On avoiding some families of arrays

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Abstract. An $n \times n$ array A with entries from $\{1, \dots, n\}$ is *avoidable* if there is an $n \times n$ Latin square L such that no cell in L contains a symbol that occurs in the corresponding cell in A. We show that the problem of determining whether an array that contains at most two entries per cell is avoidable is $\mathcal{NP}$-complete, even in the case when the array has entries from only two distinct symbols. Assuming that $\mathcal{P} \neq \mathcal{NP}$, this disproves a conjecture by Öhman. Furthermore, we present several new families of avoidable arrays. In particular, every single entry array (arrays where each cell contains at most one symbol) of order $n \geq 2k$ with entries from at most k distinct symbols and where each symbol occurs in at most $n-2$ cells is avoidable, and every single entry array of order n, where each of the symbols $1, \dots, n$ occurs in at at most $\lfloor \frac{n}{6} \rfloor$ cells, is avoidable. Additionally, if $k \geq 2$, then every single entry array of order at least $n \geq 4$, where at most k rows contain non-empty cells and where each symbol occurs in at most $n-k+1$ cells, is avoidable.

1 Introduction

Consider an $n \times n$ array A where every cell contains a subset of the symbols in $\{1, \dots, n\}$. The integer n is called the *order* of A. If each symbol occurs at most once in every column of A, then A is *column-Latin*. The concept of a *row-Latin* array is defined analogously.

The cell in position (i, j) of the array A is denoted by $(i, j)_A$, and the set of symbols in cell $(i, j)_A$ is denoted by $A(i, j)$. As a shorthand, if the cell $(i, j)_A$ contains only one symbol r, we usually write $A(i, j) = r$. Moreover, if $k \in A(i, j)$, then we say that k is an *entry* of the cell $(i, j)_A$. If each cell in A contains at most one entry, then A is a *single entry array*. Otherwise, A is a *multiple entry array*. Recall that if A is a single entry array that is both column-Latin and row-Latin, then A is a *partial Latin square*, and if no cell is empty, then A is a *Latin square*.

An $n \times n$ Latin square L *avoids* an $n \times n$ array A if for each pair of integers (i, j) such that $1 \leq i, j \leq n$, we have that $L(i, j) \notin A(i, j)$. The array A is *avoidable* if

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there is a Latin square L that avoids A. Otherwise, A is *unavoidable*. The problem of avoiding arrays was first posed by Häggkvist [12]. He also found the first family of avoidable arrays: every column-Latin single entry array of order $n = 2^k$ with empty last column is avoidable. In 1995 the second non-trivial family of avoidable arrays was given. Chetwynd and Rhodes [6] proved that all chessboard squares (arrays with cells colored in the form of a chessboard with at most one symbol per black cell and no entries in the white cells) of even order at least 4 are avoidable, and that all chessboard squares of odd order at least 5 where all corner cells are white are avoidable. Furthermore, by results of Chetwynd and Rhodes [7], Cavenagh [5] and Öhman [16], every partial Latin square of order at least 4 is avoidable. In [15], the problem of avoiding single entry arrays with entries from at most two symbols was completely solved. Therein a complete characterization of unavoidable single entry arrays with at most two symbols is given. Furthermore, Markström (personal communication) has made the following conjecture.

Conjecture 1.1. *If A is an $n \times n$ single entry array where each of the symbols $1, \dots, n$ occurs at most $n - 2$ times, then A is avoidable.*

If true, Conjecture 1.1 would be best possible, because the array A defined by letting each cell in row 1 except $(1,1)_A$ have entry 1, every cell in column 1 except $(1,1)_A$ have entry 2 and every other cell be empty, is clearly unavoidable.

For multiple entry arrays, Chetwynd and Rhodes [8] established that every $4k \times 4k$ array where each cell contains at most two entries and each symbol occurs at most twice in every row and column is avoidable, if $k > 3240$. Cutler and Öhman [10] proved that for every positive integer m, there is a $k_0 = k_0(m)$ such that if $k > k_0$ and A is a (m, m, m)-array of order $2mk$, then A is avoidable, where k_0 is of the order m^8. By results of Andrén [1], and Andrén, Casselgren and Öhman [2], there is a constant $c > 0$, such that if A is an $n \times n$ array in which every cell contains at most cn symbols and every symbol occurs at most cn times in each row and column, then A is avoidable. That there is such a constant c was conjectured by Häggkvist [12].

The problem of determining whether a general multiple entry array is avoidable is $\mathcal{NP}$-complete, because it contains the problem of completing Latin squares as a special case; and this is an $\mathcal{NP}$-complete decision problem [9]. In this paper we prove that the problem to determine whether a multiple entry array A is avoidable remains $\mathcal{NP}$-complete in the case when A has entries from only two distinct symbols. Unless $\mathcal{P} = \mathcal{NP}$, this disproves a conjecture in [17].

We also present some families of avoidable arrays related to Conjecture 1.1. Let $k \in \{1, \dots, n\}$.

- Every array of order n with entries from at most k symbols and where each symbol occurs in at most $n - k$ cells is avoidable;
- every array of order n where all non-empty cells occur in k rows and where each row has at most $n - k$ entries is avoidable;

- every single entry array of order $n \geq 2k$ with entries from at most k symbols and where each symbol occurs in at most $n-2$ cells is avoidable;
- every single entry array of order n where each of the symbols $1, \ldots, n$ occurs at most $\lfloor \frac{n}{6} \rfloor$ times is avoidable;
- if $k \geq 2$, then every single entry array of order $n \geq 4$, where at most k rows contain non-empty cells and where each symbol occurs in at most $n-k+1$ cells, is avoidable;
- Conjecture 1.1 is true for arrays with entries from at most 3 symbols.

2 Preliminaries

A partial Latin square L is said to be *completeable* if there is a Latin square L' such that for any non-empty cell $(i,j)_L$ of L, $L(i,j) = L'(i,j)$.

Hilton and Andersen [3] characterized which $n \times n$ partial Latin squares with at most n entries are completeable. In particular, the following theorem follows from their result.

Theorem 2.1. *Every $n \times n$ partial Latin square with at most $n-1$ entries is completeable.*

This theorem was also proved independently by Smetaniuk [18].

It can be useful to look at a Latin square of order n as a set of n^2 triples of the form (row, column, symbol). For each Latin square there are six *conjugate* Latin squares obtained by uniformly permuting the coordinates in each of its triples. Similarly, we can think of an array A as a collection of triples, where we include one triple for each symbol in a cell if A is a multiple entry array. Conjugacy can then be defined for arrays exactly in the same way as it was defined for Latin squares. Note then that if A is an avoidable array, then any conjugate A' of A is also avoidable. For example, suppose that L avoids A and let A' be obtained from A by exchanging the roles of the columns and the symbols in A. If L' is the Latin square obtained from L by exchanging the roles of the columns and the symbols of L, then L' avoids A'.

The observation that an array A is avoidable if some conjugate of A is avoidable will be used in Section 4 where we will present several families of avoidable arrays. We will also use the simple fact that A is avoidable if and only if an array obtained from A by relabeling the rows and/or columns and/or symbols of A is avoidable.

A *generalized diagonal* D, or just a *diagonal*, in an array A of order n is a set of n cells in A, no two of which are in the same row or column. If D is a diagonal in A then $\sigma(D)$ denotes the set of symbols that occur in cells in D. The following observation is crucial for most of our results.

Observation 2.2. *Let A be an $n \times n$ array and suppose that there is a partition of the set of cells of A into diagonals $D_1, \ldots, D_n$ and a bijective map*

$$\varphi : \{D_1, \ldots, D_n\} \to \{1, \ldots, n\}$$

such that for each $i \in \{1, \dots, n\}$, if $\varphi(D_i) = j$, then $j \notin \sigma(D_i)$. Then A is avoidable.

Proof. Define an $n \times n$ Latin square L by setting

$$L(i,j) = s, \text{ if } (i,j)_A \in D_r \text{ and } \varphi(D_r) = s,$$

for all $1 \leq i, j \leq n$. Evidently, L avoids A. □

Given an $n \times n$ array A and a partition of the set of cells of A into diagonals $D_1, \dots, D_n$, we say that a map

$$\varphi : \{D_1, \dots, D_n\} \to \{1, \dots, n\}$$

is *legal* if it satisfies the conditions in Observation 2.2.

We will also use the following well-known result due to Ryser. An $s \times t$ *Latin rectangle* (on the symbol set $\{1, \dots, n\}$) is an $s \times t$ single entry array with no empty cells and where each symbol occurs at most once in every row and column.

Theorem 2.3. *Let $n \geq \max\{s, t\}$ and let P be an $n \times n$ partial Latin square whose upper left $s \times t$ subarray is a Latin rectangle and all other cells of P are empty. Then P can be completed to a Latin square if and only if each symbol occurs at least $s + t - n$ times in the $s \times t$ subarray in the upper left corner of P.*

Corollary 2.4. *If P is an $n \times n$ partial Latin square, all of whose entries lie within an $s \times t$ subarray with $s + t \leq n$, then P can be completed to an $n \times n$ Latin square.*

For proofs of the two above results, see e.g. [4]. By conjugacy, we also have the following corollary of Theorem 2.3.

Corollary 2.5. *If P is an $n \times n$ partial Latin square where each symbol occurs either in n cells or in no cells, then P is completeable.*

3 Complexity results

Let G be a graph. A *precoloring* f of $E' \subseteq E(G)$ is a proper coloring of the edges in E'. If f is a precoloring of E', then a proper edge coloring f' of G is an *extension* of f if $f(e) = f'(e)$ for all edges $e \in E'$. Fiala [11] proved that the following problem is $\mathcal{NP}$-complete.

Problem 1. *Edge precoloring extension.*

> *Instance: A 3-regular bipartite graph G, a precoloring f of $E' \subseteq E(G)$.*
>
> *Question: Can f be extended to a proper edge coloring of G using at most 3 colors?*

In [11] a polynomial reduction from Not-All-Equal 3-SAT [14] to Problem 1 is described. However, the precoloring of the constructed bipartite graph in fact only uses two distinct colors. Therefore the following problem is also $\mathcal{NP}$-complete.

Problem 2. *Edge precoloring extension with only two colors in the precoloring.*

Instance: A 3*-regular bipartite graph* G*, a precoloring* f *of* $E' \subseteq E(G)$ *using only two colors.*

Question: Can f *be extended to a proper edge coloring of* G *using at most* 3 *colors?*

We will show that the following problem is $\mathcal{NP}$-complete.

Problem 3. *Avoiding multiple entry arrays with 2 symbols.*

Instance: A multiple entry array with entries from only two distinct symbols.

Question: Is A *avoidable?*

Theorem 3.1. *Problem 3 is* $\mathcal{NP}$*-complete.*

Proof. We give a polynomial reduction from Problem 2 to Problem 3.

Let G be a 3-regular bipartite graph with parts

$$X = \{x_1, \dots, x_n\} \text{ and } Y = \{y_1, \dots, y_n\},$$

where some edges are colored 1 and some edges are colored 2. (Note that G is balanced since it is bipartite and 3-regular.) Denote this precoloring by f. Define an $n \times n$ array A by setting $A(i,j) = \{1,2\}$ for all $i,j \in \{1,\dots,n\}$ such that $x_iy_j \notin E(G)$. Furthermore, for each precolored edge x_iy_j of G, if $f(x_iy_j) = r$ ($r \in \{1,2\}$), then we include the entry r in all cells in A that are in row i, except the cell $(i,j)_A$. Let

$$B = \{(i,j)_A : x_iy_j \in E(G)\}.$$

It is easy to see that if the precoloring f can be extended to a proper edge coloring f' of G with 3 colors, then A is avoidable. Simply define a Latin square L by setting $L(i,j) = f'(x_iy_j)$ for all cells $(i,j)_L$ such that $(i,j)_A \in B$ and then fill in the remaining symbols $4,\dots,n$ in such a way that L is a Latin square. By Corollary 2.5, this is possible.

Now suppose that A is avoidable and let L be at Latin square that avoids A. For $i \in \{1,2\}$, let L_i be the set of cells of L that have entry i. Note that cells in $L_1 \cup L_2$ correspond to cells in A that are in B. Define an edge coloring f' of G by setting, for $r \in \{1,2\}$, $f'(x_iy_j) = r$ if $(i,j)_L \in L_r$, and then coloring all uncolored edges of G with the color 3. It is not hard to see that f' is a proper coloring of $E(G)$. We show that it is an extension of the precoloring f.

Let

$$A_r = \{(i,j)_A : (i,j)_L \in L_r\},$$

for $r = 1,2$. Since L avoids A, the diagonal A_1 does not contain any cell with entry 1 and, similarly, A_2 does not contain any cell with entry 2. Now suppose that $e = x_iy_j$ is an edge of G with $f(e) = 1$ (the case $f(e) = 2$ is analogous). By the construction of A, the symbol 1 is an entry of every cell in row i of A except $(i,j)_A$. Hence we must have $(i,j)_L \in L_1$, which implies that $f'(x_iy_j) = 1$, as required. □

Remark 1. We note that the problem of avoiding a multiple entry array A on 2 symbols where each cell contains either both symbols or is empty can be solved in polynomial time, since it is equivalent to the problem of finding a 2-factor in a bipartite graph with edges corresponding to empty cells in A.

4 Some families of avoidable arrays

In this section we present some families of avoidable arrays. Our first result concerns arrays where any cell, row and column can contain any number of entries.

Theorem 4.1. *Let $k \in \{1, \dots, n\}$. If A is an $n \times n$ array with entries from at most k symbols such that each symbol occurs in at most $n-k$ cells of A, then A is avoidable.*

Proof. By Observation 2.2, to prove the theorem it suffices to find a partition of the set of cells in A into diagonals $D_1, \dots, D_n$ and find a legal map

$$\varphi : \{D_1, \dots, D_n\} \to \{1, \dots, n\}.$$

Take an arbitrary such partition

$$\mathcal{D} = \{D_1, \dots, D_n\}$$

of the cells in A; for instance, let $(i,j)_A \in D_r$ if $L(i,j) = r$, where L is some Latin square of order n, and consider a bipartite graph G with parts $\mathcal{D}$ and $\{1, \dots, n\}$, and where each D_i and j are adjacent if $j \notin \sigma(D_i)$. Clearly, a legal map φ corresponds to a perfect matching in G.

Now, note that the part $\mathcal{D}$ in G satisifes Hall's condition, $|N_G(S)| \geq |S|$ for each subset $S \subseteq \{D_1, \dots, D_n\}$, where $N_G(S)$ is the set of neighbors of S in G. To see this, suppose that there is some subset $T \subseteq \mathcal{D}$ that satisfies $|N_G(T)| < |T|$. Then $|T| \leq n-k$, because each symbol occurs at most $n-k$ times in A. On the other hand, since A contains entries from at most k distinct symbols, $|\sigma(D_i)| \leq k$ for every $i \in \{1, \dots, n\}$, and thus we must have $|N_G(T)| \geq n-k$, which contradicts our assumption. Thus G has a perfect matching, as required. □

Theorem 4.1 is essentially best possible. To see this, consider an $n \times n$ array A with $A(1,j) = \{1, \dots, k+1\}$, for $j = 1, \dots, n-k$ and with all other cells empty. If a Latin square L avoids A, then in row 1 of L, the symbols $1, \dots, k+1$ must occur in the last k columns, which is clearly not possible. We also note that the method employed in the proof of Theorem 4.1 can be used to prove a slightly stronger result:

Proposition 4.2. *If A is an $n \times n$ array where symbol r occurs at most $n-r$ times for all $r \in \{1, \dots, n\}$, then A is avoidable.*

Proof. Similar to the proof of the preceding theorem. □

By conjugacy, we also have the following corollary to Theorem 4.1.

Corollary 4.3. *If A is an $n \times n$ array where all non-empty cells are in k rows and where each row contains at most $n-k$ entries, then A is avoidable.*

Of course, we could exchange rows for columns in Corollary 4.3, and the same conclusion would still hold. Also, Proposition 4.2 yields corresponding corollaries. The details are omitted.

Next, we consider arrays where we put restrictions on both the number of rows with non-empty cells and on the number of occurences of a specific symbol.

Theorem 4.4. *Let $k \in \{1, \dots, n-1\}$. If A is an $n \times n$ single entry array where at most k rows contain non-empty cells and where each symbol occurs most $n-k$ times, then A is avoidable.*

Proof. By Observation 2.2, to prove the theorem it suffices to find a partition of the set of cells in A into diagonals $D_1, \dots, D_n$ and find a legal map

$$\varphi : \{D_1, \dots, D_n\} \to \{1, \dots, n\}.$$

Take an arbitrary such partition $\mathcal{D} = \{D_1, \dots, D_n\}$ of the cells in A. Now consider a bipartite graph G with parts $\mathcal{D}$ and $\{1, \dots, n\}$, and where each D_i and j are adjacent if $j \notin \sigma(D_i)$. Clearly, a legal map φ correspond to a perfect matching in G.

Similarly as in the proof of Theorem 4.1, we will show that the part $\mathcal{D}$ in G satisifes Hall's condition. Suppose that there is some subset S of $\mathcal{D}$ that satisfies $|N_G(S)| < |S|$. Then $|S| \leq n-k$, because each symbol occurs at most $n-k$ times in A. On the other hand, since A contains non-empty cells in at most k rows, $|\sigma(D_i)| \leq k$ for every $i \in \{1, \dots, n\}$, and thus we must have $|N_G(S)| \geq n-k$, which contradicts our assumption. Thus G has a perfect matching, as required. □

Theorem 4.4 is in general best possible. To see this consider an array A where symbol 1 occurs in every cell in the first row of A. Such an array is obviously unavoidable. For arrays of order at least 4, we can prove a bit more, as the next theorem states. To prove it, we need the following lemma.

Lemma 4.5. *Let k and m be positive integers. If G is a bipartite graph with $|E(G)| \geq km$ and the maximum degree $\Delta(G)$ of G satisfies $\Delta(G) \leq m$, then there is a proper edge coloring of G with colors $1, 2, \dots, m$ such that each color appears on at least k edges.*

Proof. Let f be a proper edge coloring of G with (at most) m colors. We will describe an algorithm for constructing an edge coloring in which each of the colors $1, \dots, m$ appears on at least k edges. For an edge coloring f of G, let $E_r(f) = \{e \in E(G) : f(e) = r\}$.

Suppose that there is some color i that appears on strictly less than k edges. Then there must be some color j that appears on at least $k+1$ edges. Consider the graph $H = G[E_i(f) \cup E_j(f)]$. Clearly, H has maximum degree 2. Moreover, since

$|E_i(f)| + 2 \le |E_j(f)|$, some component of H must be a path P with the first and last edge colored j. Define a new edge coloring f' of G by setting

$$f'(e) = \begin{cases} j & \text{if } e \in E(P) \text{ and } f(e) = i, \\ i & \text{if } e \in E(P) \text{ and } f(e) = j, \\ f(e) & \text{if } e \in E(G) \setminus E(P). \end{cases}$$

Since $|E_r(f')| \ge k$ for all colors r such that $|E_r(f)| \ge k$, and $|E_i(f')| > |E_i(f)|$, the desired result follows. □

Theorem 4.6. *Let $n \ge 4$ and $k \in \{2, \dots, n\}$. If A is an $n \times n$ single entry array where at most k rows contain non-empty cells and where each symbol occurs at most $n - k + 1$ times, then A is avoidable.*

Proof. We proceed similarly as in the proof of the preceding theorem. By Observation 2.2, to prove the theorem it suffices to find a partition of the set of cells of A into diagonals $D_1, \dots, D_n$ and find a legal map

$$\varphi : \{D_1, \dots, D_n\} \to \{1, \dots, n\}.$$

Take an arbitrary such partition $\mathcal{D} = \{D_1, \dots, D_n\}$ of the cells of A. Now consider a bipartite graph G with parts $\mathcal{D}$ and $\{1, \dots, n\}$, where D_i and j are adjacent if $j \notin \sigma(D_i)$. As before, a legal map φ corresponds to a perfect matching in G.

Suppose that the part $\mathcal{D}$ in G does not satisify Hall's condition. Then there is some subset $S \subseteq \mathcal{D}$ such that $|N_G(S)| < |S|$. Note that $|S| \le n-k+1$, because each symbol occurs at most $n-k+1$ times in A. On the other hand, since A contains non-empty cells in at most k rows, $|\sigma(D_i)| \le k$ for each diagonal D_i, and thus we must have $|N_G(S)| \ge n - k$. Hence, $|S| = n - k + 1$ and $|N_G(S)| = n - k$, which implies that there are k symbols $s_1, \dots, s_k$, each of which occurs in cells of every diagonal in S.

Without loss of generality we assume that $S = \{D_1, \dots, D_{n-k+1}\}$. We will prove that there is another partition $\mathcal{D}' = \{D'_1, \dots, D'_n\}$ of the cells of A, such that in the bipartite graph G' with parts $\mathcal{D}'$ and $\{1, \dots, n\}$, where D'_i and j are adjacent if $j \notin \sigma(D'_i)$, the part $\mathcal{D}'$ satisfies Hall's condition.

Case 1. $k = 2$:
When $k = 2$, there are two symbols s_1, s_2 such that $\{s_1, s_2\} \subseteq \sigma(D_i)$, for each $i = 1, \dots, n-1$. Suppose, for simplicity, that the non-empty cells of A lie in the first two rows. Now, note that there is one cell in row 1 and one cell in row 2 of A that are not in any of the diagonals $D_1, \dots, D_{n-1}$. Without loss of generality we assume that those cells are $(1,1)_A$ and $(2,n)_A$ and that $(2,1)_A \in D_1$. Note that $A(1,1) \notin \{s_1, s_2\}$ and $A(2,n) \notin \{s_1, s_2\}$, because each symbol occurs at most $n-1$ times in A.

Now, since $n \ge 4$, there is some positive integer $j_0 < n$ such that $(1, j_0)_A \notin D_1$. Suppose, without loss of generality, that $(1, j_0)_A \in D_2$. Define the partial Latin square L with entries only in the first two rows by setting

$$L(1,1) = 2, \; L(1, j_0) = L(2, n) = n$$

and for $(i,j) \in \{1,2\} \times \{1,\dots,n\} \setminus \{(1,j_0)\}$ setting

$$L(i,j) = r \text{ if } (i,j)_A \in D_r,\ r = 1,\dots,n-1.$$

By Theorem 2.3, L is completeable to a Latin square L'. We define the diagonals $D'_1,\dots,D'_n$ in A by including the cell $(i,j)_A$ in D'_r if $L(i,j) = r$.

Let us verify that in the bipartite graph G' with parts

$$\mathcal{D}' = \{D'_1,\dots,D'_n\} \text{ and } \{1,\dots,n\},$$

and where D'_i and j are adjacent if $j \notin \sigma(D'_i)$, the part $\mathcal{D}'$ satisfies Hall's condition. Suppose, for a contradiction, that there is some set $S' \subseteq \mathcal{D}'$ with $|N_{G'}(S')| < |S'|$. Then, similarly as above, $|S'| = n-1$ and $|N_{G'}(S')| = n-2$, which means that there are $n-1$ diagonals $D'_{r_1},\dots D'_{r_{n-1}}$ in $\mathcal{D}'$ and symbols c_1, c_2, such that $\{c_1,c_2\} \subseteq \sigma(D'_i)$ for each $i \in \{r_1,\dots,r_{n-1}\}$. By the construction of $\mathcal{D}'$, the diagonals D'_1 and $D'_3, D'_4 \dots, D'_{n-1}$ all contain the symbols s_1, s_2. Since $k=2$, these diagonals do not contain any other symbols. Moreover, D'_2 contains either s_1 or s_2 but not both, and similarly D'_n also contains exactly one of the symbols in $\{s_1,s_2\}$. This clearly contradicts our assumption and the desired result follows.

Case 2. $k > 2$:
As before, we assume that all non-empty cells of A lie in the first k rows. Let A_1 be the $k \times n$ subarray of A consisting of the first k rows of A. Let c be the column of A_1 that contains the largest number of cells that belong to diagonals in S. Clearly, there are at least two cells in column c that are in diagonals in S. Suppose that $(r,c)_{A_1}$ is a cell in column c such that $(r,c)_A$ is in some diagonal of S. Without loss of generality we assume that $(r,c)_A \in D_1$. Let c' a column in A such that $(r,c')_A$ is not in any of the diagonals in S. Since $n-k+1 < n-1$, there must exist such a column. We will proceed similarly as in Case 1 and construct new pairwise disjoint diagonals $D'_1,\dots,D'_n$ in A. For the construction of these diagonals we need to consider two different cases. Then we show that in both cases the bipartite graph with parts $\{D'_1,\dots,D'_n\}$ and $\{1,\dots,n\}$ and where D'_i and j are adjacent if $j \notin \sigma(D'_i)$, the part $\{D'_1,\dots,D'_n\}$ satisfies Hall's condition.

Case 2a. The cell of D_1 that lies in column c' is not in A_1:
We define an $n \times n$ partial Latin square L by setting $L(r,c') = 1$, and for

$$(i,j) \in \{1,\dots,k\} \times \{1,\dots,n\} \setminus \{(r,c'),(r,c)\}$$

$$\text{and } s \in \{1,\dots,n-k+1\},$$

setting

$$L(i,j) = s \text{ if } (i,j)_A \in D_s.$$

Let all other cells of L be empty. Consider now the upper $k \times n$ subarray L_1 of L. Clearly, all entries of L lie in L_1. Now, note that in L_1 every row contains exactly $k-1$ empty cells and every column contains at most $k-1$ empty cells. Thus by Lemma 4.5, there is a proper edge coloring with colors $n-k+2,\dots,n$ of the bipartite graph G with vertices for rows and columns in L_1 and edges for empty cells in L_1. Let f be such an edge coloring of G. By placing the symbols $n-k+2,\dots,n$ in the empty cells in the first k rows of L according to the edge coloring f, we obtain an $n\times n$ partial Latin square L_2 where there is no empty cell in the first k rows and no entries in the last $n-k$ rows. It follows from Theorem 2.3 that L_2 is completeable to a Latin square L'. We now define the diagonals $D'_1,\dots,D'_n$ in A by including the cell $(i,j)_A$ in D'_s if $L(i,j)=s$ for all $i,j,s\in\{1,\dots,n\}$.

Case 2b. The cell of D_1 that lies in column c' is in A_1:
Suppose that the cell of D_1 that lies in column c' is $(r',c')_A$. There are $n-k+1$ cells in row r' that are in diagonals in S and thus $k-1$ cells that do not belong to any of those diagonals. Let F be the set of cells $(r',q)_A$ in row r' for which there is a row p in A such that $(p,q)_A\in D_1\cap A_1$. Note that $|F|=k$ and $\{(r',c')_A,(r',c)_A\}\subseteq F$. Thus, there is some cell

$$(r',c'')_A\notin\bigcup_{i=1}^{n-k+1}D_i$$

in row r' such that if D_1 contains some cell of A_1 in column c'', then $c''=c$. We now define a partial Latin square L by setting $L(r,c')=L(r',c'')=1$, and for

$$(i,j)\in\{1,\dots,k\}\times\{1,\dots,n\}\setminus\{(r,c),(r',c')\},$$

$$\text{and } s\in\{1,\dots,n-k+1\},$$

we set

$$L(i,j)=s \text{ if } (i,j)_A\in D_s.$$

Let all other cells of L be empty. The diagonals $D'_1,\dots,D'_n$ are now defined similarly as in Case 2a.

We now show that in the bipartite graph G' with parts

$$\mathcal{D}'=\{D'_1,\dots,D'_n\} \text{ and } \{1,\dots,n\},$$

and where D'_i and j are adjacent if $j\notin\sigma(D'_i)$, the part $\mathcal{D}'$ satisfies Hall's condition. Suppose, for a contradiction, that there is some set $S'\subseteq\mathcal{D}'$ with $|N_{G'}(S')|<|S'|$. Then, similarly as above, $|S'|=n-k+1$ and $|N_{G'}(S')|=n-k$, which means that there are k symbols $t_1,\dots,t_k$, such that $\{t_1,\dots,t_k\}\subseteq\sigma(D'_i)$ for each diagonal $D'_i\in S'$. By the construction of the diagonals in $\mathcal{D}'$, the diagonals $D'_2,\dots,D'_{n-k+1}$ all contain cells with symbols $s_1,\dots,s_k$ (and no other symbols since all entries of A lie in the first k rows of A). Moreover, D'_1 contains all but one or two symbols from $\{s_1,\dots,s_k\}$. Now, since any symbol occurs at most $n-k+1$ times in A and $k>2$, it follows that

$$S'\subseteq\{D'_{n-k+2},\dots,D'_n\}. \tag{4.1}$$

On the other hand, since each symbol in $\{s_1,\dots,s_k\}$ occurs on at most one diagonal in $\{D'_{n-k+2},\dots,D'_n\}$, we have

$$\{t_1,\dots,t_k\}\subseteq\{1,\dots,n\}\setminus\{s_1,\dots,s_k\}. \tag{4.2}$$

The fact that $|S'| = n - k + 1$ and (4.1) imply that

$$n - k + 1 \leq k - 1 \Leftrightarrow n \leq 2k - 2,$$

and (4.2) implies that $2k \leq n$, which clearly is a contradiction. Thus we conclude that the part $\mathcal{D}'$ in G' satisfies Hall's condition. $\square$

Note that if we remove the condition that $n \geq 4$, then the above theorem is not true, as seen by the following unavoidable arrays.

1	
	2

1	2	3
1	3	2

Remark 2. If A is an single entry array where at most k rows contain non-empty cells and where each symbol occurs at most $n - k + 1$ times, then the conjugate A' of A obtained from A by exchanging the roles of rows and symbols is an array with entries from at most k symbols, where each row contains at most $n-k+1$ entries and where each symbol occurs at most once in every column. Similarly, by exchanging the roles of symbols and columns in A we obtain an array A'', where at most k rows contain non-empty cells, each column contain at most $n - k + 1$ entries and each symbol occurs at most once in every row. Note that A' and A'' are in general not single entry arrays. Instead, the property of being single entry translates to the property that each symbol occurs at most once in every row or column, depending on which type of conjugacy is used.

We conclude that by exchanging the roles of rows and symbols, or symbols and columns, in Theorems 4.4 and 4.6, we obtain some more families of avoidable arrays. These arrays are row- or column-Latin and since avoiding such arrays is not our main concern in this paper, the details are omitted.

In the following, we will give three theorems that are closely related to Conjecture 1.1. Each theorem proves a special case of this conjecture by showing that a specific family of single entry arrays is avoidable. Similarly as in Remark 2, we note that for each of these results we get corollaries for row- or column-Latin (multiple entry) arrays by using conjugacy. As such arrays are not our main concern in this paper, the details are omitted here as well.

Theorem 4.7. *If A is an $n \times n$ single entry array where each of the symbols $1, \dots, n$ appears in at most $\lfloor n/6 \rfloor$ cells, then A is avoidable.*

Proof. Since each symbol occurs at most $\lfloor n/6 \rfloor$ times in A, there are at most $n\lfloor n/6 \rfloor$ non-empty cells in A. We will show that there is partition of the cells in A into n diagonals $D_1, \dots, D_n$ and a legal map $\varphi : \{D_1, \dots, D_n\} \to \{1, \dots, n\}$. The theorem will then follow from Observation 2.2.

Suppose first that n is even. Consider the four pairwise disjoint $(n/2) \times (n/2)$ subarrays that lie in the upper left, upper right, lower left, and lower right corner of A. Obviously, one of these subarrays has at most $\left\lfloor \frac{n\lfloor n/6 \rfloor}{4} \right\rfloor$ non-empty cells. Without loss of generality we assume that this subarray lies in the upper left corner of A and denote it by B. Now, there are at least

$$\frac{n^2}{4} - \left\lfloor \frac{n^2}{24} \right\rfloor \geq \left\lceil \frac{5n^2}{24} \right\rceil$$

empty cells in B. Consider a bipartite graph G with parts $X = \{x_1, \dots x_{n/2}\}$ and $Y = \{y_1, \dots, y_{n/2}\}$ and where $x_i y_j \in E(G)$ if $(i,j)_A$ is empty. Since

$$\left\lceil \frac{5n^2}{24} \right\rceil \geq n \left\lfloor \frac{n}{6} \right\rfloor,$$

Lemma 4.5 implies that there is a proper edge coloring f of G with colors $1, \dots, n$ such that each color appears on at least $\lfloor n/6 \rfloor$ edges. We define an $n \times n$ partial Latin square L by setting

$$L(i,j) = f(x_i y_j)$$

for all $1 \leq i, j \leq n/2$ such that $x_i y_j \in E(G)$, and letting all other cells of L be empty. By Corollary 2.4, L can be completed to a Latin square L'. We define n diagonals $D_1, \dots, D_n$ in A by setting $D_r = \{(i,j)_A : L'(i,j) = r\}$ for $r = 1, \dots, n$. Set $\mathcal{D} = \{D_1, \dots, D_n\}$. Note that by the construction of L', each diagonal in $\mathcal{D}$ contains at least $\lfloor n/6 \rfloor$ empty cells.

It remains to prove that there is a legal map $\varphi : \mathcal{D} \to \{1, \dots, n\}$. Consider a bipartite graph B with parts $\mathcal{D}$ and $\{1, \dots, n\}$ and where D_i and j are adjacent if $j \notin \sigma(D_i)$. A perfect matching in B corresponds to a legal map φ. Suppose that there is no perfect maching in B. Then, by Hall's condition, there is a set $W \subseteq \mathcal{D}$ such that $|N_B(W)| < |W|$. Since each symbol occurs at most $\lfloor n/6 \rfloor$ times in A, $|W| \leq \lfloor n/6 \rfloor$. On the other hand, by the construction of the diagonals in $\mathcal{D}$, each such diagonal contains at least $\lfloor n/6 \rfloor$ empty cells, which implies that $|N_B(D_i)| \geq \lfloor n/6 \rfloor$ for each vertex D_i in $\mathcal{D}$. Since $|W| \leq \lfloor n/6 \rfloor$, this is a contradiction. Hence, there is a perfect matching in B and thus a legal map φ. The theorem now follows by invoking Observation 2.2.

Suppose now that n is odd. We first consider the case when $n \leq 9$. We may clearly assume that $n \geq 6$, which means that each symbol occurs in at most 1 cell in A. Note that this implies that A is a partial Latin square. Since all partial Latin squares of order at least 4 are avoidable [5, 16, 7], A is avoidable.

Suppose now that $n \geq 9$. We proceed similarly as in the even case. There is a permutation α, so that the array A', obtained from A by applying α to the rows of A, contains at most $(n-1)/2\lfloor n/6 \rfloor$ non-empty cells in the first $(n-1)/2$ rows. We can simply choose α by letting $\alpha^{-1}(1)$ be the row in A that contains the least number of non-empty cells, and $\alpha^{-1}(2)$ be the row in A that contains the second least number of non-empty cells etc. Similarly, we may thereafter permute the columns of A' to obtain an array C, where the upper left $(n-1)/2 \times (n-1)/2$ subarray F contains at most $(n-3)/4\lfloor n/6 \rfloor$ non-empty cells. This implies that there are at least

$$\frac{(n-1)^2}{4} - \left\lfloor \frac{(n^2-3n)}{24} \right\rfloor$$

empty cells in F. Note that this expression is greater than $n\lfloor n/6 \rfloor$ if $n \geq 9$. We may now finish the proof by repeating the same arguments as in the even case. The details are omitted. □

By results of Markström and Öhman [15], Conjecture 1.1 is true in the case when the array has entries from only 2 distinct symbols. We now show that it is true in the case when the array contains entries from 3 symbols. The following was proved in [15].

Proposition 4.8. *Let A be an $n \times n$ single entry array, all entries of which contain the same symbol. If A is unavoidable, then there is an integer $k \in \{1, \dots, n\}$, such that there is a $k \times (n-k+1)$ subarray in A with no empty cells.*

The converse is of course also true: if an $n \times n$ array A contains a $k \times (n-k+1)$ subarray each cell of which contains the same symbol, then A is unavoidable.

Theorem 4.9. *Let A be an $n \times n$ single entry array with entries from at most 3 symbols. If each symbol occurs in at most $n-2$ cells, then A is avoidable.*

Proof. Since Markström and Öhman classified all unavoidable single entry arrays of order at most 4 [15], it suffices to consider arrays of order $n \geq 5$. Moreover, by Observation 2.2, it suffices to find a partition of the set of cells in A into diagonals $D_1, \dots, D_n$ and find a legal map

$$\varphi : \{D_1, \dots, D_n\} \to \{1, \dots, n\}.$$

In fact, it suffices to find pairwise disjoint diagonals D_1, D_2, D_3, such that D_r contains no cells with entry r, for each $r \in \{1, 2, 3\}$. We may then define a partial Latin square L where $L(i,j) = r$ if $(i,j)_A \in D_r$ and all other cells are empty. By Corollary 2.5 we may complete L to a Latin square L', which clearly avoids A.

So suppose that A is a minimal unvoidable array with entries from $\{1, 2, 3\}$ and no entries from $\{4, \dots, n\}$, and such that each symbol occurs in at most $n-2$ cells, and where *minimal* here means that removing any entry from A results in an avoidable array. (If B is an unavoidable array with entries from 3 symbols then we can successively remove entries from B until we obtain a minimal unavoidable array. Hence, it suffices to consider minimal unavoidable arrays.)

Since all single entry arrays with entries from at most 2 symbols and where each symbol occurs in at most $n-2$ cells is avoidable [15], we may assume that A has entries from all the symbols $1, 2, 3$. Moreover, since A is minimal, we may without loss of generality assume that there are pairwise disjoint diagonals D_1, D_2, D_3 in A such that $1 \notin \sigma(D_1)$, $2 \notin \sigma(D_2)$ and D_3 contains precisely one cell with entry 3. Let

A_3 be the set of cells in A with entry 3. Now, note that it follows from Proposition 4.8 that a subset of the cells in $A_3 \cup D_1 \cup D_2$ form a $r \times (n-r+1)$ subarray in A for some positive integer r. To see this, consider an array A' obtained from A by putting symbol 3 in all cells of $A_3 \cup D_1 \cup D_2$ and letting all other cells be empty. By Proposition 4.8, A' is avoidable unless there is a $r \times (n-r+1)$ subarray in A', all cells of which have entry 3. Therefore we may conclude that some of the cells in $A_3 \cup D_1 \cup D_2$ form a $r \times (n-r+1)$ subarray in A for some $r \in \{1, \dots, n\}$.

The rest of the proof breaks into several different cases. In all cases we will prove that there exist disjoint diagonals D_1', D_2', such that $1 \notin \sigma(D_1')$, $2 \notin \sigma(D_2')$, and the cells in $D_1' \cup D_2' \cup A_3$ do not form an $r \times (n-r+1)$ subarray in A for some r. As explained above, this will imply the theorem. Note that since D_1 and D_2 are diagonals and $|A_3| \leq n-2$, if $r \notin \{1, n\}$, then necessarily $n = 5$ and a subset of the cells in $A_3 \cup D_1 \cup D_2$ form a 3×3 array in A.

Suppose first that a subset of the cells in $D_1 \cup D_2 \cup A_3$ form a 3×3 subarray B in A, which we assume lies in the upper left corner of A. Since $n = 5$, $|A_3| \leq 3$ and therefore D_1 contains 3 cells from B, D_2 contains 3 cells from B and all cells of A_3 lie in B. We define a new diagonal D_1' by including all cells of A_3 in D_1' and by letting $(i,j)_A \in D_1'$ if $i \geq 3$ and $(i,j)_A \in D_1$. Note that $D_1' \cap D_2 = \emptyset$ and that D_1' contains all cells of A_3. Therefore, it is easily seen that the cells of $D_1' \cup D_2 \cup A_3$ do not form any $r \times (n-r+1)$ subarray in A for some r.

Suppose now that a subset of the cells in $D_1 \cup D_2 \cup A_3$ form an $n \times 1$ or a $1 \times n$ subarray B in A. Without loss of generality we assume that B is a $n \times 1$ subarray in A that lies in the first column of A and that

$$\{(1,1)_A, (2,2)_A\} \subseteq D_1 \text{ and } (2,1)_A \in D_2.$$

Note also that all cells of A_3 lie in the first column of A; more precisely

$$A_3 = \{(3,1)_A, \dots, (n,1)_A\}.$$

Therefore it suffices to show that we can find pairwise disjoint diagonals D_1' and D_2' so that $(1,1)_A \notin D_1' \cup D_2'$ or $(2,2)_A \notin D_1' \cup D_2'$ and $i \notin \sigma(D_i')$, $i = 1, 2$.

Case 1. $(1,2)_A \in D_2$:
We first show that if A is unavoidable, then every cell in

$$\{(1,3)_A, (1,4)_A, \dots, (1,n)_A\}$$

has entry 1. Suppose to the contrary that there is some integer $j \geq 3$ such that $A(1,j) \neq 1$ and let $(i_1, j)_A$ be the cell in column j that is in D_1. We define a new diagonal D_1' by including $(1,j)_A$, $(i_1, 1)_A$ and all cells in

$$\{(i,j)_A : i \neq 1, i \neq i_1 \text{ and } (i,j)_A \in D_1\}$$

in D_1'. Note that $D_1' \cap D_2 = \emptyset$. Since $(1,1)_A \notin D_1'$ and all cells in A with entry 3 lie in the first column, the cells in $D_1' \cup D_2 \cup A_3$ do not form an $r \times (n-r+1)$ subarray for some r. Therefore there is a diagonal D_3 disjoint from D_1' and D_2, and such that $3 \notin \sigma(D_3)$. It follows from Corollary 2.5 that A is avoidable, a contradiction. Similarly, it is easily seen that each cell in

$$\{(2,3)_A, (2,4)_A, \ldots, (2,n)_A\}$$

has entry 2. Since at most $n-2$ cells in A contain a particular symbol, all cells with entry 1 are in the first row of A and all cells with entry 2 are in the second row of A. Also, $(i,j)_A$ is empty for all $i,j \in \{1,2\}$.

Let j_1 be the positive integer such that $(3,j_1)_A \in D_2$. Obviously $j_1 > 2$. We define new disjoint diagonals D_1' and D_2' by setting

$$D_1' = \{(1,2)_A, (2,1)_A\} \cup \{(i,j)_A : i > 2 \text{ and } (i,j)_A \in D_1\},$$

$$D_2' = \{(3,1)_A, (2,2)_A, (1,j_1)_A\} \cup \{(i,j)_A : i > 3 \text{ and } (i,j)_A \in D_2\}.$$

Note that $1 \notin \sigma(D_1')$ and $2 \notin \sigma(D_2')$. Moreover, $(1,1)_A \notin D_1' \cup D_2'$, which, similarly as above, contradicts that A is unavoidabe.

Case 2. $(1,2)_A \notin D_2$:
Without loss of generality, we assume that $(1,3)_A \in D_2$. It follows similarly as in Case 1 that all cells in $\{(1,4)_A, \ldots, (1,n)_A\}$ has entry 1 and all cells in $\{(2,4)_A, \ldots, (2,n)_A\}$ has entry 2. Let b_1, b_2 be integers such that $(b_1,n)_A \in D_1$ and $(b_2,n)_A \in D_2$. Now, if $A(1,2) \neq 1$, then

$$D_1' = \{(1,2)_A, (2,n)_A, (b_1,1)_A\} \cup \{(i,j)_A : i \notin \{1,2,b_1\} \text{ and } (i,j)_A \in D_1\}$$

is a diagonal disjoint from D_2 such that $1 \notin \sigma(D_1')$. Since $(1,1)_A \notin D_1' \cup D_2$, this contradicts that A is unavoidable and thus $A(1,2) = 1$. Similarly, it is not hard to see that we must have $A(2,3) = 2$ if A is unavoidable. Thus all $n-2$ cells in A that have entry 1 are in the first row of A and all $n-2$ cells that have entry 2 are in the second row of A. Additionally, $(1,3)_A, (2,2)_A$ are empty. Let k_1 be an integer such that $(k_1,3)_A \in D_1$. We now define disjoint diagonals D_1' and D_2' that do not contain cells with entries 1 and 2, respectively, by setting

$$D_1' = \{(1,3)_A, (k_1,1)_A\} \cup \{(i,j)_A : i \notin \{1,k_1\} \text{ and } (i,j)_A \in D_1\},$$

$$D_2' = \{(1,n)_A, (b_2,3)_A\} \cup \{(i,j)_A : i \notin \{1,b_2\} \text{ and } (i,j)_A \in D_2\}.$$

As before, $(1,1)_A \notin D_1' \cup D_2'$, which contradicts that A is unavoidable. □

We now show that Conjecture 1.1 holds in the special case when at most k symbols appear in an array of order at least $2k$.

Theorem 4.10. *Let k be a positive integer, and let A be an $n \times n$ single entry array with entries from at most k symbols, such that each symbol occurs in at most $n-2$ cells. If $2k \leq n$, then A is avoidable.*

Proof. By Theorem 4.9, we may assume that A contains at least 4 symbols, which implies that $n \geq 8$.

So suppose that A is a minimal unvoidable array of order $n \geq 2k \geq 8$ with entries from $\{1, 2, \ldots, k\}$ and no entries from $\{k+1, \ldots, n\}$, such that each symbol occurs in at most $n-2$ cells, where *minimal* is taken to mean that removing any entry from A results in an avoidable array. Since A is minimal, there are pairwise disjoint diagonals $D_2, \ldots, D_k$ such that $s \notin \sigma(D_s)$, $s = 2, \ldots, k$. Let A_1 be the set of cells in A with entry 1. Similarly as in the preceding proof, it follows from Proposition 4.8 that a subset of the cells in $A_1 \cup D_2 \cup \cdots \cup D_k$ form an $r \times (n-r+1)$ subarray B in A for some positive integer r, because otherwise there would be a diagonal D_1 such that $D_1 \cap (A_1 \cup D_2 \cup \ldots D_k) = \emptyset$, and A would be avoidable.

We first prove that $r = 1$ or $r = n$. By symmetry, it suffices to consider the case when $r > n/2$. The number of cells in B that are in A_1 is at least

$$r(n-r+1) - (n-r+1)(k-1),$$

which is at most $n-2$ if

$$f(r) = r^2 - r(n+k) + k(n+1) - 3 \geq 0.$$

Since the maximum of $f(r)$ is attained at $r = \lfloor n/2 \rfloor + 1$ or $r = n$, it is easily verified that we must have $r = n$ if $f(r) \geq 0$ and $r > n/2$.

So suppose, without loss of generality, that B is an $n \times 1$ subarray in A that lies in the first column. Since there are at most $k-1$ cells in B that belong to $D_2 \cup \cdots \cup D_k$, at least

$$n - k + 1 \geq n/2 + 1$$

cells in B are in A_1. We now show that all cells in A with entry 1 lie in the first column of A. Suppose, for a contradiction, that there is a cell $(i,j)_A$ such that $j > 1$ and $A(i,j) = 1$. Since A is minimally unavoidable, this implies that there are pairwise disjoint diagonals $D'_1, \ldots, D'_k$, such that $r \notin \sigma(D'_r)$, $r = 2, \ldots, k$, and D'_1 contains $(i,j)_A$, but no other cell with entry 1. (This follows from the fact that removing the entry 1 from $(i,j)_A$ results in an avoidable array.) It now follows from Proposition 4.8 that the cells in $D'_2 \cup \cdots \cup D'_k \cup A_1$ form an $s \times (n-s+1)$ subarray C in A for some s. Since $(i,j)_A$ was the only cell of D'_1 with entry 1, $(i,j)_A$ must lie in C. (If $(i,j)_A$ does not lie in C, then by the converse of Proposition 4.8, there cannot be a diagonal D'_1 disjoint from $D'_2, \ldots, D'_n$, such that $(i,j)_A$ is the only cell with entry 1 in D'_1.) Moreover, it follows similarly as before, that we must have $s = 1$ or $s = n$, and that at least $\lfloor n/2 \rfloor + 1$ cells in row i or column j has entry 1, a contradiction because $j > 1$, column 1 contains at least $\lfloor n/2 \rfloor + 1$ cells with entry 1 and there are at most $n-2$ cells in A with entry 1. Hence, we may conclude that all cells with entry 1 lie in column 1.

Now, by proceeding exactly as above for every symbol $s \in \{2, \ldots, k\}$ we may conclude that all cells of A with entry s lie in a specific column or row and that this column or row contains at least $\lfloor n/2 \rfloor + 1$ cells with entry s. Clearly, if all cells with entry s lie in a particular row or column l, then there is no other symbol t such that all cells with entry t lie in row/column l. Let l_r be the row or column that contain all cells with entry r for each $r \in \{1, \dots, k\}$.

We will now define a partial Latin square L with k non-empty cells; for each $r \in \{1, \dots, k\}$, exactly one cell in L will have entry r, the cell $(i,j)_L$ with entry r will lie in row/column l_r, and $A(i,j) \neq r$.

Consider a bipartite graph G with parts

$$\{1, \dots, k\} \text{ and } \{(i,j) : 1 \leq i, j \leq n\},$$

where $s \in \{1, \dots, k\}$ and (i,j) are adjacent if $A(i,j) \neq s$ and i is the row in A that contain all cells with entry s or j is the column in A that contain all cells with entry s. Since any symbol occurs at most $n-2$ times in A, every vertex in $\{1, \dots, k\}$ has degree at least 2 in G. Furthermore, since every cell is in exactly one row and one column, every ordered pair (i,j) has degree at most 2 in G. Thus, by Hall's condition, there is a matching M in G that saturates all vertices in $\{1, \dots, k\}$. Now define L by setting $L(i,j) = r$ if (i,j) and r are adjacent in $G[M]$, for each $r = 1, \dots, k$. Clearly, L has k non-empty cells and exactly one cell with entry r for each $r \in \{1, \dots, k\}$. Moreover, if $L(i,j) = r$, then all cells of A with entry r lie in column j or row i, and, additionally, $A(i,j) \neq r$. By Theorem 2.1, L is completeable to a Latin square L'. Evidently L' avoids A. □

References

[1] L. J. Andrén, *Avoidability by Latin squares of arrays with even side*, submitted.

[2] L. J. Andrén, C. J. Casselgren, L.-D. Öhman, *Avoiding arrays of odd order by Latin squares*, submitted.

[3] L. D. Andersen, A. J. W. Hilton, *Thank Evans!, Proc. London Math. Soc.* 47 (1983), pp. 507-522.

[4] A. S. Asratian, T. M. J. Denley, R. Häggkvist, *Bipartite graphs and their applications*, Cambridge University Pres, Cambridge, 1998.

[5] N. Cavenagh, Avoidable partial latin squares of order 4m+1, *Ars Combinatoria* 95 (2010).

[6] A. G. Chetwynd, S. J. Rhodes, Chessboard squares, *Discrete Mathematics* 141 (1995), pp. 47–59.

[7] A. G. Chetwynd, S. J. Rhodes, Avoiding partial Latin squares and inricacy *Discrete Mathematics* 177 (1997), pp. 17–32.

[8] A. G. Chetwynd, S. J. Rhodes, Avoiding multiple entry arrays, *Journal of Graph Theory* 25 (1997), pp. 257–266.

[9] C. J. Colbourn, The complexity of completing partial Latin squares, *Discrete Applied Mathematics* 8 (1984), pp. 25–30.

[10] J. Cutler, L.-D. Öhman, Latin squares with forbidden entries, *Electronic Journal of Combinatorics* 13 (2006), 9 pp. (electronic).

[11] J. Fiala, NP-completeness of the edge precoloring extension problem on bipartite graphs, *Journal of Graph Theory* 43 (2003), pp. 156–160.

[12] R. Häggkvist, A note on Latin squares with restricted support, *Discrete Mathematics* 75 (1989), pp. 253–254.

[13] P. Erdős, A. L. Rubin, H. Taylor, Choosability in graphs, *Proceedings West Coast Conf. on Combinatorics, Graph Theory and Computing, Congressus Numerantium XXVI*, 1979, pp. 125–157.

[14] M. R. Garey, D. S. Johnson, *Computers and intractability*, W.H. Freeman and Co., New York, 1979.

[15] K. Markström, L.-D. Öhman, Unavoidable arrays, *Contributions to Discrete Mathematics* (2009), pp. 90–106.

[16] L.-D. Öhman, Partial latin squares are avoidable, to appear in *Annals of Combinatorics*.

[17] L.-D. Öhman, *How to do what you want to do when you can not do what you want - On avoiding and completing partial latin squares*, Doctoral Thesis, Umeå, 2006.

[18] B. Smetaniuk, A new construction for Latin squares I. Proof of the Evans conjecture, *Ars Combinatoria* 11 (1981), pp. 155–172.

[19] V. G. Vizing, Coloring the vertices of a graph with prescribed colors (in Russian), *Metody Diskretnogo Analiza Teorii Kodov i Skhem* 29 (1976), pp. 3–10.