

On Latin squares and avoidable arrays

*For Daniel, and for what springtime brings*

Lina J. Andrén: On Latin squares and avoidable arrays

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## Abstract

This thesis consists of the four papers listed below and a survey of the research area.

- I Lina J. Andrén: *Avoiding  $(m, m, m)$ -arrays of order  $n = 2^k$*
- II Lina J. Andrén: *Avoidability of random arrays*
- III Lina J. Andrén: *Avoidability by Latin squares of arrays with even order*
- IV Lina J. Andrén, Carl Johan Casselgren and Lars-Daniel Öhman:  
*Avoiding arrays of odd order by Latin squares*

Papers I, III and IV are all concerned with a conjecture by Häggkvist saying that there is a constant  $c$  such that for any positive integer  $n$ , if  $m \leq cn$ , then for every  $n \times n$  array  $A$  of subsets of  $\{1, \dots, n\}$  such that no cell contains a set of size greater than  $m$ , and none of the elements  $1, \dots, n$  belongs to more than  $m$  of the sets in any row or any column of  $A$ , there is a Latin square  $L$  on the symbols  $1, \dots, n$  such that there is no cell in  $L$  that contains a symbol that belongs to the set in the corresponding cell of  $A$ . Such a Latin square is said to *avoid*  $A$ . In Paper I, the conjecture is proved in the special case of order  $n = 2^k$ . Paper III improves on the techniques of Paper I, expanding the proof to cover all arrays of even order. Finally, in Paper IV, similar methods are used together with a recoloring theorem to prove the conjecture for all orders.

Paper II considers another aspect of the problem by asking to what extent way a deterministic result concerning the existence of Latin squares that avoid certain arrays can be used when the sets in the array are assigned randomly.

## Sammanfattning

Denna avhandling innehåller de fyra nedan uppräknade artiklarna, samt en översikt av forskningsområdet.

- I Lina J. Andrén: *Avoiding  $(m, m, m)$ -arrays of order  $n = 2^k$*
- II Lina J. Andrén: *Avoidability of random arrays*
- III Lina J. Andrén: *Avoidability by Latin squares of arrays with even order*
- IV Lina J. Andrén, Carl Johan Casselgren and Lars-Daniel Öhman:  
*Avoiding arrays of odd order by Latin squares*

Artikel I, III och IV behandlar en förmodan av Häggkvist, som säger att det finns en konstant  $c$  sådan att för varje positivt heltal  $n$  gäller att om  $m \leq cn$  så finns för varje  $n \times n$  array  $A$  av delmängder till  $\{1, \dots, n\}$  sådan att ingen cell i  $A$  innehåller fler än  $m$  symboler, och ingen symbol förekommer i fler än  $m$  celler i någon av raderna eller kolumnerna, så finns en latinsk kvadrat  $L$  sådan att ingen cell i  $L$  innehåller en symbol som förekommer i motsvarande cell i  $A$ . En sådan latinsk kvadrat sägs *undvika*  $A$ . Artikel I innehåller ett bevis av förmodan i specialfallet  $n = 2^k$ . Artikel III använder och utökar metoderna i Artikel I till

ett bevis av förmodan för alla latinska kvadrater av jämn ordning. Förmodan visas slutligen för samtliga ordningar i Artikel IV, där bevismetoden liknar den som finns i i Artikel I och III tillsammans med en omfärgningssats.

Artikel II behandlar en annan aspekt av problemet genom att undersöka vad ett deterministiskt resultat om existens av latinska kvadrater som undviker en viss typ av array säger om arrayer där mängderna tilldelas slumpmässigt.

## Tack (acknowledgements)

Jag vill passa på och tacka alla som på olika sätt bidragit till att den här avhandlingen har blivit färdig. Det är en blandning av kollegor som har varit inblandade i arbetet på olika sätt, och familj och vänner vars stöd utanför arbetet har varit ovärderligt. Några personer hör naturligtvis till båda kategorierna. Framför allt har min man, Daniel, hela tiden funnits där både i livet utanför arbetet, och för att diskutera detaljer i min forskning. Tack! Dessutom vill jag särskilt tacka min handledare Roland Häggkvist, och min biträdande handledare Klas Markström. Mina medförfattare på avhandlingens sista artikel, Carl Johan Casselgren och Lars-Daniel Öhman förtjänar också ett stort tack för ett givande samarbete.

Jag vill också tacka mina föräldrar och systrar som alltid har trott på mig och uppmuntrat mig. Stort tack till er och till alla goda vänner som funnits där på vägen. Tack slutligen till kollegor i den diskreta gruppen och vid resten av Institutionen för matematik och matematisk statistik i Umeå. Utan er hade det varken varit lika roligt eller lika givande.



# Chapter 1

## Summary of papers

### I Lina J. Andrén: *Avoiding $(m, m, m)$ -arrays of order $n = 2^k$*

In this paper, we prove a conjecture by Häggkvist in the special case of Latin squares of order  $2^k$  for natural numbers  $k \geq 9$ . The conjecture states that there is a constant  $c$  such that for any  $n \times n$  array containing in each cell at most  $cn$  of the symbols  $1, \dots, n$ , and each symbol appearing at most  $cn$  times in each row and each column, there is a Latin square  $L$  such that every cell contains a symbol that does not appear in the corresponding cell of  $A$ .

(Submitted.)

### II Lina J. Andrén: *Avoidability of random arrays*

In this paper, we prove that any result giving a value  $m$  for which all  $(m, m, m)$ -arrays are avoidable also yields that an array where each symbol belongs to the set of forbidden symbols with probability  $p$  is almost surely

avoidable for  $p < \frac{m+9/2 \log n (1 - \sqrt{4m/(9 \log n + 1)})}{n}$ .

### III Lina J. Andrén: *Avoidability by Latin squares of arrays of even order*

The proof in this paper is an expansion of the result in Paper I to all arrays of order  $2k$  where  $k$  is a positive integer.

(Submitted.)

### IV Lina J. Andrén, Carl Johan Casselgren and Lars-Daniel Öhman: *Avoiding arrays of odd order by Latin squares*

In this paper we conclude the proof of the abovementioned conjecture by Häggkvist, proving it for arrays of order  $2k + 1$ , for positive integers  $k$ .



# Chapter 2

## Introduction

There are two main themes in the present thesis, namely the subject of Latin squares and that of colorings of graphs. Latin squares are one of the main objects in the four included papers, whereas we in this introductory survey also will talk about the same problems rephrased into the language of graph theory.

Latin squares, that is, arrangements of the numbers from 1 to  $n$  in a  $n \times n$  grid so that each number occurs exactly once in every row and in every column, were studied already by Euler. Below is an example of what a Latin square can look like.

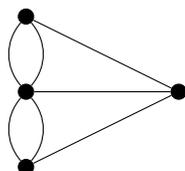
1	2	3	4	5
2	3	1	5	4
5	4	2	3	1
3	5	4	1	2
4	1	5	2	3

Euler studied the existence of pairs of Latin squares  $(L_1, L_2)$  that are *orthogonal*, by which is meant that for every pair of numbers  $(a, b)$ , with  $a = 1, \dots, n$  and  $b = 1, \dots, n$ , there is a position in  $L_1$  that contains  $a$  for which the corresponding position of  $L_2$  contains  $b$ . An example of two such Latin squares is given below.

1	2	3	1	3	2
2	3	1	2	1	3
3	1	2	3	2	1

Both such pairs of Latin squares, and Latin squares in general have been put to use in the statistical theory of design of experiments. As experimental designs, Latin squares were first used by Sir Ronald A. Fisher in his pioneering work “Design of experiments” [23]. On a less serious note, there are a few different puzzles based on Latin squares, the most popular being that of “Sudoku”.

Graphs and colorings, on the other hand, have their own range of applications and known problems. Again Euler was one of the first to study the concept, when he tried to solve the puzzle of the bridges of Königsberg. This problem asks whether the people of Königsberg can walk around their city, which is situated on the two banks of a river, with two islands in the middle, in such a way that they use each of the city's seven bridges exactly once. A schematic picture with the river banks and the islands as dots, and the bridges connecting them as lines looks like this:



This kind of picture, with lines and dots, is a common way of visualizing a graph. (The answer to the question is no.) In graph theory, the dots are called vertices and the lines between them are called edges. A coloring of a graph is an assignment of colors to the vertices so that no two vertices that are the endpoints of a common edge get the same color. The study of such colorings started with a question by Francis Guthrie who, in 1852 tried to color a map of the counties of England, and observing that he only needed 4 colors to color them all so that no two neighboring counties had the same color. He then asked if this was the case for all maps. The question was brought to the attention of the mathematics community by his brother Frederick Guthrie, and his brother's teacher, Augustus De Morgan. This question can be rephrased to ask whether it is true that for all graphs such that no two of the edges cross, it is possible to color the vertices of the graph with only 4 colors. The question was answered in the affirmative only in 1977 by the papers of Appel and Haken [4] and Appel, Haken and Koch [5]. In the meantime, the theory of colorings of graphs had grown into a large research field with many interesting other problems.

In the present thesis, we consider a problem of coloring the edges rather than the vertices. If every edge has a list of colors we can use there, is it possible to find a coloring so that no two edges that have a common endpoint get the same color? This, of course, depends both on the graph and the lists of colors. One way of thinking of this problem is to consider the scheduling of a number of school classes that need to be assigned both a classroom and a teacher for an hour of class. But every teacher only teaches certain classes, and furthermore, some of the classes can only use certain classrooms. Depending on those requirements, it may or may not be possible to assign a teacher and a classroom to every class at the same time. In the papers presented in this thesis we prove that under certain conditions, there will be a possible assignment.

# Chapter 3

## Basic definitions and tools

Both Latin squares and colorings of graphs are well-established objects of study in discrete mathematics. The main topic in this thesis is a problem introduced by Häggkvist [26], which can be formulated either as a problem for Latin squares, or as a problem of list-edge coloring complete bipartite graphs. For Latin squares, the problem amounts to constructing an array where each cell is assigned a set of forbidden symbols, and then trying to find a Latin square such that each cell in the Latin square contains a symbol that does not belong to the set of forbidden symbols in the corresponding cell of the array. The forbidden symbols should be fairly regularly distributed. Is it possible to find such a Latin square? In terms of list-coloring the edges of  $K_{n,n}$  this question becomes: If all edges of  $K_{n,n}$  are assigned a list, again fairly regularly distributed, taken from the set of symbols  $\{1, \dots, n\}$ , is it possible to color the edges of  $K_{n,n}$  according to those lists? To be able to make the questions mentioned above more precise, and to partly answer some of them, we need some definitions and a little notation.

### 3.1 Latin squares and arrays

An *array* is an arrangement of  $k \times \ell$  cells on a grid, each containing some mathematical object. Thus, an array is a kind of matrix, and while most matrix operations would be meaningless in this setting, the transpose of an array  $A$  will be taken to mean the same as a matrix transpose would. Given an array  $A$ , we call the cell in position  $(i, j)$  the *cell*  $(i, j)$  of  $A$ . The contents of this cell is denoted  $A(i, j)$ . A Latin square is a special kind of array.

**Definition 3.1.** A *Latin square of order  $n$*  on a set of symbols  $S$  with  $|S| = n$  is an  $n \times n$  array such that each symbol in  $S$  appears exactly once in each row and in each column.

For the rest of this thesis, unless explicitly stated otherwise, a Latin square uses the symbols  $[n] = \{1, 2, \dots, n\}$ .

**Example 3.2.** An example of a Latin square of order 4 is

$$A = \begin{array}{|c|c|c|c|} \hline 1 & 2 & \mathbf{3} & 4 \\ \hline 4 & 1 & \mathbf{2} & 3 \\ \hline 3 & 4 & 1 & 2 \\ \hline 2 & 3 & 4 & 1 \\ \hline \end{array}$$

where, for example, the symbol in cell  $(2, 3)$  of  $A$  is  $A(2, 3) = \mathbf{2}$ .

**Example 3.3.** The array given by the multiplication table of any quasi-group is a Latin square. The Latin square of Example 3.2 is the multiplication table of the cyclic group of order 4.

Given a Latin square  $L$ , any permutation of its rows, of its columns or of its symbols is still a Latin square. This partitions the set of all Latin squares of given order into equivalence classes:

**Definition 3.4.** We say that a Latin square  $L'$ , which can be obtained from  $L$  by permuting the rows, columns and symbols in some way, is *isotopic* to  $L$ . Isotopy is an equivalence relation on the set of all Latin squares of a given order.

For any square array of order  $n$ , a *diagonal* is a subset of  $[n] \times [n]$  such that each element in  $[n]$  occurs exactly once in each coordinate. The *main diagonal* is the cells  $(i, i)$  for  $i = 1, 2, \dots, n$ . The *antidiagonal*, or *back diagonal*, is the cells  $(i, n - i + 1)$  for  $i = 1, 2, \dots, n$ . When discussing Latin squares, we will also use the notion of an *i*-diagonal.

**Definition 3.5.** Let  $L$  be a Latin square, and let  $i$  belong to the symbol set of  $L$ . The *i*:th diagonal of  $L$  is the set of all cells in  $L$  that contain symbol  $i$ . This kind of diagonal is called a *i*-diagonal.

**Example 3.6.** Below are two isotopic Latin squares  $K$  and  $L$  of order 4. To see that  $K$  and  $L$  are isotopic, we can (for example) permute the columns of  $K$  by  $(4321)$  and the symbols by  $(13)(24)$ .

$$K = \begin{array}{|c|c|c|c|} \hline 1 & 2 & \mathbf{3} & 4 \\ \hline \mathbf{3} & 1 & 4 & 2 \\ \hline 2 & 4 & 1 & \mathbf{3} \\ \hline 4 & \mathbf{3} & 2 & 1 \\ \hline \end{array} \qquad L = \begin{array}{|c|c|c|c|} \hline 4 & 1 & 2 & \mathbf{3} \\ \hline \mathbf{3} & 2 & 4 & 1 \\ \hline 2 & \mathbf{3} & 1 & 4 \\ \hline 1 & 4 & \mathbf{3} & 2 \\ \hline \end{array}$$

The 3-diagonal is printed with a bold typeface in both  $K$  and  $L$ .

A Latin square is essentially two-dimensional. There are numerous ways to generalize Latin squares, but one way is to consider a Latin cube.

**Definition 3.7.** A *Latin cube* of order  $n$  and dimension  $m$  is a set of  $m$   $n \times n$  Latin squares  $\{L_1, \dots, L_m\}$  such that there is no cell  $(i, j)$  where  $L_k(i, j) = L_\ell(i, j)$  for any  $1 \leq k, \ell \leq m$ .

**Example 3.8.** Taking the Latin square of Example 3.2 and permuting the rows according to (1234) would result in a new Latin square, different from the first one in all cells. Permuting the rows repeatedly up to a total of 4 times, yields 4 Latin squares all different in each cell. The result is a Latin cube of order 4 and dimension 4, usually called *the cyclic Latin cube*:

1	2	3	4	2	3	4	1	3	4	1	2	4	1	2	3
4	1	2	3	1	2	3	4	2	3	4	1	3	4	1	2
3	4	1	2	4	1	2	3	1	2	3	4	2	3	4	1
2	3	4	1	3	4	1	2	4	1	2	3	1	2	3	4

One question that has been studied is whether a Latin cube of dimension  $m$  can be expanded to a Latin cube of dimension  $m+1$ , that is, given a Latin cube  $\mathcal{L}$  of dimension  $m$  and order  $n$ , is there a Latin square of the same order which, in each cell, contains a symbol not in the corresponding cell in any of the Latin squares in  $\mathcal{L}$ ? A discussion of this topic can be found in [37]. In Example 3.8 above, this is not possible, since the dimension is already as large as the order (and hence each symbol occurs in each cell  $(i, j)$  in one of the Latin squares). There are other examples where it is not possible to expand even though the dimension is not as large as the order. In the present thesis, we consider a somewhat more general notion than expanding Latin cubes.

**Definition 3.9.** An  $(m, m, m)$ -array is an array where the cells contain subsets of  $\{1, \dots, n\}$  such that each cell contains a set of at most  $m$  symbols, and each symbol appears in at most  $m$  of the sets in any row or column.

**Example 3.10.** The following arrays are all  $(3, 3, 3)$ -arrays:

1.

1, 3, 4	1, 2, 4	1, 2, 3	2, 3, 4
2, 3, 4	1, 3, 4	1, 2, 4	1, 2, 3
1, 2, 3	2, 3, 4	1, 3, 4	1, 2, 4
1, 2, 4	1, 2, 3	2, 3, 4	1, 3, 4

2.

1, 2, 3	1, 2, 3	1, 2, 3
1, 2, 3	1, 2, 3	1, 2, 3
1, 2, 3	1, 2, 3	1, 2, 3

3.

1, 2, 3		2	1, 3, 4	3, 4, 5
2, 5	1, 3, 4	2	2, 4, 5	
1	4	1	2, 5	2
1, 2, 3	1, 2, 3	2, 3, 4	4, 5	4, 5
3, 4, 5	1, 2, 3	1, 3, 4	1, 2	2

**Example 3.11.** Any Latin square is a  $(1, 1, 1)$ -array with no empty cell.

**Example 3.12.** Given a Latin cube  $\mathcal{L}$  of dimension  $m$  and any order  $n$  we can obtain an  $n \times n$   $(m, m, m)$ -array by for each cell  $(i, j)$  letting

$$A(i, j) = \bigcup_{L \in \mathcal{L}} \{L(i, j)\}.$$

**Definition 3.13.** Given an  $n \times n$   $(m, m, m)$ -array  $A$ , a Latin square  $L$  of order  $n$  and a cell  $(i, j)$  such that  $L(i, j) \in A(i, j)$  we say that  $L$  has a *conflict* with  $A$  in cell  $(i, j)$ . If  $L$  has no conflict with  $A$ , we say that  $L$  *avoids*  $A$ . If there exists a Latin square  $L$  avoiding  $A$ , then  $A$  is *avoidable*, otherwise  $A$  is *unavoidable*.

**Example 3.14.** Of the arrays from Example 3.10, array 1 can be avoided by

$$L = \begin{array}{|c|c|c|c|} \hline 2 & 3 & 4 & 1 \\ \hline 1 & 2 & 3 & 4 \\ \hline 4 & 1 & 2 & 3 \\ \hline 3 & 4 & 1 & 2 \\ \hline \end{array}.$$

Array 2 is unavoidable, since all symbols are forbidden everywhere. Array 3 is avoided by

$$L = \begin{array}{|c|c|c|c|c|} \hline 4 & 1 & 3 & 5 & 2 \\ \hline 3 & 2 & 4 & 1 & 5 \\ \hline 2 & 3 & 5 & 4 & 1 \\ \hline 5 & 4 & 1 & 2 & 3 \\ \hline 1 & 5 & 2 & 3 & 4 \\ \hline \end{array}.$$

**Example 3.15.** The  $(3, 3, 3)$ -array

$$A = \begin{array}{|c|c|c|c|c|} \hline 1, 2, 3 & 1, 2, 3 & 1, 2, 3 & & \\ \hline & & & & \\ \hline \end{array}$$

is not avoidable, since when trying to construct a Latin square  $L$  avoiding  $A$ , for the first row, we can only choose where to put two of the numbers 1, 2, 3 without putting one of them in one of the cells  $(1, 1)$ ,  $(1, 2)$  or  $(1, 3)$ .

**Example 3.16.** The  $(m, m, m)$ -array obtained from an expandable Latin cube  $\mathcal{L}$  of dimension  $m$  as described in Example 3.12 is avoidable, since the Latin square which could be used for the next “layer” of the Latin cube avoids such an array.

A  $(1, 1, 1)$ -array with some cells empty is usually called a *partial Latin square*. A partial Latin square is called *completable* if it is possible to fill the empty cells so that no symbol occurs more than once in any row or column.

**Example 3.17.** Let  $P$  be a partial Latin square of order  $n > 1$  which is completable. Such  $P$  are avoidable (see [18]), as are all Latin squares. The reason for this is that a Latin square is avoided by, for example, a Latin square where the rows all have been moved one step down, and the bottom row is placed at the top. The partial Latin square  $P$  can be completed to a Latin square, and thus avoided in the same way.

**Example 3.18.** It is impossible to complete the partial Latin square 1. below, but the partial Latin square 2. can be completed to the Latin square 3. Furthermore, the Latin square 4. avoids 2.

1.

1	2	
2	1	
		3

2.

1		3
2		
	1	2

3.

1	2	3
2	3	1
3	1	2

4.

3	1	2
1	2	3
2	3	1

Note that in this case the partial Latin square 1. is avoidable, even though it is not completable, since it can be avoided by

3	1	2
1	2	3
2	3	1

There is, in fact, up to isotopy only one partial Latin square of order 3 which is not avoidable, namely the partial Latin square depicted below [18].

1	2	3
2	1	
3		1

## 3.2 Graphs and colorings

An equivalent formulation of the problems considered in papers I–IV of this thesis can be made in terms of coloring edges in complete bipartite graphs. Note that, for the rest of this thesis, we consider only simple graphs with no loops. Let  $K_{m,n}$  be the complete bipartite graph, that is, a graph with two sets of vertices  $V_1$  and  $V_2$ ,  $|V_1| = m$ ,  $|V_2| = n$  such that each pair of vertices  $v_1 \in V_1$  and  $v_2 \in V_2$  are connected by an edge. There are no edges between two vertices both in  $V_1$ , or both in  $V_2$ .

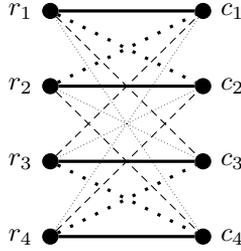
Some additional graph notation will be used in the subsequent chapters. Given a graph  $B$ , the number of edges to which a vertex  $v$  belongs is called *the degree of  $v$* . The minimum of the degrees of the vertices in  $B$  is called *the minimum degree of  $B$* . This is written as  $\delta(B)$ . The *maximum degree of  $B$*  is written as  $\Delta(B)$  and is defined similarly. A *matching* in  $B$  is a set of edges such that no vertex belongs to more than one of them, and a *perfect matching* is a matching such that every vertex belongs to exactly one of the edges in the matching.

In order to formulate the Latin square problems above in terms of graphs we need to consider graph colorings. In this case, we will work with colorings of the edges of a graph  $G$ . A (*proper*) *coloring of the edges* of  $G$  is an assignment  $\chi$  of colors to the edges of  $G$  so that each edge  $e$  of  $G$  is assigned a color  $\chi(e)$ , but no two incident edges  $e$  and  $f$  are assigned the same color, i. e.  $\chi(e) \neq \chi(f)$ . The basic problem in the theory of graph colorings is to ask how many colors are needed to color a given graph, or some particular family of graphs. A Latin square of order  $n$  corresponds to an edge-coloring of  $K_{n,n}$ , as can be seen in the following example:

**Example 3.19.** The Latin square

$$L = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\ \hline 2 & 1 & 4 & 3 \\ \hline 3 & 4 & 1 & 2 \\ \hline 4 & 3 & 2 & 1 \\ \hline \end{array}$$

corresponds to the coloring of the edges of  $K_{4,4}$  depicted below. Each row in  $L$  corresponds to a vertex on the left of the graph, and each column corresponds to a vertex on the right. Hence, cell  $(i, j)$  corresponds to the edge between vertex  $r_i$  and vertex  $c_j$ . In this picture different symbols in  $L$  correspond to different patterns on the edges.



Note that the cells containing one symbol in  $L$  correspond to a monochromatic matching in the graph.

**Definition 3.20.** The *chromatic index*  $\chi'(G)$  of a graph  $G$  is the smallest number of colors needed to color the edges of  $G$ .

A well known theorem by Vizing [45] states that

$$\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1,$$

where  $\Delta(G)$  denotes the maximum degree of  $G$ . In the special case of bipartite graphs, König's theorem states that

$$\chi'(G) = \Delta(G),$$

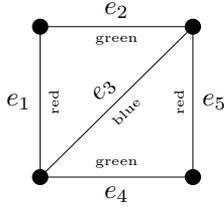
hence for  $K_{m,n}$  we have  $\chi'(K_{m,n}) = \max(m, n)$ . König's theorem remains true when considering graphs with multiple edges, but loops still have to be forbidden. A Latin square is a coloring of the edges of  $K_{n,n}$ , using the minimal number of colors, namely  $n$ . The next definition explains a concept for edge colorings of graphs similar to the notion of avoiding arrays.

**Definition 3.21.** Let  $G = (V, E)$  be a graph on a vertex set  $V$  with edge set  $E$ . To each edge  $e$  in  $E$ , assign a list  $\ell(e)$  of colors that are allowed on edge  $e$ . A  *$\ell$ -list coloring* of the edges  $G$  is an edge-coloring  $\chi(G)$  of  $G$  such that for all edges  $e \in E$  we have that  $\chi(e) \in \ell(e)$ .

**Example 3.22.** Given the list assignments

$$\begin{aligned} \ell(e_1) &= \{\text{red, blue}\}, \\ \ell(e_2) &= \{\text{red, green}\}, \\ \ell(e_3) &= \{\text{green, blue}\}, \\ \ell(e_4) &= \{\text{green}\} \text{ and} \\ \ell(e_5) &= \{\text{red, green, blue}\} \end{aligned}$$

it is possible to list-color the edges of the graph below.



**Definition 3.23.** The *list-chromatic index* of a graph  $G$  is the smallest number  $k$  such that it is possible to list-color the edges of  $G$  according to  $\ell$  for any list assignment  $\ell$  with  $|\ell(e)| \geq k$  for all edges  $e$  of  $G$ . The list-chromatic index of  $G$  is denoted  $\chi'_\ell(G)$ .

While  $\chi'(G)$  is very closely related to  $\Delta(G)$  by Vizing's theorem, it is not yet known if the value of  $\chi'_\ell(G)$  is so easily bounded. The problem of finding the list-chromatic index for general graphs leads to some of the problems discussed in the next section.

In contrast with the general problems of the list-chromatic index of graphs, to make the analogy between list-colorings of  $K_{n,n}$  and avoiding arrays by Latin squares complete, it must be noted that the total number of colors used for the lists is restricted. In general, when assigning lists to the edges (or the vertices) of a graph, any number of symbols can be used. If we want to consider a situation for  $K_{n,n}$  which would be equivalent to avoiding arrays by Latin squares, the union of all lists has to consist of only  $n$  colors in total. Since this is a rather low bound for assigning lists to the edges of a graph on  $2n$  vertices, some unexpected results for list-coloring the edges of  $K_{n,n}$  with this kind of list-assignments can be obtained.

# Chapter 4

## Problems and known results

Latin squares as an object of mathematical study go back a long time, having been studied already by Leonard Euler in the 1770's. In light of their connection to graphs, and also by merit of their own, they are still a combinatorial object of interest. Recent years have seen a large popular interest in so called “sudokus”, a problem of completing a partial Latin square that fulfills an extra condition. The specific problem covered here, namely the problem of finding Latin squares that avoid a certain kind of arrays was first suggested by Häggkvist [26]. The paper [26] explains the concept of avoiding  $(m, m, m)$ -arrays, and contributes a first result on avoiding arrays, which concerns avoiding *partial column Latin squares*, namely arrays with at most one element out of  $1, \dots, n$  in each cell and each symbol occurring at most once in each column.

**Theorem 4.1.** [26] *Let  $n = 2^k$  and let  $L$  be a partial  $n \times n$  column Latin square on  $1, \dots, n$  with empty last column. Then there exists a Latin square of order  $n$  on the same symbols which differs from  $L$  in every cell.*

In addition to proving the above result, a conjecture which can be reformulated in terms of  $(m, m, m)$ -arrays was stated:

**Conjecture 4.2.**

- (a) *There is a constant  $c > 0$  such that any  $n \times n$   $(m, m, m)$ -array  $A$  where  $m \leq cn$  is avoidable.*
- (b) *For  $n > 3$ , the largest value of  $c$  such that (a) holds is  $\frac{1}{3}$ .*

The above conjecture is the starting point for the articles in this thesis.

In terms of graphs, the problem of avoiding  $(m, m, m)$ -arrays becomes the problem of coloring  $K_{n,n}$  from lists of length  $n - m$  taken from the symbol set  $[n]$ , where the lists of allowed colors includes each color in at least  $n - m$  of the list for edges adjacent to any one vertex. The area of list-colorings of graphs was introduced independently by Vizing [46] in 1976 and by Erdős, Rubin and

Taylor [22] in 1979. Those early papers introduced a number of problems, and also stated some results. The main subject in those papers was not list-edge coloring, but rather list coloring of the vertices of graphs.

The paper [22], and much of the early research on list-colorings, was inspired by a problem by Dinitz:

**Dinitz' conjecture.** *Given an  $n \times n$  array of sets of size  $n$ , it is always possible to choose one element from each set, keeping the chosen elements distinct in every row and distinct in every column.*

The Dinitz problem remained open for quite long, even though many related results were published. For example, Janssen [33] proved the statement for  $r \times n$  arrays for any  $r < n$ :

**Theorem 4.3.** *Let  $r < n$ , and let  $S = \{S(i, j) | 1 \leq i \leq r, 1 \leq j \leq n\}$  be a collection of sets such that  $|S(i, j)| = n$  for all  $i, j$ . Then there exists an  $r \times n$  partial Latin rectangle  $L$  with  $L(i, j) \in S(i, j)$  for all  $i, j$ .*

Since this theorem holds for  $r = n - 1$  it shows that the value  $n$  in Dinitz' conjecture is at most 1 too small. The proof is based on a technique presented by Alon and Tarsi in [1] which has been useful also for list-coloring vertices of graphs. This technique will be discussed further in Section 6.1.

Note that neither in Dinitz' conjecture nor in Theorem 4.3, there is an assumption on all symbols coming from the set  $[n]$ . Dinitz' problem can be rephrased as a problem for bipartite graphs, in which case it concerns whether the list-edge chromatic number of the complete bipartite graph  $K_{n,n}$  is  $n$ . Galvin finally resolved the original question by proving the following, more general theorem:

**Galvin's Theorem.** [24] *Any bipartite multigraph  $G$  has  $\chi'_\ell(G) = \chi'(G)$ .*

Since Latin squares of order  $n$  use exactly  $n$  symbols this means that in order for there to be a Latin square that respects the restrictions, not even one symbol can be forbidden in each cell. One simple example of why this cannot work is that if symbol 1 is forbidden everywhere, but all other symbols are allowed in all cells, no Latin square can be constructed. It is not obvious though, if extreme cases like this is the only reason for the list-chromatic index of  $K_{n,n}$  to be as large as  $n$ . Restricting the total number of colors used in the lists can actually make it harder to assign lists that do not allow a coloring. In fact, Öhman [40] has shown that if all edges of  $K_{n,n}$  are assigned 8 colors uniformly at random from the colors  $1, \dots, n$ , then the expected number of colorings is larger than 1. This result relies on the fact that while most random assignments contain no valid list coloring, some assignments will contain a very high number of them, but it would still be interesting to know what kinds of constraints do allow list-colorings. To make the constraints more regular by imposing the conditions of  $(m, m, m)$ -arrays is one way to examine how much influence the bound in Galvin's Theorem has when restrictions are imposed on the lists.

On the other hand, Markström and Öhman [36] have undertaken a study of what kind of “extremal” arrays are unavoidable. They classify small unavoidable arrays with at most one symbol in each cell. Furthermore, they make a conjecture about what *all* unavoidable arrays with at most one symbol in each cell look:

**Conjecture 4.4.** [36] *For  $n \geq 5$ , an  $n \times n$  unavoidable array with at most one symbol per cell contains a subarray isotopic to one of the array-types  $A, B, C$  or  $D$ .*

**Type A** *For  $r = 1, \dots, \lceil \frac{n}{2} \rceil$ ,  $A_r$  is an  $n \times n$  array with an  $r \times (n - r + 1)$  subarray where all cells contain the same symbol.*

**Type B** *For  $r = 1, \dots, \lceil \frac{n}{2} \rceil$ ,  $B_r$  is an  $n \times n$  array such that it contains two symbols,  $s$  and  $t$ . There is an  $r \times (n - r + 1)$  subarray  $R_s$  where all cells except one cell  $c$  contain  $s$  and an  $(n - r + 1) \times r$  subarray  $R_t$  such that  $R_t \cap R_s = c$  and all cells in  $R_t$  except  $c$  contain symbol  $t$ .*

**Type C** *Arrays of type  $C_1$  are  $n \times n$  arrays where cells  $(1, 1), \dots, (n - 1, 1)$  and  $(n - 1, 3), \dots, (n - 1, n - 1)$  contain one symbol  $s$ , cells  $(1, 2), \dots, (n - 2, 2)$  and  $(n, 2), \dots, (n, n - 1)$  contain another symbol  $t$  and cells  $(1, n), \dots, (n - 2, n)$  all contain a third symbol  $u$ . Arrays of type  $C_2$  are  $n \times n$  arrays where cells  $(1, 1), \dots, (n - 1, 1)$  and  $(1, 1), \dots, (1, n - 1)$  all contain one symbol  $s$ , cells  $(n, 2), \dots, (n, n - 1)$  contain another symbol  $t$  and cells  $(2, n), \dots, (n - 1, n)$  all contain a third symbol  $u$ .*

**Type D** *Arrays of type  $D_r$ , for  $r = 1, \dots, n$  are arrays where for  $i = 1, \dots, r - 1$  column  $i$  contains symbol  $i$  in the rows  $1, \dots, n - 1$ , and row  $n$  contains symbol  $r$  in the cells in columns  $r, \dots, n$ .*

Note that arrays of type  $B_2$  always contain a subarray of type  $B_1$ , and that arrays of type  $A_1$  and  $D_1$ , and of type  $B_1$  and  $D_2$ , are isotopic. Hence, this list is not minimal.

**Example 4.5.** Let  $n = 5$ . The different types of arrays in Conjecture 4.4 are shown below.

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One generalization that arose from the study of Dinitz problem was to ask the same question for general graphs. In the general case, the corresponding statement is called the List-Coloring Conjecture.

**The List-Coloring Conjecture.** *For any graph  $G$*

$$\chi'_\ell(G) = \chi'(G).$$

The origin of the List-Coloring Conjecture is not entirely clear, see the historic overview in [27]. Most sources agree that Vizing was the first person to mention the conjecture, but many others worked on it independently. It appeared in print only in 1985 in [9], where it was attributed to Albertson and Tucker. As a starting point, a trivial upper bound can be obtained as follows: If each edge is assigned a list of length  $2\Delta(G) - 1$  it is always possible to list-color the edges, since no edge is adjacent to more than  $2\Delta - 2$  other edges. The first general sharper result was published in 1985 by Bollobás and Harris [9].

**Theorem 4.6.** *For any simple graph  $G$*

$$\chi'_\ell(G) < \frac{11\Delta(G)}{6} + o(\Delta(G)). \tag{4.1}$$

This was subsequently improved for triangle-free graphs by Chetwynd and Häggkvist [16] to

**Theorem 4.7.** *If  $G$  is a triangle-free graph, then*

$$\chi'_\ell(G) < \frac{9\Delta(G)}{5}. \quad (4.2)$$

It was pointed out by Bollobás and Hind [10] that the constant  $\frac{9}{5}$  in Theorem 4.7 can be improved to  $\frac{12}{7}$ . Kahn [34] proved that the List-Coloring Conjecture holds asymptotically:

**Theorem 4.8.** *For any simple graph  $G$  we have that*

$$\chi'_\ell(G) = \Delta(G) + o(\Delta(G)).$$

Molloy and Reed [38] improved Kahn's result by refining the  $o(\Delta)$  term to  $\text{poly}(\log(\Delta))$ :

**Theorem 4.9.** *For any graph  $G$  with maximum degree  $\Delta$ ,*

$$\chi'_\ell(G) \leq \Delta + 5\sqrt{\Delta} \log^4 \Delta.$$

The proof of this theorem uses the same methods as that of Theorem 4.8, which will be discussed further in Section 6.1. Furthermore, Häggkvist and Janssen [28] bounded the list-chromatic index for complete graphs.

**Theorem 4.10.** *For the complete graph  $K_n$  it is true that  $\chi'_\ell(K_n) \leq n$ .*

This bound is sharp for odd  $n$ , but not for even  $n$ .

So far, though, the List-Coloring Conjecture remains open. However, for the rest of this thesis we will concern ourselves mainly with bipartite graphs, where Galvin's theorem has answered the List-Coloring Conjecture altogether. What we aim to study here, though, is slightly different. Consider the following statement shown by Öhman [40]:

**Theorem 4.11.** *Let  $L$  be a collection of  $n^2$  sets of size 8 with elements chosen from  $\{1, 2, \dots, n\}$  assigned to the edges of  $K_{n,n}$ . Then the expected number of colorings of the edges of  $K_{n,n}$  using only colors from the prescribed lists is greater than one.*

Thus, in a probabilistic sense, we have that if we restrict the number of colors, we might be able to expect colorings to exist for lists a lot shorter than  $n$ . It is important to note though, that the expected value in Theorem 4.11 becomes high because some list assignments contain many colorings. With list length 8, most list-assignments still will not contain a coloring, even though the expectation is large. On the other hand, since 8 is a very small, constant number, it might be interesting to try to classify some types of list-assignments that always *do* contain

colorings. If the number of colors is restricted to  $n$ , this is the same as trying to find types of arrays that are avoidable by Latin squares.

In [27] Häggkvist and Chetwynd studied list-edge colorings of general graphs with the total number of colors restricted. They defined a number of coloring parameters and derived bounds for them. One of the most interesting theorems in this article is the following:

**Theorem 4.12.** *Let  $G$  be a complete graph of order  $2n$  and assign to each edge a list of length at least  $2n - 2$  out of the colors  $1, 2, \dots, 2n - 1$  in such a way that every color belong to at least  $n + 1$  of the lists. For any coloring of the edges of  $G$  there is a permutation of the colors that yields a coloring allowed by the lists.*

Returning to the problem of avoiding arrays by Latin squares, there has been some progress since the subject first emerged. Chetwynd and Rhodes studied the avoidability of partial Latin squares, that is,  $(1, 1, 1)$ -arrays. Any Latin square, and hence also any completable partial Latin square, is avoidable, since applying a permutation with no fixed point to the symbols of a Latin square  $L$  will generate a Latin square which in each cell contains a symbol different from the corresponding symbol in  $L$ , see also (2) and (3) of Example 3.18. Furthermore, they proved the following theorem even for non-completable partial Latin squares [18]:

**Theorem 4.13.** *Any  $2k \times 2k$  or  $3k \times 3k$  partial Latin square is avoidable for  $k \geq 2$ .*

The remaining cases of this statement, namely that  $4k+1 \times 4k+1$  and  $4k-1 \times 4k-1$  partial Latin squares are avoidable, were subsequently proved by Cavenagh [14] and Cavenagh and Öhman [15] respectively.

Kuhl and Denley [35] studied a problem similar to that of avoiding a partial Latin square. They call an  $n \times n$  array of at most  $nr$  symbols so that each cell of the array contains  $r$  symbols and each symbol appears at most once in each row and column a *partial  $r$ -multi Latin square*. An  *$r$ -multi Latin square* is a partial  $r$ -multi Latin square with no empty cell. Call a partial  $r$ -multi  $n \times n$  Latin square  $\mathcal{A}$   *$r$ -multi avoidable* if there is a  $r$ -multi Latin square  $\mathcal{L}$  of order  $n$  such that for each pair of indices  $i, j$  we have that  $\mathcal{A}_{i,j} \cap \mathcal{L}_{i,j} = \emptyset$ .

**Theorem 4.14.** [35] *An  $n \times n$  partial  $r$ -multi Latin square with at most  $(n - 1)$  cells filled is  $r$ -multi avoidable.*

**Theorem 4.15.** [35] *Let  $0 < \delta < 1$  and  $0 < \varepsilon < 1/(1 + \delta)$  and let  $R$  be a partial  $r$ -multi Latin square order  $n$ , using at most  $nr$  symbols. Suppose there is a Latin square  $L$  of order  $n$  such that  $\delta n$  of the symbols in  $R$  occupy cells in  $L$  corresponding to the filled cells in  $R$ . Then  $R$  is  $r$ -multi avoidable provided  $r$  is large enough.*

We will now return to one-dimensional Latin squares, and the problem of avoiding  $(m, m, m)$ -arrays. An early result by Chetwynd and Rhodes [17] states the following:

**Theorem 4.16.** *Let  $k > 3240$  and let  $A$  be a  $(2, 2, 2)$ -array of order  $4k$  on the symbols  $1, 2, \dots, 4k$ . Then  $A$  is avoidable.*

Cutler and Öhman [19] improved this by showing that the number of forbidden symbols in each cell, and the number of times that each symbol can appear in a row or column, can actually grow as the order of the array grows:

**Theorem 4.17.** *There exists a  $c = c(m)$  such that if  $k > c$  and if  $A$  is a  $(m, m, m)$ -array of order  $2mk$  then  $A$  is avoidable.*

The proof of this statement requires a large value for  $k$ . The authors write that their calculations are not optimized, but still a different approach is likely to be necessary for significant improvement. Their proof works if

$$k > (2m + 1) \left( m^2 + 4m^2 \binom{m}{2} + 2m^5 + \binom{2m^3}{2} + 4m^6 + 4m^7 \right),$$

which means that  $(m, m, m)$ -arrays of order  $n$  are avoidable if  $n$  grows so fast that

$$\lim_{n \rightarrow \infty} \frac{n}{m^9} = \infty,$$

that is  $m = o(n^{1/9})$ .



# Chapter 5

## Structural properties of Latin squares

### 5.1 Cycles and homogeneity

In the proofs in papers I, III and IV, we use specific Latin squares with known properties. Knowing some structural properties of this Latin square, and the structure of an  $(m, m, m)$ -array, we manage to modify the Latin square so that it avoids the array. In paper I we use the boolean Latin square on symbols  $1, \dots, 2k$ , that is, the Latin square which is the addition table of  $\mathbb{Z}_2^k$ . In papers III and IV we use a Latin square specifically constructed to mimic one of the properties of the boolean Latin square, but which has even order.

**Example 5.1.** The boolean Latin square of order 8 is the Latin square

$$B(8) = \begin{array}{|c|c|c|c|c|c|c|c|} \hline 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ \hline 2 & 1 & 4 & 3 & 6 & 5 & 8 & 7 \\ \hline 3 & 4 & 1 & 2 & 7 & 8 & 5 & 6 \\ \hline 4 & 3 & 2 & 1 & 8 & 7 & 6 & 5 \\ \hline 5 & 6 & 7 & 8 & 1 & 2 & 3 & 4 \\ \hline 6 & 5 & 8 & 7 & 2 & 1 & 4 & 3 \\ \hline 7 & 8 & 5 & 6 & 3 & 4 & 1 & 2 \\ \hline 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \\ \hline \end{array}$$

The Latin square used in papers III and IV is slightly more complicated. The Latin square  $B(2^k)$  is useful for the proof of paper I because of the behavior of a certain type of subset of the cells, which is called a *4-cycle* (in paper I) or a *subsquare of order 2* (in papers III and IV).

**Definition 5.2.** A 4-cycle in a Latin square  $L$  is a set of cells  $\{(r_1, c_1), (r_1, c_2), (r_2, c_1), (r_2, c_2)\}$  such that  $L(r_1, c_1) = L(r_2, c_2)$  and  $L(r_1, c_2) = L(r_2, c_1)$ .

Another name sometimes found in the literature is an *intercalate*.

**Example 5.3.** In Example 5.1 above, choose any two cells in the same row, for example cell  $(3, 5)$  and cell  $(3, 7)$ . There is a 4-cycle that contains those two cells, and two cells on some other row where the same symbols appear in reversed columns. In the given example, these two other cells are cell  $(1, 5)$  and cell  $(1, 7)$ .

The boolean Latin square contains a very high number of 4-cycles. In general, the number of 4-cycles is not as large as in the boolean Latin square, as explained in [29] where it was shown that the highest possible number is  $\frac{n^2(n-1)}{4}$ , and this can only be attained if  $n = 2^k$  for some positive integer  $k$ . However, the Latin square  $L$  of order  $2k$  constructed from the parts  $L_{11}$ ,  $L_{12}$ ,  $L_{21}$  and  $L_{22}$  below has a reasonably high number of 4-cycles. Similar constructions giving the same number of 4-cycles are given in [29] and [30].

To define the Latin square  $L(2k)$ , let  $L_{11}$  be the addition table of  $\mathbb{Z}_k$ , using the symbols  $1, \dots, k$ . Let  $L_{12}$  be the same Latin square, but on symbol set  $k + 1, \dots, 2k$ . Let  $L_{21} = L_{12}^T$  and  $L_{22} = L_{11}^T$ . Set

$$L(2k) = \begin{array}{|c|c|} \hline L_{11} & L_{12} \\ \hline L_{21} & L_{22} \\ \hline \end{array}.$$

**Example 5.4.** The Latin square  $L(8)$  is

1	2	3	4	5	6	7	8
4	1	2	3	8	5	6	7
3	4	1	2	7	8	5	6
2	3	4	1	6	7	8	5
5	8	7	6	1	4	3	2
6	5	8	7	2	1	4	3
7	6	5	8	3	2	1	4
8	7	6	5	4	3	2	1

Careful counting in the Latin square above yields that every cell belongs to exactly 5 4-cycles. A Latin square where every cell belongs to the same number of 4-cycles as the other cells is called a *homogenous* Latin square. Homogenous Latin squares were studied in, for example [30] and [31], the later of which surveys some constructions of homogenous Latin squares.

## 5.2 Transversals and rainbow matchings

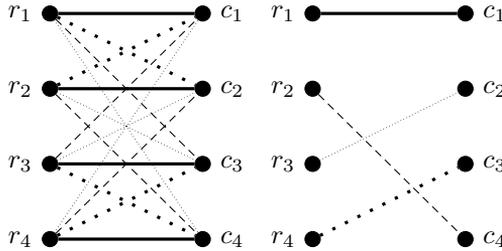
Another structural property of Latin square is that of having one or more transversals. In paper IV, we need to find a Latin square which satisfies certain other

structural conditions, and which furthermore has a transversal on the main diagonal. A *transversal* in a Latin square  $L$  of order  $n$  is a diagonal such that every symbol in  $[n]$  occurs once in the cells on the diagonal in  $L$ . A partial transversal of size  $k$  is a set of cells  $\mathcal{P}$  such that all symbols  $L(r, c)$  are different for all  $(r, c) \in \mathcal{P}$ , and  $\mathcal{P} \subset \mathcal{D}$  for some diagonal  $\mathcal{D}$ . In the more general setting of bipartite graphs, we consider *rainbow matchings*, that is matchings in edge-colored graphs such that the edges in the matching all have different color.

**Example 5.5.** One of the transversals in the Latin square  $L(8)$  is marked in boldface in the figure below.

<b>1</b>	2	3	4	5	6	7	8
4	1	<b>2</b>	3	8	5	6	7
3	4	1	2	<b>7</b>	8	5	6
2	3	4	1	6	7	<b>8</b>	5
5	8	7	6	1	<b>4</b>	3	2
6	5	8	7	2	1	4	<b>3</b>
7	<b>6</b>	5	8	3	2	1	4
8	7	6	<b>5</b>	4	3	2	1

**Example 5.6.** The graph to the left has a rainbow matching that consists of the edges drawn to the right.



Reinterpreted as a Latin square, this would be the transversal marked in bold below:

<b>1</b>	2	3	4
2	1	4	<b>3</b>
3	<b>4</b>	1	2
4	3	<b>2</b>	1

There are two conjectures on the existence of transversals in Latin squares that are still open, despite having been so for many years. The first conjecture was made by Ryser (see also [11], p. 255).

**Conjecture 5.7.** [42] *Every odd order Latin square has a transversal.*

For even order, there are Latin squares which contain no transversal, but Brualdi (see [20]), and independently Stein [44] made the following conjecture:

**Conjecture 5.8.** *Every Latin square of order  $n$  has a partial transversal of size  $n - 1$ .*

So far, some results have been presented, for example Balasubramanian [8] proved that the number of transversals in a Latin square of even order is always even. Furthermore, Shor [43] proved that every Latin square contains a partial transversal of size at least  $n - 5.53(\log n)^2$ , and Cameron and Wanless [13] proved that every Latin square contains a diagonal in which no symbol occurs more than twice.

Phrased in terms of rainbow matchings in graphs, Stein [44] proved that if  $K_{n,n}$ ,  $n \geq 3$ , is edge-colored so that no color appears more than twice, then there is a rainbow perfect matching. Furthermore, Cameron [12] proved that, even though finding matchings in bipartite graphs is algorithmically easy, finding a rainbow perfect matching is an NP-complete problem. For complete graphs, Woolbright and Fu [48] proved that if  $K_{2n}$  is colored with  $2n - 1$  colors, the colored graph has a rainbow perfect matching.

# Chapter 6

## Research on avoidability and list-edge colorings

This chapter contains an overview of some techniques that have proven to be useful in the study of avoiding arrays or similar problems related to list-edge coloring graphs. At the end of the section we also demonstrate how the main result of Paper I can be extended to avoiding arrays of even order. The proofs in papers I, III and IV is performed by applying the first and the fourth of the techniques described in Section 6.1, but a refinement of the methods could lead to somewhat better constants.

### 6.1 Some useful techniques

The three general techniques presented first all have at some point proved useful in the study of list-coloring graphs and/or avoiding arrays by Latin squares. The fourth technique is mainly a way of expanding the use of the other method, to correct some remaining colors that are still not allowed.

#### Counting bad permutations

In order to calculate bounds for a number of list-coloring related parameters, the authors of [27] use a basic counting of the number of ways a coloring process can go wrong. This technique was also used by Gustavsson in [25] for proving a number of theorems on graph decompositions. Formulated in terms of list-coloring of a graph  $G$ , the method can be summarized as follows: Given a coloring  $\chi$  of the edges of a graph  $G$  on  $n$  vertices using at most  $s$  colors, and a list-assignment  $\ell$  to the edges using  $s$  colors, we want to find out if we, by permuting

the colors, can find a coloring such that the number of edges  $e$  incident to any vertex that is colored with color  $\chi(e)$  that does not belong to  $\ell(e)$  is small, say at most  $r$ . The colors can be permuted in  $s!$  ways. The key point of this method is to bound the number of permutations  $\sigma$  of the colors that would make  $\chi$  have  $r$  or more edges  $e$  with  $\sigma(\chi(e)) \notin \ell(e)$ . An upper bound for how many such  $\sigma$  there are can be obtained by considering how such a permutation can be constructed. Fix a vertex  $v$ . A permutation  $\sigma$  with  $r$  or more edges  $e$  incident with  $v$  such that  $\sigma(\chi(e)) \notin \ell(e)$  can be constructed in at most the following number of ways:

- There are at most  $\binom{\Delta}{r}$  different sets of edges that could have  $\sigma(\chi(e)) \notin \ell(e)$  incident to  $v$ .
- For each edge at most  $s - |\ell(e)|$  colors are forbidden; hence at most  $s - |\ell(e)|$  different values for  $\sigma(\chi(e))$  can be bad for each edge.
- The value of  $\sigma$  for the remaining  $\Delta - r$  colors is not important, so any of the  $(\Delta - r)$  possibilities will work.

Altogether, this yields

$$\binom{\Delta}{r} (s - |\ell(e)|)^r (\Delta - r)! = \frac{\Delta! (s - |\ell(e)|)^r}{r!} \quad (6.1)$$

ways to construct  $\sigma$  so that at least  $r$  edges incident to  $v$  have  $\sigma(\chi(e)) \notin \ell(e)$ . There are  $n$  vertices, so there exist at most

$$n \binom{\Delta}{r} (s - |\ell(e)|)^r (\Delta - r)! = n \frac{\Delta! (s - |\ell(e)|)^r}{r!} \quad (6.2)$$

permutations that have more than  $r$  edges colored with forbidden colors at any of the vertices. Now if (6.2)  $< s!$  there are permutations where no vertex is incident to  $r$  or more edges colored with the bad colors. It will now be enough to show that the problem of having  $r$  remaining edges colored so that they have a color not in their associated list can be resolved in some way.

Even though the basic counting argument used in the calculations above is quite rough, in most applications, improving it only yields minor improvements of the results. For larger improvements a change of technique would usually be required.

## The Combinatorial Nullstellensatz

The second approach we present here was introduced by Alon and Tarsi in [1]. Similar techniques have been applied within different areas of combinatorics. An overview of some applications can be found in [3], while [2] contains a quick survey of how to use this technique on list-coloring related problems. The method is based on a theorem on polynomials, applied to the graph polynomial.

**Definition 6.1.** Let  $G$  be a graph on vertex set  $V = \{1, \dots, n\}$  with edge set  $E$ . The graph polynomial of  $G$ , denoted by  $f_G$  is defined as

$$f_G(x) = \prod_{\substack{ij \in E \\ i < j}} (x_i - x_j). \quad (6.3)$$

**Theorem 6.2.** Let  $F$  be an arbitrary field, and let  $f = f(x_1, \dots, x_n)$  be a polynomial in  $F[x_1, \dots, x_n]$ . Suppose the degree  $\deg(f)$  of  $f$  is  $\sum_{i=1}^n t_i$  where each  $t_i$  is a nonnegative integer, and suppose the coefficient of  $\prod_{i=1}^n x_i^{t_i}$  in  $f$  is nonzero. Then, if  $S_1, \dots, S_n$  are subsets of  $F$  with  $|S_i| > t_i$  there are  $s_1 \in S_1, s_2 \in S_2, \dots, s_n \in S_n$  so that  $f(s_1, \dots, s_n) \neq 0$ .

Armed with Theorem 6.2, the next step is to show that the coefficient of  $\prod_{i=1}^n x_i^{t_i}$  in the graph polynomial is the difference between the number of even and the number of odd orientations of a graph. An even orientation of a graph  $G$  on vertices  $x_1, \dots, x_n$  is an orientation with an even number of edges to which the orientation assigns the direction from  $i$  to  $j$  with  $i > j$ . An orientation which is not even is odd. If the number of even orientations and the number of odd orientations can be proven to be different this coefficient is nonzero, and hence there are  $s_1, \dots, s_n$  so that  $f_G \neq 0$ . But if the sets  $S_i$  are a list-assignment, this means that we can pick one color from each list so that no two adjacent vertices have the same color, i. e. there is a list-coloring. Using the Combinatorial Nullstellensatz like this primarily obtains results on vertex colorings, but given a graph  $G$  we can consider *line graph* of  $G$  called  $L(G)$ , which is the graph obtained by taking the edge set of  $G$  as the vertex set. Two elements of the vertex set belongs to a common edge in  $L(G)$  if and only if they are adjacent to a common vertex in  $G$ .

## The naive coloring procedure

In order to prove Theorem 4.8, Kahn used an iterative technique consisting of a coloring procedure and some probabilistic arguments about how the coloring proceeds. The method is discussed at some length in [39], where the authors also discuss other results obtained by similar methods. The coloring procedure used goes as follows:

Let  $G$  be a graph with vertex set  $V$  and edge set  $E$ . We can assume that each edge  $E$  has a list of allowed colors, since if applying the method for coloring without lists, just let all lists contain all colors.

1. Assign to each uncolored edge a color chosen randomly from its list of allowed colors.
2. Retain the color of those newly colored edges that are not adjacent to a different edge with the same color. Uncolor all of the edges just colored that ended up with a color used for an adjacent edge.

3. Modify the lists of all edges so that no list contains a color with which an adjacent edge is colored.
4. Repeat steps 1, 2 and 3 until a sufficiently large number of edges are properly colored.
5. Show that the lists for the uncolored edges still contain enough colors for a coloring to exist.

Of course, in order to make step 5 possible, caution is necessary during the earlier steps. How to do this will depend on the setting, but Kahn's proof of Theorem 4.8 relies on the following points:

- (i) For all vertices, the random variable counting the number of adjacent edges not yet colored after step 2 are at each iteration tightly concentrated around its mean.
- (ii) The number of colors allowed for any as of yet uncolored edge is tightly concentrated around its mean.
- (iii) At a given iteration, the uncolored subgraph is, with high probability, smaller than at the previous iteration. Furthermore, the number of edges in the uncolored graph that have a given color  $c$  still in their list is bounded from above with a decreasing bound.

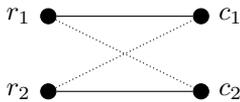
The means in (i) and (ii) do of course tend to 0, but the number of iterations performed is low enough to make sure that since the values are tightly concentrated, the probability of actually having no colors is very low. After proving this, Kahn shows that it is possible to perform a limited number of steps of the naive coloring algorithm, and then color the rest greedily. Other applications of similar techniques depend on different results for knowing that no problem occurs during iterations, and sometimes also other techniques for completing the coloring in Step (5).

A useful variation of the method is to not color all edges in step 1, but rather to color each vertex  $v$  with some probability  $p_v$ . By doing this the probabilistic analysis of steps (2)-(5) can sometimes be simplified (see [39]).

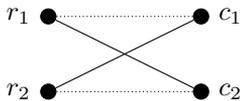
## Local recoloring

Sometimes when using probabilistic methods it is not possible to find a suitable coloring, but only to find a coloring which is allowed on some edges, or a partial allowed coloring. One way to deal with this is to construct a proper coloring, which on a small set  $C$  of edges uses colors that do not belong to their lists. If this coloring can be adjusted locally so that the edges in  $C$  get recolored with colors that belong to their lists without producing any new edges with forbidden colors, a suitable list-coloring can be found. A similar method is used for Latin squares in papers I, III and IV. This method amounts to finding cycles with four vertices such that the edges in the cycle have alternating colors. The colors on

these edges can be exchanged without disturbing the coloring on any other edge, so if we have a cycle colored as in the picture:



we can exchange the colors only on this cycle to obtain



without changing the coloring on the remaining edges.

In a Latin square, a transformation as above corresponds to exchanging the symbols on a 4-cycle, leaving the remaining Latin square as it was.

**Example 6.3.** Let  $L$  be the Latin square

$$L = \begin{array}{|c|c|c|c|c|c|} \hline 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 3 & 1 & 2 & 6 & 4 & 5 \\ \hline 2 & \mathbf{3} & 1 & 5 & 6 & \mathbf{4} \\ \hline 4 & 6 & 5 & 1 & 3 & 2 \\ \hline 5 & \mathbf{4} & 6 & 2 & 1 & \mathbf{3} \\ \hline 6 & 5 & 4 & 3 & 2 & 1 \\ \hline \end{array}$$

The operation of exchanging the symbols on the 4-cycle marked in boldface gives a Latin square  $L'$ :

$$L' = \begin{array}{|c|c|c|c|c|c|} \hline 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 3 & 1 & 2 & 6 & 4 & 5 \\ \hline 2 & \mathbf{4} & 1 & 5 & 6 & \mathbf{3} \\ \hline 4 & 6 & 5 & 1 & 3 & 2 \\ \hline 5 & \mathbf{3} & 6 & 2 & 1 & \mathbf{4} \\ \hline 6 & 5 & 4 & 3 & 2 & 1 \\ \hline \end{array}$$

In papers I, III and IV, this operation is called a *swap*.

This simple type of operation will, however, not give access to all possible colorings. In fact, if two colorings (or Latin squares) are considered equivalent if they can be obtained from each other by a sequence of such operations, the set of all edge colorings of a bipartite graph (or all Latin squares) is partitioned into equivalence classes. The number of equivalence classes for some small orders of Latin squares is studied in [47].

In a more general setting, Asratian and Mirumian [7] proved that by using two types of simple operations, any coloring of a regular bipartite graph can be transformed to any other coloring of the same graph. A simplified proof was

presented in [6]. The operations used are a *2-transformation*, an exchange of the colors on a cycle colored with only 2 colors as in the swap operation above, but on a cycle of arbitrary length, and a *3-transformation*. A 3-transformation is a transformation on a cycle  $C = v_0e_1v_1e_2 \dots 2_{2k-1}v_{2k-1}e_{2k}v_0$  in a  $t$ -regular bipartite graph  $B$  colored according to a coloring  $\chi_1$  where the even-numbered edges are colored in a color  $a$ , and the odd-numbered edges are colored either color  $b$  or color  $c$ . Let  $\chi_a$ ,  $\chi_b$  and  $\chi_c$  be the set of edges in  $B$  colored  $a$ ,  $b$  and  $c$  by  $\chi_1$  respectively, and partition the set

$$(\chi_b \cup \chi_c \cup \{e_2, e_4, \dots, e_{2k}\}) \setminus \{e_1, e_3, \dots, e_{2k-1}\}$$

into two matchings. Call these  $P_1$  and  $P_2$ . We can now define a new proper coloring  $\chi_2$  of the edges of  $B$  by setting

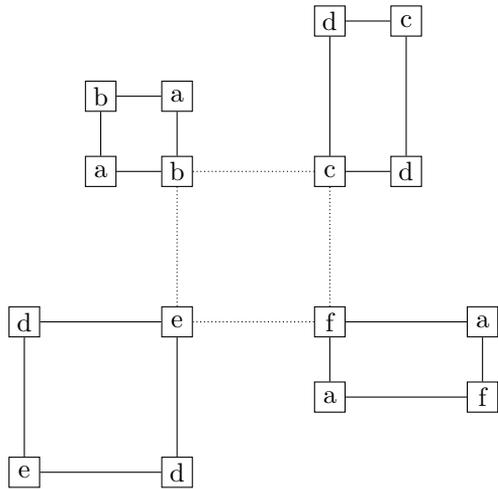
$$\chi_2(e) = \begin{cases} a & \text{if } e \in (\chi_a \setminus \{\{e_2, e_4, \dots, e_{2k}\}\}) \cup \{e_1, e_3, \dots, e_{2k-1}\}, \\ b & \text{if } e \in P_1, \\ c & \text{if } e \in P_2, \end{cases} \quad (6.4)$$

and leaving the color as in  $\chi_1$  for all other edges. This operation will change the colors on  $k$  of the edges colored  $a$ , and some edges colored  $b$  and  $c$ . All other edges will keep their colors. It is now possible to prove the following:

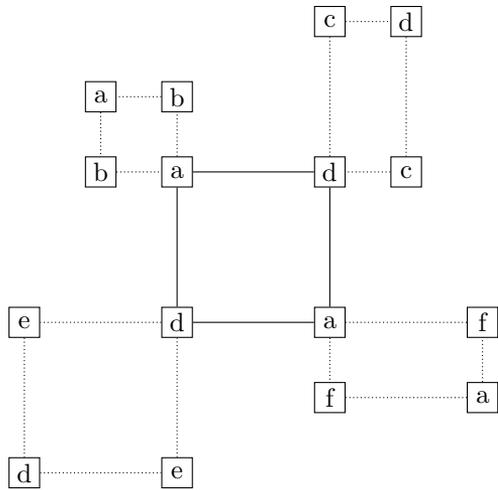
**Theorem 6.4.** [7] *Let  $t \geq 3$  and  $G$  be a  $t$ -regular bipartite graph. Every proper  $t$ -coloring of  $G$  can be obtained from any other  $t$ -coloring of  $G$  by a sequence of 2-transformations and 3-transformations.*

In terms of Latin squares, the equivalent transformations can obtain any Latin square from any other Latin square. Other sets of operations that can achieve this have been found by Donovan and Mahmoodian [21], Jacobson and Mathews [32], and Pittenger [41]. The transformations used are more complicated than simply using 2 transformation (or swaps), but they affect only a limited part of the Latin square.

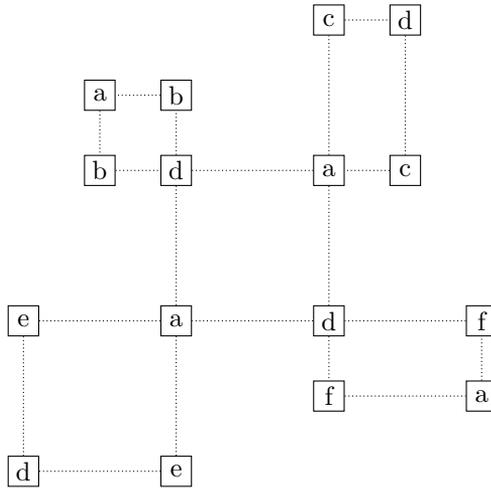
In order to expand the result on arrays of even order in paper III to a result on odd order in paper IV, a special kind of sequence of swaps in a Latin square is used. We call this operation a *major swap*, and it is set up by first swapping the symbols on some 4-cycles, so that an entirely new 4-cycle is constructed. We then swap this 4-cycle. The general method is to find a set of 4-cycles as illustrated in the picture below, and swap on the cycle drawn with solid lines. (The picture should be viewed as cells in a Latin square, where the lines represent any number of cells in the same row or column.)



After swapping on these 4-cycles, the symbols will have been exchanged, and we can swap on the new 4-cycle that has been created in the middle (drawn below with solid lines):



This final swap results in the following:



In some cases it is necessary to do even a few more swaps in order to obtain the right kind of 4-cycle in the middle. In paper IV, we do the following: In every cell of a Latin square a number of symbols are forbidden. For a small set of the cells, we want to be able to choose exactly which symbol occurs in which cell. By using major swaps, we can find a Latin square that satisfies these requirements.

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