity costs</th>
<th>sigma=0.1</th>
<th>sigma=0.3</th>
<th>sigma=0.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Percent of european C</td>
<td>0.000106</td>
<td>0.000128</td>
<td>0.000117</td>
</tr>
<tr>
<td>Percent of european P</td>
<td>0.000372</td>
<td>0.000195</td>
<td>0.000150</td>
</tr>
<tr>
<td>Percent of digital C</td>
<td>0.000926</td>
<td>0.001060</td>
<td>0.000775</td>
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<tr>
<td>Percent of digital P</td>
<td>0.000803</td>
<td>0.001071</td>
<td>0.001240</td>
</tr>
<tr>
<td>Percent of asiang C</td>
<td>0.000235</td>
<td>0.000335</td>
<td>0.000307</td>
</tr>
<tr>
<td>Percent of asiang P</td>
<td>0.000719</td>
<td>0.000452</td>
<td>0.000335</td>
</tr>
<tr>
<td>Percent of asiana C</td>
<td>0.000072</td>
<td>0.000164</td>
<td>0.000187</td>
</tr>
<tr>
<td>Percent of asiana P</td>
<td>0.000215</td>
<td>0.000235</td>
<td>0.000232</td>
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</tbody>
</table>

Table 5: The basic data are used except for the volatility which is changed.

Spot Price Sensitivity

<table>
<thead>
<tr>
<th>Liquidity costs</th>
<th>s0=10, K=10</th>
<th>s0=100, K=100</th>
<th>s0=1000, K=1000</th>
</tr>
</thead>
<tbody>
<tr>
<td>Percent of european C</td>
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<td>Percent of european P</td>
<td>0.001955</td>
<td>0.000195</td>
<td>0.000020</td>
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<td>Percent of digital C</td>
<td>0.014222</td>
<td>0.001060</td>
<td>0.000104</td>
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<tr>
<td>Percent of digital P</td>
<td>0.014298</td>
<td>0.001071</td>
<td>0.000105</td>
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<td>Percent of asiang C</td>
<td>0.003355</td>
<td>0.000335</td>
<td>0.000034</td>
</tr>
<tr>
<td>Percent of asiang P</td>
<td>0.004526</td>
<td>0.000452</td>
<td>0.000046</td>
</tr>
<tr>
<td>Percent of asiana C</td>
<td>0.001645</td>
<td>0.000164</td>
<td>0.000016</td>
</tr>
<tr>
<td>Percent of asiana P</td>
<td>0.002358</td>
<td>0.000235</td>
<td>0.000024</td>
</tr>
</tbody>
</table>

Table 6: The basic data are used except for the spot price and the strike price which are changed but has the same value.

Strike Price Relative Spot Price Sensitivity

<table>
<thead>
<tr>
<th>Liquidity costs</th>
<th>s0=100, K=90</th>
<th>s0=100, K=100</th>
<th>s0=100, K=110</th>
</tr>
</thead>
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<tr>
<td>Percent of european C</td>
<td>0.000035</td>
<td>0.000128</td>
<td>0.000651</td>
</tr>
<tr>
<td>Percent of european P</td>
<td>0.000131</td>
<td>0.000195</td>
<td>0.000445</td>
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<td>Percent of digital C</td>
<td>0.000336</td>
<td>0.001060</td>
<td>0.001406</td>
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<td>Percent of digital P</td>
<td>0.000259</td>
<td>0.001071</td>
<td>0.000950</td>
</tr>
<tr>
<td>Percent of asiang C</td>
<td>0.000052</td>
<td>0.000335</td>
<td>0.001676</td>
</tr>
<tr>
<td>Percent of asiang P</td>
<td>0.000349</td>
<td>0.000452</td>
<td>0.000557</td>
</tr>
<tr>
<td>Percent of asiana C</td>
<td>0.000008</td>
<td>0.000164</td>
<td>0.001084</td>
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<tr>
<td>Percent of asiana P</td>
<td>0.000051</td>
<td>0.000235</td>
<td>0.000394</td>
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</table>

Table 7: The basic data are used except for the spot/strike price gap which is changed.
Supply Curve Slope Sensitivity

<table>
<thead>
<tr>
<th>Liquidity costs</th>
<th>( \alpha = \beta = 10^{-2} )</th>
<th>( \alpha = \beta = 10^{-4} )</th>
<th>( \alpha = \beta = 10^{-6} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Percent of european C</td>
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<td>0.00128</td>
<td>0.000122</td>
</tr>
<tr>
<td>Percent of european P</td>
<td>0.001229</td>
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<td>0.00185</td>
</tr>
<tr>
<td>Percent of digital C</td>
<td>0.005020</td>
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<tr>
<td>Percent of digital P</td>
<td>0.005145</td>
<td>0.001071</td>
<td>0.001022</td>
</tr>
<tr>
<td>Percent of asiang C</td>
<td>0.001391</td>
<td>0.000335</td>
<td>0.000325</td>
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<tr>
<td>Percent of asiang P</td>
<td>0.001795</td>
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<td>0.000439</td>
</tr>
<tr>
<td>Percent of asiana C</td>
<td>0.001142</td>
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<td>0.001638</td>
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</table>

Table 8: The basic data are used except for the supply curve slope which is changed.

Supply Curve Bid Ask Spread Sensitivity

<table>
<thead>
<tr>
<th>Liquidity costs</th>
<th>( \gamma = 10^{-2} )</th>
<th>( \gamma = 10^{-4} )</th>
<th>( \gamma = 10^{-6} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Percent of european C</td>
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</tr>
<tr>
<td>Percent of european P</td>
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<td>0.001060</td>
<td>0.000049</td>
</tr>
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<td>Percent of digital P</td>
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<td>0.001071</td>
<td>0.000049</td>
</tr>
<tr>
<td>Percent of asiang C</td>
<td>0.032564</td>
<td>0.000335</td>
<td>0.000014</td>
</tr>
<tr>
<td>Percent of asiang P</td>
<td>0.044011</td>
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<td>0.000018</td>
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<td>0.000011</td>
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<td>0.000016</td>
</tr>
</tbody>
</table>

Table 9: The basic data are used except for the bid ask spread of the supply curve which is changed.

Timestep Sensitivity

<table>
<thead>
<tr>
<th>Liquidity costs</th>
<th>1024 timesteps</th>
<th>256 timesteps</th>
<th>64 timesteps</th>
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</thead>
<tbody>
<tr>
<td>Percent of european C</td>
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<td>0.000175</td>
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<tr>
<td>Percent of european P</td>
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<td>0.000195</td>
<td>0.000266</td>
</tr>
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<td>Percent of digital C</td>
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<td>0.001060</td>
<td>0.000945</td>
</tr>
<tr>
<td>Percent of digital P</td>
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<td>0.001071</td>
<td>0.001666</td>
</tr>
<tr>
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<td>0.000335</td>
<td>0.000436</td>
</tr>
<tr>
<td>Percent of asiang P</td>
<td>0.000428</td>
<td>0.000452</td>
<td>0.000582</td>
</tr>
<tr>
<td>Percent of asiana C</td>
<td>0.000137</td>
<td>0.000164</td>
<td>0.000290</td>
</tr>
<tr>
<td>Percent of asiana P</td>
<td>0.000196</td>
<td>0.000235</td>
<td>0.000416</td>
</tr>
</tbody>
</table>

Table 10: The basic data are used except for the timesteps which is changed.
## Time to Maturity Sensitivity

<table>
<thead>
<tr>
<th></th>
<th>0.5 year</th>
<th>1 year</th>
<th>2 years</th>
</tr>
</thead>
<tbody>
<tr>
<td>Liquidity costs</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Percent of european C</td>
<td>0.000162</td>
<td>0.000128</td>
<td>0.000107</td>
</tr>
<tr>
<td>Percent of european P</td>
<td>0.000218</td>
<td>0.000195</td>
<td>0.000195</td>
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<td>Percent of digital C</td>
<td>0.001237</td>
<td>0.001060</td>
<td>0.000632</td>
</tr>
<tr>
<td>Percent of digital P</td>
<td>0.002467</td>
<td>0.001071</td>
<td>0.000468</td>
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<td>0.000406</td>
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<td>0.000285</td>
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<tr>
<td>Percent of asiang P</td>
<td>0.000500</td>
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</tbody>
</table>

Table 11: The basic data are used except for the lifespan of the options which is changed.