

Numerical Analysis for Random Processes and Fields and Related Design Problems

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To my parents

Contents

List of papers	vi
Abstract	vii
Preface	ix
1 Introduction	1
2 Basic notation	3
3 Some classes of random functions	3
4 Regular and cross regular sequences of sampling designs	5
5 Approximation techniques	7
5.1 Piecewise linear interpolation	7
5.2 Hermite interpolation splines	8
5.3 Multivariate piecewise linear interpolation	8
5.4 Stratified Monte Carlo quadrature	9
6 Point singularities	11
6.1 Motivation	11
6.2 Quasi regular sequences of designs	12
6.3 Composite Hermite splines	12
7 A financial application: Asian option pricing	13
8 Summary of papers	14
8.1 Paper A: Spline approximation of a random process with singularity .	15
8.2 Paper B: Multivariate piecewise linear interpolation of a random field	15
8.3 Paper C: Stratified Monte Carlo quadrature for random functions . .	15
8.4 Paper D: On the error of the Monte Carlo pricing method for Asian option	16
9 Concluding remarks and future research	16
Papers A–D	

List of papers

This thesis is based on the following papers:

- A. Abramowicz, K. and Seleznev, O. (2011). Spline approximation of a random process with singularity. *Journal of Statistical Planning and Inference*. **141**, 1333–1342.
- B. Abramowicz, K. and Seleznev, O. (2011). Multivariate piecewise linear interpolation of a random field, submitted.
- C. Abramowicz, K. and Seleznev, O. (2011). Stratified Monte Carlo quadrature for random fields, submitted.
- D. Abramowicz, K. and Seleznev, O. (2008). On the error of the Monte Carlo pricing method for Asian option. Published without proofs in *Journal of Numerical and Applied Mathematics*. **96**, 1–10.

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Abstract

In this thesis, we study numerical analysis for random processes and fields. We investigate the behavior of the approximation accuracy for specific linear methods based on a finite number of observations. Furthermore, we propose techniques for optimizing performance of the methods for particular classes of random functions. The thesis consists of an introductory survey of the subject and related theory and four papers (A-D).

In paper A, we study a Hermite spline approximation of quadratic mean continuous and differentiable random processes with an isolated point singularity. We consider a piecewise polynomial approximation combining two different Hermite interpolation splines for the interval adjacent to the singularity point and for the remaining part. For locally stationary random processes, sequences of sampling designs eliminating asymptotically the effect of the singularity are constructed.

In Paper B, we focus on approximation of quadratic mean continuous real-valued random fields by a multivariate piecewise linear interpolator based on a finite number of observations placed on a hyperrectangular grid. We extend the concept of local stationarity to random fields and for the fields from this class, we provide an exact asymptotics for the approximation accuracy. Some asymptotic optimization results are also provided.

In Paper C, we investigate numerical approximation of integrals (quadrature) of random functions over the unit hypercube. We study the asymptotics of a stratified Monte Carlo quadrature based on a finite number of randomly chosen observations in strata generated by a hyperrectangular grid. For the locally stationary random fields (introduced in Paper B), we derive exact asymptotic results together with some optimization methods. Moreover, for a certain class of random functions with an isolated singularity, we construct a sequence of designs eliminating the effect of the singularity.

In Paper D, we consider a Monte Carlo pricing method for arithmetic Asian options. An estimator is constructed using a piecewise constant approximation of an underlying asset price process. For a wide class of Lévy market models, we provide upper bounds for the discretization error and the variance of the estimator. We construct an algorithm for accurate simulations with controlled discretization and Monte Carlo errors, and obtain the estimates of the option price with a predetermined accuracy at a given confidence level. Additionally, for the Black-Scholes model, we optimize

the performance of the estimator by using a suitable variance reduction technique.

Keywords: stochastic processes, random fields, approximation, numerical integration, Hermite splines, piecewise linear interpolator, local stationarity, point singularity, stratified Monte Carlo quadrature, Asian option, Monte Carlo pricing method, Lévy market models

Preface

When I arrived in Umeå in January 2006 as a Socrates Erasmus student, my intention was to stay here for five months. However, the exchange program turned into PhD studies and five months became almost six years. Now, at the end of this journey, I would like to acknowledge people who made this time at the Department of Mathematics and Mathematical Statistics such a fascinating and valuable experience.

First of all, I would like to express my gratitude to my supervisor Oleg Seleznev. Thank you for your knowledge, inspiration, patience, and for believing in me as a scientist. Without your guidance I would never be able to finish this thesis.

I would like to thank all the staff at the Department of Mathematics and Mathematical Statistics and the Department of Statistics. Special thanks to my second supervisor Xavier de Luna, Yuri Belyaev, Lennart Bondesson, Sara de Luna, and Kaj Nyström for many fruitful discussions and all your support. I am grateful to Ingrid Svensson and Peter Anton for giving me a chance to develop as a teacher. Thanks to Ingrid Westerberg, Berith Melander, and Margareta Brinkstam for your priceless help with all formalities.

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I would like to thank all my Polish friends, especially Ania, Kasia, Maciek, and Magda. Even though during last years we were miles apart, I could always rely on you in better and worse days during my stay in Umeå.

Thanks to all my family, in particular, my brother Piotr, his wife Ilona, and my cousin Ula for their support. Last but not least, I want to thank my parents Elżbieta and Wiesław, to whom I dedicate this thesis, for their constant love and encouragement.

Umeå, September 2011

Konrad Abramowicz

1 Introduction

In this thesis, we focus on numerical analysis for broad classes of random processes and fields and investigate the achievable approximation accuracy as a function of the number of observations. We study recovery of the continuous and differentiable random functions and corresponding functionals by methods based on a finite number of observations only. Problems of this type arise in many theoretical and applied areas, like random fields modelling, environmental research, geosciences, communication theory, data compression, and signal processing whenever calculations with a continuous (time, space) model are substituted by a discrete one and the discretization accuracy is evaluated.

The deterministic approximation theory provides an advanced apparatus for approximation of non-random functions and corresponding functionals. This includes techniques based on trigonometric and algebraic polynomials, rational functions, wavelets, and splines. One of central ideas when using various approximation methods is to determine the relationship between the smoothness properties of a target function and the best corresponding approximation accuracy as well as selecting the optimal method of approximation. Further, it is natural to require that a target function and the corresponding approximation function have similar smoothness properties. For an extensive summary of approximation problems in deterministic setting, we refer to, e.g., Sard (1963), Powell (1981), and Novak (1988).

Similar problems emerge in the numerical analysis for random processes and fields. Beside the methods naturally inherited from the deterministic setting, a number of techniques specific to approximation of random functions, e.g., linear regression interpolation, Karhunen-Loève approximation, and kriging, have been introduced.

In many cases, it is hard to determine the optimal method for any fixed number of observation points. An often used approach is to consider the approximation error behavior when the number of observations tends to infinity. However, to investigate the asymptotic performance, we need to construct sequences of sampling designs that can be consistently generated for any required number of observations. In the one-dimensional case, regular design sequences were introduced in Sacks and Ylvisaker (1966). In this approach, the observation points become the quantiles of a given distribution with positive and continuous (say, regular) density. Generalizations of this concept to higher dimension include cross regular designs (products of regular designs) and hyperbolic cross designs (Müller-Gronbach, 1998).

In the thesis, we study some approximation and numerical integration techniques and investigate their accuracy for particular classes of random processes and fields. For continuous locally stationary random functions and for processes with locally stationary quadratic mean (q.m.) derivatives, we derive an exact asymptotic behavior of the approximation accuracy. Optimization procedures leading to asymptotically optimal and suboptimal designs of sampling points are proposed. Additionally, we establish upper bounds for the approximation accuracy for some Hölder classes of random functions. The performance of the studied methods is also compared with the best performance among all linear methods.

Another important problem studied in the thesis is approximation of random functions with an isolated singularity point. The presence of such a point affects the global smoothness properties of a function, and consequently decreases the efficiency of standard approximation methods. We investigate some modifications of the conventional techniques. Under certain conditions, these methods eliminate the effect of the singularity and attain the optimal approximation rates.

Next, we study an approximation problem in a financial setting. We investigate the Monte Carlo method of pricing Asian options. We analyze implications of a uniform discretization of a time domain and the corresponding piecewise constant approximation of a price process for the underlying asset. Additionally, an upper bound for the simulation error is evaluated. For a wide class of Lévy market models, we provide an algorithm for estimating the price of an Asian option with a prescribed accuracy on a given confidence level.

The remaining part of the thesis consists of a survey of related theory and results followed by four included papers (A-D). In Section 2, we introduce some notation. Sections 3 and 4 describe the classes of random functions studied in the thesis and the sampling point designs, respectively. In Section 5, we discuss some techniques used in our studies. In Section 6, we consider the problem of approximating a process with an isolated singularity point at the origin and discuss possible ways of eliminating the effect of this singularity on the approximation rate. The problem of pricing Asian options using Monte Carlo techniques is discussed in Section 7. Section 8 contains a summary of the papers included in this thesis and in Section 9, we present conclusions and possible directions for future research.

2 Basic notation

Let a random field $X(\mathbf{t})$, $\mathbf{t} = (t_1, \dots, t_d) \in [0, 1]^d := \mathcal{D}$, $d \geq 1$, be defined on a probability space (Ω, \mathcal{F}, P) . Assume that for every \mathbf{t} , the random variable $X(\mathbf{t})$ lies in a normed linear space $L^2(\Omega) = L^2(\Omega, \mathcal{F}, P)$ of real-valued random variables with finite second moment and identified equivalent elements with respect to P . We set

$$\|\xi\| := (\mathbb{E}\xi^2)^{1/2}$$

for any $\xi \in L^2(\Omega)$. We investigate approximation methods based on the normed linear space $\mathcal{C}^m(\mathcal{D})$ of random fields with continuous q.m. derivatives up to order $m = 0, 1, \dots$. We also write $\mathcal{C}(\mathcal{D}) = \mathcal{C}^0(\mathcal{D})$ to denote the space of q.m. continuous random functions. For any $X \in \mathcal{C}^m(\mathcal{D})$, we define the mean integrated norm by setting

$$\|X\|_p := \left(\int_{\mathcal{D}} \|X(\mathbf{t})\|^p \right)^{1/p}, \quad 1 \leq p < \infty,$$

and the mean uniform norm by $\|X\|_{\infty} := \max_{\mathbf{t} \in \mathcal{D}} \|X(\mathbf{t})\|$. The space $\mathcal{C}^m(\mathcal{D})$, $m \geq 0$, of non-random functions with continuous derivatives up to order m can be considered as a linear subspace of $\mathcal{C}^m(\mathcal{D})$ by the usual embedding.

3 Some classes of random functions

In this section, we introduce some general classes of random functions used throughout the thesis. For a random process $X \in \mathcal{C}([a, b])$, we say that:

(i) X is *Hölder continuous* if there exists $0 < \alpha \leq 2$ such that

$$\|X(t+s) - X(t)\|^2 \leq C|s|^\alpha \text{ for all } t, t+s, \in [a, b],$$

for a positive constant C ;

(ii) X is *locally stationary* if there exist $0 < \alpha \leq 2$ and a positive continuous function $c(t)$, $t \in [a, b]$, such that

$$\frac{\|X(t+s) - X(t)\|^2}{|s|^\alpha} \rightarrow c(t) \quad \text{as } s \rightarrow 0 \text{ uniformly in } t \in [a, b].$$

In our studies, we also consider q.m. differentiable stochastic processes with q.m. derivatives satisfying the above conditions. The concept of local stationarity is in-

troduced in Berman (1974). An investigation of some properties of random processes from this class can be found, e.g., in Hüsler (1990).

Now we extend the definitions of the introduced classes to the (time) multivariate case. Let $X(\mathbf{t})$, $\mathbf{t} = (t_1, \dots, t_d) \in \mathcal{D} = [0, 1]^d$, $d \geq 2$, be a q.m. continuous real-valued random field. For $k \leq d$, let $\mathbf{l} = (l_1, \dots, l_k)$ be a vector of positive integers such that $\sum_{j=1}^k l_j = d$, and let $L_i := \sum_{j=1}^i l_j$, $i = 1, \dots, k$, $L_0 = 0$, be the sequence of its cumulative sums. Then the vector \mathbf{l} defines the decomposition of \mathcal{D} into $\mathcal{D}^1 \times \dots \times \mathcal{D}^k$, with the l_j -cube $\mathcal{D}^j = [0, 1]^{l_j}$, $j = 1, \dots, k$. For any $\mathbf{s} \in \mathcal{D}$, we denote by \mathbf{s}^j the coordinates vector corresponding to the j -th component of the decomposition, i.e.,

$$\mathbf{s}^j = \mathbf{s}^j(\mathbf{l}) := (s_{L_{j-1}+1}, \dots, s_{L_j}) \in \mathcal{D}^j, \quad j = 1, \dots, k.$$

For a vector $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_k)$, $0 < \alpha_j \leq 2$, $j = 1, \dots, k$, and the decomposition vector $\mathbf{l} = (l_1, \dots, l_k)$, let

$$\|\mathbf{s}\|_{\boldsymbol{\alpha}} := \sum_{j=1}^k \|\mathbf{s}^j\|^{\alpha_j} \quad \text{for all } \mathbf{s} \in \mathcal{D}$$

with the Euclidean norms $\|\mathbf{s}^j\|$, $j = 1, \dots, k$.

For a hyperrectangle $\mathcal{A} = [a_1, b_1] \times \dots \times [a_d, b_d] \subset \mathcal{D}$ and a random field $X \in \mathcal{C}(\mathcal{A})$, analogously to the one-dimensional case, we say that:

(i) X is *Hölder continuous* if for some $\boldsymbol{\alpha}$, \mathbf{l} , and a positive constant C ,

$$\|X(\mathbf{t} + \mathbf{s}) - X(\mathbf{t})\|^2 \leq C \|\mathbf{s}\|_{\boldsymbol{\alpha}} \quad \text{for all } \mathbf{t}, \mathbf{t} + \mathbf{s} \in \mathcal{A};$$

(ii) X is *locally stationary* if for some $\boldsymbol{\alpha}$, \mathbf{l} , and a vector function $c(\mathbf{t}) = (c_1(\mathbf{t}), \dots, c_k(\mathbf{t}))$,

$$\frac{\|X(\mathbf{t} + \mathbf{s}) - X(\mathbf{t})\|^2}{\sum_{j=1}^k c_j(\mathbf{t}) \|\mathbf{s}^j\|^{\alpha_j}} \rightarrow 1 \quad \text{as } \mathbf{s} \rightarrow 0 \text{ uniformly in } \mathbf{t} \in \mathcal{A},$$

with positive and continuous functions $c_1(\cdot), \dots, c_k(\cdot)$. We assume additionally that for $j = 1, \dots, k$, the function $c_j(\cdot)$ is invariant with respect to permutations of coordinates within the j -th component.

4 Regular and cross regular sequences of sampling designs

In this section, we describe sequences of sampling designs determining the location of observation points used to construct the approximation methods studied in the thesis. Let us begin with the one-dimensional setting, i.e., $\mathcal{D} = [0, 1]$. Consider designs consisting of $n + 1$ distinct points $T_n = \{t_0, t_1, \dots, t_n\}$, including the ends of the unit interval, i.e., $0 = t_0 < t_1 < \dots < t_n = 1$. Following Sacks and Ylvisaker (1966), we introduce the *regular sequences* of designs $T_n = T_n(h)$ generated by a positive continuous density function $h(t)$, $t \in [0, 1]$, via

$$\int_0^{t_i} h(t) dt = \frac{i}{n}, \quad i = 0, \dots, n.$$

Define the related distribution functions

$$H(t) := \int_0^t h(u) du, \quad G(t) := H^{-1}(t) = \int_0^t g(v) dv, \quad t \in [0, 1], \quad (1)$$

i.e., $G(\cdot)$ is a quantile function for the distribution H . Then by definition,

$$t_i = G\left(\frac{i}{n}\right), \quad i = 0, \dots, n,$$

i.e., the knots are i/n -quantile points of H . In particular, if $h(\cdot)$ is uniform over $[0, 1]$ ($h(t) \equiv 1$, $t \in [0, 1]$), then $G(t) = t$, $t \in [0, 1]$, and the regular sampling becomes the conventional equidistant sampling including the endpoints.

Consider now $\mathcal{D} = [0, 1]^d$, $d \geq 2$. Let $h_j(s)$, $s \in [0, 1]$, $j = 1, \dots, d$, be positive and continuous density functions, say, *withindimensional densities*, and set

$$h(\mathbf{t}) := (h_1(t_1), \dots, h_d(t_d)),$$

Moreover, let $\pi : \mathbb{N} \rightarrow \mathbb{N}^d$ be a vector function determining the *interdimensional knot distribution*:

$$\pi(N) := (n_1(N), \dots, n_d(N)),$$

such that $\lim_{N \rightarrow \infty} n_j(N) = \infty$, $j = 1, \dots, d$, and the condition

$$\prod_{j=1}^d (n_j(N) + 1) = N \quad (2)$$

is satisfied. The *cross regular sequence* of sampling designs corresponding to $h(\cdot)$ and $\pi(\cdot)$ is defined by

$$T_N = T_N(h, \pi) = \{\mathbf{t}_i = (t_{1,i_1}, \dots, t_{d,i_d}) : \mathbf{i} = (i_1, \dots, i_d), 0 \leq i_k \leq n_k(N), k = 1, \dots, d\},$$

where

$$\int_0^{t_{j,i}} h_j(v) dv = \frac{i}{n_j(N)}, \quad i = 0, \dots, n_j(N), \quad j = 1, \dots, d.$$

Equation (2) ensures that the total number of observation points is N . We suppress the argument N for $n_j = n_j(N)$, $j = 1, \dots, d$, when doing so causes no confusion. Observe that a cross regular design T_N may be expressed as a product of regular designs, i.e.

$$T_N(h, \pi) = T_{n_1}(h_1) \times \dots \times T_{n_d}(h_d).$$

The hypercube \mathcal{D} is partitioned into $\prod_{j=1}^d n_j$ disjoint hyperrectangles \mathcal{D}_i , $\mathbf{i} \in \mathbf{I}$, where $\mathbf{I} := \{(i_1, \dots, i_d), 0 \leq i_k \leq n_k - 1, k = 1, \dots, d\}$. Let $\mathbf{1}_d = (1, \dots, 1)$ denote a d -dimensional vector of ones. The hyperrectangle \mathcal{D}_i is determined by the vertex

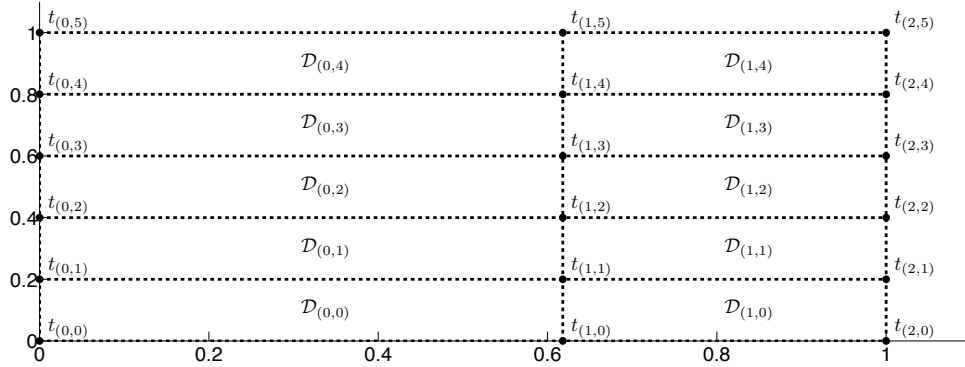


Figure 1: A two dimensional cross regular grid with $h(\mathbf{t}) = (1 + 1/2t_1, 1)$, $\mathbf{t} \in [0, 1]^2$, $n_1 = 2$ and $n_2 = 5$.

$\mathbf{t}_i = (t_{1,i_1}, \dots, t_{j,i_d})$ and the main diagonal $\mathbf{r}_i = \mathbf{t}_{i+\mathbf{1}_d} - \mathbf{t}_i$, i.e.,

$$\mathcal{D}_i := \{\mathbf{t} : \mathbf{t} = \mathbf{t}_i + \mathbf{r}_i * \mathbf{s}, \mathbf{s} \in [0, 1]^d\},$$

where $'*$ ' denotes the coordinatewise multiplication, i.e., for $\mathbf{x} = (x_1, \dots, x_d)$ and $\mathbf{y} = (y_1, \dots, y_d)$, $\mathbf{x} * \mathbf{y} := (x_1 y_1, \dots, x_d y_d)$. Figure 1 presents an example of two dimensional cross regular grid with $h(\mathbf{t}) = (1 + 1/2t_1, 1)$, $\mathbf{t} \in [0, 1]^2$, $n_1 = 2$ and $n_2 = 5$.

The introduced notation for the observation points and the coordinates of a vector \mathbf{t} is not completely unambiguous. As an example, t_i in the one-dimensional setting refers to the i -th element of a sampling design, while in the multivariate case, it refers to the i -th coordinate of a parameter \mathbf{t} . Nevertheless, in what follows, the meaning of the symbol should be clear from the context.

5 Approximation techniques

In this section, we define the approximation techniques used throughout the thesis and present shortly our results obtained for the introduced classes of random functions. We begin with a one-dimensional piecewise linear interpolator and present its generalization, namely, Hermite interpolation spline and a multivariate piecewise linear interpolator. Further, we discuss a stratified Monte Carlo quadrature in the context of numerical integration of random fields.

5.1 Piecewise linear interpolation

Let $X(t)$, $t \in [0, 1]$, be a continuous stochastic process. The conventional linear interpolator based on two observations in points $t = 0$ and $t = 1$ is given by

$$H_1(X, \{0, 1\})(t) := X(0)(1 - t) + tX(1), \quad t \in [0, 1].$$

Moreover, for any design $T_n = \{t_0, \dots, t_n\}$, $0 = t_0 < \dots < t_n = 1$, a *piecewise linear interpolator* (PLI) is defined by

$$H_1(X, T_n)(t) := X(t_i)(1 - \bar{t}_i) + \bar{t}_i X(t_{i+1}), \quad t \in [t_i, t_{i+1}], \quad (3)$$

where $\bar{t}_i = (t - t_i)/h_i$, $h_i = t_{i+1} - t_i$, $i = 0, \dots, n - 1$. We express (3) in a probabilistic setting by introducing an auxiliary random variable, i.e.,

$$H_1(X, T_n)(t) = E_{\eta_i} X(t_i + h_i \eta_i), \quad t \in [t_i, t_{i+1}], \quad (4)$$

where η_i is a Bernoulli random variable with mean \bar{t}_i , $i = 0, \dots, n-1$. Piecewise linear interpolation and its application to approximation of random processes have been widely studied. For locally stationary processes, an exact asymptotic behavior of the approximation accuracy is derived in Seleznev (2000). Seleznev (1999) shows that for a wide class of processes satisfying a Hölder condition, PLI attains the optimal rate of convergence among all linear methods.

5.2 Hermite interpolation splines

To obtain approximations attaining the optimal approximation rates for smooth functions, we need more advanced methods. If besides values of the process, it is possible to sample its q.m. derivatives, then *Hermite interpolation splines* are a feasible technique. Recall that for any $f \in C^m([0, 1])$, $m \geq 0$, the piecewise Hermite polynomial $H_k(t) = H_k(f, T_n)(t)$ of degree $k = 2r + 1$, $0 \leq r \leq m$, is the unique solution of the interpolation problem

$$H_k^{(j)}(t_i) = f^{(j)}(t_i), \quad i = 0, \dots, n, \quad j = 0, \dots, r.$$

Analogously, assuming that for $X \in \mathcal{C}^m([0, 1])$, the process and its first $r \leq m$ derivatives can be sampled, we write $H_k(X, T_n)$ with $k = 2r + 1 \leq 2m + 1$ to denote a corresponding stochastic Hermite spline. Observe that the piecewise linear interpolator is a special case of Hermite interpolation splines ($r = 0$). Hermite spline interpolation for q.m. continuous and smooth random functions is studied in Seleznev (1999) and Hüsler et al. (2003). Seleznev (2000) determines an exact asymptotics of the corresponding approximation error for processes with locally stationary q.m. derivatives. General results for approximation of q.m. continuous and differentiable random processes by linear methods can be found, e.g., in Ritter (2000) and Buslaev and Seleznev (1999).

5.3 Multivariate piecewise linear interpolation

Now we turn to the approximation of real-valued random fields. The piecewise linear interpolator expressed in terms of an auxiliary random variable by (4) is naturally extendable to higher dimensions. Consider a random field $X(\mathbf{t})$, $\mathbf{t} \in \mathcal{D} = [0, 1]^d$, $d \geq 2$. Let T_N be a cross regular sequence of designs. We define a *multivariate*

piecewise linear interpolator (MPLI) with knots T_N

$$X_N(X, T_N)(\mathbf{t}) := E_{\boldsymbol{\eta}_i} X(\mathbf{t}_i + \mathbf{r}_i * \boldsymbol{\eta}_i), \quad \mathbf{t} \in \mathcal{D}_i, \quad \mathbf{t} = \mathbf{t}_i + \mathbf{r}_i * \mathbf{s},$$

where $\boldsymbol{\eta}_i = (\eta_{i,1}, \dots, \eta_{i,d})$ and $\eta_{i,1}, \dots, \eta_{i,d}$ are auxiliary independent Bernoulli random variables with means s_1, \dots, s_d , respectively, i.e., $\eta_{i,j} \in Be(s_j)$, $j = 1, \dots, d$, $i \in \mathbf{I}$. Such an interpolator is continuous and piecewise linear along all coordinates. We study the behavior of MPLI for continuous locally stationary random fields in Paper B. For such fields, we derive the exact asymptotics of the approximation error. A method is proposed for determining the asymptotically optimal interdimensional knot distribution for such error. We also study optimality of knot allocation along coordinates of the sampling grid. An explicit formula for the asymptotically optimal density is derived for one-dimensional components. Additionally, for some classes of q.m. continuous and continuously differentiable random fields, we determine upper bounds for the approximation accuracy.

The performance of MPLI is optimal in a certain sense: the corresponding approximation error is of the same order as the minimal error for linear methods for random fields satisfying Hölder type conditions (see, e.g., Ritter, 2000). An upper bound for the approximation error for isotropic random fields satisfying Hölder type conditions is given in Ritter et al. (1995). An optimal allocation of the observations for Gaussian random fields with product type kernel is investigated in Müller-Gronbach and Schwabe (1996). Benhenni (2001) provides exact asymptotics of MPLI approximation accuracy for stationary spatial process when the estimator is based on an equidistant sampling.

5.4 Stratified Monte Carlo quadrature

In this section, we consider the numerical integration problem, i.e., an approximation of

$$I := \int_{\mathcal{D}} X(\mathbf{t}) d\mathbf{t},$$

where $X(\mathbf{t})$, $\mathbf{t} \in \mathcal{D} = [0, 1]^d$, $d \geq 1$, is a q.m. continuous random field. We evaluate the performance of a stratified Monte Carlo quadrature, introduced in a deterministic setting by Haber (1966) to overcome approximation rate limitations of conventional Monte Carlo methods. The quadrature is defined by N stratified random observations with the partition generated by a cross regular design. To control the number

of the hyperrectangular strata, rather than the number of grid vertices, we replace equation (2) in the definition of a cross regular sampling with

$$\prod_{j=1}^d n_j(N) = N. \quad (5)$$

Recall that $\mathbf{I} = \{\mathbf{i} = (i_1, \dots, i_d) : 0 \leq i_j \leq n_j - 1, j = 1, \dots, d\}$ and let $|\mathcal{D}_{\mathbf{i}}|$ denote the volume of the hyperrectangle $\mathcal{D}_{\mathbf{i}}, \mathbf{i} \in \mathbf{I}$. For a random field $X \in \mathcal{C}(\mathcal{D})$, define a *stratified Monte Carlo quadrature* (sMCQ) on the partition generated by a cross regular design T_N :

$$I_N(X, T_N) := \sum_{\mathbf{i} \in \mathbf{I}} X(\boldsymbol{\eta}_{\mathbf{i}}) |\mathcal{D}_{\mathbf{i}}|,$$

where $\boldsymbol{\eta}_{\mathbf{i}}, \mathbf{i} \in \mathbf{I}$, are independent random variables and $\boldsymbol{\eta}_{\mathbf{i}}$ is uniformly distributed in the hyperrectangle $\mathcal{D}_{\mathbf{i}}, \mathbf{i} \in \mathbf{I}$. Such a quadrature is a modification of the well known midpoint quadrature (see, e.g., Evans and Swartz, 2000). In paper C, we study the behavior of the corresponding approximation accuracy for q.m. continuous random fields. For locally stationary random fields, we provide the exact asymptotics of the mean squared error of the integral approximation. We discuss the optimal interdimensional distribution of knots resulting in an increased rate of convergence, compared to the uniform distribution. For stochastic processes, the optimal within-dimensional density which minimizes the constant for the asymptotic approximation error is determined. An upper bound for the approximation accuracy for the class of Hölder continuous random functions is derived.

Analogously to the multivariate piecewise interpolation problem, the obtained approximation rate is optimal for the class of random fields satisfying Hölder condition (see, e.g., Wasilkowski, 1994). Monte Carlo numerical integration techniques have an advantage over deterministic product rules, namely, these techniques do not suffer the curse of dimensionality. While the approximation rate of the deterministic methods decreases when dimension d increases, the Monte Carlo methods guarantee convergence not slower than $N^{-1/2}$. Regular sampling designs for estimating the integrals of stochastic processes are studied in Benhenni and Cambanis (1992). Random designs of sampling points, including stratified sampling for stochastic processes, are investigated in Schoenfelder and Cambanis (1982); Cambanis and Masry (1992). Approximation of integrals of stationary random fields using the observations on a lattice is discussed in Stein (1995b). Quadratures for smooth isotropic random functions are investigated in Ritter and Wasilkowski (1997); Stein (1995a). Multi-

variate numerical integration of random fields satisfying Sacks-Ylvisaker conditions is studied in Ritter et al. (1995).

6 Point singularities

6.1 Motivation

Regular sequences of designs are powerful and convenient tools in approximation theory. However, in the case of an isolated point singularity it is beneficial to consider slightly different designs. We illustrate this with a deterministic example of a piecewise linear interpolation presented in Rice (1969).

Example 1. Let $f(t) = \sqrt{t}$, $t \in [0, 1]$, and consider the piecewise linear interpolator $H_1(f, T_n)(t)$, $t \in [0, 1]$ with $n + 1$ knots T_n . We compare the maximum approximation error $e_n(T_n, f) = \max_{t \in [0, 1]} |f(t) - H_1(f, T_n)(t)|$ for uniformly spaced knots $T_{n,1} = \{t_i = i/n : i = 0, \dots, n\}$ and for $T_{n,2} = \{t_i = (i/n)^4 : i = 0, \dots, n\}$. Figure 2 demonstrates the plots of the function $f(\cdot)$ together with the piecewise linear interpolators $H_1(f, T_{n,1})(\cdot)$ and $H_1(f, T_{n,2})(\cdot)$ for $n = 4$. For both designs, the obser-

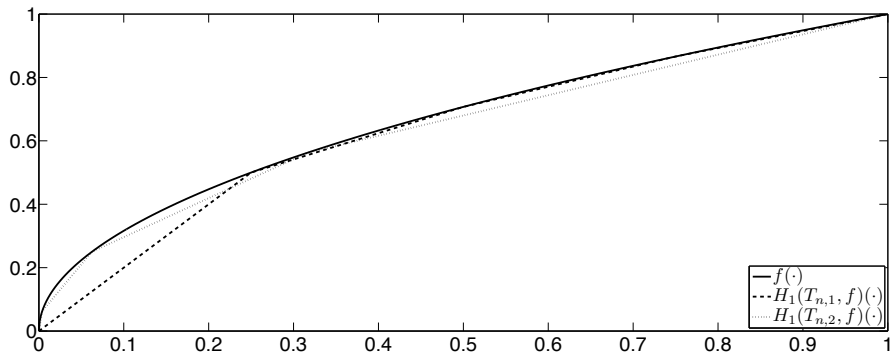


Figure 2: Plots of $f(\cdot)$ (solid line), $H_1(f, T_{n,1})(\cdot)$ (dashed line), and $H_1(f, T_{n,2})(\cdot)$ (dotted line) for $n = 4$.

vation points are the percentile points of generating densities $h_1(t) = 1$, $t \in [0, 1]$, and $h_2(t) = 1/4t^{-3/4}$, $t \in (0, 1]$, respectively. Direct calculation shows that

$$e_n(T_{n,1}, f) = \frac{1}{4n^{1/2}},$$

$$e_n(T_{n,2}, f) \leq \frac{1}{2n^2}.$$

Hence, relaxing the condition of continuity of a generating density on the whole unit interval and allowing unboundedness in the singular point result in a significant gain in the approximation rate. In principle, the main idea is well-known for nonlinear approximation of deterministic functions (see, e.g., de Boor, 1973; DeVore, 1998, and references therein). We use a finer mesh where the target function is singular and a coarser mesh where it is smooth.

6.2 Quasi regular sequences of designs

In Paper A, we introduce a modification of the regular sequences, namely, the *quasi regular sequences* (qRS) of sampling designs $T_n = T_n(h)$. We assume that $h(t)$ is continuous for $t \in (0, 1]$, and allow $h(\cdot)$ to be unbounded in $t = 0$. If $h(t)$ is unbounded in $t = 0$, then $h(t) \rightarrow +\infty$ as $t \rightarrow 0+$. The corresponding quantile density function $g(t)$ defined in (1) is assumed to be continuous for $t \in [0, 1]$ with the convention that $g(0) = 0$ if $h(t) \rightarrow +\infty$ as $t \rightarrow 0+$.

In Paper C, we study numerical integration of random functions with a point singularity. For random processes, we provide some conditions for the density generating the quasi regular sequences of designs. Under these conditions, the sMCQ asymptotically attains the convergence rate corresponding to the numerical integration of a process without the singularity. Moreover, for one component random fields ($d \geq 2$), we show that the effect of the singularity might be eliminated by the sMCQ with sampling points determined by cross regular sequences of designs.

6.3 Composite Hermite splines

In Paper A, we examine the case when due to the point singularity, the construction of a Hermite spline attaining the optimal convergence rate on $[0, 1]$ is not possible. Consider a stochastic process $X(t)$, $t \in [0, 1]$, and let $X \in \mathcal{C}^l([0, 1])$, $l \geq 0$. Furthermore, let $X \in \mathcal{C}^m([a, b])$ for any $[a, b] \subset (0, 1]$, $l < m$. Assume also that existing q.m. derivatives are sampled. Using the introduced definition of the interpolation Hermite spline, we can only construct the splines of maximum order $q = 2l + 1$. To solve this problem, in Paper A, we introduce the following adaptation of a Hermite spline. We define $H_{q,k}(X, T_n)$, $q = 2l + 1 < k = 2m + 1$, to be a *composite Hermite spline*

$$H_{q,k}(X, T_n)(t) := \begin{cases} H_q(X, T_n)(t), & t \in [0, t_1], \\ H_k(X, T_n)(t), & t \in [t_1, 1]. \end{cases}$$

In this way, we are able to use all available information about a target function. Nevertheless, such an interpolator does not attain the optimal rate of approximation when regular design sequences are used, since the main input to the error comes from the interval $[0, t_1]$. On the other hand, combining composite Hermite splines and appropriate quasi regular sequences of designs leads to elimination of the effect of the singular point. In Paper A, we investigate the accuracy of such an approximation for a wide class of stochastic processes. Further, the formula for the asymptotically optimal density is derived. Under certain conditions, such density not only attains the optimal rate of convergence but also minimizes the asymptotic constant.

7 A financial application: Asian option pricing

The log-returns model based on the Wiener process, called the Black-Scholes model, was developed by Black and Scholes (1973). The price process describing the evolution of a risky asset on the market is modelled by

$$S(t) = S(t; r, \sigma, S_0) := S(0)e^{(r-\sigma^2/2)t+\sigma W(t)}, \quad S(0) = S_0, \quad t \geq 0,$$

where $W(t)$, $t \geq 0$, is a Brownian motion, S_0 is a given initial price of the asset, $r > 0$ denotes a risk free rent, and $\sigma > 0$, called volatility, is an asset specific parameter measuring the variation. To ensure that there is no arbitrage on the market, a discounted price process $e^{-rt}S(t)$, $t \geq 0$, must be a martingale. Recent studies reveal discrepancy between empirical data and theoretical properties of the normal approximation of log-returns. The imperfection of Gaussian models is studied, e.g., in Schoutens (2003). Modern modelling is based on a wide class of Lévy processes with stationary and independent increments. Such models capture the negative skewness, exceeding kurtosis, and heavy tails properties. A stochastic process $L(t) = L(t; \Xi)$, $t \geq 0$, $L(0) = 0$, with a set of parameters Ξ is called a *Lévy process* if it has independent and stationary increments and it is stochastically continuous (i.e., $\lim_{s \rightarrow t} P(|L(t) - L(s)| > \epsilon) = 0$ for all $t \geq 0$, $\epsilon > 0$). For Lévy models, we assume that the price process can be written as

$$S(t) = S(t; \Xi, S_0, m) := S(0)e^{mt+L(t,r;\Xi)}, \quad S(0) = S_0, \quad t \geq 0.$$

The requested martingale property is obtained by setting a location parameter $m^* = m^*(\Xi, r)$ such that $ES(t) = e^{rt}S_0$, $t \in \mathbb{R}^+$.

Options are among the most widely used derivatives on the stock market. An option is a right, not an obligation, to buy or sell an asset at a strike price at a given time with respect to an initial price. An execution of a classic option, called vanilla option, depends on the price only at a terminal time. To prevent manipulations of a market close to an execution date, *Asian options* are constructed. An Asian option takes the average (arithmetic or geometric) of prices during some period. Formally, the price of an arithmetic call Asian option with European type of execution is given by

$$A = A(\Xi, S_0, m^*, r, T, K) := e^{-rT} \mathbb{E} \left(\frac{1}{T} \int_0^T S(t; \Xi, S_0, m^*) dt - K \right)_+,$$

where $T > 0$ refers to a maturity time measured in years, $K > 0$ is an exercise price, and for any $a \in \mathbb{R}$, $(a)_+ := \max(a, 0)$.

Even in the Black Scholes model, pricing arithmetic Asian options is still an open question. Approximate closed-form solutions are suggested in, e.g., Milevsky and Posner (1998) and Zhang (2004). In Paper D, the accuracy of Monte Carlo pricing method for the class of Lévy markets is studied. We investigate the implication of a uniform time discretization and corresponding piecewise constant approximation of the price process. We calculate an upper bound for the variance of the estimator and consequently obtain a bound for the simulation error. This allows us to perform accurate simulations with controlled discretization and simulation errors. Furthermore, the sufficient numbers of discretization points and Monte Carlo replications to obtain the option price estimates with a prescribed accuracy (at a given confidence level) are provided. Moreover, for the Black Scholes model, we investigate the control variate variance reduction technique and obtain a more efficient pricing algorithm. For an extensive summary of Monte Carlo methods in financial engineering, we refer to Glasserman (2004).

8 Summary of papers

In this section, we present the four papers included in the thesis. A Hermite spline approximation of stochastic processes with a singularity is investigated in Paper A. In Paper B and C, we deal with approximation and numerical integration of continuous random fields, respectively. In paper D, we consider an approximation problem applied to financial mathematics, namely, Monte Carlo evaluation of prices of arithmetic Asian options. Each study is completed with numerical experiments

illustrating the obtained results.

8.1 Paper A: Spline approximation of a random process with singularity

Let a random process with finite second moment be observed in finite number of points and have different quadratic mean smoothness at the origin (i.e., an isolated singularity). In this paper, we consider approximation of such processes by a composite Hermite interpolation spline. A quasi regular sequence of sampling designs is constructed to improve the asymptotic approximation performance eliminating the effect of the singularity point. Furthermore, we provide the formula for the generating density that under certain condition minimizes the constant for the asymptotic approximation accuracy. The proposed technique can be applied also to random processes with a finite number of isolated singularity points.

8.2 Paper B: Multivariate piecewise linear interpolation of a random field

In this paper, we consider multivariate piecewise linear interpolation of a continuous random field on a unit hypercube. The approximation performance is measured by the integrated mean squared error. A multivariate piecewise linear interpolator is defined by N field observations located on a cross regular grid (design). We investigate the class of locally stationary random fields and find the asymptotic approximation accuracy for large N . Moreover, for certain classes of continuous and continuously differentiable fields satisfying Hölder type conditions, we provide the upper bound for the approximation accuracy.

8.3 Paper C: Stratified Monte Carlo quadrature for random functions

In this paper, we consider the problem of numerical approximation of integrals of random fields over a unit hypercube. We use a stratified Monte Carlo quadrature rule and measure the approximation performance by the mean squared error. The quadrature is defined by N stratified randomly chosen observations with the strata generated by a cross regular design. For the class of locally stationary random fields the asymptotic approximation accuracy for large N is derived. For the Hölder class

of random functions, we provide an upper bound for the approximation error. Additionally, for a certain class of isotropic random functions with an isolated singularity at the origin, a sequence of designs eliminating the effect of the singularity point is constructed.

8.4 Paper D: On the error of the Monte Carlo pricing method for Asian option

In this paper, we consider a Monte Carlo method for pricing a continuous arithmetic Asian option with a given precision. Piecewise constant approximation and plain simulations are used for a wide class of models based on Lévy processes. We give bounds on the possible errors due to discretization and simulation. According to some criteria, we give the sufficient numbers of discretization points and simulations to obtain the price estimate with the required precision on a given confidence level. To demonstrate the developed general approach, the Black-Scholes model is studied in more detail.

9 Concluding remarks and future research

In this thesis, we study methods for approximation and numerical integration of stochastic processes and random fields. Further, we investigate some applications in the context of financial mathematics. The obtained results can be used for developing asymptotically optimal sampling designs in many applied fields. We performed numerical studies based on simulations, however it could be rewarding to examine the results in connection to real data.

The construction of the interpolating Hermite splines requires the knowledge of q.m. derivatives. As presented in Paper A, it is possible to obtain an exact asymptotics for the corresponding approximation accuracy in case of a finite number of singular points. One of the possible directions for future research is developing similar results for higher order approximation techniques without sampling of derivatives.

In Paper B, we investigate the multivariate piecewise linear interpolator. The method is in a certain sense optimal for the q.m. continuous and continuously differentiable random functions. It would be of interest to investigate the exact asymptotics for higher order approximation techniques and generalize the concept of the local stationarity for smooth random fields.

The performance of the stratified Monte Carlo quadrature for smooth functions can be improved by making use of antithetic variables (Haber, 1967). It is relevant to investigate this modification in the context of numerical integration of random fields and to obtain asymptotic expressions for the numerical integration accuracy for specific classes of random fields.

The Monte Carlo pricing method with controlled accuracy investigated in Paper D can be applied for a wide class of financial derivatives. Possible future work would be to develop a robust collection of results for various types of options and contracts in a Lévy market model setting.

The techniques used to obtain the results of Paper A are based on the Peano kernel theorem. The theorem provides a representation for approximation errors for a general class of linear methods. It would be of great interest to apply this general approach to problems in a random setting and derive results for various linear operators used in approximation, numerical integration, and numerical differentiation.

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