On Some Graph Coloring Problems

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För min egen blekhets skull älskar jag rött, blått och gult,
den stora vitheten är vemodig som snöskymningen
då Snövits moder satt vid fönstret och önskade sig svart och rött därtill.
Färgernas längtan är blodets. Om du törstar efter skönhet
skall du sluta ögonen och blicka in i ditt eget hjärta.
Dock fruktar skönheten dagen och alltför många blickar,
dock tål ej skönheten buller och alltför många rörelser -
du skall icke föra ditt hjärta till dina läppar,
vi böra icke störa tystnadens och ensamhetens förnäma ringar, -
vad är större att möta än en olöst gåta med sällsamma drag?

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Färgernas längtan, Edith Södergran

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Papers I–IX

List of Papers

This thesis is based on the following papers, which are referred to in the text by their roman numerals.

I. C. J. Casselgren, Coloring graphs from random lists of fixed size, manuscript.

II. C. J. Casselgren, Coloring graphs from random lists of size 2, submitted.

III. C. J. Casselgren, Vertex coloring complete multipartite graphs from random lists of size 2, *Discrete Mathematics* (in press).

IV. Lina J. Andrén, C. J. Casselgren, Lars-Daniel Öhman, Avoiding arrays of odd order by Latin squares, submitted.

V. C. J. Casselgren, On avoiding some families of arrays, manuscript.

VI. A. S. Asratian, C. J. Casselgren, On interval edge colorings of (α, β)-biregular bipartite graphs, *Discrete Mathematics* 307 (2007), 1951–1956.

VII. A. S. Asratian, C. J. Casselgren, Jennifer Vandenbussche, Douglas B. West, Proper path-factors and interval edge-coloring of $(3, 4)$-biregular bigraphs, *Journal of Graph Theory* 61 (2009), 88–97.

VIII. A. S. Asratian, C. J. Casselgren, On path factors of $(3, 4)$-biregular bigraphs, *Graphs and Combinatorics* 24 (2008), 405–411.

IX. C. J. Casselgren, A note on path factors of $(3, 4)$-biregular bipartite graphs, submitted.

Abstract

The two main topics of this thesis are list coloring and interval edge coloring. This thesis consists of nine papers and an introduction to these research areas. In Papers I-III we study random list assignments for a graph $G = G(n)$ on n vertices, where the list for each vertex is chosen uniformly at random from all k-subsets of a color set $\mathcal{C}$ of size $\sigma = \sigma(n)$, and where k is any fixed positive integer. In particular, we prove that if G has bounded maximum degree and $\sigma = \omega(n^{1/k^2})$, then the probability that G has a proper coloring from such a random list assignment tends to 1 as $n \to \infty$. We also obtain some results for complete graphs and complete multipartite graphs.

In Papers IV-V we consider the problem of avoiding arrays. An array A is avoidable if there is a Latin square L such that no cell in L contains a symbol that occurs in the corresponding cell of A. Paper IV contains a proof of the last open case of a conjecture by Häggkvist saying that there is a constant c such that, for any positive integer n, if A is an $n \times n$ array where each cell contains at most cn symbols, and each symbol is repeated at most cn times in every row and column, then A is avoidable. Using this result, we also prove a theorem on avoiding arrays where each cell contains a random set of symbols. In Paper V we prove that the problem of determining whether an $n \times n$ array A with at most two entries per cell is avoidable is $\mathcal{NP}$-complete, even in the case when A has entries from only two distinct symbols. We also present several new families of avoidable arrays.

In Papers VI-IX we consider the problem of coloring the edges of a graph G with colors $1, 2, 3, \dots$, so that the colors on the edges incident to any vertex of G form an interval of integers. Such a proper edge coloring is called an interval edge coloring. An (a, b)-biregular graph is a bipartite graph where all vertices in one part have degree a and all vertices in the other part have degree b. In Paper VI we show that if $b > a \geq 3$ and a divides b, then the problem of determining whether an (a, b)-biregular graph has an interval edge coloring with b colors is $\mathcal{NP}$-complete. We also give an upper bound on the number of colors used in an interval edge coloring of any (a, b)-biregular graph. In Paper VII we give a criterion for a $(3, 4)$-biregular graph G to have an interval edge coloring: G has an interval edge coloring if it has a spanning subgraph consisting of paths with endpoints at 3-valent vertices and lengths in $\{2, 4, 6, 8\}$. We call this a proper path factor of G. Additionally, we conjecture that every $(3, 4)$-biregular graph has a proper path factor, and we provide several sufficient conditions for a $(3, 4)$-biregular graph to have a proper path factor. In Paper VIII we provide some evidence for the conjecture in Paper VII by showing that every $(3, 4)$-biregular graph G has a path factor such that the endpoints of each path have degree 3 in G. This result is extended in Paper IX, where it is shown that every $(3, 4)$-biregular graph has such a path factor with the additional restriction that each path has length at most 22.

Preface

In the spring of 2004 I took a first course in discrete mathematics at Linköping University. At the same time I was looking for an appropriate project for my Master thesis in Applied Physics and Electrical Engineering. Professor Armen Asratian, the course leader, suggested a topic in graph theory, one of the areas of the course that I had enjoyed the most. Not long thereafter, I found myself sinking deeper and deeper into the world of graphs and soon I would prove my first result in graph theory – the main result of Paper VII in this thesis. I have learned that research in mathematics can be very rewarding, but that there are also demanding periods with barely no progress at all. Needless to say, this thesis would not have been possible without the help and support from many people.

First and foremost, I am very grateful to my supervisor Professor Roland Häggkvist for providing me with interesting research problems to work on, for all your good advice, and for generously sharing your wide knowledge in graph theory.

I would like to express my gratitude to my assistant supervisor and co-author Professor Armen Asratian for introducing me to the world of graphs and for all your support during the last couple of years.

Thanks to the research group in discrete mathematics at the Department of Mathematics and Mathematical statistics, and especially to my co-authors Lina J. Andrén and Lars-Daniel Öhman, and to my assistant supervisor Robert Johansson. Thanks also to Lars-Daniel Öhman for reading and improving parts of this thesis.

Many thanks to Konrad, Jonas, David and Magnus for being such great friends and especially for helping me to forget about mathematics sometimes. Thanks also to Jonas for reading and improving the introduction of this thesis.

To my mother and father, Astrid and Åke, thank you for always believing in me. To my brothers, Magnus and Marcus, and all other friends, thanks for all your patience and encouragement.

Thank you Tina for all your love, patience and support. Last but not least, I would like to thank our wonderful son Ludwig for enriching my life in so many ways.

Umeå, April 2011
Carl Johan Casselgren

1 Introduction

Informally, a graph is just a collection of points (called *vertices*) some pairs of which are connected by lines (called *edges*). The mathematical investigation of this kind of seemingly innocent-looking structures was initiated by the famous Swiss mathematician Leonhard Euler in the 18th century. Euler's paper *Solutio problematis ad geometriam situs pertinentis*[1] on the *Königsberg bridge problem* is regarded as the first paper in the history of graph theory. The Königsberg bridge problem asks if it is possible to find a walk through the city of Königsberg (now Kaliningrad, Russia) in such a way that we cross every bridge exactly once. Euler observed that the choice of route inside each land mass is irrelevant and thus the problem can be modeled in abstract terms by representing land masses with points (or capital letters as in Euler's original solution) and bridges between them with links between pairs of points. Such an abstract description of the problem naturally leads to the notion of a graph.[2]

The Königsberg bridge problem is an illustration of one of the most natural applications of graphs, namely for modeling transportation networks. This is just one out of numerous situations in engineering and computer science where graph theory offers a natural mathematical model. Another example is electrical circuits, where the German physicist Gustav Kirchoff was the first to use graphs to describe such networks [11]. Perhaps less intuitively, scheduling and assignment problems can be formulated in graph theoretical terms.

Graph coloring is one of the early areas of graph theory. Its origins may be traced back to 1852 when Augustus de Morgan in a letter to his friend William Hamilton asked if it is possible to color the regions of any map with four colors so that neighboring regions get different colors. This is the famous *four color problem.* The problem was first posed by Francis Guthrie, who observed that when coloring the counties of an administrative map of England only four colors were necessary in order to ensure that neighboring counties were given different colors. He asked if this was the case for every map and put the question to his brother Frederick, who was then a mathematics undergraduate in Cambridge.[3] Frederick in turn informed his teacher Augustus de Morgan about the problem.

In 1878 the four color problem was brought to the attention of the scientific community when Arthur Cayley presented it to the London Mathematical Society. It was proved that five colors are always sufficient, but despite heavy efforts it was not until 1977 that a generally accepted solution of the four color problem was published [4, 5].

The problem of coloring a map so that adjacent regions get different colors translates into a graph coloring problem in the following way: Given a map with regions, we form a graph G by representing each region with a vertex and putting an edge

[1] *Commentarii Academiae Scientiarum Imperialis Petropolitanae* 8 (1736), 128–140. For an English translation of Euler's paper, see [11].

[2] For a more detailed account on the history of the Königsberg bridge problem and its solution, and a more elaborate discussion on the origins of graph theory, see e.g. [11].

[3] It has been argued that Guthrie was in fact not the first to ask this question, see e.g. [11, 43].

between two vertices if the corresponding regions are adjacent on the map. There is a coloring of the map such that neighboring regions get different colors if and only if there is an assignment of labels to the vertices of G such that vertices which are joined by an edge are assigned different labels. The labels are usually called *colors* and are often represented by positive integers, and the assignment is called a *proper vertex coloring* (or just a *proper coloring*) of G.

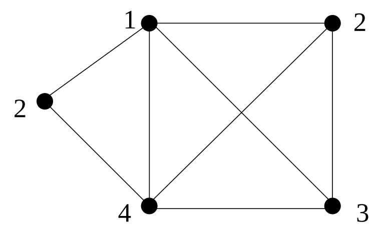

Figure 1: A graph with a proper coloring.

While much of the early research on graph coloring was directly inspired by the four color problem, the area has today grown into a mathematical research field in its own right, with its own concepts and questions. Indeed, graph coloring is one of the most popular subjects in graph theory. New directions have been motivated both by pure theoretical interest and from the perspective of possible practical applications. Besides coloring maps, which might seem a bit particular, there are many other questions that can be formulated as graph coloring problems. Consider, for instance, the following examples:

- **Example 1. Scheduling committee meetings.** Suppose that an organization has a number of committees and we want to schedule meetings for all of these committites. If a person is part of several different committees, then the meetings of these committees must be assigned to different time slots. Consider a graph G where each committee is represented by a vertex and two vertices are connected by an edge if the corresponding committees have at least one common member. We must assign colors (representing time periods) to the vertices of G so that if two vertices are joined by an edge, then they get different colors. The meetings of the committees can be scheduled for k time periods if and only if there is a proper coloring of G with k colors.

- **Example 2. Frequency assignment.** Consider a set of radio transmitters, each of which must be assigned an operating frequency. If two nearby transmitters are assigned the same frequency then they have a potential to interfere with each other. On the other hand, since the number of available frequencies is a limited (expensive) resource, we want to use as few frequencies as possible for a given set of transmitters. This problem can be modeled as a graph coloring problem in the following way: The vertices in a graph G represent the transmitters and two vertices are joined by an edge if the corresponding transmitters are so close that they might interfere with each other. If each available frequency is represented by a color, then there is a proper coloring of G with k distinct colors if and only if we can assign k frequencies to the transmitters so that no two of them have the potential to interfere with each other.

- **Example 3. Timetable design.** The problem of constructing school timetables has a natural interpretation as a graph coloring problem. Suppose that in a school there are m teachers $T_1, \dots, T_m$ and n classes $C_1, \dots, C_n$. Assume further that during a week teacher T_i should teach class C_j for b_{ij} time periods and that there are k available time periods during a week. Form a graph G by letting $C_1, \dots, C_n$ and $T_1, \dots, T_m$ be the vertices of G and adding b_{ij} edges between T_i and C_j. An assignment of k labels to the edges of the graph so that no two edges that share a common endpoint get the same label corresponds to a complete weekly timetable for all teachers and classes using k time periods. Such an assignment to the edges of a graph is called a *proper edge coloring*.

- **Example 4. Scheduling parent-teacher conferences.** Suppose that we want to schedule parent-teacher conferences at a school. Each meeting between parent(s) and a teacher lasts for the same fixed amount of time. For convenience, we want to find a schedule so that every person's conferences occur in consecutive time slots. A solution exists if and only if the graph with vertices for people and edges for the required meetings has a proper edge coloring with colors $1, 2, 3, \dots$, such that the colors on the edges incident to any vertex are consecutive.

- **Example 5. The open shop problem.** In the open shop problem we have m processors $P_1, \dots, P_m$ and n jobs $J_1, \dots, J_n$. Each job is a set of s_i tasks. Suppose that each task has to be processed for one time unit on a specific processor. Different tasks of the same job cannot be processed simultaneously and no processor can work on two tasks at the same time. The problem of constructing a schedule with k time periods so that all jobs are completed is equivalent to finding a proper edge coloring with k colors of the graph with vertices for processors and jobs, and where a task of J_i that has to be processed on P_j is represented by an edge between J_i and P_j.

Although the examples above might convince the reader that graph coloring actually is useful, in real-life applications there are often extra complications. For instance, in Example 1 it might be the case that a member of one or several committees is available only for certain time periods; or that in Example 3 a teacher works part-time and is away for some days during a week. These kinds of restrictions give rise to the notion of *list coloring*. In the language of graph theory, we can model this situation by assigning a list of acceptable colors to each vertex (or edge) of a graph. Our task is then to find a proper (edge) coloring subject to the condition that each vertex (edge) gets a color from its list of allowed colors.

List colorings were introduced by Vizing [47] and independently by Erdős et al. [22] and are one of the main topics of this thesis. An overview of this research area and a summary of the contributions of this thesis are given in Section 2.

The second main topic of this thesis is *interval edge colorings*. An *interval edge coloring* of a graph is a proper edge coloring with colors $1, 2, 3, \dots,$ such that the colors on the edges incident to any vertex form a set of consecutive integers. Interval edge coloring is a useful graph coloring model for studying scheduling problems with compactness requirements. Example 4 above is a description of an interval edge coloring problem that originally arose from a practical scheduling problem [36]. If every parent has the same number of meetings a, and every teacher has the same number of meetings b, then we say that the graph with vertices for people and edges for the required meetings is (a, b)*-biregular*. It has been conjectured that every (a, b)-biregular (bipartite) graph has an interval edge coloring (see e.g. Paper VI).

Interval edge colorings are also useful in school timetabling problems. Suppose that we want to construct a daily school timetable so that the lectures for each teacher and each class are scheduled consecutively; that is, for both teachers and classes there are no idle hours. The problem of constructing such school timetables can be formulated in terms of finding an interval edge coloring of the graph described in Example 3 above. Another situation that has a natural interpretation as a question on interval edge colorings is the open shop problem (Example 5 above) with additional compactness requirements: waiting periods are forbidden for every job and on each processor no idles are allowed. In other words, the time periods assigned to the tasks of a job must be consecutive, and each processor must be active during a set of consecutive time periods.

The notion of interval edge colorings was introduced by Asratian and Kamalian [8]. Section 3 gives an overview of this field and also contains a summary of the results presented in this thesis.

The terminology and notation used in this thesis is hopefully rather standard. In particular, the word "graph" usually means graph without multiple edges, unless otherwise explicitly stated, whereas "multigraph" indicates that multiple edges are allowed.[4] $V(G)$ and $E(G)$ denote the set of vertices and edges of a graph G, respectively. The natural logarithm of a positive number x is denoted by $\log x$ and e denotes the base of the natural logarithm. The falling factorial is denoted by $(x)_n$.

Our asymptotic notation and assumptions are also standard. In particular, all our asymptotic notation in Section 2.1 refers to the parameter n. Additionally, in that section we will assume that n is large enough whenever necessary.

2 List colorings

The concept of list coloring was introduced by Vizing [47] and independently by Erdős et al. [22]. Given a graph G, assign to each vertex v of G a set $L(v)$ of colors (positive integers). Such an assignment L is called a *list assignment* for G and the sets $L(v)$ are referred to as *lists* or *color lists*. If all lists have equal size k, then L is called a *k-list assignment*. We then want to find a proper vertex coloring φ of G, such

[4] The exception to this rule are Papers VI-IX, where a "graph" may have multiple edges while "simple graphs" do not have any multiple edges.

that $\varphi(v) \in L(v)$ for all $v \in V(G)$. If such a coloring φ exists then G is *L-colorable* and φ is called an *L-coloring*. Furthermore, G is called *k-choosable* if G is L-colorable for every k-list assignment L. The least number k for which G is k-choosable is the *list-chromatic number* $\chi_l(G)$ of G. We denote by $\chi(G)$ the chromatic number of G, i.e. the least number k so that G has a proper coloring using k colors.

Note that list coloring generalizes ordinary vertex coloring, since the latter is just a special case of the former where all lists contain the same colors. There is a wealth of results on list colorings in the literature. Let us here just point out a few of the most important ones.

Trivially, we must have $\chi_l(G) \geq \chi(G)$ for every graph G. Some cases where we have equality are complete graphs and even cycles. On the other hand, there are well-known examples where the list-chromatic number is arbitrarily much larger than the chromatic number; one such example is complete bipartite graphs, which can have arbitrarily large list-chromatic number but whose chromatic number is 2 [22]. In fact, it was proved by Alon [1] that the list-chromatic number of any graph with average degree d is at least $\Omega(\log d/\log\log d)$; that is, the list-chromatic number of a graph must grow with its average degree, unlike the chromatic number.

As regards general upper bounds on the list-chromatic number, the well-known theorem due to Brooks on ordinary vertex colorings is also true in the list coloring setting: for any graph G that is not a complete graph or an odd cycle, we have $\chi_l(G) \leq \Delta(G)$, where $\Delta(G)$ is the maximum degree of G. This was proved both by Vizing [47] and Erdős et al. [22].

Another situation where an upper bound on the chromatic number of a graph also holds for the list coloring analogue is the case of *degenerate* graphs. A graph G is called *d-degenerate* if any subgraph of G contains a vertex of degree at most d. By a simple inductive argument, it is not hard to prove that $\chi_l(G) \leq d+1$ if G is a d-degenerate graph. Since every planar graph has a vertex of degree at most 5, this implies that every planar graph is 6-choosable. In fact, it was proved by Thomassen [46] that every planar graph is 5-choosable and by an example due to Voigt [48], this is best possible. Thomassen's result, which was conjectured in [22], is a major achievement in list coloring theory. His proof is surprisingly short and can be found in many textbooks on graph theory, e.g. [12, 21].

Looking from the perspective of complexity theory, deciding if a given graph with a list assignment L is L-colorable is in general $\mathcal{NP}$-complete. Since ordinary vertex coloring is just a special case of list coloring, this follows from the well-known fact that deciding if a given graph has a proper k-coloring is $\mathcal{NP}$-complete when $k \geq 3$. A characterization of all 2-choosable graphs is given in [22].

Let us now turn our attention to list edge coloring. In this model we instead consider a list assignment L for the edges of a graph G and then want to find a proper edge coloring f so that $f(e) \in L(e)$ for every edge $e \in E(G)$. The *list-chromatic index* $\chi'_l(G)$ of G is the least number k such that G has a proper edge coloring from any list assignment where each list has size k. The *chromatic index* $\chi'(G)$ of G is the least number k such that G has a proper edge coloring using k colors.

In view of the fact that there are graphs G for which the gap between the two parameters $\chi(G)$ and $\chi_l(G)$ is arbitrarily large, the following conjecture, which is the most important open problem in list edge coloring theory, is somewhat surprising.

Conjecture 2.1. *(The list coloring conjecture) For every graph G, $\chi'_l(G) = \chi'(G)$.*

This conjecture has been suggested independently by various researchers, but Vizing was probably the first one to state it in 1975 (see [18]). Note that if the list coloring conjecture is true, then by the well-known edge coloring theorem due to Vizing [12, 21], we have $\Delta(G) \leq \chi'_l(G) \leq \Delta(G) + 1$, for every graph G.

A special case of Conjecture 2.1 is Dinitz' conjecture, which states that the list-chromatic index of a complete balanced bipartite graph equals the chromatic index. Dinitz' conjecture was originally formulated in the language of arrays and Latin squares. Nevertheless, this conjecture was the original motivation for Erdős et al. to study list colorings [22].

Dinitz' conjecture was proved in 1995 by Galvin [24]. Indeed, he showed that Conjecture 2.1 is true for all bipartite multigraphs (see [7] for a very readable proof of Galvin's celebrated theorem). Besides bipartite graphs, the list coloring conjecture has only been proved for particular families of graphs, e.g. complete graphs of odd order [34], and is still very much open. Finally, let us also mention that Kahn [37] has proved that Conjecture 2.1 is asymptotically correct by showing that for any $\varepsilon > 0$, every graph G with large enough maximum degree $\Delta(G)$ satisfies $\chi'_l(G) \leq (1+\varepsilon)\Delta(G)$.

2.1 Coloring graphs from random lists

Assign lists of colors to the vertices of a graph $G = G(n)$ with n vertices by choosing for each vertex v its list $L(v)$ uniformly at random from all k-subsets of a color set $\mathcal{C} = \{1, 2, \ldots, \sigma\}$, where k is a fixed positive integer. Such a list assignment L is called a *random $(k, \mathcal{C})$-list assignment* for G. Intuitively, it should hold that the larger σ is, the more spread are the colors chosen for the lists and thus the more likely it is that we can find a proper coloring of G with colors from the lists. The question that we address in Papers I-III is how large $\sigma = \sigma(n)$ should be in order to guarantee that **whp**[5] there is a proper coloring of the vertices of G with colors chosen from the lists. For future reference we state this formally.

Problem 2.2. Let $G = G(n)$ be a graph on n vertices, k a fixed positive integer and $\mathcal{C} = \{1, \ldots, \sigma\}$. Suppose that L is a random $(k, \mathcal{C})$-list assignment for G. How large should $\sigma = \sigma(n)$ be in order to guarantee that **whp** G is L-colorable?

This problem was first studied by Krivelevich and Nachmias [40, 41] for the case of powers of cycles and the case of complete bipartite graphs where the parts have equal size n. Let C_n^r denote the rth power of a cycle with n vertices. For powers of cycles, Krivelevich and Nachmias proved the following theorem.

[5] An event A_n occurs *with high probability*, or **whp** for brevity, if $\lim_{n\to\infty} \mathbb{P}[A_n] = 1$. All our asymptotic terminology and notation in this section is with respect to n.

Theorem 2.3. *[40] Assume r, k are fixed positive integers satisfying $r \geq k$ and let L be a random $(k, \mathcal{C})$-list assignment for C_n^r, where $\mathcal{C} = \{1, \dots, \sigma\}$. If we denote by $p_C(n)$ the probability that C_n^r is L-colorable, then*

$$p_C(n) = \begin{cases} o(1), & \sigma(n) = o(n^{1/k^2}), \\ 1 - o(1), & \sigma(n) = \omega(n^{1/k^2}). \end{cases}$$

Consider a graph $G = G(n)$ on n vertices with bounded maximum degree (i.e. there is a constant C such that $\Delta(G(n)) \leq C$ for all n) and let L be a random $(1, \mathcal{C})$-list assignment for G. Since G has $\Theta(n)$ edges it follows from an easy application of the first moment method that if $\sigma = \omega(n)$, then **whp** the assignment L induces an L-coloring of G. Hence, in the case of random $(1, \mathcal{C})$-list assignments, we can replace C_n^r in Theorem 2.3 by any graph $G = G(n)$ on n vertices with bounded maximum degree and the second part of the theorem would still hold.

Now let $H = H(n)$ be a graph on n vertices consisting of $\lfloor n/2 \rfloor$ independent edges. Suppose that $\sigma = o(n)$ and let L be a random $(1, \mathcal{C})$-list assignment for H. The probability that two adjacent vertices in H are assigned the same color by L is σ^{-1}. Since the edges of H are independent, the events that different edges get the same color for its ends are independent. Thus the probability that there is no edge in H whose ends are assigned the same color by L is at most

$$(1 - \sigma^{-1})^{\lfloor n/2 \rfloor} = o(1),$$

if $\sigma = o(n)$. Thus for any graph on n vertices with $\Omega(n)$ independent edges, if $\sigma(n) = o(n)$ then **whp** there is no proper coloring from a random $(1, \mathcal{C})$-list assignment.

From the preceding two paragraphs, we conclude that for the case of random $(1, \mathcal{C})$-list assignments, Theorem 2.3 generalizes to a much larger family of graphs. In Papers I and II we prove a generalization of Theorem 2.3 for the case of random $(k, \mathcal{C})$-list assignments where k is any fixed integer satisfying $k \geq 2$. The obtained results are summarized in the following theorem.

Theorem 2.4. *Let $G = G(n)$ be a graph on n vertices with bounded maximum degree and suppose that L is a random $(k, \mathcal{C})$-list assignment for G, where k is any fixed positive integer and $\mathcal{C} = \{1, \dots, \sigma\}$. If $\sigma(n) = \omega(n^{1/k^2})$, then* **whp** *$G$ has an L-coloring.*

By Theorem 2.3, Theorem 2.4 is best possible. In Paper II we also consider random $(2, \mathcal{C})$-list assignments for graphs of girth greater than 3. In this case it is possible to establish a better bound on σ and still guarantee **whp** an L-coloring from a random $(2, \mathcal{C})$-list assignment. In particular, we prove that if g is a fixed odd positive integer, $\sigma(n) = \omega(n^{1/(2g-2)})$, and L is a random $(2, \mathcal{C})$-list assignment for a graph $G = G(n)$ on n vertices with bounded maximum degree and girth g, then **whp** G has an L-coloring. It is also shown that this is best possible in the "coarse threshold sense". Corresponding results are given for graphs with even girth.

Let us now investigate Problem 2.2 for the case of complete graphs. Denote by K_n the complete graph on n vertices and consider a $(1, \mathcal{C})$-list assignment L for K_n.

It is not hard so see that the probability that K_n is L-colorable is $(\sigma)_n/\sigma^n$. Some calculations show that

$$(\sigma)_n/\sigma^n = \begin{cases} o(1), & \text{if } \sigma = o(n^2), \\ 1 - o(1), & \text{if } \sigma = \omega(n^2). \end{cases}$$

Hence, for a complete graph on n vertices the property of being colorable from a random $(1, \mathcal{C})$-list assignment has a coarse threshold at $\sigma(n) = n^2$. Paper II deals with random $(2, \mathcal{C})$-list assignments for K_n, and in Paper I we consider $(k, \mathcal{C})$-list assignments for K_n, for $k \geq 3$. The following theorem is a summary of the obtained results.

Theorem 2.5. *Let k be a fixed integer satisfying $k \geq 2$ and let L be a random $(k, \mathcal{C})$-list assignment for K_n.*

(i) If $k = 2$ and we denote by $p_K(n)$ the probability that K_n is L-colorable, then for any $\epsilon > 0$,

$$p_K(n) = \begin{cases} o(1), & \sigma \leq (2-\epsilon)n, \\ 1 - o(1), & \sigma \geq (2+\epsilon)n. \end{cases}$$

(ii) If $k \geq 3$ and $\sigma(n) \geq 1.223n$, then **whp** *K_n is L-colorable.*

We now turn our attention to complete multipartite graphs. For complete bipartite graphs where the parts have equal size n, Krivelevich and Nachmias showed that for all fixed $k \geq 2$, the property of having a proper coloring from a random $(k, \mathcal{C})$-list assignment exhibits a sharp threshold, and that the location of that threshold is exactly $\sigma(n) = 2n$ for $k = 2$ [41]. In Paper III we generalize the second part of this result and show that for a complete multipartite graph with s parts (fixed $s \geq 3$) where the parts have equal size n, the property of having a proper coloring from a random $(2, \mathcal{C})$-list assignment has a sharp threshold at $\sigma(n) = 2(s-1)n$. Denote by $K_{s \times n}$ the complete multipartite graphs with s parts and n vertices in each part.

Theorem 2.6. *Let s be a fixed integer satisfying $s \geq 3$, and let L be a random $(2, \mathcal{C})$-list assignment for $K_{s\times n}$. Denote by $p_s(n)$ the probability that $K_{s\times n}$ is L-colorable. For any $\epsilon > 0$,*

$$p_s(n) = \begin{cases} o(1), & \sigma \leq (2-\epsilon)(s-1)n, \\ 1 - o(1), & \sigma \geq (2+\epsilon)(s-1)n. \end{cases}$$

Let us now consider a slightly different problem.

Problem 2.7. Suppose that $G = G(n)$ is a graph with list-chromatic number n and $f(n)$ is some function of n, and let L be a random $(f(n), \{1, \dots, n\})$-list assignment for G. How large should $f(n)$ be in order to guarantee that **whp** G has an L-coloring?

We first consider the complete graph K_n on n vertices. Recall that $\chi_l(K_n) = n$, and let L be a random $(c \log n, \{1, \dots, n\})$-list assignment[6] for K_n, where c is some

[6] In order for this to make proper sense, $c \log n$ should of course be replaced by $\lceil c \log n \rceil$ or $\lfloor c \log n \rfloor$. However, for simplicity of notation, we usually omit ceiling and floor signs.

fixed constant. We will show that if $c > 1$, then **whp** K_n has an L-coloring, and if $c < 1$, then **whp** K_n is not L-colorable. To this end, we form a bipartite graph B by letting $V(K_n)$ and $\{1, \dots, n\}$ be the parts of B and letting $v \in V(K_n)$ and $i \in \{1, \dots, n\}$ be adjacent if $i \in L(v)$. Clearly, K_n is L-colorable if and only if there is a perfect matching in B. Note that the degree of a vertex in $V(K_n)$ is $c \log n$ in B. We first show that if $c < 1$, then **whp** B contains some isolated vertex, and if $c > 1$, then **whp** B has no isolated vertex. For the second statement, let X be a random variable counting the number of isolated vertices in B and note that

$$\mathbb{E}[X] = n\frac{\binom{n-1}{c\log n}^n}{\binom{n}{c\log n}^n} \sim n\exp(-c\log n) = n^{1-c}, \tag{2.1}$$

which tends to zero as $n \to \infty$ if $c > 1$. To show that **whp** B has some isolated vertex if $c < 1$, we use Chebyshev's inequality in the following form:

$$\mathbb{P}[Y = 0] \leq \frac{\mathbb{E}[Y^2] - \mathbb{E}[Y]^2}{\mathbb{E}[Y]^2}, \tag{2.2}$$

valid for all non-negative random variables Y. Since X is a sum of n identically distributed indicator random variables, we have

$$\mathbb{E}[X^2] = \mathbb{E}[X] + n(n-1)\mathbb{E}[X_1 X_2],$$

where X_1 and X_2 are the indicator random variables for the events that vertex 1 and 2 in B are isolated, respectively. Since

$$\mathbb{E}[X_1 X_2] = \frac{\binom{n-2}{c\log n}^n}{\binom{n}{c\log n}^n} \sim n^{-2c},$$

we get that

$$\mathbb{E}[X^2] \leq \mathbb{E}[X] + \frac{n(n-1)}{n^{2c}}. \tag{2.3}$$

By (2.1), $\mathbb{E}[X] \to \infty$ if $c < 1$, so it now follows from (2.1)-(2.3) that $\mathbb{P}[X = 0] = o(1)$ when $c < 1$.

We now show that if $c > 1$, then **whp** B has a perfect matching. If B does not have a perfect matching, then there is some set S of vertices in B that violates Hall's condition. We choose such a minimal set S. Then

- $|N(S)| = |S| - 1$, where $N(S)$ is the set of vertices in B that are adjacent to at least one vertex in S,
- $|S| \leq \lceil n/2 \rceil$, and
- each vertex in $N(S)$ is adjacent to at least two vertices in S.

Let $s = |S|$ and let Z_s be a random variable counting the number of minimal sets $S \subseteq \{1, \dots, n\}$ in B of size s that violates Hall's condition, and let Y_s be a random variable counting the number of minimal sets $S \subseteq V(K_n)$ in B of size s that violates Hall's condition. We set

$$Z = \sum_{s=1}^{\lceil n/2 \rceil} Z_s \text{ and } Y = \sum_{s=1}^{\lceil n/2 \rceil} Y_s,$$

and first consider the random variable Z. If $s = 1$, then S is an isolated vertex, and if $s = 2$, then S consists of two vertices of degree 1 adjacent to the same vertex. We have already shown that **whp** B contains no isolated vertices if $c > 1$. In the case when $s = 2$, we have

$$\begin{aligned}\mathbb{E}[Z_2] &\leq \frac{n^3 \binom{n-2}{c\log n-2} \binom{n-2}{c\log n}^{n-1}}{\binom{n}{c\log n}^n} \\ &= O(n \log^2 n) \exp\left(-2\frac{n-1}{n} c \log n\right),\end{aligned}$$

which tends to 0 as $n \to \infty$, because $c > 1$. Consider now the case when $s \geq 3$. We then have

$$\begin{aligned}\sum_{s=3}^{\lceil n/2 \rceil} \mathbb{E}[Z_s] &\leq \sum_{s=3}^{\lceil n/2 \rceil} \frac{\binom{n}{s}\binom{n}{s-1}\binom{s}{2}^{s-1}\binom{n-2}{c\log n-2}^{s-1}\binom{n-s}{c\log n}^{n-s+1}}{\binom{n}{c\log n}^n} \\ &\leq n \sum_{s=3}^{\lceil n/2 \rceil} e^{2s} (c \log n)^{2s} \exp\left(-\frac{s}{n}(n-s+1)c\log n\right) \\ &\leq n \sum_{s \geq 3} \left(\frac{(ce)^2 \log^2 n}{n^{c/2}}\right)^s = o(1).\end{aligned}$$

Hence, $P[Z > 0] = o(1)$. Let us now verify that there is no set $S \subseteq V(K_n)$ in B that violates Hall's condition. Proceeding as with Z we conclude

$$\begin{aligned}\sum_{s=c\log n+1}^{\lceil n/2 \rceil} \mathbb{E}[Y_s] &\leq \sum_{s=c\log n+1}^{\lceil n/2 \rceil} \frac{\binom{n}{s}\binom{n}{s-1}\binom{s-1}{c\log n}^s\binom{n}{c\log n}^{n-s}}{\binom{n}{c\log n}^n} \\ &= O(1) \sum_{s=c\log n+1}^{\lceil n/2 \rceil} \left(\frac{en}{s}\right)^s \left(\frac{en}{s-1}\right)^{s-1} \left(\frac{s-1}{n}\right)^{sc\log n}.\end{aligned}$$

Let us split this sum into the two terms

$$\Delta_1 = \sum_{s=c\log n+1}^{n/w_n+1} \left(\frac{en}{s}\right)^s \left(\frac{en}{s-1}\right)^{s-1} \left(\frac{s-1}{n}\right)^{sc\log n} \quad \text{and}$$

$$\Delta_2 = \sum_{s=n/w_n+2}^{\lceil n/2 \rceil} \left(\frac{en}{s}\right)^s \left(\frac{en}{s-1}\right)^{s-1} \left(\frac{s-1}{n}\right)^{sc\log n},$$

where w_n is some function such that $w_n \to \infty$ arbitrarily slowly. The first sum Δ_1 is easily seen to satisfy

$$\Delta_1 \leq \sum_{s \geq c \log n + 1} \frac{e^{2s-1} n^{2s-1}}{s^{2s-1} n^{sc \log w_n}} = o(1),$$

provided that n is large enough. As for Δ_2, we have

$$\Delta_2 \leq \sum_{s \geq n/w_n + 1} \frac{e^{2s-1} w_n^{2s-1}}{n^{sc \log 2}} = o(1).$$

Hence, we conclude that $\mathbb{P}[Y > 0] = o(1)$. In other words, **whp** there is no subset of $V(K_n)$ in B that does not satisfy Hall's condition. Summing up, we have proved the following theorem.

Theorem 2.8. *Suppose that L is a random $(c \log n, \{1, \dots, n\})$-list assignment for K_n. If $c > 1$, then* **whp** *K_n is L-colorable and if $c < 1$, then* **whp** *K_n is not L-colorable.*

This result should not be too surprising once it is recalled that in a random bipartite graph $\mathcal{G}(n, n, p)$, with parts of size n and where each edge occurs with probability p independently of all other edges, the property of having a perfect matching has a sharp threshold at $p = \log n/n$ (see e.g. [35]). Additionally, an immediate consequence of the sharp threshold in $\mathcal{G}(n, n, p)$ is the following: Consider a list assignment L for K_n, such that each color in $\{1, \dots, n\}$ appears in each list with probability p, independently of each other. By forming a bipartite graph as above, we get exactly a random bipartite graph distributed as $\mathcal{G}(n, n, p)$, and thus the property of being colorable from such a list assignment has a sharp threshold at $p = \log n/n$ for the complete graph K_n.

In Paper IV we consider Problem 2.7 for line graphs of balanced complete bipartite graphs; that is, we consider the problem of list edge coloring complete bipartite graphs from random lists. Let $K_{n,n}$ denote the complete bipartite graph on $n + n$ vertices. Galvin's theorem states that $\chi'_l(K_{n,n}) = \chi'(K_{n,n}) = n$. However, for random list assignments the situation is quite different as the following theorem states. Note that in Paper IV, this theorem is formulated in the language of arrays and Latin squares.

Theorem 2.9. *There is a constant $\rho > 0$, such that if L is a random $((1 - \rho)n, \{1, \dots, n\})$-list assignment for the edges of the complete bipartite graph $K_{n,n}$, then* **whp** *$K_{n,n}$ is L-colorable.*

2.2 Avoiding arrays

Let n be a positive integer and consider an $n \times n$ array A in which every cell contains a subset of the symbols in $\{1, \dots, n\}$. An $n \times n$ Latin square L *avoids* an $n \times n$ array A if for each $(i, j) \in \{1, \dots, n\} \times \{1, \dots, n\}$, we have that the symbol $L(i, j)$

in position (i, j) in L does not occur in the corresponding cell of A. An array A is *avoidable* if there is a Latin square L that avoids A. Otherwise, A is *unavoidable*. If each symbol occurs at most once in every column of A, then A is *column-Latin*. The concept of a *row-Latin* array is defined analogously. If each cell in A contains at most one entry, then A is a *single entry array*. Otherwise, A is a *multiple entry array*. Recall that if A is a single entry array that is both column-Latin and row-Latin, then A is a *partial Latin square*.

Since there is a direct correspondence between arrays and complete bipartite graphs with list assignments, and between Latin squares and proper edge colorings of complete bipartite graphs, the problem of avoiding $n \times n$ arrays is nothing else than a list edge coloring problem for the complete bipartite graph $K_{n,n}$. By Galvin's theorem we know that if all lists have size n, then we can always find a proper edge coloring with support in the lists. The idea here is to study list assignments L for the edges of $K_{n,n}$ such that $|L(e)| < n$ for some or all edges e of $K_{n,n}$, but there is still an L-coloring of $K_{n,n}$. More precisely, we are interested in finding families of list assignments L for $E(K_{n,n})$, such that each list is a subset of $\{1, \ldots, n\}$, $|L(e)| < n$ for some or all edges e of $K_{n,n}$, and $K_{n,n}$ is L-colorable.

The problem of avoiding arrays was first posed by Häggkvist [33]. He also found the first (non-trivial) family of avoidable arrays: every column-Latin single entry array of order $n = 2^k$ with empty last column is avoidable. In 1995 the second non-trivial family of avoidable arrays was given. Chetwynd and Rhodes [15] proved that all chessboard squares (arrays with cells colored in the form of a chessboard with at most one symbol per black cell and no entries in the white cells) of even order at least 4 are avoidable, and that all chessboard squares of odd order at least 5 where all corner cells are white are avoidable. Furthermore, by results of Chetwynd and Rhodes [16], Cavenagh [14] and Öhman [50], every partial Latin square of order at least 4 is avoidable.

1

1	
	2

1	2	3
2	1	
3		1

Figure 2: Unavoidable partial Latin squares of order 1, 2 and 3.

In [42], the problem of avoiding single entry arrays with entries from at most two symbols was completely solved. Therein a complete characterization of unavoidable arrays with at most two symbols is given. Moreover, the authors present several infinite families of unavoidable single entry arrays and a complete list of all unavoidable single entry arrays of order at most 4. Additionally, they conjecture that the presented families of unavoidable arrays in fact constitute a complete list of all unavoidable single entry arrays of order at least 5 (up to permuting rows/columns and/or switching the roles played by rows and columns). Markström (personal communication) has made the following more modest conjecture.

Conjecture 2.10. *If A is an $n \times n$ single entry array where each of the symbols $1, \ldots, n$ occurs at most $n-2$ times, then A is avoidable.*

If true, Conjecture 2.10 would be best possible, because the array A defined by letting each cell in row 1 except $(1,1)_A$ have entry 1, every cell in column 1 except $(1,1)_A$ have entry 2 and every other cell be empty, is clearly unavoidable. In Paper V we present some families of avoidable arrays related to Conjecture 2.10. Let $k \in \{1, \ldots, n\}$.

(i) Every array of order n with entries from at most k symbols and where each symbol occurs in at most $n-k$ cells is avoidable;

(ii) every array of order n where all non-empty cells occur in k rows (columns) and where each row (column) has at most $n-k$ entries is avoidable;

(iii) every single entry array of order $n \geq 2k$ with entries from at most k symbols and where each symbol occurs in at most $n-2$ cells is avoidable;

(iv) every single entry array of order n where each of the symbols $1, \ldots, n$ occurs at most $\lfloor \frac{n}{6} \rfloor$ times is avoidable;

(v) if $k \geq 2$, then every single entry array of order $n \geq 4$, where at most k rows contain non-empty cells and where each symbol occurs in at most $n-k+1$ cells, is avoidable;

(vi) Conjecture 2.10 is true for arrays with entries from at most 3 symbols.

It should be noted that (i) and (ii) are best possible.

Let us now consider multiple entry arrays. An $n \times n$ array A is called an (m, m, m)-*array* if each cell contains a set of size at most m and each symbol in $\{1, \ldots, n\}$ occurs at most m times in every row and column. In his original paper on avoiding arrays [33] Häggkvist made the following conjecture.[7]

Conjecture 2.11. *There is a constant $c > 0$ such that, for every positive integer n, any (cn, cn, cn)-array of order n is avoidable.*

The first progress on this conjecture was made by Chetwynd and Rhodes [17], who established that every $4k \times 4k$ array where each cell contains at most two entries and each symbol occurs at most twice in every row and column is avoidable, if $k > 3240$. Cutler and Öhman [20] proved that for every positive integer m, there is a $k_0 = k_0(m)$ such that if $k > k_0$ and A is a (m, m, m)-array of order $2mk$, then A is avoidable, where k_0 is of the order m^8. Finally, Andrén [3] established that Conjecture 2.11 holds for arrays of even order by showing the following theorem.

Theorem 2.12. *There is a constant $a > 0$ such that, for every positive integer k, any (ak, ak, ak)-array of order $2k$ is avoidable.*

[7] Slightly reformulated; for the original version, see Paper IV.

The main result of Paper IV is the following theorem, which implies that Conjecture 2.11 holds for arrays of odd order as well.

Theorem 2.13. *There is a constant $b > 0$ such that, for every positive integer k, any (bk, bk, bk)-array of order $2k + 1$ is avoidable.*

Putting the last two results together, we get the following corollary, which settles Conjecture 2.11.

Corollary 2.14. *There is a constant $c > 0$ such that, for every positive integer n, any (cn, cn, cn)-array of order n is avoidable.*

The problem of determining whether a general multiple entry array is avoidable is $\mathcal{NP}$-complete, because it contains the problem of completing a partial Latin square as a special case, and this is an $\mathcal{NP}$-complete decision problem [19]. In Paper V we prove that the problem of determining whether a multiple entry array A is avoidable remains $\mathcal{NP}$-complete in the case when A has entries from only two distinct symbols. Unless $\mathcal{P} = \mathcal{NP}$, this disproves a conjecture in [49].

3 Interval edge colorings

A proper edge coloring of a graph G with colors $1, 2, 3, \ldots$ is called an *interval coloring* if the colors on the edges incident with any vertex of G are consecutive. An interval coloring of G with colors $1, 2, \ldots, t$ is called an *interval t-coloring* if at least one edge is colored i, for $i = 1, \ldots, t$.

The notion of interval colorings was introduced by Asratian and Kamalian [8] (available in English as [9]), motivated by the problem of finding compact school timetables, that is, timetables such that the lectures of each teacher and each class are scheduled at consecutive periods. Hansen [30] suggested another scenario which arose from a practical scheduling problem: At a high school in Odense, Denmark, we wish to schedule parent-teacher conferences in time slots so that every person's conferences occur in consecutive slots. A solution exists if and only if the bipartite graph with vertices for people and edges for the required meetings has an interval coloring.

Not every graph has an interval coloring, since a graph G with an interval coloring must have a proper edge coloring with $\Delta(G)$ colors [8]. Hence, there are trivial examples of graphs without interval colorings, such as odd cycles.

It is not as easy to find a bipartite graph having no interval coloring. The first such example is due to Sevastjanov [45]. Other bipartite graphs without interval colorings include exampes given by Erdős, by Hertz and de Werra and by Malafiejski. All these examples have at least 19 vertices, maximum degree at least 13 and more than three vertices in the smallest part of the bipartition. Indeed, in [25] it was proved that if the smallest part of a bipartite graph G contains $k \leq 3$ vertices, then G has an interval coloring. It is also known [7, 30] that bipartite graphs with maximum degree $\Delta \leq 3$ always have interval colorings. The cases $\Delta \in \{4, \ldots, 12\}$ are still unsolved. For a more extensive account on bipartite graphs without interval colorings, see [25].

Generally, it is an $\mathcal{NP}$-complete problem to determine whether a given bipartite graph has an interval coloring [45]. Nevertheless, trees [30, 38], regular and complete bipartite graphs [30, 38], doubly convex bipartite graphs [38], grids [27], and outerplanar bipartite graphs [26, 10] all have interval colorings. Giaro [28] showed that one can decide in polynomial time whether bipartite graphs with maximum degree 4 have interval 4-colorings.

3.1 Interval colorings of (a, b)-biregular graphs

Recall that a bipartite graph is (a, b)-*biregular* if all vertices in one part have degree a and all vertices in the other part have degree b. The first non-trivial result on interval colorings of (a, b)-biregular graphs is due to Hansen [30]. (Note that the cases when $a = 1$, $b = 1$ or $a = b$ are trivial.) He proved that $(2, b)$-biregular graphs have interval colorings when b is even. The key to his proof is to realize that this is in fact equivalent to Petersen's 2-factor theorem: Every $2r$-regular graph has a decomposition into 2-regular edge-disjoint spanning subgraphs.[8] The vertices of degree 2 in a $(2, b)$-biregular graph G correspond to subdivisions of the edges of a $2r$-regular graph, where $b = 2r$. Hence, G has a decomposition into 2-regular edge-disjoint subgraphs $T_1, \dots, T_r$. Since every component of T_1 is an even cycle we may, in order to provide T_1 with an interval coloring, color the edges of T_1 by colors 1 and 2 alternately. Similarly, we can color the edges of T_2 by colors 3 and 4 alternately. By continuing in the same way, we obviously obtain an interval coloring of G. For the case of odd b, Hanson, Loten and Toft [32] proved that every $(2, b)$-biregular graph has an interval coloring.

Only a little is known about (a, b)-biregular graphs with $b > a \geq 3$. In Paper VI we show that if $b > a \geq 3$, then the problem to determine whether an (a, b)-biregular graph has an interval b-coloring is $\mathcal{NP}$-complete in the case when a divides b. The proof relies on the observation that such an (a, b)-biregular graph G has an interval b-coloring if and only if G has a decomposition into b/a edge-disjoint a-regular subgraphs, such that every a-regular subgraph covers the vertices of degree b in G. One can then prove that it is $\mathcal{NP}$-complete to decide whether an (a, b)-biregular graph admits such a decomposition, when $b > a \geq 3$ and a divides b. In Paper VI, a direct proof is given only for the case when $a = 3, b = 6$. The argument can, however, easily be seen to hold for any (a, b)-biregular graph, such that $b > a \geq 3$ and a divides b.

The result of Paper VI implies that there are $(3, 6)$-biregular graphs without interval 6-colorings. However, it is still an open question whether all (a, b)-biregular graphs have interval colorings (using any number of colors). In particular, it is unknown whether all $(3, 4)$-biregular graphs have interval colorings.

Pyatkin [44] proved that if a $(3, 4)$-biregular graph has a 3-regular subgraph covering the vertices of degree 4, then it has an interval coloring. In Paper VII another sufficient condition for the existence of an interval coloring of a $(3, 4)$-biregular graph

[8] This theorem was proved in Petersen's paper *Die theorie der regulären Graphs*, published in *Acta Mathematica* in 1891. For an English translation of this paper, see [11].

G is obtained: G admits an interval coloring if it has a spanning subgraph F such that every component of F is a non-trivial path with endpoints at 3-valent vertices and the length of each path is at most 8. We call this a *proper path factor* of G. It is conjectured in Paper VII that every $(3,4)$-biregular graph has a proper path factor. Note that $(3,4)$-biregular multigraphs need not have proper path factors [13].

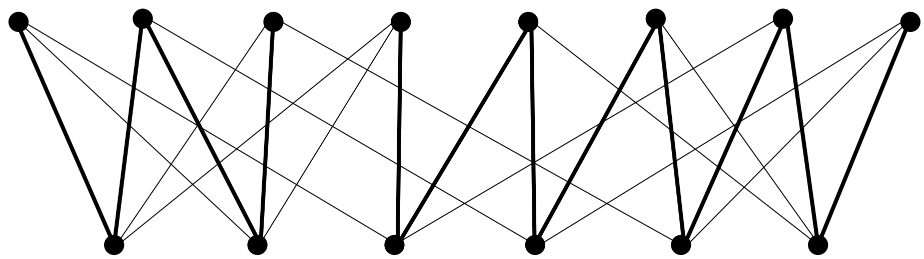

Figure 3: A $(3,4)$-biregular graph with a proper path factor.

In general, a *path factor* of a graph G is a spanning subgraph of G whose components are non-trivial paths. In Paper VIII we provide some evidence for the conjecture in Paper VII by showing that every $(3,4)$-biregular graph has a path factor such that the endpoints of each path have degree three. This result is extended in Paper IX, where it is shown that every $(3,4)$-biregular graph has such a path factor with the additional restriction that each path has length at most 22.

3.2 The number of colors in an interval coloring of an (a,b)-biregular graph

In [8, 9] Asratian and Kamalian showed that if a bipartite graph G has an interval t-coloring, then $t \le |V(G)| - 1$. Kamalian [38] proved that the complete bipartite graph $K_{b,a}$ has an interval t-coloring if and only if $a+b-\gcd(a,b) \le t \le a+b-1$. This implies that the upper bound in [8, 9] is tight for arbitrary (a,b)-biregular graphs. Suppose now that G is a connected (a,b)-biregular graph with $|V(G)| \ge 2(a+b)$. In Paper VI we prove that the upper bound on the number of colors used in an interval t-coloring of G can be improved to $t \le |V(G)| - 3$. As a corollary of this result it is proved that if $\gcd(a,b) = 1$ and G is a connected (a,b)-biregular graph such that $G \neq K_{a,b}$ and G has an interval t-coloring, then $t \le |V(G)| - 3$.

In [6] the present author and Asratian showed that the upper bound $t \le |V(G)|-3$ is tight, by constructing, for every integer $n \ge 1$, a connected (a,b)-biregular graph G with $m = n(a+b)$ vertices which admits an interval t-coloring for every t satisfying $a+b-\gcd(a,b) \le t \le m-3$.

A lower bound on the number of colors used in an interval coloring of an (a,b)-biregular graph was found by Hanson and Loten [31]. They proved that no (a,b)-biregular graph has an interval coloring with fewer than $a+b-\gcd(a,b)$ colors. It is worth noting that in the proofs of the fact that $(2,b)$-biregular graphs have interval colorings, the constructed interval colorings use exactly $2+b-\gcd(2,b)$ colors.

4 Concluding remarks and possible directions for future research

In Papers I-III, various results on coloring graphs from random lists of fixed size are obtained. In particular, we prove that if $G = G(n)$ is a graph on n vertices with bounded maximum degree, k is a positive integer, $\mathcal{C} = \{1, \ldots, \sigma\}$ and $\sigma(n) = \omega(n^{1/k^2})$, then **whp** G has a proper coloring from a random $(k, \mathcal{C})$-list assignment. If we only require that G should have bounded maximum degree, then this is best possible. For classes of graphs where the maximum degree increases as the number of vertices increases, e.g complete graphs, the situation is quite different; for example, for a complete graph on n vertices the property of being colorable from a random $(2, \mathcal{C})$-list assignment has a sharp threshold at $\sigma(n) = 2n$. This is proved in Paper II and a similar result for complete multipartite graphs is obtained in Paper III.

One possible direction for future research is of course to investigate what might be true for other classes of graphs. For example, what is the best possible bound on $\sigma = \sigma(n)$ which guarantees that a triangle-free or bipartite graph on n vertices and with bounded maximum degree **whp** has a proper coloring from a random $(k, \mathcal{C})$-list assignment? And what about planar or other families of sparse graphs; or line graphs of balanced complete bipartite graphs (which is particularly interesting since the maximum degree of such a graph grows sublinearly with the number of vertices of the graph)? Papers I and II contain some partial results in some of these directions, but it is quite clear that there are many interesting open problems in this area.

In Paper IV, Problem 2.7 is studied for the case of the line graph of the complete bipartite graph $K_{n,n}$ on $n + n$ vertices. It is proved that there is a constant $\varepsilon > 0$ such that if $k = (1 - \varepsilon)n$, then **whp** there is a proper edge coloring of $K_{n,n}$ from a random $(k, \{1, \ldots, n\})$-list assignment for $E(K_{n,n})$. However, this is probably far from being best possible. In view of Theorem 2.8, it is perhaps not unreasonable to believe that even lists of size $k = c \log n$, where c is some constant satisfying $c > 1$, would suffice to guarantee a proper edge coloring from the random lists **whp**.

Problem 2.7 can of course be considered for other families of graphs and, additionally, we could replace the list-chromatic number by any appropriate graph invariant in Problem 2.7. Indeed, Problems 2.2 and 2.7 are special cases of the following more general problems. Let $\mu(G)$ be any graph invariant of G.

Problem 4.1. Let $G = G(n)$ be a graph with $\mu(G) = n$, k a fixed positive integer and $\mathcal{C} = \{1, \ldots, \sigma\}$. Suppose that L is a random $(k, \mathcal{C})$-list assignment for G. How large should $\sigma = \sigma(n)$ be in order to guarantee that **whp** G is L-colorable?

Problem 4.2. Let $G = G(n)$ be a graph with $\mu(G) = n$ and let $f(n), \sigma = \sigma(n)$ be functions of n, and $\mathcal{C} = \{1, \ldots, \sigma\}$. Suppose that L is a random $(f(n), \mathcal{C})$-list assignment for G. How large should $f(n)$ be in order to guarantee that **whp** G is L-colorable?

Note that the result of Paper III (Theorem 2.6) together with the result of Krivelevich and Nachmias [41] for complete bipartite graphs solve the special case of

Problem 4.1 when $k = 2$ and G is the complete multipartite graph with s parts, where s is any fixed positive integer, and $\mu(G)$ is the independence number of G.

Looking from the perspective of ordinary graph (list) coloring, some of the methods developed in Papers I-III might be useful; it would be interesting to apply techniques from these papers to various list coloring problems, e.g. the list coloring conjecture (Conjecture 2.1). Additionally, methods from these papers could be helpful for finding coloring algorithms for classes of graphs where we know the chromatic number (and/or list-chromatic number) of the graph, but do not know a polynomial algorithm for finding a coloring with the minimum number of colors (or from the lists of the minimum size required). This is the case for e.g. cycle-plus-triangles graphs [23], where the Alon-Tarsi Theorem [2] guarantees a proper coloring from lists of size 3.

As regards the problem of avoiding arrays, the proof of the last open case of Conjecture 2.11 in Paper IV can be seen as the closing point of one line of research. On the other hand, the more general problem of finding proper vertex colorings subject to the condition that each vertex is given a (small) set of unavailable colors has been studied to a much lesser extent. This problem may be considered for many other classes of graphs; some natural cases to begin with being line graphs of complete graphs and line graphs of regular bipartite graphs.

In Paper V it is proved that the problem of determining whether an array with at most two entries per cell is avoidable is $\mathcal{NP}$-complete, even in the case when the array has entries from only two symbols. A natural question arises: what about single entry arrays? It is our belief that this instance is much more tractable than the general case. Several families of avoidable single entry arrays are presented in Paper V. Although this is far from giving a complete solution, perhaps it is possible to obtain a complete (or at least partial) characterization of which single entry arrays are avoidable. A natural first step would be to prove Conjecture 2.10. The variant when each edge (or vertex) of a graph is assigned at most one unavailable color would of course also be interesting to study for other classes of graphs. It should be mentioned that Chetwynd and Häggkvist [18] proved that if at most one color from $\{1, \ldots, 2n-1\}$ is forbidden on each edge of the complete graph K_{2n}, subject to the condition that no color is forbidden more than $n-1$ times, then there is a proper edge coloring of K_{2n} with colors $1, \ldots, 2n-1$ such that no edge is assigned a forbidden color.

The most natural continuation of the results on interval colorings obtained in this thesis is further work on the conjecture that every $(3,4)$-biregular graph has an interval coloring. This is the simplest unsolved case of the following more general problem [36].

Problem 4.3. Does every (a, b)-biregular graph have an interval coloring?

The complexity result of Paper VI and the seemingly difficulty of proving that $(3,4)$-biregular graphs have interval colorings are certainly reasons for believing that interval coloring problems are intrinsically hard for many (simple) classes of graphs. However, there are several variants where one might hope for a more tractable problem. Indeed, there are different ways of relaxing the condition that the colors on the edges incident to any vertex in a graph must form an interval of integers; for example, we could require that only the vertices in one part of a biregular graph get colors which form an interval on its incident edges; or perhaps that only a fraction of the vertices in such a graph have this property. Of course, these variants might be considered for other families of (bipartite) graphs.

The problem of finding "near-interval colorings", where "near-interval colorings" should be interpreted as proper edge colorings where we at every vertex v have "few gaps" in the set of colors present on edges incident to v, is also interesting; not least for applicational reasons (e.g. in school timetabling where we want to find schedules with as few idle hours as possible for the classes and teachers). An important task here is to find an appropriate interpretation of the concept of having "few gaps" that leads to tractable problems; see e.g. [13] for a global interpretation of this concept and a polynomial-time algorithm for finding the corresponding "near-interval coloring".

The results of [6, 31] provide us with sharp lower and upper bounds on the number of colors used in an interval coloring of a connected (a, b)-biregular graph. An interesting study would be to investigate if the existence of an interval t-coloring of an (a, b)-biregular graph G implies that there is an interval $(t + 1)$-coloring of G, provided that $t \leq |V(G)| - 4$. In this context, it is worth noting that Sevastjanov [45] proved that there is a bipartite graph G and integers $t_1 < t_2 < t_3$, such that G admits an interval coloring with t_1 or t_3 colors, but not with t_2 colors.

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