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P-HARMONIC FUNCTIONS NEAR THE BOUNDARY

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Contents

Abstract

We study boundary behaviour of solutions to the p-Laplacian

$$\nabla \cdot (|\nabla u|^{p-2}\nabla u) = 0,$$

in domains Ω and for $p \in (1,\infty]$. The case $p=\infty$ corresponds to the so called ∞-Laplacian $\sum_{i,j=1}^{n} u'_{x_i}u'_{x_j}u''_{x_ix_j} = 0$. More general equations of p-Laplace type will also be considered. Besides the five appended papers, we give an introduction to the p-Laplacian including a brief explanation of weak solutions, viscosity solutions and some well-known properties of the p-Laplacian and its solutions. We also discuss geometric assumptions such as δ-Reifenberg flat domains, NTA-domains, $C^{k,\alpha}$-domains and the relation between them. Moreover, we give a chapter about applications of the p-Laplacian including power-law fluids, Hele-Shaw flow, plastic moulding, image restoring, a characterization of solutions in a stochastic setting and also in the setting of a minimization problem. Papers I, II and III are about the p-Laplacian, while papers IV and V concern the more general equation

$$\nabla \cdot A(x,\nabla u) + B(x,\nabla u) = 0, \tag{$\star$}$$

of p-Laplace type and the corresponding limit equation when $p \to \infty$. In Paper I we prove, by constructing certain barrier functions, a boundary Harnack inequality for p-harmonic functions in $\mathbf{R}^n$ vanishing on a portion of an m-flat, that is, an m-dimensional plane, $0 \le m \le n-1$. In Paper II we prove, using barrier-type arguments, a growth condition for a p-harmonic measure in plane domains satisfying a generalized interior ball condition. In Paper III we solve, using a blow-up-type argument, a two-phase free boundary problem in the setting of δ-Reifenberg flat domains; that is, roughly speaking, $\partial\Omega$ approximates well by hyperplanes locally. In Paper IV we prove a boundary Harnack inequality for solutions to ($\star$) with $B \equiv 0$. By using a limiting argument, as $p \to \infty$, we also obtain the same result for solutions to equations of Aronsson type. Concerning Ω, we assume that $\Omega \subset \mathbf{R}^2$ and that $\partial\Omega$ is a quasicircle. In Paper V we prove a boundary Harnack inequality for solutions to ($\star$) as well as proving that the ratio u/v is Hölder continuous, where u and v are solutions which vanish on a portion of a δ-Reifenberg flat domain. We also solve the Martin problem in the analogue setting.

Sammanfattning

Vi studerar randbeteenden för lösningar till p-Laplaces ekvation

$$\nabla \cdot (|\nabla u|^{p-2} \nabla u) = 0,$$

i områden Ω och för $p \in (1, \infty]$. Fallet $p = \infty$ motsvaras av den så kallade ∞-Laplaces ekvation $\sum_{i,j=1}^{n} u'_{x_i} u'_{x_j} u''_{x_i x_j} = 0$. Vi studerar också mer generella ekvationer av p-Laplaces typ. Utöver de fem bifogade artiklarna ger vi en introduktion till svaga lösningar, viskositetslösningar och välkända egenskaper hos p-Laplaces ekvation och dess lösningar. Introduktionen tar även upp geometriska definitioner så som δ-Reifenberg-platthet, NTA-områden, $C^{k,\alpha}$-områden och diskuterar relationer mellan dessa. Vi presenterar också ett kapitel om tillämpningar för p-Laplaces ekvation innehållande icke-Newtonska vätskor, Hele-Shaw flöden, plastgjutning, bildanalys och karakterisering av lösningar i en stokastisk mening samt som lösningar till ett minimeringsproblem. Artiklarna I, II och III behandlar p-Laplaces ekvation, medan artiklarna IV och V behandlar den mer generella ekvationen

$$\nabla \cdot A(x, \nabla u) + B(x, \nabla u) = 0, \qquad (\star)$$

av p-Laplaces typ samt motsvarande gränsekvation då $p \to \infty$. I Artikel I visar vi, genom att konstruera barriärfunktioner, en så kallad ”boundary Harnack” olikhet för p-harmoniska funktioner i $\mathbf{R}^n$ som försvinner på ett m-dimensionellt plan, $0 \leq m \leq n-1$. I Artikel II visar vi, med hjälp av barriärargument, en grundläggande egenskap hos ett p-harmoniskt mått i tvådimensionella områden som uppfyller ett generaliserat inre bollvillkor. I Artikel III löser vi, med hjälp av ett sorts ”blow-up” argument, ett tvåfas frirandsproblem under antagandet att fria randen är δ-Reifenberg platt. Kort sagt betyder det att $\partial\Omega$ lokalt approximeras väl med hyperplan. I Artikel IV visar vi en ”boundary Harnack” olikhet för lösningar till $(\star)$ med $B \equiv 0$. Med hjälp av ett gränsvärdesargument då $p \to \infty$, visar vi också samma typ av resultat för lösningar till ekvationer av Aronsson typ. Här antar vi att $\Omega \subset \mathbf{R}^2$ samt att $\partial\Omega$ är en kvasicirkel. I Artikel V visar vi en ”boundary Harnack” olikhet för lösningar till $(\star)$, samt att kvoten u/v är Hölderkontinuerlig, där u och v är lösningar som försvinner på en del av ett δ-Reifenberg-platt område. Vi löser även Martin-problemet i ovanstående sättning.

Preface

After finished a Master of Science in engineering physics at Luleå university of technology in 2004, I continued as a Ph.D student in mechanical engineering at the same university. When completed a Licentiate degree in engineering in 2006, I decided to change direction to mathematics. In spring 2007 I began to study the p-Laplace equation with Kaj Nyström as a supervisor. During the following four years I have been involved in writing seven papers in the field, of which five are appended in this thesis. All papers are focused on basic research of boundary behaviour of p-harmonic functions. Beside the appended papers, my intention was to write the thesis in an "easy-to-read" way by including illustrations, examples and few technical details. I have also included a chapter about applications, and I hope this chapter will help to give the reader a good intuition of how solutions may behave, and also make people from other areas of research interested.

The four-year period 2007–2011 has been very challenging, interesting and fun and has given me the chance to improve my way of thinking when it comes to problem solving. I will now thank some persons who have inspired and helped me during this period and also during earlier parts of my life.

We start with my supervisor Kaj Nyström. I thank you for giving me the possibility to work with you during this period, for sharing your great knowledge and way of thinking, for always putting my learning and future possibilities first, for giving me a lot of suggestions of relevant problems to choose from and for never forcing me to do something noneducating. Your positive attitude and deep interest in pure mathematics have been inspiring.

Anton Grafström, my best friend at the department, thank you for all the time we have spent together during this period. Especially I have to mention our at least 231 lunches, and also that you made me feel welcome to the department in 2006.

When it comes to friends at work, thanks to Lina Schelin, Thomas Önskog and again Anton for all the interesting night-long discussions and hours of challenging strategy games we have had. Thanks to Benny Avelin for frequently trying to discuss mathematics with me, I have learned a lot from you and I appreciate that you introduced the p-Laplacian in the coffee room.

Thanks to Jonatan Vasilis for the time when we created Paper II. Besides your good ideas and knowledge in math, you were easy to work with. I think that our collaboration was probably the best I have had.

Now to my family. I will proceed in chronological order, so let's start with my parents. Thank you for playing, drawing, traveling, repairing and building toys, playing games and creating various things with me and my brother. Thanks for pointing out that school was important and helping us with our homework. When we did things such as building a dam in the forest to change the direction of the flow in a creek, walking around a whole summer to draw a map, building a ropeway that you had to climb over to enter the TV room and so on, you showed interest in what we did and gave us the ingredients we needed, instead of asking why and leading us into less interesting activities. You allowed us to build and also frequently use crossbows, cannons, catapults and other fun weapons. Hardly any unmotivated rules were put up. I think that was perfect, because the activities mentioned above are creative interesting projects and have helped me and my brother to produce a good way of thinking. I will always be thankful to you for the first 30 years of my life.

If my brother Joakim had not been interested in spending time with me and the activities mentioned above, my life would not have been so interesting, exciting and educating. What I mean is, for example, that to do the above-mentioned projects alone would have been much more difficult and definitely less fun. Probably most of them would not have been successfully completed or even started by me alone. Moreover, you did well in school which inspired me to do the same. Thank you for all the time you have spent with me.

My girlfriend Maria, thank you for being such a wonderful interesting person full of energy and love, and caring for me and others so much. You never become mad at me when I am unreachable because I am thinking of how to construct a proof. Instead, you appreciate that I am doing something that I love to do. You seem to read me like a book, because you do things that I really appreciate before I have understood that I would like it. You add a lot of energy to my life. I love you!

There are some more persons to be mentioned. At the department, thanks to Margareta Brinkstam, Ingrid Westerberg-Eriksson and Berith Melander at the administration for always being nice, caring and helpful. Thanks to Gunnar Aronsson for making me aware of his and others work concerning applications to the p-Laplacian. Thanks to Mikhail Surnachev for helping me with Paper V and for the days we spent in Lecco walking in the Alps. Thanks to my previous supervisors Jan-Olov Aidanpää and Gunnar Söderbacka for teaching and inspiring me to do research.

Thanks to Benny Avelin, Åke Brännström, Marie Frentz, Anton Grafström, Elin Götmark, Lisa Hed, Kaj Nyström, Lina Schelin and Thomas Önskog for reading early versions of this thesis and for giving comments which helped me to improve the final version.

List of papers

Paper I

Estimates for p-harmonic functions vanishing on a flat
Nonlinear Analysis: Theory, Methods & Applications,
volume 74, number 18, 2011, pages 6852–6860.

Paper II

Decay of a p-harmonic measure in the plane
(with Jonatan Vasilis)
Submitted for journal publication.

Paper III

On a two-phase free boundary condition for p-harmonic measures
(with Kaj Nyström)
Manuscripta Mathematica, volume 129, 2009, pages 231–249.

Paper IV

The boundary Harnack inequality for solutions to equations of Aronsson type in the plane (with Kaj Nyström)
Annales Academiæ Scientiarum Fennicæ Mathematica,
volume 36, 2011, pages 261–278.

Paper V

Boundary estimates for solutions to operators of p-Laplace type with lower order terms (with Benny Avelin and Kaj Nyström)
Journal of Differential Equations, volume 250, 2011, pages 264–291.

Papers I–V are reprinted in new editions with permission from each journal.

Notation

$\mathbf{R}^n$	n-dimensional Euclidean space.
$x = (x_1, \dots, x_n)$	a point $x \in \mathbf{R}^n$, also denoted by $x = (x', x_n)$.
Ω	a domain, that is, an open connected set in $\mathbf{R}^n$.
$\bar{E}$	closure of the set E.
∂E	boundary of the set E.
E^c	complement of the set E.
$d(x, E)$	Euclidean distance from $x \in \mathbf{R}^n$ to $E \subset \mathbf{R}^n$.
$h(E, F)$	Hausdorff distance between $E, F \subset \mathbf{R}^n$, defined in (1.7).
$\langle \cdot, \cdot \rangle$	standard inner product on $\mathbf{R}^n$.
$\lvert x \rvert$	Euclidean norm of x, that is, $\lvert x \rvert = \langle x, x \rangle^{1/2}$.
$B(x, r)$	open ball in $\mathbf{R}^n$ with center x and radius r.
$\Delta(x, r)$	the set $\partial\Omega \cap B(x, r)$.
dx	n-dimensional Lebesgue measure.
$C^k(E)$	space of all functions on E having continuous k-th derivative.
$C_0^k(E)$	$C^k(E)$ with compact support in E.
$C^{k,\alpha}(E)$	$C^k(E)$ with Hölder continuous k-th derivative of exponent α.
$W^{1,k}(E)$	first order Sobolev space on E.
$W_0^{1,k}(E)$	$W^{1,k}(E)$ with compact support in E.
∇u	weak gradient of u.
$\nabla \cdot$	divergence operator.
Δ_p	p-Laplace operator as defined in Section 1.1.
Δ	Laplace operator, that is, Δ_2.
ω_p	the p-harmonic measure in Definition 1.10.
μ	the p-harmonic measure in Definition 1.11.
$a_r(w)$	the corkscrew point introduced in Definition 1.13.
M	the NTA constant introduced in Definition 1.13.
$c(a_1, a_2 \dots a_n)$	a constant ≥ 1 depending only on $a_1, a_2, \dots a_n$.
$u \approx v$	there exists c such that $c^{-1}u \leq v \leq cu$.

Chapter 1

Introduction

1.1 The p-Laplace equation

In this thesis we study the p-Laplace equation for $p \in (1, \infty]$. If $p \in (1, \infty)$, then the equation is on divergence form and yields

$$\Delta_p u := \nabla \cdot (|\nabla u|^{p-2} \nabla u) = 0.$$

When $p = \infty$, the equation is no longer on divergence form. In this case the p-Laplace equation may be written as

$$\Delta_\infty u := \sum_{i,j=1}^{n} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \frac{\partial^2 u}{\partial x_i \partial x_j} = 0,$$

which is the so called ∞-Laplacian. In the rest of this thesis, we assume $p \in (1, \infty]$ if nothing else is mentioned, and we usually write the *p-Laplacian* in place of the p-Laplace equation. The p-Laplacian is an elliptic partial differential equation, which is degenerate if $p > 2$ and singular if $p < 2$. If $p = 2$, then the p-Laplacian reduces to the simpler classical linear Laplace equation

$$\Delta u := \nabla \cdot \nabla u = 0.$$

The connection between the p-Laplacian, $p \in (1, \infty)$, and the ∞-Laplacian rely on the calculation

$$\begin{aligned}\Delta_p u =& \nabla \cdot (|\nabla u|^{p-2} \nabla u) \\ =& |\nabla u|^{p-4} \left\{ |\nabla u|^2 \Delta u + (p-2) \sum_{i,j=1}^{n} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \frac{\partial^2 u}{\partial x_i \partial x_j} \right\} = 0.\end{aligned}$$

Let $p \to \infty$ and divide by $|\nabla u|^{p-4}$, then we obtain the ∞-Laplacian. In particular, it can be proved, see Jensen [J93], that under suitable conditions, a solution to the ∞-Laplacian is the limit of a sequence of solutions to the p-Laplacian as $p \to \infty$.

The p-Laplacian arises as a useful tool in for example fluid mechanics, minimization problems and image restoring. See Chapter 2 for applications and for a motivation for studying the p-Laplacian.

In $\mathbf{R}^n$, the simplest solution to the p-Laplacian is probably

$$u(x) = x_n.$$

Given any solution u to the p-Laplacian, then $a + b\,u$, where a, b are constants, is also a solution. Now, let $p \in (1, \infty]$ and consider the following function in $\mathbf{R}^n$,

$$\hat{f}(x) = \begin{cases} -\log|x| & \text{if } p = n \\ |x| & \text{if } p = \infty \\ |x|^{(p-n)/(p-1)} & \text{otherwise.} \end{cases} \tag{1.1}$$

The function $\hat{f}$ solves the p-Laplacian in $\mathbf{R}^n \setminus \{0\}$ and is usually referred to as a *fundamental solution.* Several useful facts can be derived using the simple solutions mentioned above. For example, when studying solutions to the p-Laplacian vanishing on a hyperplane, or vanishing on a boundary which is well approximable by hyperplanes, $u(x) = x_n$ can be used as a barrier to bound the more general solution, see [Paper III, Lemma 2.7 and Lemma 2.8]. The barrier functions in [Paper I, Lemma 3.1] are built from versions of $\hat{f}$. Moreover, proving results concerning capacity, implying boundary convergence of solutions to the p-Laplacian, is another example, see the book by Heinonen, Kilpeläinen and Martio [HKM93, Chapter 2].

Figure 1.1 shows the solution $a + b\hat{f}(x)$ for $n = 2$ and some different values of p. Compare to the previously mentioned fact that a solution to the ∞-Laplacian is the limit of a sequence of solutions to the p-Laplacian as $p \to \infty$. Observe also the difference between the cases $1 < p \le n$ and $n < p \le \infty$.

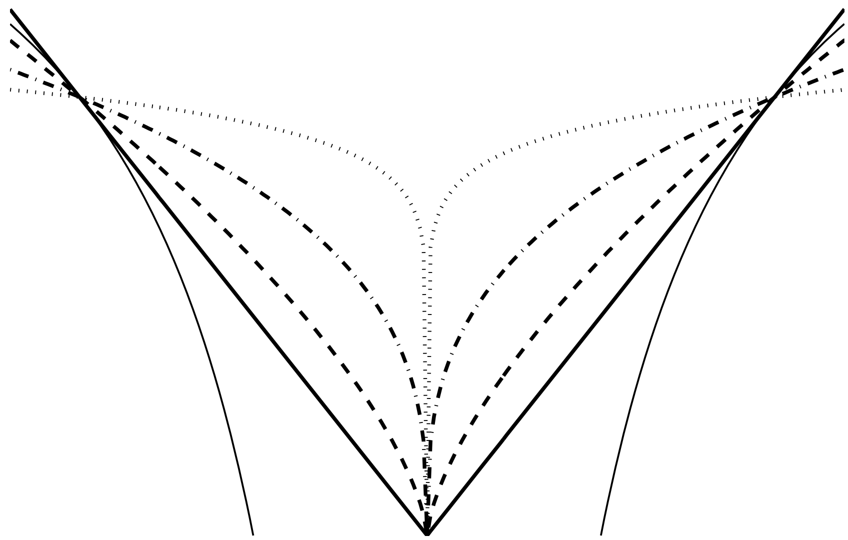

Figure 1.1: The function $a + b\hat{f}(x)$ for suitable a and b. Here, $n = 2$ and $p = 1.5$ (thin solid) approaches $-\infty$ as $|x| \to 0$, $p = 2.1$ (dotted), $p = 2.5$ (dash-dot) , $p = 4$ (dashed) and $p = \infty$ (thick solid) is the cone function.

We end this section by noting that the solutions presented here are so called *classical solutions*. That is, solutions which can be verified by differentiating. Definitions of "weaker" solutions will be recalled in Section 1.3.

1.2 Equations of p-Laplace type

In Papers I, II and III we study the p-Laplacian, while in Papers IV and V, we consider the more general equation

$$\nabla \cdot A(x, \nabla u) + B(x, \nabla u) = 0, \tag{1.2}$$

where $A(x, \eta)$ and $B(x, \eta)$ satisfy certain conditions of p-Laplace type. Many results in the papers [LLuN08] and Paper V by Avelin, Lewis, me and Nyström concerning (1.2) are proved by using corresponding results for the

p-Laplacian and some approximation arguments. In paper IV, we study (1.2) with $B \equiv 0$ and the corresponding generalized ∞-Laplacian, which may be written,

$$\big\langle \nabla F(x, \nabla u), \nabla_\eta F(x, \nabla u) \big\rangle = 0,$$

where $F(x, \eta)$ satisfies certain conditions. This type of equation was introduced by Aronsson, see [A65], [A66], [A67] and [A68], and is usually referred to as an equation of Aronsson type. As in the case of the ∞-Laplacian, a solution to the above Aronsson type equation is the limit of a sequence of solutions to the p-Laplace type equation (1.2) as $p \to \infty$. A short description of the relation between the above equations and their solutions is given in Section 3.4.

For applications and for a motivation for studying the above p-Laplace type equations, we refer the reader to Chapter 2.

In Chapter 1 and Chapter 2, we usually consider the p-Laplacian instead of the more general type equation to avoid a lengthy notation. However, most of the definitions and basic results presented in the Introduction also hold for (1.2) and its limit equation, and we give some references to this more general case along the way.

1.3 Notion of solutions

In this section, we present standard definitions of solutions to the p-Laplacian. We start by subsolutions and supersolutions. A *classical subsolution (supersolution)* u in a domain Ω is a function satisfying

$$\Delta_p u \geq 0 \quad (\Delta_p u \leq 0) \qquad \text{in} \quad \Omega.$$

Roughly speaking, this implies larger (smaller) second derivative than a solution, and hence the subsolution (supersolution) will be below (above) a solution with the same boundary values. In particular, see Lemma 1.3 below; hence, subsolutions (supersolutions) are useful when constructing barriers, see for example [Paper I, Lemma 3.1].

It is usually to narrow to consider only classical solutions. To be able to prove existence in a general setting, we need to use weaker definitions of solutions, and therefore we continue by recalling *weak solutions* to the p-Laplacian, $1 < p < \infty$, also known as *variational solutions* due to Theorem 2.1 in Chapter 2.

Definition 1.1. *If $p \in (1, \infty)$, then we say that u is a weak subsolution (supersolution) to the p-Laplacian in Ω provided $u \in W^{1,p}_{loc}(\Omega)$ and*

$$\int_\Omega |\nabla u|^{p-2} \langle \nabla u, \nabla \theta \rangle \, dx \leq \ (\geq) \, 0 \tag{1.3}$$

whenever $\theta \in C_0^\infty(\Omega)$ is non-negative. A function u is a weak solution to the p-Laplacian if it is both a weak subsolution and a weak supersolution.

It is often desirable to deal with a larger class of test functions in (1.3). If we assume in addition that $u \in W^{1,p}(\Omega)$, then we may use test functions from the larger space $W_0^{1,p}(\Omega)$, see [HKM93, Lemma 3.11].

If $p = \infty$, then the equation is no longer on divergence form and the above definition, which rely on integration by parts, is no longer suitable. Instead, the concept of viscosity solutions is standard.

Definition 1.2. *An upper semicontinuous function u is a viscosity subsolution of the ∞-Laplacian in Ω provided for each function $\psi \in C^2(\Omega)$, such that $u - \psi$ has a local maximum at a point $x_0 \in \Omega$, implies*

$$\Delta_\infty \psi(x_0) \geq 0.$$

A lower semicontinuous function u is a viscosity supersolution of the ∞-Laplacian in Ω provided for each function $\psi \in C^2(\Omega)$, such that $u - \psi$ has a local minimum at a point $x_0 \in \Omega$, implies

$$\Delta_\infty \psi(x_0) \leq 0.$$

A function u is a viscosity solution of the ∞-Laplacian if it is both a viscosity subsolution and a viscosity supersolution.

Figure 1.3 gives a sketch for the definition of viscosity solutions. Subsolutions are tested from above by testfunctions, while supersolutions are tested from below. We remark that the definition of viscosity solutions also applies to the p-Laplacian, $1 < p < \infty$; just replace Δ_∞ with Δ_p. Moreover, continuous weak solutions are also viscosity solutions, as proved by Juutinen in [Ju98, Theorem 1.29].

We next introduce the notation of p-harmonic functions. If u is an upper (lower) semicontinuous weak subsolution (supersolution) to the p-Laplacian in Ω, then we say that u is *p-subharmonic (p-superharmonic)* in Ω. If u

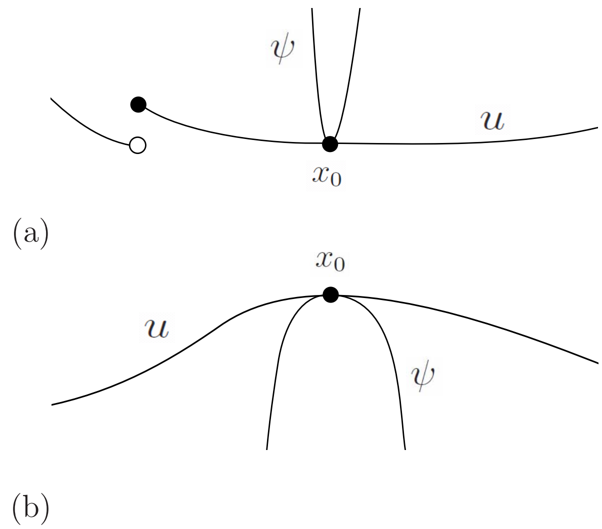

Figure 1.2: (a) viscosity subsolution, (b) viscosity supersolution.

is a continuous weak solution to the p-Laplacian in Ω, then u is called *p-harmonic* in Ω. In the case of viscosity solutions, note that solutions are by definition continuous, and hence ∞-sub-, ∞-super- or ∞-harmonic. We mention that sometimes, for example in Papers III–V, the continuity assumption is removed and authors denote weak solutions by p-harmonic functions.

For more on weak solutions, viscosity solutions, p-harmonicity and p-superharmonicity, see for instance the book by Heinonen, Kilpeläinen and Martio [HKM93], Lindqvist [L06] and the paper [CIL92] by Crandall, Ishii, and Lions.

1.4 Basics of p-harmonic functions

In this section we present some well known basic results for p-harmonic functions, frequently used in Papers I–V. In fact, the below lemmas are used in nearly all appended papers and are therefore fundamental for our results. We only shortly discuss the proofs by giving references.

We start with the comparison principle.

Lemma 1.3 (Comparison principle)**.** *Suppose that u is p-superharmonic and that v is p-subharmonic in a bounded domain Ω. If*

$$\limsup_{x\to y} v(x) \leq \liminf_{x\to y} u(x)$$

for all $y \in \partial\Omega$, and if both sides of the above inequality are not simultaneously ∞ or $-\infty$, then $v \leq u$ in Ω.

Proof. If $p \in (1,\infty)$, this follows from [HKM93, Theorem 7.6]. For the case $p = \infty$, the lemma was proved by Jensen, see [J93, Theorem 3.11]. Later, Armstrong and Smart presented a shorter proof for $p = \infty$, see [AS10]. Concerning more general equations, we refer to the book by Gilbarg and Trudinger [GT98, remark following the proof of Theorem 10.7]. □

We next consider existence of weak solutions.

Lemma 1.4. *Let $\Omega \subset \mathbf{R}^n$ be a bounded domain and let $f \in W^{1,p}(\Omega)$. Then there exists a p-harmonic function u in Ω satisfying $u - f \in W_0^{1,p}(\Omega)$.*

Proof. The proof of existence of a weak solution u to the p-Laplacian, $1 < p < \infty$, is based on variational principles, see [HKM93, Chapter 5]. See also the more general proof based on monotone operators [HKM93, Appendix I]. By [HKM93, Theorem 3.70], u is p-harmonic after a redefinition on a set of Lebesgue measure zero. Moreover, for equations of the form (1.2), we refer to Michael and Ziemer [MZ91, Theorem 1.2]. If $p = \infty$, we may use the limit of a sequence of p-harmonic functions as $p \to \infty$, see Jensen [J93, Theorem 1.22]. □

To be able to solve the Dirichlet problem, we recall the definition of p-regularity.

Definition 1.5. *A boundary point $y \in \partial\Omega$ is said to be p-regular if, for each function $f \in W^{1,p}(\Omega) \cap C(\bar{\Omega})$, the p-harmonic function u in Ω, guaranteed by Lemma 1.4 above, with $u - f \in W_0^{1,p}(\Omega)$, satisfies*

$$\lim_{x\to y} u(x) = f(y).$$

Furthermore, we say that Ω is p-regular if each $y \in \partial\Omega$ is p-regular.

At this moment we mention that if Ω is an NTA-domain, see Definition 1.13 in Section 1.6, then $\partial\Omega$ is p-regular for $1 < p \leq \infty$. Moreover, if $p > n$, then all points $\{x\} \in \mathbf{R}^n$ are p-regular. See [HKM93, Chapter 6] and [GZ77, Theorem 2.5] for proofs of these statements and for more on this subject. We are now ready to consider solutions to the Dirichlet problem for the p-Laplacian.

Lemma 1.6. *Let $\Omega \subset \mathbf{R}^n$ be a bounded p-regular domain and let $f \in W^{1,p}(\Omega) \cap C(\bar{\Omega})$. Then the Dirichlet problem for the p-Laplacian is uniquely solvable, that is, there exists a unique p-harmonic function u in Ω satisfying*

$$\lim_{x \to y} u(x) = f(y) \quad \textit{for all} \quad y \in \partial\Omega.$$

Proof. The existence follows from Lemma 1.4 and the assumption that Ω is p-regular. The uniqueness follows from Lemma 1.3. □

From the symmetries in the p-Laplace equation, we obtain the possibility to rotate, translate and scale the coordinates without changing the equation.

Lemma 1.7. *Assume that $\Omega \subset \mathbf{R}^n$ and that u is a p-harmonic function in Ω. Let $F : \mathbf{R}^n \to \mathbf{R}^n$ be the composition of a translation, a rotation and a scaling. Define $\hat{u}(x) = u(F(x))$ whenever $F(x) \in \Omega$. Then $\hat{u}$ is p-harmonic in $F^{-1}(\Omega)$.*

Proof. The proof of Lemma 2.7 in Paper V yields the details, and applies also to more general equations of p-Laplace type. □

Lemma 1.8 (Interior estimates)**.** *Assume that u is a positive p-harmonic function in $B(w, 2r)$. Then there exist $\alpha \in (0, 1]$ and c, depending only on n and p, such that*

$$(i) \qquad r^{p-n} \int\limits_{B(w,r)} |\nabla u|^p dx \leq c \Big(\sup_{B(w,2r)} u \Big)^p,$$

$$(ii) \qquad \sup_{B(w,r)} u \leq c \inf_{B(w,r)} u,$$

$$(iii) \qquad |u(x) - u(y)| \leq c \left(\frac{|x-y|}{r} \right)^\alpha \sup_{B(w,2r)} u, \quad \textit{whenever} \quad x, y \in B(w, r).$$

If in addition $1 < p < \infty$, or if $B(w, 2r) \subset \mathbf{R}^2$, then u has a representative with Hölder continuous partial derivatives of first order in $B(w, 2r)$. In particular, there exist $\sigma \in (0, 1]$ and c, depending only on n and p, such that

$$
\begin{aligned}
(iv) \qquad & |\nabla u(x) - \nabla u(y)| \le c r^{-1} \left(\frac{|x-y|}{r} \right)^{\sigma} \sup_{B(w,2r)} u, \\
& |\nabla u(x)| \le c r^{-1} \sup_{B(w,2r)} u, \quad \textit{whenever} \quad x, y \in B(w, r/2).
\end{aligned}
$$

Proof. The proof of (i) follows by taking $\psi = \eta^p u$, where η is an appropriate smooth cut-off function, as a test function in (1.3). The statements (ii) and (iii) were proved by Serrin in [S64, Theorems 5-9] for more general operators. If $1 < p < \infty$, then (iv) follows from Tolksdorf [T84]. If $p = \infty$ and $B(w, 2r) \subset \mathbf{R}^2$, then (iv) follows from Evans and Savin [ES08]. $\square$

To comment on Lemma 1.8, we mention that (i) is usually referred to as an *energy estimate* or a *Caccioppoli estimate*, while (ii) is the fundamental *Harnack's inequality.* Statements (iii) and (iv) are *Hölder continuity* of solutions and of the gradient of solutions respectively.

Concerning boundary estimates, (i), (ii) and (iii) have a boundary version as the next lemma shows. Harnack's inequality up to the boundary is sometimes referred to as a *Carleson estimate*. For the definition of an NTA-domain, we refer the reader to Definition 1.13 in Section 1.6.

Lemma 1.9 (Boundary estimates)**.** *Assume that $\Omega \subset \mathbf{R}^n$ is an NTA-domain, let $w \in \partial\Omega$, $0 < r < r_0$ and suppose that u is a positive p-harmonic function in $\Omega \cap B(w, 2r)$, continuous on $\bar{\Omega} \cap B(w, 2r)$ with $u = 0$ on $\Delta(w, 2r)$. Then there exist c and $\alpha \in (0, 1]$, depending only on M, n and p, such that*

$$
\begin{aligned}
(i) \qquad & r^{p-n} \int_{\Omega \cap B(w,r)} |\nabla u|^p dx \le c \left(\sup_{\Omega \cap B(w,2r)} u \right)^p, \\
(ii) \qquad & \sup_{\Omega \cap B(w,r/c)} u \le c u(a_{r/c}(w)), \\
(iii) \qquad & |u(x) - u(y)| \le c \left(\frac{|x-y|}{r} \right)^{\alpha} \sup_{\Omega \cap B(w,2r)} u,
\end{aligned}
$$

whenever $x, y \in B(w, r)$.

Proof. To prove (i) we use, in addition to the proof of (i) in Lemma 1.8, a standard subsolution argument, see the proof of Lemma 2.2 in Paper IV for details. A proof of (ii) for linear elliptic partial differential equations can be found in the paper [CFMS81] by Caffarelli, Fabes, Mortola and Salsa. The proof uses only analogues of Harnack's inequality, (iii) in Lemma 1.9 and the comparison principle for linear equations as well as the NTA-properties of the domain. In particular, the proof also applies in our situation, and Lemma 2.5 in Paper IV repeats the proof. By observing that Ω is p-regular by the NTA-assumption, statement (iii) follows by the same arguments as in [HKM93, Theorem 6.44, Lemma 6.47]. $\square$

1.5 p-harmonic measures

Let $\Omega \subset \mathbf{R}^n$ be a bounded 2-regular domain and recall that the solution to the Dirichlet problem for the Laplace operator in Ω is the unique smooth function $u \in C^\infty(\Omega)$ which is harmonic in Ω and satisfies $u = f$ continuously on $\partial\Omega$, where $f \in C(\bar{\Omega})$. The maximum principle and the Riesz representation theorem yield the representation formula

$$u(x) = \int_{\partial\Omega} f(y) d\omega^x(y), \quad \text{whenever} \quad x \in \Omega. \tag{1.4}$$

Here, $d\omega^x(y) = \omega(dy, x, \Omega)$ is referred to as the *harmonic measure* of dy at x with respect to Ω. Let also $g(\cdot, x)$ be the Green function for Ω with pole at $x \in \Omega$ and extend g to $\mathbf{R}^n$ by putting $g \equiv 0$ on $\mathbf{R}^n \setminus \Omega$. Then ω is the Riesz measure associated to g in the sense that

$$\int_{\mathbf{R}^n} \langle \nabla g, \nabla \phi \rangle \, dy = -\int_{\mathbf{R}^n} \phi(y) \, d\omega^x(y), \tag{1.5}$$

whenever $\phi \in C_0^\infty(\mathbf{R}^n \setminus \{x\})$. It turns out that for $E \subset \partial\Omega$, $\omega(E, x, \Omega)$ is the unique harmonic function in Ω, $0 \leq \omega(E, x, \Omega) \leq 1$, with boundary values 1 on E and 0 on $\partial\Omega \setminus E$. For more on harmonic measures, see for example the book by Capogna, Kenig and Lanzani [CKL05], the book by Garnett and Marshall [GM05] and the thesis by Vasilis [V10].

Two ways to generalize the harmonic measure to a *p-harmonic measure*, $p \neq 2$, will be considered in this thesis. We start by noting that unfortunately, both generalizations, which result in two different tools, are sometimes named p-harmonic measure in the literature. Moreover, the harmonic measure is both a measure and a harmonic function, and one of the p-harmonic measures (Definition 1.10) is a p-harmonic function but not a measure, while the other p-harmonic measure (Definition 1.11) is not a p-harmonic function but a measure. In this thesis we make use of both definitions, and we will not try to "correct the literature" by introducing new names. However, it will be clear from the context which p-harmonic measure we mean when this name is used. We next present the definitions. The following definition is from the book by Heinonen, Kilpeläinen and Martio [HKM93, Chapter 11].

Definition 1.10. *Let $\Omega \subseteq \mathbf{R}^n$ be a domain, $E \subseteq \partial\Omega$, $p \in (1, \infty)$, and $x \in \Omega$. The p-harmonic measure of E at x with respect to Ω, denoted $\omega_p(E, x, \Omega)$, is defined as $\inf_u u(x)$, where the infimum is taken over all p-superharmonic functions $u \geq 0$ in Ω such that $\liminf_{z \to y} u(z) \geq 1$, for all $y \in E$.*

The ∞-harmonic measure is usually defined in a similar manner, but with p-superharmonicity replaced by absolutely minimizing (AM), see [PSSW09, page 173]. As in the case of harmonic measure $\omega(E, x, \Omega) = \omega_2(E, x, \Omega)$, it turns out that $\omega_p(E, x, \Omega)$ is the unique p-harmonic function in Ω, $0 \leq \omega(E, x, \Omega) \leq 1$, with boundary values 1 on E and 0 on $\partial\Omega \setminus E$. For these and other properties of p-harmonic measures, we refer to [HKM93, Chapter 11]. However, $\omega_p(E, x, \Omega)$ is not a measure when $p \neq 2$, and no representation formula similar to (1.4) is known, see Llorente, Manfredi, Wu [LMW05]. The p-harmonic measure introduced in Definition 1.10 is studied in Paper II.

Next, we recall the other generalization of harmonic measure, which originates from (1.5). To do so we observe that a version of (1.5) holds in the nonlinear setting. Let $\Omega \subset \mathbf{R}^n$ be a domain, $w \in \partial\Omega$, $r > 0$ and $p \in (1, \infty)$ be given. Suppose that u is a positive p-harmonic function in $\Omega \cap B(w, r)$, continuous on $\bar{\Omega} \cap B(w, r)$ with $u = 0$ on $\Delta(w, r)$. Extend u to $B(w, r)$ by defining $u \equiv 0$ in $B(w, r) \setminus \Omega$. It can be shown (see for example Paper IV, Lemma 2.2 and Lemma 2.6) that then u is p-subharmonic in $B(w, r)$, and that $\nabla \cdot (|\nabla u|^{p-2} \nabla u)$ is a non-negative distribution in $B(w, r)$. Hence, $\nabla \cdot (|\nabla u|^{p-2} \nabla u)$ is represented by a measure μ. In analogue with (1.5), μ is the Riesz measure associated to u in the sense that

$$\int_{\mathbf{R}^n} \langle |\nabla u|^{p-2}, \nabla \phi \rangle \, dy = -\int_{\mathbf{R}^n} \phi(y) \, d\mu(y), \tag{1.6}$$

whenever $\phi \in C_0^\infty(B(w,r))$. The p-harmonic measure is defined as follows.

Definition 1.11. *The measure μ in* (1.6) *is the p-harmonic measure of u.*

It turns out that the p-harmonic measure μ has support in $\Delta(w,r)$, and satisfies the following useful relation to u, which in turn implies that μ is a doubling measure. For the definition of an NTA-domain, we refer the reader to Definition 1.13 in Section 1.6.

Lemma 1.12. *Let Ω be an NTA-domain and let w, r, p, u and μ be as above. There exists $c = c(M, n, p)$ such that*

$$c^{-1} r^{p-n} \mu\big(\Delta(w, r/c)\big) \leq \big(u(a_{r/c}(w))\big)^{p-1} \leq c r^{p-n} \mu\big(\Delta(w, r/c)\big).$$

Proof. This follows from Kilpeläinen and Zhong [KZ03]. Alternatively, see the proof of Lemma 2.7 in Paper IV. □

The p-harmonic measure in Definition 1.11 was studied by Kilpeläinen and Zhong in [KZ03] and was used as a fundamental tool in Paper III, Paper IV and in several related papers by Lewis and Nyström. Moreover, the dimension of this measure has been studied by Bennewitz and Lewis in [BL05].

We end this section by noting that when $p = 2$, then as expected, both Definition 1.10 and Definition 1.11 yield the harmonic measure.

1.6 Geometric assumptions

In this section, we recall some geometric definitions which are frequently used when studying boundary behaviour of solutions to partial differential equations. We also give some examples and try to explain connections between the different geometric assumptions. Before recalling the definition of NTA-domains, we note that if $w_1, w_2 \in \Omega$, then a *Harnack chain* from w_1 to w_2 in Ω is a sequence of nontangential balls such that the first ball contains w_1, the last ball contains w_2, and consecutive balls intersect. The following definition is from Jerison and Kenig [JK82].

Definition 1.13 (NTA-domain)**.** *A domain Ω is called non-tangentially accessible (NTA) if there exist constants $M > 1$ and $r_0 > 0$ such that the following are fulfilled:*

(i) corkscrew condition: For any $w \in \partial\Omega$ and any $r \in (0, r_0)$, there exists a point $a_r(w) \in \Omega$ such that

$$\frac{r}{M} < |a_r(w) - w| < r \qquad \textit{and} \qquad d\big(a_r(w), \partial\Omega\big) > \frac{r}{M}.$$

(ii) $\mathbf{R}^n \setminus \Omega$ satisfies the corkscrew condition.

(iii) Harnack chain condition: If $\epsilon > 0$ and $w_1, w_2 \in \Omega$ satisfy $d(w_1, \partial\Omega) > \epsilon$, $d(w_2, \partial\Omega) > \epsilon$ and $|w_1 - w_2| < c\epsilon$, then there exists a Harnack chain from w_1 to w_2 whose length depends on c, but not on ϵ.

A useful consequence of condition (ii) above is that boundary points are p-regular in the range $1 < p \le n$, see [HKM93, Theorem 6.31] for a proof. Roughly speaking, the reason is that the corkscrew in the complement gives space enough to construct a barrier from below and a barrier from above to obtain boundary convergence. If $p > n$, then the corresponding barriers need no such space, which can be proved by capacity estimates from the fundamental solution $\hat{f}$ introduced in (1.1). The difference between the cases $1 < p \le n$ and $p > n$ may be seen in $\hat{f}$, which is infinite in the first case and finite in the second, see Figure 1.1. We next give examples of simple NTA-domains, and simple domains which fail to be NTA-domains.

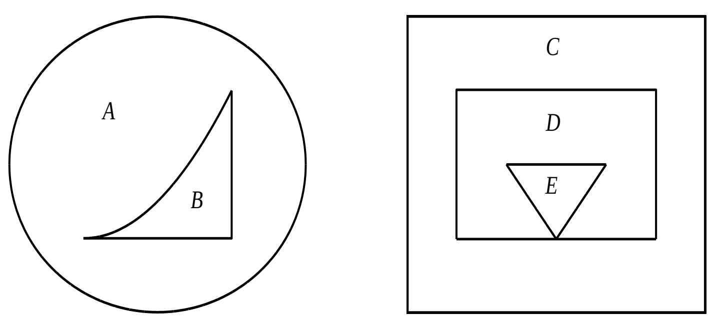

Figure 1.3: The domains A, B, C, D and $E \subset \mathbf{R}^2$ in Example 1.14.

Example 1.14. *The curve in domain A in Figure 1.3 is $y = x^2$. Therefore, A and B fail to be NTA-domains due to the corkscrew conditions. In fact, A fails due to (ii) and B fails due to (i) in Definition 1.13. Domains C and E satisfy the NTA-assumption, but not domain D, which fails due to the Harnack chain condition.*

We next present δ-Reifenberg flat domains. Roughly speaking, in such domains, $\partial\Omega$ can be well approximated within δ by hyperplanes in the Hausdorff distance sense, in a scale invariant manner. δ-Reifenberg flat domains were introduced by Reifenberg [R60] and are considered in Papers III and V. To set up the definition, we need the notation of Hausdorff distance. We let

$$h(E, F) = \max\big(\sup\{d(x, E) : x \in F\}, \sup\{d(x, F) : x \in E\}\big) \tag{1.7}$$

be the Hausdorff distance between the sets $E, F \subset \mathbf{R}^n$.

Definition 1.15. *Let $\Omega \subset \mathbf{R}^n$ be a domain. Then $\partial\Omega$ is said to be uniformly (δ, r_0)-approximable by hyperplanes, provided there exists, whenever $w \in \partial\Omega$ and $0 < r < r_0$, a hyperplane Λ containing w such that*

$$h\big(\partial\Omega \cap B(w, r), \Lambda \cap B(w, r)\big) \leq \delta r.$$

In the above definition, note that Λ may depend on both w and r. We let $\mathcal{F}(\delta, r_0)$ denote the class of all domains Ω which satisfy Definition 1.15. Let $\Omega \in \mathcal{F}(\delta, r_0)$, $w \in \partial\Omega$, $0 < r < r_0$, and let Λ be as in Definition 1.15. We say that $\partial\Omega$ separates $B(w, r)$, if

$$\{x \in \Omega \cap B(w, r) : d(x, \partial\Omega) \geq 2\delta r\} \subset \text{one component of } \mathbf{R}^n \setminus \Lambda. \tag{1.8}$$

Definition 1.16 (δ-Reifenberg flat domain)**.** *Let $\Omega \subset \mathbf{R}^n$ be a domain. Then Ω and $\partial\Omega$ are said to be (δ, r_0)-Reifenberg flat provided $\Omega \in \mathcal{F}(\delta, r_0)$ and (1.8) hold whenever $0 < r < r_0$ and $w \in \partial\Omega$.*

Figure 1.4 shows the definition of a δ-Reifenberg flat domain in an example. For short we say that Ω and $\partial\Omega$ are δ-Reifenberg flat whenever Ω and $\partial\Omega$ are (δ, r_0)-Reifenberg flat for some $r_0 > 0$. We note that an equivalent definition of Reifenberg flat domains is given by Kenig and Toro in [KT97]. It can be shown, see [KT97, Theorem 3.1], that a δ-Reifenberg flat domain is an NTA-domain with constant $M = M(n)$, provided $0 < \delta < \hat{\delta}(n)$ for some $\hat{\delta}(n)$ small enough.

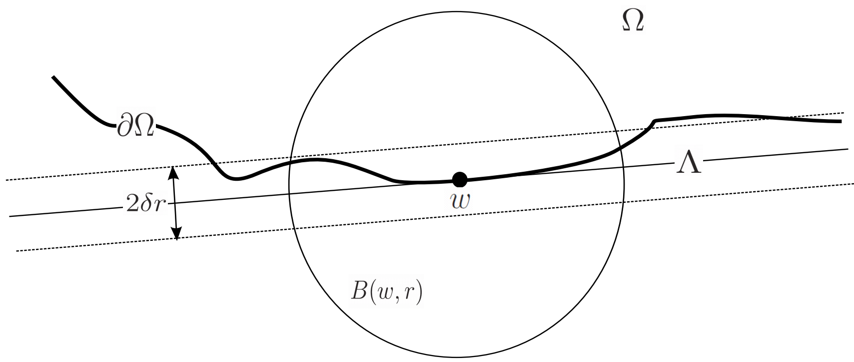

Figure 1.4: δ-Reifenberg flat domain.

In Paper III we use the following stronger flatness assumption.

Definition 1.17 (δ-Reifenberg flat domain with vanishing constant)**.** *We say that Ω and $\partial\Omega$ are Reifenberg flat with vanishing constant if for every $\epsilon > 0$, there exists $\tilde{r} = \tilde{r}(\epsilon) > 0$ such that whenever $w \in \partial\Omega$ and $0 < \rho < \tilde{r}$, then* (1.8) *holds and there exists a plane $P' = P'(w, \rho)$ containing w satisfying*

$$h\big(\partial\Omega \cap B(w, \rho), P' \cap B(w, \rho)\big) \leq \epsilon\rho.$$

For short, we say that Ω and $\partial\Omega$ are *vanishing Reifenberg* when Ω is Reifenberg flat with vanishing constant. We next present some examples of δ-Reifenberg flat domains, and domains which are vanishing Reifenberg.

Example 1.18. *The cone with aperture ψ, $0 \leq \psi \leq \pi$ is δ-Reifenberg flat. When $\psi = \pi$, the cone becomes a plane and is therefore vanishing Reifenberg. Moreover, a ball is vanishing Reifenberg, since on decreasing scales, the ball becomes flat in the limit. By the same reason, C^1-domains (see Definition 1.21) are vanishing Reifenberg.*

Example 1.19. *Consider the function*

$$a(x) = \sum_{k=1}^{\infty} \frac{\cos(2^k x)}{2^k \sqrt{k}},$$

which is in the Zygmund class λ_*. This class is vanishing Reifenberg, see [CKL05, Example 4.3]. The function $a(x)$ is almost nowhere differentiable, see [Z59, Page 47], which shows that a vanishing Reifenberg domain may be very irregular.

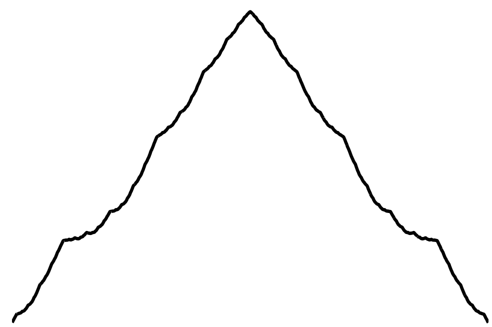

Figure 1.5: The function $a(x)$, which is vanishing Reifenberg and almost nowhere differentiable.

Example 1.20. *The Koch snowflake is δ-Reifenberg flat. The curve is constructed as follows. Consider $\mathbf{R}^2$, the interval $[0,1]$ in the x_1-axis and divide it into the segments $[0,a]$, $[a,1-a]$ and $[1-a,1]$. Let $[a,1-a]$ be the base for a triangle whose opposite sides both have side length a and are located above the x_1-axis. It follows that each side makes an angle θ with the base, where $\cos\theta = (1/2-a)/a$, provided $1/4 < a < 1/2$. The intervals $[0,a]$, $[1-a,1]$, and the two sides of the triangle form the first step in the construction. Repeat this construction on each of the four sides. The construction should be done in such a way that if we takes a side s in the first step and translates and dilates it to $[0,1]$, then the translated and dilated construction on s coincides with the four intervals in the original construction. Repeating this procedure yields a Koch snowflake curve in the limit, see Figure 1.6. Let Ω consist of points in $\mathbf{R}^2$ lying above both the snowflake and the x_1-axis. It follows that as $a \to 1/4$ $(\theta \to 0)$, then Ω and Ω^c are δ-Reifenberg flat with $\delta \to 0$. We observe that $\partial\Omega$ do not have finite length and is nowhere differentiable, so either the unit normal nor the surface measure are well defined. Moreover, if we let the angle decrease $(\theta \to 0)$ in the above construction as the scale decreases, then Ω and $\partial\Omega$ becomes vanishing Reifenberg.*

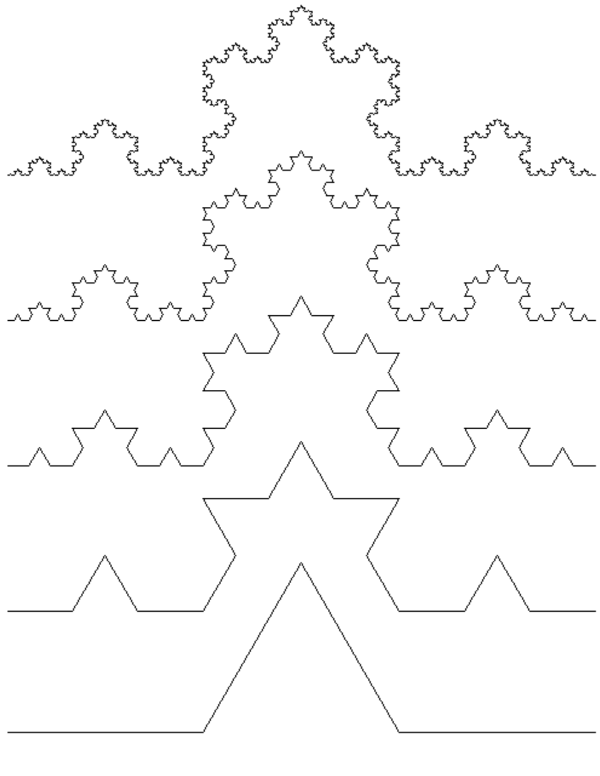

Figure 1.6: Construction of the Kock snowflake curve which is δ-Reifenberg flat. In the figure, $a = 1/3$.

We proceed by recalling $C^{k,\alpha}$-domains, quasiballs and domains satisfying different ball conditions. We also relate these definitions to each other and to δ-Reifenberg flat, vanishing Reifenberg and NTA-domains.

Definition 1.21 ($C^{k,\alpha}$-domain)**.** *Let $k \in [0, \infty]$ and $\alpha \in (0, 1]$. A domain Ω is said to be a $C^{k,\alpha}$-domain if there exist, for any compact subset $K \subset \mathbf{R}^n$, balls $\{B(x_i, r_i)\}$, with $x_i \in \partial\Omega$ and $r_i > 0$, such that $\{B(x_i, r_i)\}$ constitutes a covering of an open neighborhood of $\partial\Omega \cap K$ and such that, for each i,*

$$\begin{aligned} \Omega \cap B(x_i, r_i) &= \{x = (x', x_n) \in \mathbf{R}^n : x_n > \phi_i(x')\} \cap B(x_i, r_i), \\ \partial\Omega \cap B(x_i, r_i) &= \{x = (x', x_n) \in \mathbf{R}^n : x_n = \phi_i(x')\} \cap B(x_i, r_i), \end{aligned}$$

in an appropriate coordinate system and for a $C^{k,\alpha}$-function ϕ_i.

Usually, a $C^{0,1}$-domain is called a *Lipschitz domain* and one writes C^k-domain for $C^{k,0}$-domains. If Ω is a bounded Lipschitz domain, then the Lipschitz constant of Ω is given by

$$L = \max_i \left\{ \sup_{y,z \in B(x_i,r_i)} \frac{|\phi_i(y) - \phi_i(z)|}{|y-z|} \right\}.$$

If Ω is a Lipschitz domain with L small enough, then Ω is also δ-Reifenberg flat for some $\delta(L)$ with $\delta(L) \to 0$ as $L \to 0$. Moreover, if Ω is a $C^{k,\alpha}$-domain, $1 \leq k \leq \infty$, $0 \leq \alpha \leq 1$, then Ω is by definition vanishing Reifenberg, since on decreasing scales, such domains become flat in the limit. Further, a Lipschitz domain is an NTA-domain, which follows more or less directly from the definitions.

We next consider domains satisfying certain ball conditions.

Definition 1.22 (The ball condition)**.** *A domain Ω is said to satisfy the interior (exterior) ball condition if there exists $r_1 > 0$ satisfying the following: For every $w \in \partial\Omega$ there exists a point $w' \in \Omega$ ($w' \in \Omega^c$) such that $B(w', r_1) \subset \Omega$ ($B(w', r_1) \subset \Omega^c$) and $w \in \partial B(w', r_1)$. We say that Ω satisfies the ball condition (with radius r_1) if Ω satisfies both the exterior and the interior ball condition.*

It is known that a domain Ω satisfies the ball condition if and only if Ω is a $C^{1,1}$-domain. For a proof, see [AKSZ07, Lemma 2.2] by Aikawa, Kilpeläinen, Shanmugalingam and Zhong. Hence, if Ω satisfies the ball condition then Ω is vanishing Reifenberg and hence also an NTA-domain.

In Paper II, we consider the following generalized interior ball condition in $\mathbf{R}^2$. In the following, (r, ϕ) are polar coordinates for $(x, y) \in \mathbf{R}^2$.

Definition 1.23 (Generalized interior ball condition)**.** *Let $\Omega \subseteq \mathbf{R}^2$ be a domain and let, for $\gamma, r_1 > 0$ and $v \geq 1/2$,*

$$E(\gamma, r_1, v) = \left\{ (x,y) \in \mathbf{R}^2;\ \cos(v\phi) > \left(\frac{r}{r_1}\right)^\gamma, |\phi| < \frac{\pi}{2v} \right\}.$$

A boundary point $w \in \partial\Omega$ is said to satisfy the generalized interior ball condition of type (γ, r_1, v), if there exists a rigid transformation T of the plane such that $T\big(E(\gamma, r_1, v)\big) \subseteq \Omega$ and $w = T(0,0)$. We then say that $w \in \partial\Omega$ is of type (γ, r_1, v) for Ω.

The set $E(\gamma, r_1, v)$ is by definition contained in a sector with aperture π/v and apex at the origin. On the boundary $\partial E(\gamma, r_1, v)$ we have that $r = r_1 \cos^{1/\gamma}(v\phi)$, see Figure 1 in Paper II for some illustrations. If $v = \gamma = 1$, this is just a circle with radius $r_1/2$ and center $(r_1/2, 0)$ and hence, in this case Definition 1.23 yields the usual interior ball condition. We also see that if $w \in \partial\Omega$ is of type (γ, r_1, v), then it is also of type (γ', r_1', v'), for all $\gamma \geq \gamma' > 0$, $r_1 \geq r_1' > 0$, and $1/2 \leq v \leq v'$.

Finally, we recall the definition of quasiballs. Quasicircles (quasiballs in $\mathbf{R}^2$) are considered in paper IV.

Definition 1.24 (Quasiball)**.** *A domain Ω is said to be a quasiball provided $\Omega = f(B(0,1))$, where $f = (f_1, f_2, ..., f_n) : \mathbf{R}^n \to \mathbf{R}^n$ is a $K > 1$ quasiconformal mapping of $\mathbf{R}^n$ onto $\mathbf{R}^n$. That is, $f_i \in W^{1,n}(B(0,\rho)), 0 < \rho < \infty, 1 \leq i \leq n$, and for almost every $x \in \mathbf{R}^n$ with respect to Lebesgue n-measure the following hold,*

$$
\begin{aligned}
&(i) \qquad |Df(x)|^n = \sup_{|h|=1} |Df(x)h|^n \leq K|J_f(x)|, \\
&(ii) \qquad J_f(x) \geq 0 \ \text{ or } \ J_f(x) \leq 0.
\end{aligned}
$$

In this display we have written $Df(x) = (\partial f_i/\partial x_j)$ for the Jacobian matrix of f and $J_f(x)$ for the Jacobian determinant of f at x.

If Ω is a quasiball, then one can show that Ω is δ-Reifenberg flat, with $r_0 = 1$, where $\delta \to 0$ as $K \to 1$, see Reshetnyak [R89, Theorems 12.5–12.7] for a proof. Finally, if Ω is a quasiball then Ω is an NTA-domain, see [JK82] and the references therein.

Chapter 2

Applications

In this chapter we consider applications for the p-Laplacian and for equations of p-Laplace type. We start with power-laws and their connection to the p-Laplacian. We also consider image restoring, stochastic characterization and minimization problems.

2.1 Connection with power-laws

A *power-law* states that one quantity, say y, is related to another quantity x by certain power, for example $y = c\,x^{\,q}$ for some constants c and q. When a power-law is assumed to hold, the governing equation may be of p-Laplace type, as the following derivation shows. Let u denote the density of some quantity in equilibrium, let Ω be a domain and $E \subset \Omega$ be a C^1-domain so that the divergence theorem can be applied. Due to the equilibrium, the net flux of u through ∂E is zero, that is

$$\oint_{\partial E} \langle \boldsymbol{F}, \boldsymbol{n} \rangle \, ds \;=\; 0, \tag{2.1}$$

where $\boldsymbol{F}$ denotes the flux density, $\boldsymbol{n}$ the normal to ∂E and ds is the surface measure. Here and in the rest of this chapter, we denote vectors by bold symbols. The divergence theorem gives

$$\int_E \nabla \cdot \boldsymbol{F} \, dx \;=\; \oint_{\partial E} \langle \boldsymbol{F}, \boldsymbol{n} \rangle \, ds \;=\; 0.$$

Since E was arbitrary, we conclude

$$\nabla \cdot \boldsymbol{F} = 0 \quad \text{in} \quad \Omega. \tag{2.2}$$

In many situations it is physically reasonable to assume that the flux vector $\boldsymbol{F}$ and the gradient ∇u are related by a power-law of the form

$$\boldsymbol{F} = -c\,|\nabla u|^{q}\,\nabla u, \tag{2.3}$$

for some constants c and q. The reason is that flow is usually from regions of higher concentration to regions of lower concentration. From this assumption, with $q = p-2$, and from (2.2), we obtain the p-Laplace equation

$$\nabla \cdot (|\nabla u|^{p-2}\nabla u) = \Delta_p u = 0 \quad \text{in} \quad \Omega.$$

To obtain the more general equation (1.2) of p-Laplace type, we generalize the above derivation as follows. Instead of (2.3) we make the weaker assumption $\boldsymbol{F} = -\boldsymbol{A}(x, \nabla u)$. In addition, assume that there is a source $B = B(x, \nabla u)$ present in Ω, increasing or decreasing the concentration of the quantity u. Then (2.1) should be replaced by

$$-\oint_{\partial E} \langle \boldsymbol{A}(x, \nabla u), \boldsymbol{n} \rangle \, ds = \int_{E} B(x, \nabla u)\, dx,$$

and again by the divergence theorem and since E was arbitrary, we obtain (1.2), that is

$$\nabla \cdot \boldsymbol{A}(x, \nabla u) + B(x, \nabla u) = 0 \quad \text{in} \quad \Omega.$$

As expected, the linear case $p = 2$ ($q = 0$) in (2.3) is dominating in applications. In fact, the Laplacian is everywhere. We list some examples where the corresponding assumption

$$\boldsymbol{F} = -c\,\nabla u \tag{2.4}$$

arises as a physical law: If u denotes a chemical concentration, then the assumption (2.4) is the well known Fick's law of diffusion, if u denotes a temperature, then (2.4) is Fourier's law of heat conduction, if u denotes electrostatic potential, then (2.4) is Ohm's law of electrical conduction, and if u denotes pressure, then (2.4) is Darcy's law of fluid flow through a porous media. For more on basics about applications to the Laplacian, as well as the heat- and wave-equation, see for example the book by Sparr and Sparr [SS00, Chapter 1] and the book by Feynman, Leighton and Sands [FLS66, Chapter 12].

Concerning examples of power-laws, $p \neq 2$, we first note that the above linear laws are usually approximations, and a natural next step to obtain a more accurate analysis may be to consider some nonlinear generalization.

Moreover, a problem involving the p-Laplace operator studied by Aronsson, Evans and Wu in [AEW96] is fast/slow diffusion of sandpiles. Here, p is large and we think of u as the height of a growing sandpile. If $|\nabla u| > 1 + \delta$ for some $\delta > 0$, then $|\nabla u|^{p-2}$ is very large, and hence the transport of sand is also large, and if $|\nabla u| < 1 - \delta$, then $|\nabla u|^{p-2}$ is very small. Therefore, when adding sand particles to a sandpile, they accumulate as long as the slope of the pile does not exceed one. If the slope exceeds one, then the sand becomes unstable and instantly slides.

Next, we focus on fluid dynamics. If u denotes pressure in some liquid and if u satisfies (2.3) in a plane domain, then the corresponding flow may describe a so called Hele-Shaw flow of a power-law fluid.

Hele-Shaw flow is a flow between two plates where the gap between the plates is assumed to be small. Such flow appears for example when producing plastic products, see Section 2.1.1. We also present an example where exact solutions to the p-Laplacian models Hele-Shaw flow near a corner, see Section 2.1.2. We proceed by recalling some more standard terminology in fluid dynamics.

A *Newtonian fluid* is a fluid whose stress versus strain rate curve is linear and passes through the origin. The constant of proportionality is known as the viscosity. Examples of Newtonian fluids are water, oil and air.

A *non-Newtonian fluid* is a fluid whose flow properties differ in any way from those of Newtonian fluids. Examples of non-Newtonian fluids are ketchup, custard, toothpaste, paint, blood, shampoo, some salt solutions, molten polymers and starch suspensions.

In *power-law fluids* the viscosity η is modeled according to

$$\eta(\dot{\gamma}) = c\,\dot{\gamma}^{\hat{n}-1}, \tag{2.5}$$

where $c > 0$ and $\hat{n} > 0$ are material constants. Here, $\dot{\gamma}$ is a scalar measure for the rate of deformation defined as a function of the rate-of-strain tensor D according to $\dot{\gamma} = \sqrt{\langle 2D, D\rangle}$, see Aronsson and Janfalk [AJ92, page 344] for details. Power-law fluids are examples of non-Newtonian fluids. As expected from the derivation in the beginning of this section, the governing equation is usually of p-Laplace type in the setting of power-law fluids. Concerning the connection between power-law fluids, the p-Laplacian, (2.5) and the parameter $\hat{n}$, as well as for more on Hele-Shaw flow of power-law fluids, we refer the reader to the papers by Aronsson and Janfalk [AJ92], Aronsson [A91], Henriksen and Hassager [HH89], Martinez and Kindelan [MK09] and the references therein.

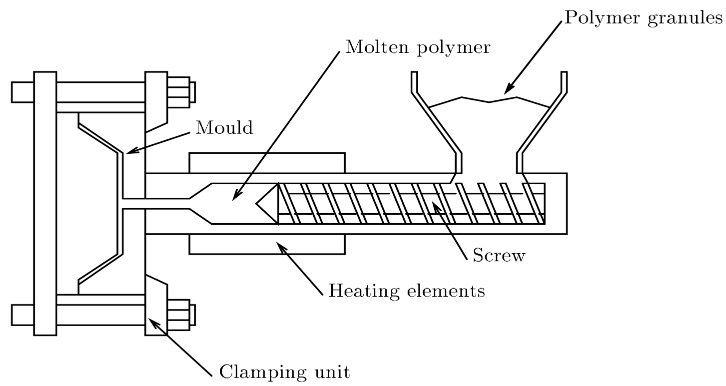

Figure 2.1: An injection moulding machine. This picture is from Johansson [J97, Page 3].

2.1.1 Injection moulding of plastic products

One reason for studying Hele-Shaw flow of power-law fluids is injection moulding of plastic products. The following material is from the thesis [J97] by Johansson and we refer the reader to his thesis for more on this subject. When producing thin-walled plastic products (which are dominating, thickness of about 0.5–4 mm is standard) by injection moulding, the heated plastic fluid (often modeled as a power-law fluid) is forced to flow in a mould having approximately parallel walls close together. The injection moulding process has three major steps, filling, packing and cooling. It is in the filling stage, when melted polymer is forced to flow in a mould, the p-Laplacian appears. Figure 2.1 shows an injection moulding machine. In short, the filling stage works as follows. Polymer granules are fed forward in the barrel while the screw retreats, and are melted by the heating elements. When sufficient molten material is in the front of the screw, it stops rotating and starts to move forward. As the screw moves forward it pushes the polymer melt into the mould through a nozzle, and the flow is thereby produced.

2.1.2 Hele-Shaw flow of a power-law fluid near a corner

Everything presented in this section originates from Aronsson and Janfalk [AJ92], and we refer to this paper for additional comments, details and derivations. Let (r, ϕ) be polar coordinates for (x, y) and consider a Hele-Shaw flow of a power-law fluid in the sector

$$\mathcal{S} = \Big\{(x, y) \in \mathbf{R}^2;\ 0 < r, |\phi| < w\Big\}.$$

Recall that the distance between the plates is assumed to be small, and hence, a three-dimensional flow is well modeled in two dimensions. Recall also that $\hat{n}$ is a material constant as introduced in (2.5). Under reasonable simplifications, it turns out that the pressure $\hat{p}$ is $(1 + 1/\hat{n})$-harmonic and the corresponding *stream function* ψ is $(\hat{n}+1)$-harmonic in $\mathcal{S}$. In particular

$$\psi'_x = -|\nabla \hat{p}|^{1/\hat{n}-1}\hat{p}'_y, \qquad \text{and} \qquad \psi'_y = |\nabla \hat{p}|^{1/\hat{n}-1}\hat{p}'_x,$$

and

$$\begin{aligned} \nabla \cdot (|\nabla \hat{p}|^{1/\hat{n}-1}\nabla \hat{p}) = 0 \quad \text{in} \quad \mathcal{S}, \qquad & \frac{\partial \hat{p}}{\partial \nu} = 0 \quad \text{on} \quad \partial\mathcal{S}, \\ \nabla \cdot (|\nabla \psi|^{\hat{n}-1}\nabla \psi) = 0 \quad \text{in} \quad \mathcal{S}, \qquad & \end{aligned} \tag{2.6}$$

where ν denotes the normal direction to $\partial\mathcal{S}$. The *streamlines* (lines indicating the direction a fluid element will travel in) of the flow are level curves of the $(\hat{n}+1)$-harmonic stream function to the $(1 + 1/\hat{n})$-harmonic pressure. Note that the flow is due to the pressure differences and is perpendicular to the level curves of $\hat{p}$. Moreover, (2.6) can be solved analytically and it follows that $\hat{p}$ and ψ behave like *quasi-radial* solutions near the corner. That is

$$\psi = r^k f(\phi), \qquad \text{and} \qquad \hat{p} = r^l g(\phi),$$

where

$$k = \frac{\beta}{\beta - 1}, \qquad l = \frac{\beta + \hat{n} - 1}{\beta - 1},$$

and

$$\beta = \frac{1}{2}\left(1 - \hat{n} \pm \sqrt{(\hat{n}-1)^2 + \frac{4\hat{n}\pi^2}{(\pi - w)^2}}\right),$$

where the sign is chosen as $+$ if $w < \pi$, and $-$ if $w > \pi$. The functions $f(\phi)$ and $g(\phi)$ are given implicitly. Put $a = (p-1)/(p-2)$, then

$$\begin{aligned} f &= \frac{c}{k}\left(1 - \frac{\cos^2\vartheta}{ak}\right)^{\frac{k-1}{2}} \cos\vartheta, \\ g &= \frac{c}{l}\left(1 - \frac{\cos^2\vartheta}{ak}\right)^{\frac{l-1}{2}} \sin\vartheta, \\ \phi(\vartheta) &= \vartheta + \left(\frac{w}{\pi} - 1\right) \arctan\left(\sqrt{\frac{ak}{ak-1}}\tan\vartheta\right), \end{aligned}$$

for a suitable choice of the parameter ϑ.

Figure 2.2 shows the streamlines of the flow for $\hat{n} = 0.4$. The pressure is therefore 3.5-harmonic and the corresponding stream function is 1.4-harmonic. Streamlines are marked solid. Level curves of $\hat{p}$ are dotted, and the flow is perpendicular to these lines. It is a constant step in height between the level curves for each function.

Next, we note that the speed v of the flow is related to the pressure gradient according to

$$v = c|\nabla\hat{p}|^{1/\hat{n}} = c|\nabla\psi| \approx r^{k-1},$$

where c depends on the material and the width of the small gap between the plates.

The above result concerning analytic solutions describing Hele-Shaw flow may be used to improve numerical simulations of the flow near sharp corners.

We end this section by noting that quasi-radial solutions by Aronsson [A86] and their corresponding stream functions by Persson [Pe89], similar to those appearing in this application, are used as a main tool in Paper II to estimate a p-harmonic measure.

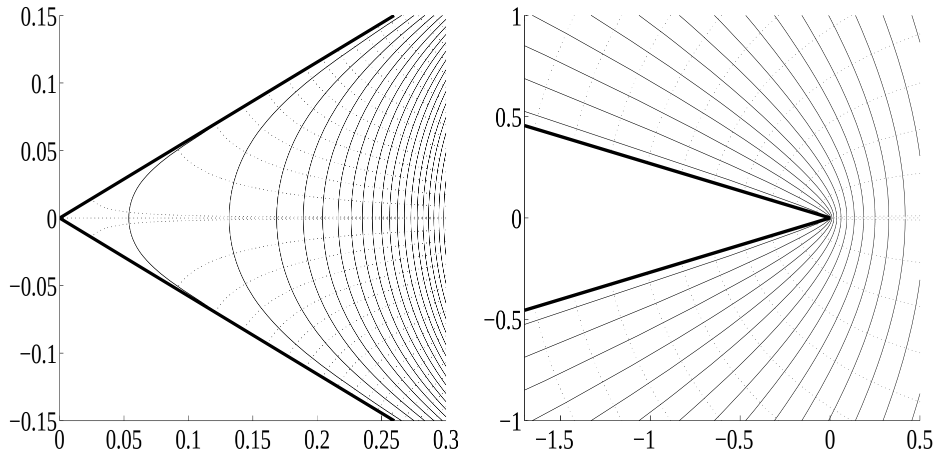

Figure 2.2: Level lines of the stream function ψ, that is, streamlines of the flow (solid). Level lines of the pressure $\hat{p}$ (dotted). $\hat{n} = 0.4$ and the opening angle w of the sector $\mathcal{S}$ is (a) $w = \pi/3$, (b) $w = 11\pi/6$.

2.1.3 A free boundary problem

An interesting physical problem concerning a free boundary problem involving the p-Laplacian and power-laws was studied by Acker and Meyer in [AM95]. In their paper, several theoretic results concerning solutions to the problem are proved. We will repeat the problem and some physical interpretations here. Free boundary problems for partial differential equations are frequently studied, and we note that Paper III in this thesis deals with a two-phase free boundary problem concerning flatness of the boundary as a consequence of a condition imposed on a p-harmonic measure.

The problem studied in [AM95] is, roughly speaking, the following.

The free boundary problem.
Let $a(x)$ be a positive function, $E \subset \mathbf{R}^n$ a bounded C^2-domain and let $p \in (1, \infty)$. We seek a domain Ω such that $\bar{E} \subset \Omega$ and

$$|\nabla u| = a(x) \quad on \quad \partial\Omega,$$

where u denotes the p-harmonic function in $\Omega \setminus \bar{E}$ satisfying $u \equiv 1$ on $\bar{E}$ and $u = 0$ on $\partial\Omega$.

We note that u is usually referred to as the p-capacitary function for $\bar{E}$ in Ω. Figure 2.3 shows the free boundary problem.

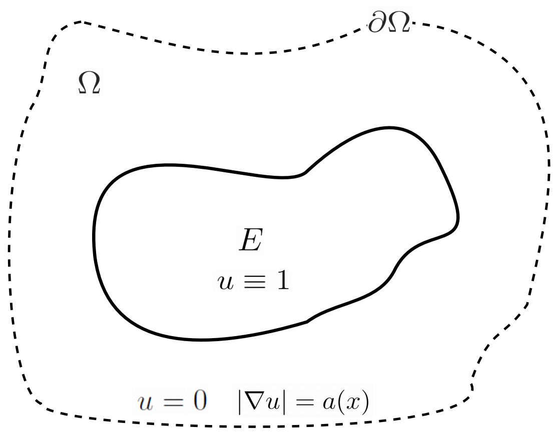

Figure 2.3: The setting of the free boundary problem.

The above problem arises in the study of fluid flow through a porous media. In particular, consider two reservoirs of fluid at different constant pressure, separated by a homogeneous porous medium occupying $\Omega \setminus \bar{E}$, through which the fluid flows due to the pressure difference. If we choose the power-law relation between the flow $\boldsymbol{F}$ and the pressure u mentioned in Section 2.1, that is $\boldsymbol{F} = c|\nabla u|^{p-2}\nabla u$, then the free boundary can be interpreted as a surface on which the flow magnitude is given by the function $a(x)$ of position.

Next, if $a(x) = c$, it has been shown by Acker in [A77] that the solution to the above problem can be interpreted as the minimization of a heat flow through the annular domain $\Omega \setminus E$ subject to a fixed given volume of $\Omega \setminus E$.

Moreover, by Lasey and Shillor [LS87] it has been shown that $\partial\Omega$ can be interpreted as the equilibrium surface, resulting from an electrochemical machining process, in which there is a threshold of current, corresponding to $|\nabla u| = a(x)$, below which etching does not occur. Etching is, roughly speaking, the process of using for example strong acid or electric current to cut into parts of a material surface to create a design.

2.1.4 Membrane in equilibrium

It is well known that the Laplacian approximates the governing equation of certain membrane in equilibrium, see for example the book by Sparr and Sparr [SS00, Chapter 1]. In this section, we start by repeating this derivation and end by giving some comments on the nonlinear case $p \neq 2$. We note that the following derivation is, as the derivation of the p-Laplacian in the beginning of Section 2.1, based on an equilibrium and a power-law.

Consider a membrane spanned over a region $\Omega \subset \mathbf{R}^2$ as Figure 2.4 shows.

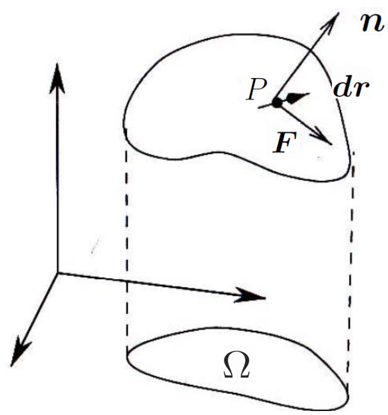

Figure 2.4: The membrane spanned over $\Omega \subset \mathbf{R}^2$.

Let $u = u(x_1, x_2)$ denote the membrane. Consider a small cut in the membrane at point P. Assume that the membrane is flexible, then there is a force perpendicular to the cut and tangent to the membrane, holding it together. If $\boldsymbol{n}$ is the surface normal in P, we denote the force acting on the part $\boldsymbol{dr}$ of the small cut by

$$\boldsymbol{F} = a(x_1, x_2)\boldsymbol{dr} \times \boldsymbol{n},$$

where $\times$ denotes the standard cross-product. Here, $a(x_1, x_2)$ denotes the surface tension (force/length) and is assumed to be independent of the direction of the small cut. If we in addition assume that $u_{x_1} = \partial u / \partial x_1$ and $u_{x_2} = \partial u / \partial x_2$ are small, then

$$\boldsymbol{n} = \frac{(-u_{x_1}, -u_{x_2}, 1)}{\sqrt{1 + u_{x_1}^2 + u_{x_2}^2}} \approx (-u_{x_1}, -u_{x_2}, 1).$$

Assume also that the membrane is loaded by an external force $f(x_1, x_2)\boldsymbol{e_{x_3}}$ (force/area) directed perpendicular to Ω. Consider a smooth subset $E \subset \Omega$ and let E' denote the part of the membrane above E. Since the membrane is in equilibrium we obtain

$$\oint_{\partial E'} a(x_1, x_2)\boldsymbol{dr} \times \boldsymbol{n} + \boldsymbol{e_{x_3}} \int_E f(x_1, x_2)dx = 0.$$

Hence, in the x_3-direction

$$\oint_{\partial E} a(x_1, x_2)(-u_{x_2}dx_1 + u_{x_1}dx_2) + \int_E f(x_1, x_2)dx = 0,$$

and by Green´s theorem

$$\int_E \nabla \cdot \big(a(x_1, x_2)\nabla u\big)dx = -\int_E f(x_1, x_2)dx. \tag{2.7}$$

Since E was arbitrary, we conclude

$$\nabla \cdot \big(a(x_1, x_2)\nabla u\big) = -f(x_1, x_2) \quad \text{in} \quad \Omega.$$

If we assume that a is constant, $a(x_1, x_2) = c$, and that $f(x_1, x_2) = 0$, then we obtain the Laplacian

$$\Delta u = 0 \quad \text{in} \quad \Omega,$$

which in this situation approximates for example the governing equation for soap-bubbles.

We next give some comments on the $p \neq 2$ case. In the paper [CEP06] by Cuccu, Emamizadeh and Porru, the authors indicate that the p-Laplacian, $p \neq 2$, models a nonlinear membrane. We have not found any good derivation for this, but if we assume the power-law $a(x_1, x_2) = c|\nabla u|^{p-2}$, then from (2.7) we end up with

$$\nabla \cdot \big(|\nabla u|^{p-2}\nabla u\big) = -c^{-1}f(x_1, x_2) \quad \text{in} \quad \Omega.$$

However, we do not discuss the relevance of assuming the surface tension $a(x_1, x_2)$ according to the above power-law.

2.2 Image restoring

Consider the problem of restoring a damaged image. A natural way of doing this is to interpolate data from the remaining parts of the image. This interpolation process can be done by using the p-Laplacian. Roughly speaking, one may proceed as follows. Assume for simplicity that the image is in greyscale and put the grey levels in the interval $[0, 1]$, that is, 0 is white and 1 corresponds to black. The remaining part of the image (RI) may be described by a function $f : RI \to [0, 1]$. Consider the missing pieces of the image as a union of domains Ω_i, $i = 1, 2, 3, \ldots$. Now for any of the domains Ω_i, the function f is defined on $\partial\Omega_i$ and yields boundary data for the Dirichlet problem

$$\Delta_p u_i = 0 \quad \text{in} \quad \Omega_i, \qquad \lim_{x\to y} u_i(x) = f(y) \quad \text{for all} \quad y \in \partial\Omega_i.$$

We may assume that f is sufficiently smooth, that is $f \in W^{1,p}(\Omega) \cap C(\bar{\Omega}_i)$, to guarantee a solution via Lemma 1.6. The interpolated values which will restore the damaged part of the image is now given by the solutions u_i above.

A few reasons for using the p-Laplacian is the comparison principle, translation, rotation, scale and normalization invariance, possibility of adding constants to solutions and the fact that points are p-regular for the p-Laplacian, if we assume $p > n$. The last fact is necessary when using values valid only at single points, and this fact disqualifies the Laplacian. For more on this interpolation method and for a clear motivation why using the p-Laplacian in this problem we refer the reader to a paper by Caselles, Morel and Sbert [CMS98]. Another contribution to this field is the thesis by Almansa [A02], where a p-Laplacian model is used for interpolation of terrain elevation maps. Comparisons are made with other methods of interpolation and the outcome is favorable for the p-Laplacian approach.

2.3 Stochastic characterization

Concerning applications to the p-Laplacian, it is convenient to consider different characterizations of solutions. A p-harmonic function has a stochastic characterization. When $p = 2$ it is given by Brownian motion, and for $p \neq 2$ by the two player game tug-of-war.

2.3.1 The case $p = 2$: Brownian motion

Consider a 2-regular domain $\Omega \subset \mathbf{R}^n$ with boundary data $f \in C(\bar{\Omega})$ and the corresponding unique solution $u(x)$ to the Dirichlet problem. Fix $x \in \Omega$ and let B_t^x be Brownian motion started at x. Then u can be represented as the expected value of $f(B_\tau^x)$, where τ denotes the hitting time of $\partial\Omega$, that is

$$u(x) = E^x\left(f(B_\tau^x)\right),$$

see Figure 2.5. The connection to Brownian motion leads to several applications for the Laplacian. To mention a few, particle dispersion, population dynamics, optimal stopping problems and other problems in mathematical finance. Concerning more general linear equations, Brownian motion should be replaced by a more general process X_t, having a drift and noise term, solving a stochastic differential equation. For more on the connection between stochastic differential equations and linear partial differential equations of Laplace type, and for applications in this setting, we refer to the book by Øksendal [O03].

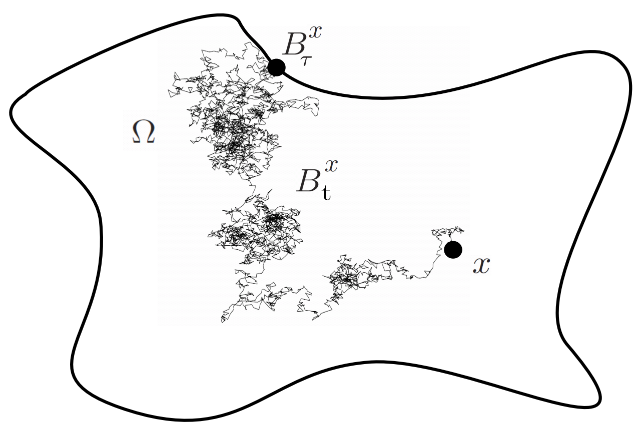

Figure 2.5: Brownian motion B_t^x started at x and hitting $\partial\Omega$ at B_τ^x.

2.3.2 The case $p \neq 2$: Tug-of-war

In this situation, no simple process replacing Brownian motion in the former case is known. The interpretation is instead given by the two-player game *tug-of-war*, introduced by Peres, Schramm, Sheffield and Wilson in [PSSW09] and [PS08]. We here only briefly describe the game. Consider a p-regular domain $\Omega \subset \mathbf{R}^n$ with boundary data $f \in W^{1,p}(\Omega) \cap C(\bar{\Omega})$ and the corresponding unique solution u to the Dirichlet problem. Fix a starting point $x \in \Omega$ and a step length $\epsilon > 0$. At every step, a fair coin is tossed and the winner gets to move a shared marker, which is initially placed at x. The chosen step $\boldsymbol{\nu}$ must satisfy $|\boldsymbol{\nu}| \leq \epsilon$, and the step is then perturbed by an orthogonal mean-zero noise vector of length $|\boldsymbol{\nu}|\sqrt{(n-1)/(p-1)}$. If $\partial\Omega$ gets within reach, then the winner of the coin toss chooses a terminating point $x_\tau \in \partial\Omega$, and the game ends. Player one now receives a payoff of value $f(x_\tau)$ from player two. As $\epsilon \to 0$, the value of the game tends to the p-harmonic function $u(x)$ solving the Dirichlet problem. When $p \to \infty$, the length of the noise vector tends to zero, and thus the game without any added noise gives an interpretation of ∞-harmonic functions. It is clear that player one (two) wants to end the game at a point x_τ so that $f(x_\tau)$ has a maximum (minimum).

Tug-of-war may serve as a useful tool when studying the p-Laplacian. In the case $p = \infty$, the tug-of-war game without noise was used to prove new uniqueness results for the ∞-Laplacian and for the equation $\Delta_\infty u = g$, see [PSSW09]. One difficulty with using tug-of-war to estimate p-harmonic functions is to find strategies for the players of which the value of the game (that is, the solution u) can be controlled. In some cases this has been successfully done, for example, in [PS08] several bounds of a p-harmonic measure (Definition 1.10) is presented.

Concerning applications in the setting of tug-of-war, we have not found any concrete example. However, it is noteworthy to mention that although tug-of-war was applied to solve problems in analysis, the founders of the game were originally motivated by the game itself ,see [PS08, page 95]. They expect that tug-of-war has applications to political and economical modeling. The game may serve as a natural model for situations in which opposing parties tries to improve their positions through incremental "tugs". Tug-of-war is related to many of the differential games (in which players choose drift and diffusion terms at each point in a domain) already used in economic modeling.

2.4 Minimization problems

In this section we discuss another characterization of p-harmonic functions, namely minimization problems. It is well known that for $1 < p < \infty$, the p-Laplacian is the Euler-Lagrange equation arising when minimizing the functional

$$G_p(v) := \int_\Omega |\nabla v|^p dx,$$

among certain functions having prescribed boundary values. In particular, see Theorem 2.1 below. As expected, the ∞-Laplacian plays a similar role: Consider the functional

$$G_\infty(v) := \sup_{x\in\Omega} |\nabla v|.$$

A *minimal Lipschitz extension* is a function $u \in W^{1,\infty}(\Omega)$ such that

$$G_\infty(u) \le G_\infty(v) \quad \text{whenever} \quad u - v \in W_0^\infty(\Omega), \tag{2.8}$$

where $\Omega \subset \mathbf{R}^n$ is a bounded domain. It follows from (2.8) that the minimal Lipschitz extension is a solution to the variational problem: Find a minimizer of the L^∞-norm of the gradient over all functions with prescribed boundary values. The ∞-Laplacian is the Euler-Lagrange equation of the minimal Lipschitz extension problem.

Concerning more general equations, the above functionals should be replaced by

$$G_{p,F}(v) = \int_\Omega F(x,\nabla v)^{p/2} dx \quad \text{and} \quad G_{\infty,F}(v) = \sup_{x\in\Omega} F(x,\nabla v) \tag{2.9}$$

respectively, where $F(x,\eta)$ satisfies certain conditions. In fact, if $1 < p < \infty$, then the Euler-Lagrange equation yields

$$\nabla \cdot \big(F(x,\eta)^{(p-2)/2} \nabla_\eta F(x,\eta)\big) = 0 \quad \text{in} \quad \Omega,$$

which is of type (1.2). If $p = \infty$, then we obtain the equation

$$\big\langle \nabla F(x,\nabla u), \nabla_\eta F(x,\nabla u)\big\rangle = 0 \quad \text{in} \quad \Omega.$$

The above equations are studied in Paper IV. The relation between the functionals in (2.9) as $p \to \infty$ has been studied by Juutinen in [Ju98] and has been used in Paper IV.

For more on Lipschitz extensions and their generalizations, see for example Aronsson [A65], [A66], [A67], Jensen [J93], Juutinen [Ju98], Crandall, Evans, Gariepy [CEG01], Barron, Evans, Jensen [BEJ08] and the references therein.

Concerning applications in this setting we mention the following. A material rod loaded by a torsional moment for an extended period of time and at sufficiently high temperature may exhibit a permanent plastic deformation called *creep*. Modeling of creep involves power-laws, the p-Laplacian as well as the functional G_p for large p. Here, G_p is the p-torsional stiffness, that is, the energy of the system. See Bhattacharya, DiBenedetto, Manfredi [BDM89].

The importance of variational problems in general should also be mentioned, since minimization problems arise in many contexts. For example, in the problem of the deflection of a loaded beam one seeks to minimize the maximum deflection. When constructing a machine, one often seeks to minimize the maximal stress applied to the construction. In chemotherapy (treatment of cancer with certain drug), one seeks to minimize the maximum tumor load. In numerical analysis, one is often interested in finding the best approximation to a given function, in certain function spaces.

We end Chapter 2 by recalling the following well known theorem and its proof.

Theorem 2.1. *Let $p \in (1, \infty)$. The function $u \in W^{1,p}(\Omega)$ is minimizing, that is*

$$G_p(u) \leq G_p(v) \quad \textit{whenever} \quad u - v \in W_0^{1,p}(\Omega),$$

if and only if u is a weak solution to the p-Laplacian in Ω.

Proof. Assume that $u \in W^{1,p}(\Omega)$ is minimizing, let $w \in W^{1,p}(\Omega)$ be given and consider the function

$$J(\epsilon) = \int_\Omega |\nabla(u + \epsilon w)|^p dx.$$

Since u is minimizing, we must have $J'(0) = 0$, which gives

$$J'(0) = p \int_\Omega |\nabla u|^{p-2} \nabla w \, dx = 0.$$

Since $w \in W^{1,p}(\Omega)$ was arbitrary, u is a weak solution to the p-Laplacian.

Next, assume that u is a weak solution to the p-Laplacian. Since $|\cdot|^p$ is a convex function, we have, for any vectors x and y

$$|x|^p + p|x|^{p-2}\langle x, x-y\rangle \leq |y|^p.$$

Hence

$$\int_\Omega |\nabla u|^p dx + p\int_\Omega |\nabla u|^{p-2}\langle \nabla u, \nabla(u-v)\rangle dx \leq \int_\Omega |\nabla v|^p dx.$$

Since $u - v \in W^{1,p}(\Omega)$ and u is a weak solution, the result follows. $\square$

Chapter 3

On the appended papers

In this chapter we give, for each of the five appended papers, the background of the study, the main theorems, informal proofs of the main theorems, contribution from authors and some thoughts about future work. Concerning the order of the appended papers, we consider the papers about the p-Laplacian (Papers I, II, and III) first, followed by papers treating more general equations of p-Laplace type (Papers IV and V). We note that the appended Papers I - V are slightly updated versions of previously published papers. Moreover, we point out that in Papers I–V, authors are sorted in alphabetic order.

Since boundary Harnack inequalities are the main focus in this thesis, we recall that a *boundary Harnack inequality* states the following: Consider two positive functions u and v which vanish on a portion of the boundary of a domain; then these functions vanish at the same rate, that is, close to the boundary we have $u \approx v$.

3.1 Paper I Estimates for p-harmonic functions vanishing on a flat

This paper was created during the periods January–June 2010 and December 2010–Mars 2011, submitted to Nonlinear analysis: Theory, Methods and Applications 2011–03–16, and accepted 2011–06–25.

3.1.1 Background

Recently several breakthroughs have been made in the study of boundary behavior of p-harmonic functions. In [LN07], [LN10] Lewis and Nyström establish a number of new results, for example the boundary Harnack inequality and Hölder continuity for ratios of positive p-harmonic functions, $1 < p < \infty$, vanishing on a portion of the boundary of a Lipschitz domain $\Omega \subset \mathbf{R}^n$. Moreover, the p-Martin boundary problem was resolved under the assumption that Ω is either a convex domain, a C^1-domain or a Lipschitz domain with small constant. In [LN08*], again by Lewis and Nyström, these questions were resolved in the setting of δ-Reifenberg flat and Ahlfors regular NTA-domains, and in [LLuN08] parts of these results were extended to more general equations of p-Laplace type allowing for variable coefficients. Furthermore, in Paper V the results by Lewis, Nyström and me in [LLuN08] were extended to hold also for equations with lower order terms, in particular, for equations of type (1.2). To motivate the research in the above direction, we mention that boundary Harnack inequalities are useful, for example, when studying free boundary problems as in Paper III, and the Martin problem solved in Paper V.

In all the above cases, the conditions imposed on the domains imply that the boundary is $(n-1)$-dimensional. In Paper I, we begin to study the above problems for a low-dimensional boundary. In particular, it is well known that for $n < p$, points are p-regular. Hence, in this case it is possible to study p-harmonic functions which vanish on a boundary having dimension m, $0 \leq m \leq n-1$. A natural first step in this setting is to consider an m-dimensional plane, here called an *m-flat*. The borderline case $p = n$ has been solved by Lindqvist in [L85], but to our knowledge, the following results are new for $p \neq n$.

3.1.2 Main results

We prove a boundary Harnack inequality for p-harmonic functions vanishing on a portion of an m-flat Λ_m, where $0 \leq m \leq n-1$. Moreover, we establish the growth-exponent of p-harmonic functions near Λ_m. Let N denote the set of natural numbers. In particular, we prove the following theorem.

Paper I: Theorem 1.1. *Suppose that $m, n \in N$ such that $m \in [0, n-1]$, let Λ_m be an m-flat, and suppose that $w \in \Lambda_m$, $r \in (0, \infty)$ and $p \in (n-m, \infty]$. Assume that u is a positive p-harmonic function in $B(w, 2r) \setminus \Lambda_m$, continuous in $B(w, 2r)$ with $u = 0$ on $B(w, 2r) \cap \Lambda_m$, and suppose that $\beta = (p-n+m)/(p-1)$ with $\beta = 1$ if $p = \infty$. Then there exists a constant $c(n, p)$ such that*

$$(i) \qquad d(x, \Lambda_m)^\beta \leq c\, \frac{u(x)}{u(a_r(w))} \quad \text{whenever} \quad x \in B(w, r/c).$$

If in addition $n \leq p$, then

$$(ii) \qquad \frac{u(x)}{u(a_r(w))} \leq c\, d(x, \Lambda_m)^\beta \quad \text{whenever} \quad x \in B(w, r/c).$$

Here, $a_r(w)$ is a point satisfying $d(a_r(w), \Lambda_m) = r$ and $a_r(w) \in \partial B(w, r)$. Moreover, there exists $\hat{p}$ such that if $\hat{p} \leq p \leq \infty$, then the constant c can be chosen independent of p but depending on n and $\hat{p}$.

A corollary yields that $u \in C^{0,\beta}(B(w, r/c))$, and β is the optimal Hölder exponent for u. Figure 3.1 shows an example of a situation where Theorem 1.1 from Paper I applies.

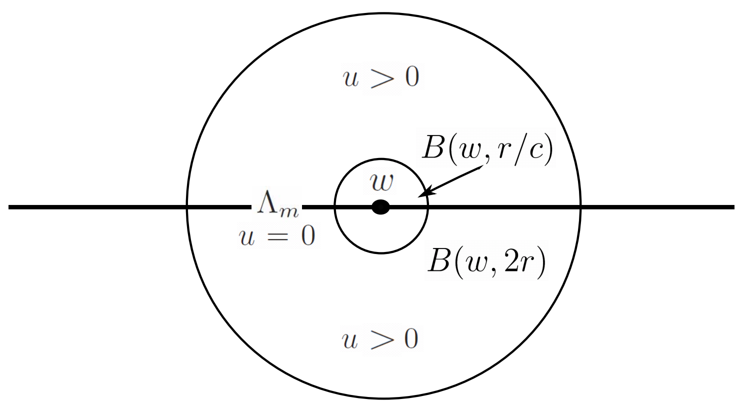

Figure 3.1: The setting of Theorem 1.1 in Paper I.

3.1.3 Informal proof

The main argument in the proof lies in the construction of two comparison functions $\hat{u}$ and $\check{u}$. Besides these functions, the proof uses only the comparison principle, Harnack's inequality and (ii) in Lemma 1.9. The argument is approximately the same for the upper and lower bound, so we only explain the upper bound.

In the following we denote points $x \in \mathbf{R}^n$ by $x = (x', x'')$, where $x' = (x_1, x_2, \ldots, x_{n-m})$ and $x'' = (x_{n-m+1}, x_{n-m+2}, \ldots, x_n)$. We assume, as we may due to Lemma 1.7, that $w = 0$ and that $\Lambda_m = \{x \in \mathbf{R}^n : |x'| = 0\}$. To construct the comparison function $\hat{u}$ for the upper bound, we use a prototype solution f, as starting point. Consider the function

$$\bar{f} = |x'|^{\frac{p-n+m}{p-1}} \quad \text{where} \quad x' \in \mathbf{R}^{n-m}.$$

This is the fundamental solution to the p-Laplacian in $\mathbf{R}^{n-m}$ and thus $\bar{f}$ is p-harmonic in $\mathbf{R}^{n-m} \setminus \{0\}$. Extend $\bar{f}$ to $\mathbf{R}^n$ by "stretching" $\bar{f}$ in the m directions spanning the m-flat Λ_m. Denote this extended function by f, then f is p-harmonic in $\mathbf{R}^n \setminus \Lambda_m$ and satisfies Theorem 1.1. Next, we know from the comparison principle and Harnack's inequality that u can be bounded above by the fundamental solution $\hat{f}$ introduced in (1.1), giving $u \leq cd(x, \Lambda_m)^{(p-n)/(p-1)}$ close to Λ_m. Inspired by the fact that $|x'|^{(p-n)/(p-1)}$ is a supersolution in $\mathbf{R}^{n-m}$, we build $\hat{u}$ according to

$$\hat{u} = |x'|^{\frac{p-n+m}{p-1}} + |x''|^2 |x'|^{\frac{p-n}{p-1}} - \frac{1}{2}|x'|^2.$$

Then $\hat{u}$ grows fast enough to dominate u on the boundary of a cylinder, and $\hat{u}$ has the desired growth on the plane $|x''| = 0$. Hence, to find the upper bound, we apply the comparison principle to $\hat{u}$ and u on the cylinder. Finally, to prove that $\hat{u}$ is a supersolution, it suffices to do a tedious calculation. See Lemma 3.1 in Paper I. $\square$

3.1.4 Future work

(i) Replace the plane Λ_m with a more general set, for example a set that is well approximable by m-flats; a generalization of δ-Reifenberg flat domains may do. (ii) Prove that the ratio u/v of p-harmonic functions is Hölder continuous close to Λ_m. (iii) Apply the results for the low-dimensional set to very thin domains with $(n-1)$-dimensional boundary. For example, consider a p-harmonic function in $\mathbf{R}^3$, which vanish on a thin cable.

3.2 Paper II
Decay of a p-harmonic measure in the plane

This paper was created during the period October 2009–June 2010, and submitted to Potential Analysis 2011–01–31.

3.2.1 Background

Initially we were interested in the p-Green function and 3G-inequalities. By Hirata [H08, Theorem 2.7], the 3G-inequality relies on asymptotic behaviour of a p-harmonic measure. In particular, a 3G inequality would be possible if we could prove that the p-Green function was quasi-symmetric, and our result of Paper II implies that this is not true in $C^{1,1}$-domains in the plane, see Corollary 2.5 and Remark 2.6 in Paper II. We first tried to understand p-harmonic measure by using the stochastic characterization of p-harmonic functions tug-of-war, introduced by Peres, Schramm, Sheffield and Wilson in [PSSW09] and [PS08]. We did not succeed. In fact, we did not find strategies for the players in the game good enough of which it was possible to do some analysis giving barriers for the p-harmonic measure. In $\mathbf{R}^2$ however, we did prove a result using some singular p-harmonic functions discovered by Aronsson [A86] and Persson [Pe89].

3.2.2 Main results

We study the asymptotic behavior of the p-harmonic measure ω_p introduced in Definition 1.10. Let $v \in [1/2, \infty)$, $p \in (1, \infty]$ and

$$q(v,p) = \frac{(2v+1)(2-p) + pv^2 + (v+1)\sqrt{(2v+1)(2-p)^2 + p^2v^2}}{2(p-1)(2v+1)}, \quad (3.1)$$

interpreted as a limit when $p = \infty$, so that $q(v, \infty) = v^2/(2v+1)$. Our main result is the following characterization of p-harmonic measure of plane domains. Roughly speaking, the theorem implies that if $\partial\Omega$ fits in between a sector and a generalized ball, both with aperture π/v and apex at $w \in \partial\Omega$, see Figure 3.2, then

$$\omega_p\big(\Delta(w,\delta), w_0, \Omega\big) \approx \delta^{\,q},$$

when $\delta > 0$ is small.

If the domain is unbounded, then the point at infinity is by definition in the boundary. For definitions of generalized interior ball, Harnack chain and exterior corkscrew conditions, we refer the reader to Definitions 1.13 and 1.23 in Section 1.6.

Paper II: Theorem 2.1. *Let $\Omega \subseteq \mathbf{R}^2$ be a domain, $w_0 \in \Omega$, $p \in (1,\infty]$, $v \in [1/2,\infty)$, $r_1 \in (0,1)$, and let $q = q(v,p)$ be as in (3.1). There exist constants C_1 and C_2 such that the following is true.*

(i) If $w \in \partial\Omega \setminus \{\infty\}$ is such that there exists a sector, with aperture π/v and apex at w, which contains Ω, then

$$\omega_p\big(\Delta(w,\delta), w_0, \Omega\big) \le C_1 \left(\frac{\delta}{|w_0 - w|}\right)^q$$

for all $\delta > 0$.

(ii) Assume that Ω satisfies the Harnack chain condition and that either $p > 2$ or that Ω satisfies the exterior corkscrew condition on $\Delta(w,\delta)$. There exists a constant $\delta_0 > 0$, such that if $w \in \partial\Omega \setminus \{\infty\}$ is of type (q, r_1, v), then

$$\delta^{\,q} \le C_2\, \omega_p\big(\Delta(w,\delta), w_0, \Omega\big)$$

for all $0 < \delta < \delta_0$.

The constant C_1 depends only on p and v; C_2 depends only on p, v, r_1, $d(w_0, \partial\Omega)$, $|w - w_0|$, and if $p \in (1,2]$ also on the exterior corkscrew condition, that is, on r_0 and M; δ_0 depends only on p, v, r_1, and if $p \in (1,2]$ also on r_0. Moreover, C_1 decreases in p, δ_0 increases in p and, if $p > 2$, then C_2 is decreasing in p.

The exponent $q(v,p)$ is decreasing in p and increasing in v. Moreover

$$\lim_{p\to 1} q = \infty, \qquad \lim_{p\to\infty} q = \frac{v^2}{2v+1} \qquad \text{and} \qquad \lim_{v\to\infty} q = \infty.$$

For the classical case $p = 2$, we have that $q(v,2) = v$.

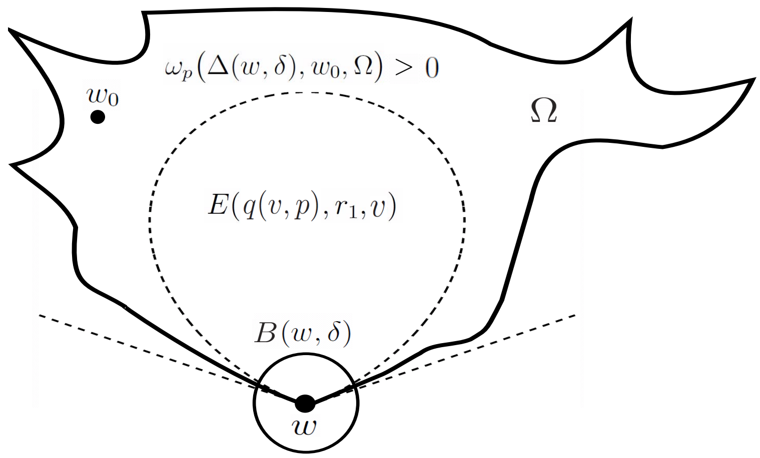

Figure 3.2: Theorem 2.1 with $v = 3/2$ and $p = 10$. The boundary $\partial\Omega$ fits in between a sector with aperture π/v and a generalized ball.

In the case $p = \infty$, the theorem generalizes to domains $\Omega \subset \mathbf{R}^n$ which is rotationally invariant around an axis. This follows from the $\mathbf{R}^2$ result and from properties of the ∞-Laplacian. In fact, by rotating the ∞-harmonic function in $\mathbf{R}^2$ around the axis, we obtain the result in higher dimensions. See Corollary 2.4 in Paper II. Moreover, by Hirata [H08, Theorem 2.7], our result implies that the p-Green function for $p \in (1,2)$ is not quasi-symmetric in plane $C^{1,1}$-domains. See Corollary 2.5 and Remark 2.6 in Paper II.

3.2.3 Informal proof

The proof is based on comparison with certain positive p-harmonic functions in the plane of the form $H_{v,p}(x,y) = r^{-q(v,p)} h_{v,p}(\phi)$, where (r,ϕ) are polar coordinates for $(x,y) \in \mathbf{R}^2$. These functions were discovered by Aronsson [A86] and Persson [Pe89]. Besides the solutions $H_{v,p}$, the proof uses Harnack's inequality, the comparison principle and boundary regularity for p-harmonic measure. The functions $H_{v,p}$ are positive, have pole at $(0,0)$ and are p-harmonic in the sectors

$$\mathcal{S}_v = \left\{(x,y) \in \mathbf{R}^2 \setminus \{(0,0)\};\ |\phi| < \frac{\pi}{2v}\right\}, \quad v \geq \frac{1}{2}.$$

We start with the upper bound. Since the p-harmonic measure $\omega_p = \omega_p(\Delta(w,\delta), w_0, \Omega)$ is an infimum over p-superharmonic functions which are

≥ 1 on $\Delta(w, \delta)$ and ≥ 0 on $\partial\Omega \setminus \Delta(w, \delta)$, we only need to create one such function, then by definition we have an upper bound of ω_p. $H_{v,p}$ is p-harmonic and hence also p-superharmonic in the sector $\mathcal{S}_v$, and also in Ω since by assumption $\Omega \subset \mathcal{S}_v$. Observe that thanks to Lemma 1.7, Ω may have been translated and rotated to satisfy this assumption. Moreover, $0 < H_{v,p}$, so we focus on $\Delta(w, \delta)$. To obtain a good bound, we need to carefully normalize and translate $H_{v,p}$, and it turns out that we obtain a nice comparison function $\widetilde{H}_\delta$ by

$$\widetilde{H}_\delta(x, y) = (2\delta)^{q(v,p)} H_{v,p}(x + \delta, y).$$

By examining $H_{v,p}$ in detail, see Lemma 4.1 in paper II, we conclude the existence of c so that $c\widetilde{H}_\delta \geq 1$ on $\Delta(w, \delta)$. Since in addition $c\widetilde{H}_\delta > 0$ and $c\widetilde{H}_\delta$ is p-harmonic in Ω, the upper bound now follows from the definition of p-harmonic measure and some calculations on $\widetilde{H}_\delta$.

To prove the lower bound we use, in addition to the functions $H_{v,p}$, Harnack's inequality, the comparison principle and boundary regularity for p-harmonic functions. Assume that $w = 0$ and that the generalized ball $E(q(v, p), r_1, v) \subset \Omega$. We start by concluding that ω_p is not too small close to $\Delta(0, \delta)$. In fact, from boundary regularity it follows that $\omega_p > c^{-1}$ on $\bar{\widetilde{B}}_\delta$ where $\widetilde{B}_\delta$ is a ball in the domain close to $\Delta(0, \delta)$. Next, we introduce H^*_δ, which is, similar to $\widetilde{H}_\delta$, the barrier function constructed from $H_{v,p}$ by translating and normalizing $H_{v,p}$. In fact, we put

$$H^*_\delta(x, y) \approx \delta^{\,q(v,p)} H_{v,p}(x - 2\delta, y),$$

so it has pole at the center of $\widetilde{B}_\delta$, which is $(2\delta, 0)$. We prove a bound from above for H^*_δ on $\partial\widetilde{B}_\delta$, and also a bound from above for H^*_δ on the boundary of the generalized ball, that is $\partial E(q(v, p), r_1, v)$. From these two bounds and the bound from below for ω_p on $\widetilde{B}_\delta$, we obtain, by standard applications of the comparison principle and the Harnack inequality, the lower bound of ω_p.
. $\square$

3.2.4 Contribution from authors

Jonatan Vasilis did half of the paper, and I did the rest. The arguments and ideas where given equally by both authors.

3.2.5 Future work

(i) Prove this in $\mathbf{R}^n$; if we proceed as in $\mathbf{R}^2$ then one way is to try some generalizations of Aronssons explicit solutions to $\mathbf{R}^n$. Another way may be to use tug-of-war; given the result in $\mathbf{R}^2$, we may find hints of strategies in the game. (ii) Theorem 2.1 in Paper II may imply additional properties of the p-Green function.

3.3 Paper III
On a two-phase free boundary condition for p-harmonic measures

This paper was created during the period September 2007–May 2008, submitted to Manuscripta Mathematica 2008–05–07, and accepted 2009–02–16.

3.3.1 Background

In Paper III we generalize parts of the results proved by Kenig and Toro in [KT06], concerning two-phase free boundary problems in the linear case $p = 2$, to hold also in the nonlinear setting $1 < p < \infty$. In particular, we generalize the first part of [KT06] which is the following.

Let $\Omega^1, \Omega^2 \subset \mathbf{R}^n$ be NTA-domains such that $\Omega^1 \cap \Omega^2 = \emptyset$, $w \in \partial\Omega^1 \cap \partial\Omega^2$ and that

$$\partial\Omega^1 \cap B(w, 2r) = \partial\Omega^2 \cap B(w, 2r). \tag{3.2}$$

Put $\Delta(w, 2r) = \partial\Omega^1 \cap B(w, 2r) = \partial\Omega^2 \cap B(w, 2r)$ and let $x_i \in \Omega^i$, $i \in \{1, 2\}$ and ω^i be the harmonic measure defined with respect to x_i and Ω^i. Assume that ω^2 is absolutely continuous with respect to ω^1 on $\Delta(w, 2r)$, $d\omega^2 = k d\omega^1$ for ω^1-almost every point in $\Delta(w, 2r)$ and assume that $\log k \in VMO(\Delta(w, r), \omega^1)$, where VMO is the space of functions of vanishing mean oscillation recalled below (1.8) in Paper III.

In [KT06] it is proved that if $\Delta(w, 2r)$ is δ-Reifenberg flat for some δ small enough, then $\Delta(w, r/2)$ is Reifenberg flat with vanishing constant. Figure 3.3 illustrates the setting of the two-phase free boundary problem.

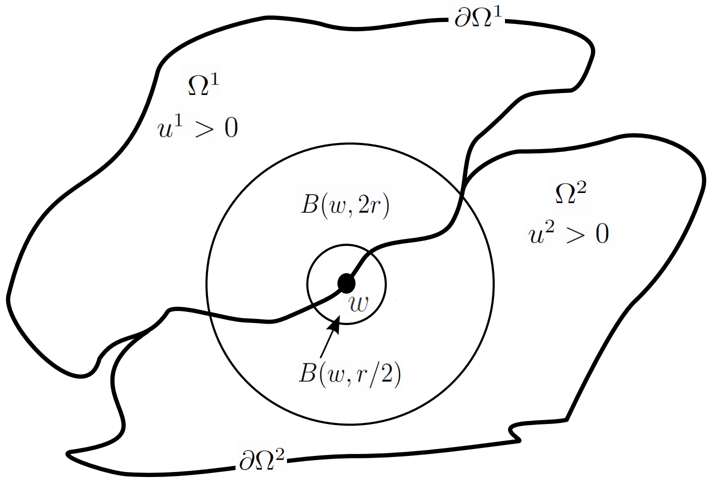

Figure 3.3: The setting of the two-phase free boundary problem.

3.3.2 Main results

In Paper III, we prove the same result for the case $1 < p < \infty$, and in this case the harmonic measure is replaced by the p-harmonic measure as defined in Definition 1.11. In particular, we let u^i, $i \in \{1, 2\}$, denote a positive p-harmonic function in Ω^i. We assume that u^i is continuous in $\bar{\Omega}^i \cap B(w, 2r)$ and that $u^i = 0$ on $\partial\Omega^i \cap B(w, 2r)$. We extend u^i to $B(w, 2r)$ by defining $u^i \equiv 0$ on $B(w, 2r) \setminus \Omega^i$. Then there exists a unique finite positive Borel measure μ^i, generalizing the harmonic measure to $1 < p < \infty$, referred to as a p-harmonic measure, recall Definition 1.11. The measure μ^i has support in $\partial\Omega^i \cap B(w, 2r)$ and satisfies

$$\int\limits_{\mathbf{R}^n} |\nabla u^i|^{p-2} \langle \nabla u^i, \nabla \phi \rangle dx = - \int\limits_{\mathbf{R}^n} \phi d\mu^i,$$

whenever $\phi \in C_0^\infty(B(w, 2r))$.

In Paper III, we prove the following theorem. Here, $\hat{\delta}(n)$ ensures that the domains are NTA-domains whenever $0 < \delta < \hat{\delta}$.

Paper III: Theorem 1.10. *Let Ω^1 and Ω^2, be two (δ, r_0)-Reifenberg flat domains, for some $0 < \delta < \hat{\delta}$ and $r_0 > 0$, and assume that (3.2) holds for some $w \in \mathbf{R}^n$, $0 < r < r_0$. Let, for given $p \in (1, \infty)$, u^1 and u^2 be positive p-harmonic functions as above and let μ^1 and μ^2 be corresponding p-harmonic measures defined as in Definition 1.11. Assume that μ^2 is absolutely continuous with respect to μ^1 on $\Delta(w, 2r)$, $d\mu^2 = kd\mu^1$ for μ^1-almost every point in $\Delta(w, 2r)$ and that $\log k \in VMO(\Delta(w, r), \mu^1)$. Then there exists $\tilde{\delta} = \tilde{\delta}(p, n) < \hat{\delta}$, such that if $0 < \delta < \tilde{\delta}$, then $\Delta(w, r/2)$ is Reifenberg flat with vanishing constant.*

Concerning corollaries, we note that Theorem 1.10 applies also if the (δ, r_0)-Reifenberg flat domains are replaced by Lipschitz domains with small enough constant L, or by quasiballs with K close enough to 1. See the definitions and discussions in Section 1.6 for more about geometric assumptions.

3.3.3 Informal proof

The proof is based on a blow-up-type argument. To describe the procedure, let $\Omega = \Omega^1$ be as in Theorem 1.10, $w \in \partial\Omega$ and $0 < r < r_0$. Let $\{w_j\}$ and $\{r_j\}$ be sequences such that $w_j \in \Delta(w, r/2)$, $r_j \to 0$, and assume that $\Delta(w_j, r_j) \subset \Delta(w, r)$ for every j. Define

$$\Omega_j = \{r_j^{-1}(x - w_j) : \quad x \in \Omega\}. \tag{3.3}$$

Let $u = u^1$ and $\mu = \mu^1$ be as in Theorem 1.10 and define, for $j \geq 1$,

$$u_j(z) = \left(\frac{r_j^{n-p}}{\mu(\Delta(w_j, r_j))}\right)^{1/(p-1)} u(w_j + r_j z), \tag{3.4}$$

whenever $z \in \Omega_j$. Moreover, we define a measure μ_j on $\mathbf{R}^n$ in the following way. Given a Borel set $E \subset \mathbf{R}^n$ we define

$$\mu_j(E) = \frac{\mu(\{z \in \partial\Omega : \ (z - w_j)/r_j \in E\})}{\mu(\Delta(w_j, r_j))}. \tag{3.5}$$

It follows that Ω_j, u_j and μ_j has similar relations as Ω, u and μ, and in Lemmas 3.1 and 3.4 in Paper III, we prove that the quantities also converge desirable. In particular, in Lemma 3.1 we prove, by repeating the argument from [KT03, Theorem 4.1], the following. There exists a subsequence (which we relabel) satisfying

$$\Omega_j \to \Omega_\infty \qquad \text{and} \qquad \partial\Omega_j \to \partial\Omega_\infty \quad \text{as} \quad j \to \infty$$

in the Hausdorff distance sense, uniformly on compact subsets of $\mathbf{R}^n$, and Ω_∞ is a 4δ-Reifenberg flat domain. For a definition of convergence in the Hausdorff distance sense, see the beginning of Section 3 in Paper III. Moreover, in Lemma 3.4 in Paper III we prove that there exists a subsequence (which we relabel) satisfying

$$u_j \to u_\infty \qquad \text{and} \qquad \mu_j \to \mu_\infty \quad \text{as} \quad j \to \infty$$

where $u_j \to u_\infty$ uniformly on compact subsets and u_∞ is a non-negative p-harmonic function in Ω_∞ with $u_\infty = 0$ continuously on $\partial\Omega_\infty$. Moreover, $\mu_j \to \mu_\infty$ weakly and whenever $\phi \in C_0^\infty(\mathbf{R}^n)$ we have

$$\int_{\mathbf{R}^n} |\nabla u_\infty|^{p-2} \nabla u_\infty, \nabla\phi\rangle dx = -\int_{\partial\Omega_\infty} \phi d\mu_\infty.$$

Next, we use the assumption that μ^2 is absolutely continuous with respect to μ^1 on $\Delta(w, 2r)$, that is $d\mu^2 = k d\mu^1$ for μ^1-almost every point in $\Delta(w, 2r)$, and the assumption $\log k \in VMO(\Delta(w, r), \mu^1)$ imposed on the p-harmonic measures in Theorem 1.10. In particular we conclude, see Lemma 3.6 in Paper III, that then there exist $\alpha = \alpha(p, n, \hat{\delta}) \in (0, 1)$, and $c = c(p, n, \hat{\delta})$ such that if $\hat{w} \in \Delta(w, r/2)$, $\hat{r} < r/2$ and $E \subset \Delta(\hat{w}, \hat{r})$, then

$$\frac{\mu^2(E)}{\mu^2(\Delta(\hat{w}, \hat{r}))} \leq c \left(\frac{\mu^1(E)}{\mu^1(\Delta(\hat{w}, \hat{r}))} \right)^\alpha. \tag{3.6}$$

Consider now the sequences $\{w_j\}$ and $\{r_j\}$ introduced above (3.3). Let $\{\Omega_j^i\}$, $\{u_j^i\}$ and $\{\mu_j^i\}$, $i \in \{1, 2\}$, be defined according to (3.3), (3.4) and (3.5) respectively. By using (3.6), the weak convergence $\mu_j^i \to \mu_\infty^i$ and again that $d\mu^2 = k d\mu^1$ and $\log k \in VMO(\Delta(w, r), \mu^1)$, we prove, see Lemma 3.7 in Paper III, that

$$\mu_\infty^1 \equiv \mu_\infty^2. \tag{3.7}$$

To proceed, we will make use of Lemma 2.8 in Paper III. In fact, this lemma states that if $\Omega \subset \mathbf{R}^n$ is a (δ, r_0)-Reifenberg flat domain, $w \in \partial\Omega$, $0 < r < r_0$, and u is a non-negative continuous p-harmonic function in $\Omega \cap B(w, r)$ and that $u = 0$ continuously on $\Delta(w, r)$. Then there exist, for $\epsilon > 0$ given, $\tilde{\delta} = \tilde{\delta}(p, n, \epsilon) > 0$ and $c = c(p, n, \epsilon)$, such that

$$c^{-1}\left(\frac{\hat{r}}{r}\right)^{1+\epsilon} \leq \frac{u(a_{\hat{r}}(w))}{u(a_r(w))} \leq c\left(\frac{\hat{r}}{r}\right)^{1-\epsilon}, \tag{3.8}$$

whenever $0 < \delta \leq \tilde{\delta}$ and $0 < \hat{r} < r/16$.

Now, let $u_\infty(x) = u^1_\infty(x)$ if $x \in \Omega^1_\infty$ and $u_\infty(x) = -u^2_\infty(x)$ if $x \in \Omega^2_\infty$. By using (3.7), the fact that $u^i_j \to u^i_\infty$, u_∞ is a non-negative p-harmonic function in Ω_∞ with $u_\infty = 0$ continuously on $\partial\Omega_\infty$, Hölder continuity of the gradient of p-harmonic functions (see (iv) in Lemma 1.8) and finally (3.8), we can prove, see Lemma 3.22 in Paper III, that

$$u_\infty \text{ is a linear function on } \mathbf{R}^n \qquad \text{and} \qquad \Omega^1_\infty, \Omega^2_\infty \text{ are halfspaces.} \tag{3.9}$$

From (3.9) it is realized that Ω^1 and Ω^2 becomes more flat on smaller scales, and the conclusion of Theorem 1.10 in Paper III follows by a simple contradiction argument. $\square$

3.3.4 Contribution from authors

The main ideas in this paper were given by Kaj Nyström. The writing was done by me and Kaj Nyström.

3.3.5 Future work

(i) By using the boundary Harnack inequality proved in Paper V, we believe that it is possible to generalize Paper III to equations of the type studied in Paper V. (ii) The paper by Kenig and Toro [KT06] also shows that in the linear case $p = 2$, the δ-Reifenberg flatness assumption and the assumption $\log k \in VMO(\Delta(w, r), \mu^1)$, can be replaced by the assumption that Ω^1 and Ω^2 are chord-arc domains, and $\log(d\omega^1/d\sigma), \log(d\omega^2/d\sigma) \in VMO(\Delta(w, r), d\sigma)$, where σ is surface measure. A next step is to generalize this to hold also in the nonlinear setting $1 < p < \infty$.

3.4 Paper IV
The boundary Harnack inequality for solutions to equations of Aronsson type in the plane

This paper was created during the period April 2007–April 2008, submitted to Mathematische Zeitschrift 2008–04–17 and refused two years later, 2010–04–29. Re-submitted to Annales Academiæ Scientiarum Fennicæ Mathematica 2010–04–30, and accepted 2010–09–10.

3.4.1 Background

As in Paper I, we continue our progress in proving boundary Harnack inequalities for p-Laplace type problems, and in this study we focus on the case $p = \infty$. In [BL05], Bennewitz and Lewis proved a boundary Harnack inequality for p-harmonic functions in a domain $\Omega \subset \mathbf{R}^2$, assuming that $\partial\Omega$ is a quasicircle. In [LN08], Lewis and Nyström proved that the constants in the former proof could be chosen independent of p if p is large. They used this to prove a boundary Harnack inequality for ∞-harmonic functions by letting $p \to \infty$. A natural next step is to generalize the equation, and in this paper, we prove results in that direction.

3.4.2 Main Results

In this paper we prove a boundary Harnack inequality for solutions to the p-Laplace type equation

$$\nabla \cdot A(x, \nabla u) = 0, \tag{3.10}$$

in a domain $\Omega \subset \mathbf{R}^2$ under the assumption that $\partial\Omega$ is a quasicircle. The function $A(x, \eta)$ is assumed to be of p-Laplace character. We call solutions to (3.10) A-harmonic functions, see Definitions 1.1 and 1.2 in Paper IV. We conclude that our result is independent of p when p is large, and from this fact we derive a boundary Harnack inequality for solutions to the Aronsson type equation

$$\big\langle \nabla F(x, \nabla u), \nabla_\eta F(x, \nabla u) \big\rangle = 0. \tag{3.11}$$

Solutions to (3.11) will be called F-∞-harmonic functions, see Definitions 1.5 and 1.6 in paper IV. Before stating our results, we note that if $\Omega \subset \mathbf{R}^2$ and $\partial\Omega$ is a Jordan curve, then the conditions $\partial\Omega$ is a quasicircle, Ω is an NTA-domain, Ω is a uniform domain are all equivalent. See (1.1)–(1.3) and the following discussion in Paper IV for details concerning the geometry in $\mathbf{R}^2$. See also Section 1.6. We first present the theorem concerning A-harmonic functions.

Paper IV: Theorems 1.3 and 1.4. *Assume that $\Omega \subset \mathbf{R}^2$, $\partial\Omega$ is a Jordan curve and that Ω is a uniform domain with constant $\widehat{M}$. Let $w \in \partial\Omega$, $0 < r \leq r_0$ and suppose that u and v are positive A-harmonic functions in $\Omega \cap B(w,r)$, continuous on $\bar{\Omega} \cap B(w,r)$ with $u = 0 = v$ on $\Delta(w,r)$. Then there exists a constant c, depending only on $\widehat{M}, p, \Lambda/\lambda$, such that if $\tilde{r} = r/c$, then*

$$c^{-1}\frac{u(a_{\tilde{r}}(w))}{v(a_{\tilde{r}}(w))} \leq \frac{u(x)}{v(x)} \leq c\frac{u(a_{\tilde{r}}(w))}{v(a_{\tilde{r}}(w))} \quad \textit{whenever} \quad x \in \Omega \cap B(w,\tilde{r}).$$

Moreover, there exists $\hat{p} > 2$ such that if $\hat{p} \leq p < \infty$, then the constant c can be chosen independent of p but depending on $\widehat{M}, \hat{p}, \Lambda/\lambda$.

We note that $\widehat{M}$ is introduced in the definition of a uniform domain and that Λ, λ are introduced in Definition 1.1 in Paper IV, giving the conditions of $A(x,\eta)$. Figure 3.4 illustrates the boundary Harnack inequality.

From the above theorem, we derive the corresponding result for solutions to equations of Aronsson type.

Paper IV: Theorem 1.7. *Let Ω, w and r be as in the preceding theorem. Suppose that u and v are positive F-∞-harmonic functions in $\Omega \cap B(w,r)$, continuous on $\bar{\Omega} \cap B(w,r)$ with $u = 0 = v$ on $\Delta(w,r)$. Then there exists a constant c, depending only on $\alpha, \beta, \gamma, \bar{p}, \bar{c}, \widehat{M}$, such that if $\tilde{r} = r/c$, then*

$$c^{-1}\frac{u(a_{\tilde{r}}(w))}{v(a_{\tilde{r}}(w))} \leq \frac{u(x)}{v(x)} \leq c\frac{u(a_{\tilde{r}}(w))}{v(a_{\tilde{r}}(w))} \quad \textit{whenever} \quad x \in \Omega \cap B(w,\tilde{r}).$$

We note that $\alpha, \beta, \gamma, \bar{p}, \bar{c}$, were introduced in Definition 1.5 in Paper IV, giving the conditions of $F(x,\eta)$.

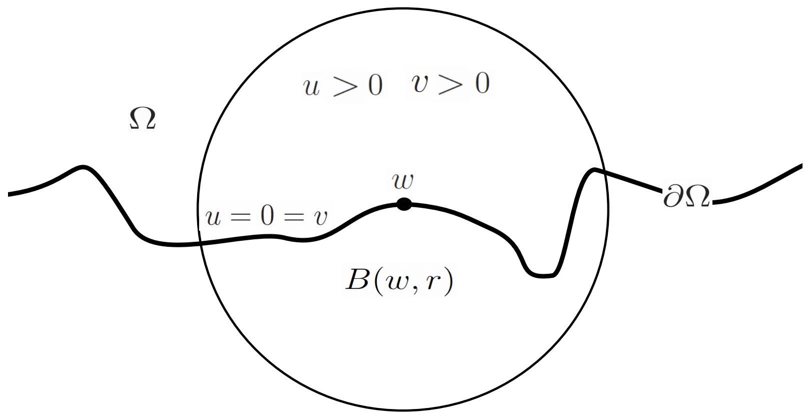

Figure 3.4: The setting of the boundary Harnack inequality.

3.4.3 Informal proof

Theorems 1.3 and 1.4. This proof originates from Bennewitz, Lewis and Nyström [BL05, Lemma 2.16] and [LN08], and we therefore only briefly describe it. The proof uses that $\Omega \subset \mathbf{R}^2$, $\partial\Omega$ is a quasicircle, the comparison principle, Harnack's inequality, (ii) in Lemma 1.9 and the doubling property of the p-harmonic measure μ defined in Definition 1.11. To prove the theorem we need to generalize the proofs of the above mentioned estimates to hold for solutions of (3.10), independent of p if p is large. To do so we have followed approximately the arguments from the p-independent p-Laplacian case already established in [LN08]. $\square$

Theorem 1.7. The main argument here is to use a limiting process as $p \to \infty$ mentioned in the Introduction, and a uniqueness result by Juutinen [Ju98, Theorem 4.25], which is a generalization of the corresponding uniqueness proof by Jensen for the ∞-Laplacian [J93]. To understand the proof of the theorem we observe the relation between solutions of (3.10) and (3.11). Consider a function $F(x, \eta)$ satisfying Definition 1.5 in paper IV and define

$$A(x, \eta) = F(x, \eta)^{(p-2)/2} \nabla_\eta F(x, \eta).$$

It follows that then $A(x,\eta)$ satisfies Definition 1.1 in Paper IV for some λ and Λ independent of p. Let $p \in (1,\infty)$, Ω be a bounded domain and let $f \in W^{1,\infty}(\Omega) \cap C(\bar{\Omega})$. Then there exists, see Lemma 2.A in Paper IV, a unique weak solution u_p to the Dirichlet problem

$$\begin{aligned} \nabla \cdot A(x, \nabla u_p) = \nabla \cdot \big(F(x, \nabla u_p)^{(p-2)/2} \nabla_\eta F(x, \nabla u_p)\big) = 0 \quad \text{in} \quad \Omega, \\ \lim_{x \to y} u_p(x) = f(y) \quad \text{for all} \quad y \in \partial\Omega. \end{aligned}$$

Let $u_\infty \in W^{1,\infty}(\Omega) \cap C(\bar{\Omega})$ be the unique viscosity solution to (3.11) with boundary data defined by f guaranteed by [Ju98, Theorem 4.25]. Then, arguing as in [Ju98, Theorem 1.15, Proposition 2.5 and Corollary 3.8] we see that there exists a sequence $\{p_j\}$, $p_j \to \infty$ as $j \to \infty$, such that $u_{p_j} \to u_\infty$ uniformly in Ω as $j \to \infty$. In particular, the unique solution u_∞ is the uniform limit of the corresponding unique solutions $\{u_{p_j}\}$. This conclusion allows us to derive Theorem 1.7 from Theorem 1.4 in paper IV. $\square$

3.4.4 Contribution from authors

Kaj Nyström gave a majority of the main arguments, and I did most of the writing and details in the proofs.

3.4.5 Future work

(i) Does this argument hold if we allow the parameter p in the p-Laplacian to vary in space? (ii) Is there any possibility to remove parts of the lengthy conditions on the function F imposed in the definition of the Aronsson type equations? In fact, Armstrong and Smart [AS10] have presented a simpler proof of the uniqueness of ∞-harmonic functions, perhaps this proof generalizes to a broader class of Aronsson type equations. (iii) Does this argument apply also for equations with lower order terms as studied in Paper V? Either Jensen's, Jutinnen's or the proof by Armstrong and Smart may generalize to equations with lower order terms.

3.5 Paper V
Boundary estimates for solutions to operators of p-Laplace type with lower order terms

Now to the last appended paper, created during the period August 2008–June 2009, submitted to Journal of Differential Equations 2009–12–21, and accepted 2010–06–02.

3.5.1 Background

As in Paper I and Paper IV, we continue our progress in proving boundary Harnack inequalities for p-Laplace type problems. In [LLuN08], Lewis, Nyström and I studied boundary behavior for solutions to equations of the form

$$\nabla \cdot A(x, \nabla u) = 0, \tag{3.12}$$

in δ-Reifenberg flat domains. We now study (3.12) with lower order terms.

3.5.2 Main results

In this paper we study the boundary behaviour of solutions to equations of the form

$$\nabla \cdot A(x, \nabla u) + B(x, \nabla u) = 0, \tag{3.13}$$

in a domain $\Omega \subset \mathbf{R}^n$, assuming that Ω is a δ-Reifenberg flat domain for sufficiently small δ. The functions $A(x, \eta)$ and $B(x, \eta)$ are assumed to be of p-Laplace character and we call solutions to (3.13) (A, B)-harmonic functions, see Definitions 1.1 and 1.2 in Paper V. Moreover, $\hat{\delta}(n)$ ensures that the domains in the below theorems are NTA-domains when $0 < \delta < \hat{\delta}$. Paper V has two main theorems, we start with the following.

Paper V: Theorem A. *Assume that $\Omega \subset \mathbf{R}^n$ is a (δ, r_0)-Reifenberg flat domain, Let $p \in (1, \infty)$, $w \in \partial\Omega$ and $r \in (0, r_0)$. Suppose that u and v are positive (A, B)-harmonic functions in $\Omega \cap B(w, r)$, continuous on $\bar{\Omega} \cap B(w, r)$,*

$u \geq v$ in $\bar{\Omega} \cap B(w,r)$, with $u = 0 = v$ on $\partial\Omega \cap B(w,r)$. Then there exist $\tilde{\delta} \in (0, \hat{\delta})$ and c, both depending only on n and M_p such that if $0 < \delta < \tilde{\delta}$ and $\tilde{r} = r/c$, then

$$c^{-1}\frac{u(a_r(w)) - v(a_r(w))}{v(a_r(w))} \leq \frac{u(x) - v(x)}{v(x)} \leq c\frac{u(a_r(w)) - v(a_r(w))}{v(a_r(w))}$$

whenever $x \in \Omega \cap B(w, \tilde{r})$.

Here, M_p denotes dependence of the class of equations, see the beginning of Section 2 in Paper V. Theorem A is stronger than the usual boundary Harnack inequality $u/v \approx c$. In particular, Theorem A implies that the quotient u/v is Hölder continuous, see Corollary 1.6 in Paper V, which in turn implies $u/v \approx c$. We also remark that for linear equations the above three statements are equivalent.

Concerning the Martin problem, described below, we have the following theorem in Paper V.

Paper V: Theorem B. *Assume that $\Omega \subset \mathbf{R}^n$ is a bounded (δ, r_0)-Reifenberg flat domain and let $p \in (1, \infty)$. Then there exists r_1, $0 < r_1 << r_0$, depending only on n, M_p and r_0, such that if $w \in \partial\Omega$ and if $0 < r' < r_1$, then the following is true. Suppose that u, v are positive (A, B)-harmonic functions in $\Omega \setminus B(w, r')$, continuous on $\bar{\Omega} \setminus B(w, r')$, with $u = 0 = v$ on $\partial\Omega \setminus B(w, r')$. Then there exist $\tilde{\delta} \in (0, \hat{\delta})$, $\sigma \in (0, 1]$ and c, all depending only on n and M_p, such that if $0 < \delta < \tilde{\delta}$, then*

$$\left|\log\frac{u(x)}{v(x)} - \log\frac{u(y)}{v(y)}\right| \leq c\left(\frac{r'}{\min\{\min\{r_0, 1\}/c, |x-w|, |y-w|\}}\right)^{\sigma}$$

whenever $x, y \in \Omega \setminus B(w, cr')$.

Again, M_p denotes dependence of the class of equations. Using Theorem B we solve the Martin problem for (A, B)-harmonic functions in bounded (δ, r_0)-Reifenberg flat domains for δ sufficiently small.

To describe the Martin problem, let $w \in \partial\Omega$ and assume that u is a positive (A, B)-harmonic function in Ω with $u = 0$ continuously on $\partial\Omega \setminus \{w\}$; then u is called a minimal positive (A, B)-harmonic function in Ω relative to w. Moreover, the (A, B)-Martin boundary of Ω is the set of equivalence classes of positive minimal (A, B)-harmonic functions relative to all boundary

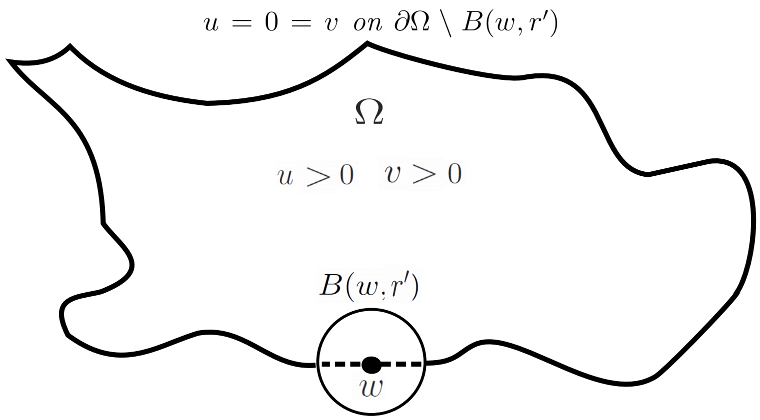

Figure 3.5: The setting of Theorem B.

points of Ω. Two minimal positive (A, B)-harmonic functions are in the same equivalence class if they correspond to the same boundary point and one function is a constant multiple of the other function. By letting $r' \to 0$ in Theorem B, we solve the Martin problem. That is, we prove that the (A, B)-Martin boundary is identifiable with $\partial\Omega$, see Corollary 1.7 in Paper V.

3.5.3 Informal proof

The roadmaps to the proofs of Theorem A and of Theorem B are quite lengthy. Therefore, we included an informal proof of Theorem A already in Paper V, to which we refer the reader. Here, we just mention that the idea is to use results from the earlier paper [LLuN08] by Lewis, me and Nyström, and a crucial scaling argument. In particular, by changing coordinates according to $x \to \tilde{r}x$ for some small $\tilde{r}$, we obtain a new equation in the same class of equations, but with a smaller lower order term. Hence, locally, we may assume that the lower order terms are small. $\square$

3.5.4 Contribution from authors

The paper was created with Benny Avelin and Kaj Nyström. I did most of the writing and details in the proofs.

3.5.5 Future work

(i) Avelin, I and Nyström have studied Optimal doubling of an A-harmonic measure, see [ALuN11]. We believe that by using the boundary Harnack inequality from Paper V, it is not to hard to generalize the paper [ALuN11] to hold for (A, B)-harmonic functions. (ii) Similarly, Paper III may generalize to hold for (A, B)-harmonic functions. (iii) As stated, the functions A and B are independent of u. At least for B, we believe that it is not so difficult to add such a dependence. Concerning A, a dependence of u will result in a re-investigation of [LLuN08] and is probably more difficult.

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