Statistical Analysis and Simulation
Methods Related to Load-Sharing Models

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Abstract

We consider the problem of estimating the reliability of bundles constructed of several fibres, given a particular kind of censored data. The bundles consist of several fibres which have their own independent identically distributed failure stresses (i.e., the forces that destroy the fibres). The force applied to a bundle is distributed between the fibres in the bundle, according to a load-sharing model. A bundle with these properties is an example of a load-sharing system. Ropes constructed of twisted threads, composite materials constructed of parallel carbon fibres, and suspension cables constructed of steel wires are all examples of load-sharing systems. In particular, we consider bundles where load-sharing is described by either the Equal load-sharing model or the more general Local load-sharing model.

In order to estimate the cumulative distribution function of failure stresses of bundles, we need some observed data. This data is obtained either by testing bundles or by testing individual fibres. In this thesis, we develop several theoretical testing methods for both fibres and bundles, and related methods of statistical inference.

Non-parametric and parametric estimators of the cumulative distribution functions of failure stresses of fibres and bundles are obtained from different kinds of observed data. It is proved that most of these estimators are consistent, and that some are strongly consistent estimators. We show that resampling, in this case random sampling with replacement from statistically independent portions of data, can be used to assess the accuracy of these estimators. Several numerical examples illustrate the behavior of the obtained estimators. These examples suggest that the obtained estimators usually perform well when the number of observations is moderate.¹


Key words and phrases: Non-parametric and parametric estimation, load-sharing models, asymptotic distribution, martingale, resampling, life testing, reliability.

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List of papers

This thesis is based on the following papers, referred to in the text by the letters A-C.


Parts of this thesis have been published in the following places:


List of conferences

Parts of this thesis have been presented at the following conferences:

Engineering Structures and Extreme Events.

The Second Scandinavian-Ukrainian Conference in Mathematical Statistics,
Umeå University, Sweden, 8-13 June, 1997.

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Papers A-C
1 Introduction

“Statistical theory attempts to answer these basic questions:

1. How should I collect my data?
2. How should I analyze and summarize the data I’ve collected?
3. How accurate are my data summaries?”


The reliability of a system is the ability of that system to fulfil what is required of it. The reliability of a cable can be characterized by the probability that the cable does not break under a certain load. In this thesis, we present statistical methods for estimating the reliability of so-called load-sharing systems. Ropes constructed of several twisted threads, composite materials constructed of parallel carbon fibres, and suspension cables constructed of steel wires are all examples of load-sharing systems. We start with a short informal introduction to the problem.

Figure 1. The Golden Gate bridge, San Francisco, U.S.A.

Suspension bridges can span distances of 600 to 2000 meters, far longer than any other kind of bridge. A suspension bridge suspends the roadway from huge main cables that extend from one end of the bridge to the other, see Figures 1 and 2. The cables are made up of several parallel strands, each strand consisting of several twisted wires. One of the world’s longest suspension bridges is the Japanese Akashi Kaikyo bridge, with a center span of 1990 meters. The Akashi Kaikyo bridge has two main cables each
constructed of 290 strands, with every strand made up of 127 steel wires, see e.g. Kashima and Kitagawa (1997). The dimensions and design of cables depend on several factors, for example the possibility of extreme weather, earthquakes, the ageing of materials due to fatigue, and the random strengths of materials.

Figure 2. The High Coast-bridge under construction, Ångermanland Sweden. This 1800 meter long suspension bridge has two suspension cables, each constructed of over 11000 steel wires.

Suppose we consider cables with no ageing of materials, and suppose we know the most extreme load that will be applied to the cables. Then the problem of constructing the cables depends essentially on estimating the probability of a cable collapsing under a given extreme load. It is clear that we must construct cables in such a way that the probability of the cable collapsing will be very small. For economical reasons, this probability cannot be made too small. One of the problems we consider is to estimate the failure stresses of cables (i.e. the forces that destroy the cables) constructed of several parallel strands. Figure 3a shows one of the two main cables in the High Coast-bridge. Note that the strands are parallel. We also consider the problem of estimating the failure stresses of strands constructed of several twisted steel wires, see Figure 3b. It should be pointed out that we approach these problems of estimating the failure stresses of cables and wires from a purely theoretical point of view.

The failure stress of a system, e.g. a cable, depends on the strength of its components, e.g. strands, and on the arrangement of these components,
e.g. if the strands are parallel or if they are twisted. A load-sharing model describes how a load applied to a system, at any time, is distributed between the components of the system. Different physical arrangements of the components, e.g. if the components are parallel or twisted, give different types of load-sharing models. If the failure stress of a system is decided by the failure stresses of its components and by the load-sharing model of the system, then the system is called a load-sharing system. Henceforth, a system will be referred to as a bundle and its components will be called fibres.

In the following sections we give a more formal presentation of the problems we consider. However, our objective is not to give a complete exposition of our research, but to present some of the main ideas of our research in a short and interesting way. In Section 2, we introduce some model assumptions, and in Section 3 we define several load-sharing models. The testing procedure of fibres and bundles are described in section 4. In Section 5, we present methods for obtaining the strength of a bundle when we know the strengths of its fibres. Statistical methods for estimating the strength of bundles, given the data obtained from the tests described in Section 4, are presented in Section 6. Section 7 discusses possible areas of further research.

Figure 3a
Figure 3b

Figure 3a. One of the main cables of the High Coast-bridge under construction. The cable is constructed of parallel steel strands.

Figure 3b. One of the steel strands used in the construction of the High Coast-bridge. The strand is constructed of several twisted steel wires.
2 The model of a bundle

We consider a bundle constructed of $m$ "statistically similar" fibres of length $l_{\text{fibre}}$, see Figure 4. Henceforth, we consider only fibres with non-elastic behavior, and fibres with no aging. We say that fibres are statistically similar if they follow the assumptions presented below.

Figure 4. A bundle with 4 fibres. $\{l_p^-, l_p^+\}$ denotes the endpoints of a piece of the fourth fibre of length $l_p$. $u_b(t)$ denotes the load applied to the bundle at time $t$.

If we cut a piece of length $l_p$, $l_p \in (0, l_{\text{fibre}}]$ from any of the fibres in the bundle, and test this piece, then we assume that:

a.1. The failure stress of the piece, denoted by $U_{f,p}$, is a random variable (r.v.) with continuous cumulative distribution function (c.d.f.) $F_{f,l_p}(\cdot)$, where $F_{f,l_p}(u) := P(U_{f,p} \leq u), u > 0$.

The corresponding cumulative hazard function (c.h.f.) is denoted by $H_{f,l_p}(\cdot)$, where $H_{f,l_p}(u) := -\ln (1 - F_{f,l_p}(u)), u > 0$.

a.2. The position of the break of the piece is an r.v., uniformly distributed over the length of the piece.

a.3. If the piece is divided into $k$ smaller pieces, then the failure stresses of these pieces, denoted by $U_{f,p_1}, ..., U_{f,p_k}$, are independent r.v.'s, and $U_{f,p} = \min \{U_{f,p_1}, ..., U_{f,p_k}\}$.

In addition we assume that:

a.4. If we cut two pieces from different fibres, then the failure stresses of these pieces are independent r.v.'s.
It is not always necessary to assume \((a.1) - (a.4)\), for some of our results we need only assumption \((b.1)\):

b.1 The failure stresses of individually tested fibres in a bundle are independent identically distributed (i.i.d.) r.v.'s with a continuous c.d.f. \(F_{f_{\text{fibre}}} (\cdot)\).

Note that assumption \((b.1)\) follows from assumptions \((a.1) - (a.4)\).

From assumptions \((a.1) - (a.4)\), it follows that the c.h.f. for the failure stress of a piece of length \(l_p\) can be defined as

\[
H_{f,l_p} (u) := \frac{l_p}{l_{\text{fibre}}} H_{f_{\text{fibre}}} (u), \ u > 0,
\]

where \(H_{f_{\text{fibre}}} (\cdot)\) is the c.h.f. for the failure stress of a fibre of length \(l_{\text{fibre}}\).

The failure stress of a bundle is an r.v. and is denoted by \(U_b\). The c.d.f. of the r.v. \(U_b\) is denoted by \(F_b (\cdot)\), where

\[
F_b (u) := P (U_b \leq u), \ u > 0. \tag{2.1}
\]

### 3 Load-sharing models

We consider a bundle constructed of \(m\) statistically similar fibres of length \(l_{\text{fibre}}\). A load-sharing model describes how the load applied to the bundle, denoted by \(u_b (t)\) at any moment \(t\), is distributed between the fibres in the bundle, see Figure 4. In this section, we present three different load-sharing models: the Equal load-sharing (ELS) model, the Local load-sharing (LLS) model, and the General load-sharing (GLS) model. Where the ELS model is a special case of the LLS model, and where the LLS model is a special case of the GLS model. The results in Paper (A) are obtained for the ELS model, and the results in Paper (C) are obtained for the LLS model. The GLS model is introduced in Paper (C).

We recall that we consider only fibres with non-elastic behavior. There are several load-sharing models, closely related to the ELS-model, that allow the fibres to be elastic, see for example Phoenix (1979). There are also load-sharing models that allow extra sources of reliability, such as random change from elastic to plastic behavior or the possibility of initial random slack in the fibres, see for example Phoenix and Taylor (1973).
3.1 The ELS model

The ELS model, often referred to as the Daniels model, was originally studied by Peirce (1926), but the mathematical theory was developed by Daniels (1945). The ELS model is defined in the following way: the applied load on the bundle is, at any time, equally distributed between the unbroken fibres in the bundle, i.e.

\[ u_{f,i}(t) = \begin{cases} \frac{u(t)}{b(t)} & \text{if the } i\text{th fibre is unbroken} \\ 0 & \text{if the } i\text{th fibre is broken} \end{cases} \]

where \( u_{f,i}(t) \) is the load shared by the \( i\)th fibre at time \( t \), and where \( b(t) \) is the number of unbroken fibres at time \( t \), \( i = 1, ..., m, \ t > 0 \). In the ELS model, a broken fibre does not share any of the load applied to the bundle. It follows that the ELS model can serve as a good approximation of the true load-sharing in a bundle if frictional forces between neighboring fibres are small or absent. A bundle with its load-sharing described by the ELS model will be called an ELS bundle. There has been a lot of probabilistic work related to the ELS model, see for example Suh, Bhattacharyya and Grandage (1970), Barbour (1981) or Smith (1982). Some of these results will be discussed in Section 4. In Paper (A), we present several statistical results for the ELS model.

3.2 The LLS model

In the ELS model, a broken fibre does not share any of the load applied to the bundle, therefore ELS models do not allow for any form of physical interaction among fibres. However, there are often frictional forces between neighboring fibres in a bundle, e.g. when the fibres are twisted. If there are frictional forces between neighboring fibres, we can expect broken fibres to share some of the load applied to the bundle, at least locally. In this case, one possible model to consider is the LLS model.

We begin by considering a special case of the LLS model: the chain of bundles model. In this model, a bundle constructed of \( m \) fibres, each of length \( l_{\text{fibre}} \), is regarded as if it is constructed of several shorter bundles of length \( \gamma \), see Figure 5. The load applied to the bundle of length \( l_{\text{fibre}} \) is identical with the loads applied to all of the shorter bundles of length \( \gamma \). The bundle of length \( l_{\text{fibre}} \) breaks when the weakest bundle of length \( \gamma \) breaks.
Each bundle of length $\gamma$ is regarded as a load-sharing bundle, for example an ELS bundle. The chain of bundles model, where each chain is an ELS bundle, was studied by Smith (1982). Several other load-sharing models similar to the chain of bundles model also exist, see e.g. Harlow-Phoenix (1978a, 1978b).

The LLS model we consider is a generalization of the model studied by Smith (1982). We do not consider the bundle as a chain of shorter bundles, but assume that a broken fibre in the bundle is damaged only around the position of the break. More formally, if a fibre is broken at position $z'$, then it is only damaged in the region $(z' - \delta, z' + \delta)$, where $\delta$ is called the damage parameter, $\delta > 0$, $z' \in (0, l_{\text{fibre}})$, see Figure 6.

Let $u_{f,i,z}(t)$ be the load shared by the $i$th fibre, at position $z$, time $t$, and let $b_z(t)$ be the number of undamaged fibres at position $z$, time $t$, $i = 1, \ldots, m$, $z \in (0, l_{\text{fibre}})$, $t > 0$. The LLS model is defined as

$$u_{f,i,z}(t) := \begin{cases} 
  u_b(t) & \text{if the $i$th fibre is undamaged at position $z$, time $t$} \\
  b_z(t) & \text{if the $i$th fibre is damaged at position $z$, time $t$} \\
  0 & \text{if the $i$th fibre is damaged at position $z$, time $t$} 
\end{cases}$$

(3.3)

$i = 1, \ldots, m$, $z \in (0, l_{\text{fibre}})$, $t > 0$.}

**Figure 5.** The chain of bundles model. This bundle contains 5 shorter bundles.
A bundle with its load-sharing described by the LLS model will be called an LLS bundle. Note that if the damage parameter \( \delta \geq \ell_{\text{fibre}} \), then this model is identical to the ELS model. In Paper (C), we present several statistical results related to the LLS model we consider.

### 3.3 The GLS model

It is possible to introduce more general load-sharing models than the LLS model. We introduce the capacity variable \( c_{i,z} (t) \), where \( c_{i,z} (t) \in [0,1] \), \( i = 1, \ldots, m, \ z \in (0, \ell_{\text{fibre}}), \ t > 0 \). The variable \( c_{i,z} (t) \) describes how well, compared with an unbroken fibre, the \( i \)th fibre is functioning at position \( z \), time \( t \). For example, if \( c_{i,z} (t) = 0.5 \), then the \( i \)th fibre, at position \( z \), time \( t \), shares half as much of the load applied to the bundle as an unbroken fibre shares. We give a small example to clarify. A bundle constructed of two fibres is considered. At position \( z \), time \( t \), the capacity of fibre 1 is 0.5, and the capacity of fibre 2 is 0.75, i.e. \( c_{1,z} (t) = 0.5 \) and \( c_{2,z} (t) = 0.75 \). The load applied to the bundle at time \( t \) is \( u_b (t) \), so it follows that the loads applied to the fibres, at position \( z \), time \( t \), are

\[
 u_{f,1,z} (t) = \frac{2u_b (t)}{5} \quad \text{and} \quad u_{f,2,z} (t) = \frac{3u_b (t)}{5}.
\]

For a bundle with \( m \) fibres, the load shared by the \( i \)th fibre, at position \( z \), time \( t \), is defined as

\[
 u_{f,i,z} (t) := \frac{c_{i,z} (t) u_b (t)}{\sum_{i=1}^{m} c_{i,z} (t)}, \quad (3.4)
\]

\( i = 1, \ldots, m, \ z \in (0, \ell_{\text{fibre}}), \ t > 0 \). We call this model the GLS model, and it is presented more fully in Paper (C).

For many load-sharing models, the capacity \( c_{i,z} (t) \) of the \( i \)th fibre, time \( t \), broken at position \( z' \), is a function \( \{ f (z, z') : z, z' \in (0, \ell_{\text{fibre}}) \} \) increasing with the distance between \( z \) and the break \( z' \), i.e. \( |z - z'| \), \( i = 1, \ldots, m, \)
\( z' \in (0, l_{\text{fibre}}) \). Figure 7 shows the capacity of a fibre broken at one position, for several different load-sharing models.

![Graphs showing capacity](image)

Figure 7. The top left plot shows the capacity of an unbroken fibre. The other plots show the capacity of a fibre broken at position \( z' \) for different types of load-sharing models. The upper right plot shows the capacity for an LLS model.

### 4 Tests of bundles and fibres

In this section, we introduce different methods for testing fibres and bundles. In Section 6, we show how the observed data from these tests can be used to obtain estimators of c.d.f.'s of failure stresses of ELS bundles and LLS bundles. We emphasize that these methods are theoretical, and that there have been just a few attempts to try these methods in practice. In Paper (A) we introduce a testing method for ELS bundles, and in Paper (C) this method is generalized for LLS bundles. In Paper (B) a special kind of testing of single fibres is introduced: the Binary tree structured test.

#### 4.1 Tests of fibres

For some load-sharing bundles, e.g. ELS bundles, the data necessary to obtain an estimator of the c.d.f. \( F_b(\cdot) \) can be obtained by testing several fibres independently. In the next section we describe a special method for testing fibres: the Binary tree structured test.
4.1.1 The Binary tree structured test

We consider a fibre of any length \( l_0 \), e.g. \( l_0 = l_{\text{fibre}} \), for which assumptions (a.1.) – (a.4.) hold, see Section 2. Any piece of fibre of length \( l_p \), \( l_p \leq l_0 \), is tested in the following way: we use some length \( \nu \) to fix each end of the piece in the testing equipment, see Figure 8. An increasing force is applied to the piece until the piece breaks. The failure stress of the piece and the position of the break are registered.

![Figure 8. A piece of fibre of length \( l_p \) with the endpoints \( \{l_p^-, l_p^+\} \). The dashed parts of the piece are the parts used to fix the piece in the testing equipment.](image)

We describe the binary tree structured test for a fibre of length \( l_0 \). The fibre of length \( l_0 \) is tested as described above. The position of the break and the failure stress are registered, and denoted by \( z_0 \) and \( u_0 \). As a result of this test, we obtain two new pieces of lengths \( l_1 = z_0 \) and \( l_2 = l_0 - z_0 \). Each new piece longer than \( 4\nu \) is tested as described above. If both \( l_1 \) and \( l_2 \) are longer than \( 4\nu \), then the position of the breaks, denoted by \( z_1 \) and \( z_2 \), and the failure stresses, denoted by \( u_1 \) and \( u_2 \), of both new pieces are registered. From these two new breaks we obtain four new pieces, and each of them longer than \( 4\nu \) is tested as described above. This testing process continues until all remaining pieces are shorter than \( 4\nu \).

The data generated from \( m \) tested fibres of length \( l_0 \) is denoted by

\[
x_B(m) := \{x_1, ..., x_m\},
\]

where

\[
x_i := \{\{z_{i,0}, u_{i,0}\}, ..., \{z_{i,k_i}, u_{i,k_i}\}\},
\]

and where \( z_{i,l} \) denotes the position of the \( l \)th break of the \( i \)th tested fibre, and where \( u_{i,l} \) denotes the force that caused this break, \( i = 1, ..., m \).

The Binary tree structured test was introduced in Paper (B). In Section 6, we show how the data \( x_B(m) \) can be used to obtain a Nelson-Aalen type estimator of the c.d.f. of failure stresses of fibres.
4.2 Tests of ELS bundles

An ELS bundle is tested in the following way: a force \( u_b(t) \), increasing with time \( t \), is applied to the bundle constructed of \( m \) statistically similar fibres, until the bundle breaks, see Figure 4. The bundle is broken when all \( m \) fibres in the bundle are broken. Let \( u_f = u_f(t) \) denote the load applied to an unbroken fibre at time \( t \), and let \( b(t) \) denote the number of unbroken fibres just before time \( t \). It follows from the definition of the ELS model that \( u_f(t) = \frac{u_b(t)}{b(t)}, t \geq 0 \).

The test of a bundle generates censored and uncensored observations of the failure stresses of fibres, and one observation of the failure stress of the bundle. A test starts at time 0. At the start we have \( u_b(0) = u_f(0) = 0 \). The applied force \( u_b(t) \) increases until the weakest fibre in the bundle breaks at some time \( t_1 \). The failure stress of this fibre is registered, and its value is \( u_f = \frac{u_b(t_1)}{m} \). When a fibre breaks, the force \( u_b \) will immediately be distributed equally among the remaining unbroken fibres in the bundle. It follows that the value of the force \( u_f \), i.e. the force applied to unbroken fibres, will immediately jump to a higher value \( u_f^+ \), where \( u_f^+ = \frac{u_b(t_1)}{m-1} \). If the force \( u_f^+ \) is larger than the smallest failure stress of the remaining unbroken fibres, another break will occur immediately, and the force \( u_f \) will jump to an even higher value \( \frac{u_b(t_1)}{m-2} \). This immediate process continues until all remaining fibres have failure stresses higher than the redistributed force \( u_f \). If no such fibres exist, the bundle will break. If several fibres break simultaneously in time, we can only register the force that caused the weakest of these fibres to break, and the number of simultaneously broken fibres. We call this event an observable break. Therefore, during the testing of one bundle, only \( n_o \leq m \) failure stresses of the fibres are registered. The observed data are denoted:

\[
x := \left\{ \{ u_f,(1), \Delta b(1) \}, ..., \{ u_f,(n_o), \Delta b(n_o) \} \right\},
\]

where \( \Delta b(i) \) is the number of registered simultaneously destroyed fibres at the \( i \)th observable break, and where \( u_f,(i) \) is the value of force causing this break. Note that the number of unbroken fibres just before the \( i \)th observable break is defined as

\[
y(i) := m - \sum_{i=1}^{i-1} \Delta b(1).
\]

We give an example to clarify the process described above. In this example, the applied load \( u_b \) was increased linearly with time, and \( b(0) = m = 6 \).
\( u_b(0) = u_f(0) = 0 \). The tested bundle was constructed of 6 fibres, with the following ordered unknown failure stresses of its fibres:

\[
\begin{align*}
  u_{f, \text{fibre}, 1} &= 0, 10, \quad u_{f, \text{fibre}, 2} = 0, 24, \quad u_{f, \text{fibre}, 3} = 0, 26, \\
  u_{f, \text{fibre}, 4} &= 0, 60, \quad u_{f, \text{fibre}, 5} = 1, 20, \quad u_{f, \text{fibre}, 6} = 3, 00.
\end{align*}
\] (4.7)

Figure 9 shows the applied forces \( u_b \) and \( u_f \) in this example. There were five observable breaks, i.e. \( n_0 = 5 \), and we observed the data

\[ x = \{\{0.10, 1\}, \{0.24, 2\}, \{0.60, 1\}, \{1.20, 1\}, \{3.00, 1\}\}. \]

We describe in detail the process started by the second observable break. The second observable break occurred at the force \( u_b = 1.2 \), and \( u_f = \frac{1.2}{3} = 0.24 \). Immediately as this break occurred, the force \( u_f \) jumped to a higher value of force \( u_f = \frac{0.24}{3} = 0.30 \). However, since \( u_{f, \text{fibre}, 3} < 0.30 \), another break immediately occurred. This break generated an interval censored observation because we could only observe that the random failure stress of the third weakest fibre was in the interval \((0.24, 0.30]\). As a consequence of this censored break, the force \( u_f \) immediately jumped to an even higher value of force \( u_f = \frac{0.30}{3} = 0.40 \). After this the applied force on the bundle, i.e. \( u_b \), continued to grow until the third observable break occurred. This concludes the example.

![Figure 9](image.png)

Figure 9. The forces \( u_b \) and \( u_f \) during the test of one bundle. \( u_b \) is the force applied to the bundle, and \( u_f \) is the force applied to an unbroken fibre. For this test, we observed 5 observable breaks, i.e. 5 uncensored observations of the failure stresses of fibres.
In one experiment, we independently test \( n \) ELS bundles, each constructed of \( m \) statistically similar fibres of length \( l_{\text{fibre}} \). The data generated from this experiment is denoted by

\[
x_{\text{ELS}} (n) := \{x_1, \ldots, x_n\}, \tag{4.8}
\]

where \( x_j \) denotes the data obtained from the \( j \)th tested ELS bundle, where \( x_j \) is defined similar to relation (4.6).

The tests of ELS bundles where introduced in Paper (A). In Section 6, we show how the data \( x_{\text{ELS}} (n) \) can be used to obtain parametric and non-parametric estimators of the failure stresses of fibres and bundles.

### 4.3 Tests of LLS bundles

The method used for testing LLS bundles is much more complicated than the method for testing ELS bundles. Therefore, we do not describe the complete testing of LLS bundles here. Instead we refer to Paper (C) where this testing is described in detail.

We recall that an observable break occurs when a fibre, or piece of a fibre, breaks in a tested bundle, and we can observe the failure stress of this fibre (or piece of a fibre). An observable break can generate several censored breaks. In the testing of ELS bundles, we observe for each observable break the failure stress of the weakest fibre (we do not need to know which fibre this is), and the number of simultaneously broken fibres at this observable break. In the testing of LLS bundles we also need, for each break (observable or censored), to observe which fibre broke and at which position this break occurred. We introduce some notation. Let \( x \) denote the data generated from one tested LLS bundle with no observable breaks, where

\[
x := \left\{ \{u_{b,(1)}, u_{f,(1)}, \{f_{(1),1}, z_{(1),1}\}\}, \ldots, \{f_{(1),b_1}, z_{(1),b_1}\}\right\}, \ldots, \tag{4.9}
\]

and where \( u_{f,(k)} \) is the failure stress that caused the \( k \)th observable break of a piece of a fibre. \( u_{b,(k)} \) is the force applied to the bundle at the time of this break. \( f_{(k),l} \) denotes the number of the \( l \)th broken fibre caused by the applied force \( u_{b,(k)} \), and \( z_{(k),l} \) denotes the position at which this break occurred. In one experiment we independently test \( n \) LLS bundles, each constructed of \( m \) statistically similar fibres of length \( l_{\text{fibre}} \). The data generated from this experiment is denoted by

\[
x_{\text{LLS}} (n) := \{x_1, \ldots, x_n\}, \tag{4.10}
\]
where \( x_j \) denotes the data obtained from the \( j \)th tested LLS bundle, and where \( x_j \) is defined similar to relation (4.9).

We give a small example to clarify. Consider a bundle made of two fibres of lengths \( l_{\text{fibre}} = 12 \), for which the damage parameter \( \delta = 1.5 \). A force increasing with time was applied to the bundle until the bundle broke. During this test, two observable breaks occurred, the first at time \( t_1 \) and the second at time \( t_2 \). Figure 10 shows the status of the bundle during this test. The following data were registered:

\[
\begin{align*}
x &= \{ u_{b,(1)}, u_{f,(1)}, f_{(1),1} = 1, z_{(1),1} = 9 \}, \\
&\quad \{ u_{b,(2)}, u_{f,(2)}, f_{(2),1} = 2, z_{(2),1} = 5, f_{(2),2} = 1, z_{(2),2} = 6 \}.
\end{align*}
\]

Note that at the second observable break, at the applied force \( u_{b,(2)} \), we also had one censored break.

The tests of LLS bundles were introduced in Paper (C). In Section 6, we show how the data \( x_{ELS}(n) \) can be used to obtain non-parametric estimators of the failure stresses of fibres and bundles.

![Figure 10](image_url)

Figure 10. The tested bundle at three breaks, one observable break at time \( t_1 \), one observable break at time \( t_2 \), and one censored break at time \( t_2 \). Note that the two last breaks occurred simultaneously. The dashed lines show the damaged parts of the tested bundle.
5 Probabilistic methods and simulation techniques

We consider a bundle constructed of \( m \) statistically similar fibres of length \( l_{\text{fibre}} \). If we know the load-sharing model of the bundle and the c.d.f. of its fibres, i.e. \( F_{f_{\text{fibre}}} (\cdot) \), then we can use the available information of the single fibres to evaluate the c.d.f. of the failure stress of the bundle, i.e. \( F_{b}(\cdot) \).

5.1 Methods for ELS bundles

We consider ELS bundles where the failure stresses of individually tested fibres in the bundle are i.i.d. r.v.'s with a known continuous c.d.f. \( F_{f_{\text{fibre}}} (\cdot) \). In this section, we present three methods for obtaining the c.d.f. \( F_{b}(\cdot) \) of an ELS bundle, when the c.d.f. of its fibres, i.e. \( F_{f_{\text{fibre}}} (\cdot) \), is known.

5.1.1 The Recursive method

There is a recursive relation between the c.d.f. of a bundle, i.e. \( F_{b}(u) \), and the c.d.f. of its fibres, i.e. \( F_{f_{\text{fibre}}} (\cdot) \). Let \( F_{b}(\cdot, k) \) denote the c.d.f. of a bundle containing \( k \) fibres, and define \( F_{b}(\cdot, 0) := 1 \). Then the c.d.f. \( F_{b}(\cdot) \) of a bundle, with \( m \) fibres, can be obtained by the following recursive formula:

\[
F_{b}(u) := \sum_{q=1}^{m} (-1)^{q+1} \binom{m}{q} F_{f_{\text{fibre}}}^{q} \left( \frac{u}{m} \right) F_{b}(u, m - q), \tag{5.11}
\]

suggested by Suh, Bhattacharyya and Grandage (1970). This recursive formula becomes numerically unstable when \( m > 40 \), but there exists a numerical improved method that can handle larger \( m \), suggested by McCartney and Smith (1983).

5.1.2 The Simple simulation method

The failure stress of an ELS bundle \( U_{b} \) is defined as

\[
U_{b} := \max \left\{ mU_{f,(1)}, (m - 1)U_{f,(2)}, \ldots, U_{f,(m)} \right\}, \tag{5.12}
\]

where \( U_{f,(1)} \leq U_{f,(2)} \leq \ldots \leq U_{f,(m)} \) are the random ordered failure stresses of the individually tested fibres of the bundle, see for example Crowder et al. (1991) Since the c.d.f. \( F_{f_{\text{fibre}}} (\cdot) \) is known, we can use relation (5.12) to simulate a large number of failure stresses of bundles, and the empirical distribution function (e.d.f.) obtained from these simulations is an approximation of the c.d.f. \( F_{b}(\cdot) \).
5.1.3 Asymptotic approximations

If the number of fibres is large, then it is possible to approximate the c.d.f. \( F_b(\cdot) \) with a normal distribution. Let \( \tilde{F}_{f, fibre, m}(\cdot) \) denote the e.d.f. obtained from the \( m \) failure stresses of the individually tested fibres of the bundle, i.e.

\[
\tilde{F}_{f, fibre, m}(u_f) = \begin{cases} 
0 & \text{if } u_f < U_{f,(1)} \\
\frac{i}{m} & \text{if } U_{f,(i)} \leq u_f < U_{f,(i+1)} \\
1 & \text{if } u_f \geq U_{f,(m)}
\end{cases}
\]

It follows that \( m \tilde{F}_{f, fibre, m}(u_f) \) is the number of failed fibres when the load applied to the unbroken fibres in the bundle is \( u_f \). The load supported by the bundle at this time is \( \tilde{U}_b = u_f m \left( 1 - \tilde{F}_{f, fibre, m}(u_f) \right) \). It follows that the bundle will break at

\[
U_b = \max_{u_f > 0} \left( u_f \left( 1 - \tilde{F}_{f, fibre, m}(u_f) \right) \right). \tag{5.13}
\]

For any fixed value \( u_f \) we have

\[
\lim_{m \to \infty} u_f \left( 1 - \tilde{F}_{f, fibre, m}(u_f) \right) = u_f \left( 1 - F_{f, fibre}(u_f) \right),
\]

and

\[
m \tilde{F}_{f, fibre, m}(u_f) \sim Bin \left( m, F_{f, fibre}(u_f) \right),
\]

where \( Bin \left( m, p \right) \) denotes the binomial distribution with mean \( mp \) and variance \( mp(1-p) \). We assume that the function \( u_f \left( 1 - F_{f, fibre}(u_f) \right) \) has a unique maximum at \( u_f = u_f^* \), and that the second derivative of this function is positive at the point \( u_f^* \). Furthermore, we assume that the maximum of \( u_f \left( 1 - \tilde{F}_{f, fibre, m}(u_f) \right) \) is achieved at \( u_f^* \). It follows, by the Central limit theorem, that for a bundle with a large number of fibres, the c.d.f. of the failure stress of the bundle, i.e. \( F_b(\cdot) \), can be approximated by a normal distribution \( N(\mu, \sigma^2) \) with mean value

\[
\mu = m u_f^* \left( 1 - F_{f, fibre}(u_f^*) \right), \tag{5.14}
\]

and variance

\[
\sigma^2 = m \left( u_f^* \right)^2 F_{f, fibre}(u_f^*) \left( 1 - F_{f, fibre}(u_f^*) \right), \tag{5.15}
\]

see Daniels (1945) and Phoenix and Taylor (1973). There exist several other improved normal approximations of the c.d.f. \( F_b(\cdot) \), see for example Barbour (1981), Daniels (1989), and Smith (1982).
5.2 A method for LLS bundles

The idea behind this method is to imitate the true testing of a bundle. We consider an LLS bundle with \( m \) fibres of lengths \( l_{\text{fibre}} \), for which assumptions (a.1) – (a.4) hold, see Section 2. For any piece of length \( l_p \), \( l_p \leq l_{\text{fibre}} \), it follows that the c.h.f. of the piece, denoted by \( H_{f,l_p}(\cdot) \), is defined by

\[
H_{f,l_p}(u) := \frac{l_p}{l_{\text{fibre}}} \ln \left( 1 - F_{f,l_p}(u) \right),
\]

where \( F_{f,l_{\text{fibre}}}(\cdot) \) is the known c.d.f. for a fibre of length \( l_{\text{fibre}} \), \( u > 0 \). Furthermore, the c.d.f. of a piece of length \( l_p \), denoted \( F_{f,l_p}(\cdot) \), is defined by \( F_{f,l_p}(u) := 1 - e^{-H_{f,l_p}(u)} \). By assumption (a.2), we know that the position of a break in a piece is uniformly distributed over the length of the piece. Therefore, if we know the c.d.f. \( F_{f,l_{\text{fibre}}}(\cdot) \), we can simulate the failure stress and the position of the break for any piece with arbitrary length. This property can be used to simulate the testing of LLS bundles. As a result of one simulation, we obtain the simulated failure stress of one bundle. The empirical distribution function obtained from a large number of simulated failure stresses of bundles is an approximation of the desired c.d.f. \( F_b(\cdot) \).

In Paper (C) we introduce a method that imitates the true testing of bundles. This method can be defined by an algorithm that can be easily implemented on a computer.

6 Statistical methods

In the previous section we saw that in the case when the c.d.f. of the failure stresses of the single fibres, i.e. \( F_{f,l_{\text{fibre}}}(\cdot) \), in a bundle is known, there exist methods to determine the c.d.f. of the failure stress of the bundle, i.e. \( F_b(\cdot) \). In general, the c.d.f. \( F_{f,l_{\text{fibre}}}(\cdot) \) is unknown, and so needs to be estimated from the observed data introduced in Section 4.

In the simplest case, we can test a number of fibres individually, and obtain an estimator of the c.d.f. \( F_{f,l_{\text{fibre}}}(\cdot) \). This estimator can be used in one of the methods suggested in Section 5 to obtain an estimator of the c.d.f. \( F_b(\cdot) \). This approach has been used by Perry (1998), for estimating the strengths of the suspension cables in the old Williamsburg bridge in New York, U.S.A. This estimation was done under the assumption that the cables were ELS systems. Another approach, the case we usually consider, is to test a series of bundles where we observe the failure stresses of the tested bundles, and where we also observe uncensored and censored observations.
of the failure stresses of fibres. In general, an estimator of the c.d.f. $F_b(\cdot)$ is obtained in two steps. First, we use the data to obtain an estimator of the c.d.f. $F_{f,\text{fibre}}(\cdot)$, and then we use one of the methods suggested in Section 5 to obtain an estimator of the desired c.d.f. $F_b(\cdot)$.

Two techniques are used to estimate the c.d.f. $F_{f,\text{fibre}}(\cdot)$ from the observed data. One parametric approach where we use the Maximum likelihood (ML) estimator, and a non-parametric approach where we use the Nelson-Aalen estimator. These methods are presented in Sections 6.1.1 – 6.1.2. The accuracy of the estimators are assessed by using resampling techniques similar to the technique described below in Section 6.1.3. The results related to the Binary tree structured test are presented in Section 6.2. These results where obtained in Paper (B). Statistical results for ELS bundles, both parametric and non-parametric, are presented in Sections 6.3 — 6.4. These results where obtained in Paper (A). In Section 6.5, we present statistical results for LLS bundles. These results were obtained in Paper (C).

6.1 General methods

Life data, e.g. failure stresses of fibres, frequently contain ‘incomplete’ observations. This commonly occurs when the exact lifetime data of a unit is not observed, but it is known to exceed a certain force, say $x^-$. Such an observation is referred to as right censored. Suppose that the exact lifetime data of a unit is not observed, but it is known to be between two values, say $x^-$ and $x^+$. Such an observation is referred to as interval censored.

We test $n$ systems independently, each with $m$ statistically similar units. The c.d.f. and the c.h.f. of the random failure stress of a unit is denoted by $F_X(\cdot)$ and $H_X(\cdot)$, respectively. The $j$th tested system generates $k_j$ uncensored observations of failure stresses of units, denoted by $x_{j}^u = \{x_{j,1}, \ldots, x_{j,k_j}\}$, and $m-k_j$ right censored observations of failure stresses of units, denoted by $x_{j}^c = \{x_{j,k_j+1}, \ldots, x_{j,m}\}$, $j = 1, \ldots, n$. Let $x(n)$ denote the data obtained from the $n$ tested systems. In the following three sections we define the ML estimator, the Nelson-Aalen estimator, and a resampling method for the data $x(n)$.

6.1.1 The Maximum likelihood estimator

The ML method is probably the most popular general method of estimation, and it can be traced back to Lambert (1760) and Bernoulli (1777). Later, it was reproposed and further developed by Fisher (1912, 1922).
Suppose that the c.d.f. $F_X(\cdot)$ belongs to a family of distributions characterized by a set of parameters $\theta = \{\theta_1, ..., \theta_p\}$, for example the family of Weibull distributions. Let $F_X(\cdot | \theta)$ denote the c.d.f. of the r.v. $X$, and let $f_X(\cdot | \theta)$ denote the density function of this r.v. The likelihood function, denoted by $L(\theta | x(n))$, is defined as

$$L(\theta | x(n)) := \prod_{j=1}^{k_n} \left( \prod_{i=1}^{k_j} f_X(x_{j,i}^- | \theta) \prod_{i=k_j+1}^{n} \left( 1 - F_X(x_{j,i}^- | \theta) \right) \right),$$

where $\theta$ belongs to some parameter space $\Theta$. The idea behind the ML estimator is to choose the set of parameters $\hat{\theta}_1, ..., \hat{\theta}_p$ for which the data $x(n)$ is most likely to be observed. The ML estimators of the parameters $\theta_1, ..., \theta_p$, denoted by $\hat{\theta}_1, ..., \hat{\theta}_p$, are those values that maximize the likelihood function $L(\theta | x(n))$. In Paper (A) an ML estimator is obtained for interval censored data.

### 6.1.2 The Nelson-Aalen estimator


We introduce two r.v.'s related to the $j$th tested system. These r.v.'s are obtained from the random data $x(n)$. Let $N_j(x)$ denote the number of uncensored observations in the $j$th tested system at the force $x$, and let $Y_j(x)$ denote the number of units at risk of generating an uncensored observation, in the $j$th tested system, just before the force $x$, $j = 1, ..., n$, $x > 0$. Let $N_n(x)$ denote the total number of uncensored observations at $x$, and let $Y_n(x)$ denote the total number at risk just before $x$, i.e.

$$N_n(x) := \sum_{j=1}^{n} N_j(x), \quad Y_n(x) := \sum_{j=1}^{n} Y_j(x), \quad x > 0.$$  

The Nelson-Aalen estimator is defined as

$$\hat{H}_{X,n}(x) := \int_0^x \frac{dN_n(u)}{Y_n(u)}, \quad x > 0. \quad (6.16)$$

Note that, $\int_{\frac{Y_n(u)>0}{Y_n(u)}}$ = 0 whenever $Y_n(x) = 0$, $x > 0$. If the process $N_n := \{N_n(x) : x > 0\}$ is a counting process with respect to some filtration
and if \( Y_n := \{Y_n(x) : x > 0\} \) is a predictable process with respect to the same filtration, then the Nelson-Aalen estimator \( \hat{H}_{X,n}(\cdot) \) is a reasonable estimator of the c.h.f. \( H_X(\cdot) \), see e.g. Andersen et al. (1993). The Nelson-Aalen estimator is used in Paper (A), Paper (B), and Paper (C).

### 6.1.3 Resampling techniques

In 1958 Tukey introduced the jackknife method to estimate biases and variances of various statistics of i.i.d. r.v.'s. The bootstrap was introduced by Efron (1979) as a method that could potentially be applied to problems of statistical error assessment beyond biases and variances. In recent years, the bootstrap and other resampling methods have been increasingly popular, and a number of books and lecture notes have been published, e.g. Efron and Tibshirani (1979), Hall (1992), Belyaev (1995), and Davison and Hinkley (1997).

We define a resampling method used for estimating the accuracy of the Nelson-Aalen estimator \( \hat{H}_{X,n}(\cdot) \), defined by (6.16). Let \( J^*(n) = \{J_1^*, ..., J_n^*\} \) be \( n \) i.i.d. r.v.'s with the probability function \( P(J_j^* = h) = \frac{1}{n}, h = 1, ..., n \). Let \( D_h^* \) be the number of times the \( h \)th member is chosen, i.e.

\[
D_h^* := \sum_{j=1}^{n} I(J_j^* = h), \ h = 1, ..., n. \tag{6.17}
\]

The total number of uncensored observations and the total number of units at risk for the resampled data are defined as

\[
N_n^{(*)}(x) := \sum_{j=1}^{n} D_j^*N_j(x), \ Y_n^{(*)}(x) := \sum_{j=1}^{n} D_j^*Y_j(x), \tag{6.18}
\]

where the processes \( N_j(\cdot) \) and \( Y_j(\cdot) \) were introduced in the previous section, \( x > 0, j = 1, ..., n \). The resampling copy of the estimator of \( \hat{H}_{X,n}(\cdot) \) is defined as

\[
\hat{H}_{X,n}^{(*)}(x) := \int_0^x \frac{dN_n^{(*)}(u)}{Y_n^{(*)}(u)}, \ x > 0. \tag{6.19}
\]

The idea behind resampling is that the conditional distribution law (d.l.)

\[
\mathcal{L}\left(\sqrt{n}\left(\hat{H}_{X,n}^{(*)}(\cdot) - \hat{H}_{X,n}(\cdot) \mid x(n)\right)\right)
\]

can be estimated by a sufficiently large number of simulated resampling copies, and can be used to estimate the
desired d.l. $\mathcal{L}\left(\sqrt{n}\left(\hat{H}_{X,n} (\cdot) - H_X (\cdot)\right)\right)$, where these d.l.'s are defined on a metric space, e.g. the Skorohod space. This resampling method, and similar methods, are used in Paper (A), Paper (B), and Paper (C).

6.2 Estimation related to the Binary tree structured test

In this section, we show how a Nelson-Aalen type estimator of the c.h.f. $H_{f,fibre} (\cdot)$ can be derived from the data $x_B (m)$, defined by (4.5). We recall that the data $x_B (m)$ were obtained from $m$ independently Binary tree structured tested fibres. The results in this section were obtained in Paper (B), where they are treated in more detail. We show how the random data $x_B (m)$ can be used to define the two random processes of interest, denoted by

$$N_m := \{N_m (u) : u \geq 0\}, \quad Y_m := \{Y_m (u) : u \geq 0\},$$

where $N_m (u)$ is the total number of observable breaks at force $u$, and where $Y_m (u)$ is the total length at risk of breaking just before force $u$, $u \geq 0$. We obtain a Nelson-Aalen type estimator of the c.h.f. $H_{f,fibre} (\cdot)$, defined as

$$\hat{H}_{f,fibre}^B (u) := \int_0^u \frac{dN_m (v)}{Y_m (v)}, \quad u \in (0, u_S),$$

where $u_S$ is an arbitrary finite value. It is proved that this estimator is a strongly consistent estimator of the c.h.f. $H_{f,fibre} (\cdot)$. It is also proved that resampling can be used to asymptotically consistently estimate d.l.'s of continuous (in Skorohod space) functionals of the random deviation between the estimator and the true c.h.f. Numerical examples suggest that resampling works well for a moderate number of tested fibres.

6.3 Non-parametric estimation of ELS bundles

In this section, we show how a Nelson-Aalen type estimator of the c.h.f. $H_{f,fibre} (\cdot)$ can be derived from the data $x_{ELS} (n)$, defined by (4.8). We recall that the data $x_{ELS} (n)$ were obtained from $n$ independently tested ELS bundles constructed from $m$ fibres. The results in this section were obtained in Paper (A), where they are treated in more detail. We show how the random data $x_{ELS} (m)$ can be used to define the two random processes of interest, denoted by

$$N_m := \{N_m (u_f) : u_f \geq 0\}, \quad Y_m := \{Y_m (u_f) : u_f \geq 0\},$$
where \( N_m(u_f) \) is the total number of observable breaks at force \( u_f \), and where \( Y_m(u_f) \) is the total length at risk of breaking and generating an observable break just before force \( u_f, u_f \geq 0 \). \( j = 1,...,n \), \( i = 1,...,n_0j \).

Note that \( u_f \) is the force applied to unbroken fibres. We obtain a Nelson-Aalen type estimator of the c.h.f. \( H_{f,fibre}(\cdot) \), defined as

\[
\hat{H}_{f,fibre,n}^{ELS}(u_f) := \int_0^u \frac{dN_n(u)}{Y_n(u)}, \quad u_f \in (0,u_S),
\]

where \( u_S \) is an arbitrary finite value. It is proved that this estimator is a strongly consistent estimator of the c.h.f. \( H_{f,fibre}(\cdot) \). We show that resampling can be used to obtain asymptotically correct estimators of the d.l.

\[
L\left( \sqrt{n}\left( \hat{H}_{f,fibre,n}^{ELS}(u_f) - H_{f,fibre}(u_f) \right) \right), \quad u_f \in (0,u_S).
\]

By using the method suggested in Section 5.1.2, we obtain an estimator of the desired c.d.f. \( F_f(\cdot) \), i.e. the c.d.f. of failure stresses of bundles. Several numerical examples illustrate the behavior of the obtained estimators. These examples suggest that the obtained estimators usually perform well for moderate numbers of tested bundles.

### 6.4 Parametric estimation of ELS bundles

In this section, we show how an ML estimator of the c.d.f. \( F_{f,fibre}(\cdot) \) can be derived from the data \( x_{ELS}(n) \), defined by (4.8). We recall that the data \( x_{ELS}(n) \) were obtained from \( n \) independently tested ELS bundles constructed from \( m \) fibres. The results in this section were obtained in Paper (A), where they are treated in more detail.

We show that the data \( x_{ELS}(n) \) can be reorganized in such a way that it contains uncensored and interval censored observations of failure stresses of fibres. In this section, we assume that the c.d.f. \( F_{f,fibre}(\cdot) \) belongs to a known family of distributions denoted by \( F_{f,fibre}(\cdot|\theta) \), where \( \theta \) is a set of unknown parameters. The likelihood function, denoted by \( L(\theta|x_{ELS}(n)) \), is defined for the reorganized data \( x_{ELS}(n) \) and the c.d.f. \( F_X(\cdot|\theta) \). In the case where the failure stresses of fibres are described by the 2-parameter Weibull distribution, we obtain strongly consistent ML estimators of the parameters \( \theta \). By using the method suggested in Section 5.1.2, we obtain an estimator of the desired c.d.f. of failure stresses of bundles. Several numerical examples illustrate the behavior of the obtained estimators. These examples suggest that the obtained estimators usually perform well for moderate numbers of tested bundles.
6.5 Non-parametric estimation of LLS bundles

In this section, we show how a Nelson-Aalen type estimator of the c.h.f. \( H_{f,fibre}(\cdot) \) can be derived from the data \( x_{LLS}(n) \), defined by (4.10). We recall that the data \( x_{LLS}(n) \) were obtained from \( n \) independently tested LLS bundles constructed from \( m \) fibres. The results in this section were obtained in Paper (C), where they are treated in more detail. We show how the random data \( x_{ELS}(m) \) can be used to define the two random processes of interest, denoted by

\[
N_n := \{N_n(u_f) : u_f \geq 0\}, \quad Y'_n := \{Y'_n(u_f) : u_f \geq 0\},
\]

where \( N_n(u_f) \) is the total number of observable breaks at force \( u_f \), and \( Y'_n(u_f) \) is an approximation of the total length at risk of breaking and generating an observable break just before force \( u_f, u_f \geq 0 \). In order to obtain the process \( Y'_n \) from the observed data \( x_{LLS}(n) \), we treat the force variable \( u_f \) and the “position” variable \( z \) as if they were discrete variables, see Paper (C), Section 7.2. As a result, we obtain an algorithm for calculating the process \( Y'_n \) from the observed data \( x_{LLS}(n) \). The processes \( N_n \) and \( Y'_n \) are used to obtain a Nelson-Aalen type estimator of the c.h.f. \( H_{f,fibre}(\cdot) \), defined as

\[
\hat{H}_{f,fibre,n}(u_s) := \int_0^{u_s} \frac{dN_n(u)}{Y'_n(u)}, u \in (0,u_S),
\]

where \( u_S \) is an arbitrary finite value. We present theoretical arguments which suggest that this estimator is a reasonable estimator of the c.h.f. \( H_{f,fibre}(\cdot) \). By using the method suggested in Section 5.2 we obtain an estimator of the desired c.d.f. of failure stresses of bundles. Several numerical examples illustrate the behavior of the obtained estimators. These examples suggest that the obtained estimators usually perform well for moderate numbers of tested bundles.

7 Further research

Several questions and potential areas have come up during the work of this thesis. Some of the questions have been answered, but many are still unanswered. In this section, we mention a few of the most interesting areas for further research.
All of the results in this thesis are obtained under the assumption that the fibres have no elastic behavior. In general, this is not true. Therefore, it would be interesting to develop methods that allow the fibres to be elastic.

In Paper (C), we developed statistical methods for estimating the strengths of LLS bundles under the assumption that the damage parameter is known. It would be very interesting to develop methods for the case when the damage parameter is unknown. A simpler problem, but still interesting, would be to estimate the damage parameter in the case when the c.d.f. of the failure stresses of the fibres is known.

An area of further research would be to allow the fibres in the bundles to be of different sizes.

A very interesting extension of the theory is to develop statistical tests for testing different load-sharing assumptions. There has been some research in this direction, see Volf and Linka (2000).

One obvious generalization is to consider more general load-sharing models than ELS models and LLS models.

The most interesting extension of this research would be to apply the results of this thesis to real data.

The author would be grateful for comments from readers interested in analysis of life testing with failure stresses, especially those who have any additional information or references on this topic.

References


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