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Finite Element Methods for Threads and Plates with Real-Time Applications

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Abstract

Thin and slender structures are widely occurring both in nature and in human creations. Clever geometries of thin structures can produce strong constructions while using a minimal amount of material. Computer modeling and analysis of thin and slender structures has its own set of problems stemming from assumptions made when deriving the equations modeling their behavior from the theory of continuum mechanics. In this thesis we consider two kinds of thin elastic structures; threads and plates.

Real-time simulation of threads are of interest in various types of virtual simulations such as surgery simulation for instance. In the first paper of this thesis we develop a thread model for use in interactive applications. By viewing the thread as a continuum rather than a truly one dimensional object existing in three dimensional space we derive a thread model that naturally handles both bending, torsion and inertial effects. We apply a corotational framework to simulate large deformation in real-time. On the fly adaptive resolution is used to minimize corotational artifacts.

Plates are flat elastic structures only allowing deflection in the normal direction. In the second paper in this thesis we propose a family of finite elements for approximating solutions to the Kirchhoff-Love plate equation using a continuous piecewise linear deflection field. We reconstruct a discontinuous piecewise quadratic deflection field which is applied in a discontinuous Galerkin method. Given a criterion on the reconstruction operator we prove a priori estimates in energy and L^2 norms. Numerical results for the method using three possible reconstructions are presented.

Keywords: finite element method, real-time simulation, absolute nodal continuous formulation, large deformation, corotation, adaptive resolution, Kirchhoff-Love plate, reconstruction, discontinuous Galerkin, a priori error estimation

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This thesis consists of the following two papers:

Paper I. K. Larsson, G. Wallgren, and M. Larson : *Interactive Simulation of a Continuum Mechanics based Torsional Thread*, To appear in: Proceedings of the 7:th Workshop on Virtual Reality Interaction and Physical Simulation (VRIPHYS), Copenhagen, 2010

Abstract: This paper introduces a continuum mechanics based thread model for use in real-time simulation. The model includes both rotary inertia, shear deformation and torsion. It is based on a three-dimensional beam model, using a corotational approach for interactive simulation speeds as well as adaptive mesh resolution to maintain accuracy. Desirable aspects of this model from a numerical and implementation point of view include a true constant and symmetric mass matrix, a symmetric and easily evaluated tangent stiffness matrix, and easy implementation of time-stepping algorithms. From a modeling perspective interesting features are deformation of the thread cross section and the use of arbitrary cross sections without performance penalty.

Paper II. K. Larsson and M. Larson : *Continuous Piecewise Linear Finite Elements for the Kirchhoff-Love Plate Equation*, Preprint, 2010

Abstract: A family of continuous piecewise linear finite elements for thin plate problems is presented. Standard linear interpolation of the deflection field is used for reconstructing a discontinuous piecewise quadratic displacement field. This allows the use of discontinuous Galerkin methods for the Kirchhoff-Love plate equation. Three example reconstructions into quadratic functions from linear interpolation triangles are presented: a reconstruction into the space of Morley basis functions, a fully quadratic reconstruction, and a more general least squares approach to a fully quadratic reconstruction. The Morley reconstruction is shown to be equivalent to the Basic Plate Triangle. Given a condition on the reconstruction operator, a priori error estimates are proved in energy norm and L_2 norm. Numerical results show that the Morley reconstruction/Basic Plate Triangle does not converge on unstructured meshes while the fully quadratic reconstruction show optimal convergence.