Topics on subelliptic parabolic equations structured on Hörmander vector fields

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To my family

Contents

List of papers

Paper I

The obstacle problem for parabolic non-divergence form operators of Hörmander type, with E. Götmark, K. Nyström, Submitted.

Paper II

Regularity in the obstacle problem for parabolic non-divergence operators of Hörmander type, Submitted.

Paper III

Non-divergence form parabolic equations associated with non-commuting vector fields: Boundary behavior of nonnegative solutions, with N. Garofalo, E. Götmark, I. Munive, K. Nyström. To appear in Annali della Scuola Normale Superiore di Pisa, Classe di Scienze, series V, XI (2) (2012).

Paper IV

Adaptive stochastic weak approximation of degenerate parabolic equations of Kolmogorov type, with K. Nyström. Journal of Computational and Applied Mathematics, 234 (1) (2010), 146-164.

Abstract

This thesis collects new contributions to the theory of subelliptic parabolic equations. The approach used varies from computational issues and Malliavin calculus to methods from the theory of partial differential equations. Before we present the papers we introduce the topic and account for the link between the different methods and problems considered. In particular, we will explain the proper geometric setting, which is not the Euclidean one. Hence, in the following, Sobolev spaces and Hölder spaces are the intrinsic ones.

Paper I-II deals with issues concerning the obstacle problem for operators

$$\mathcal{H} = \sum_{i,j=1}^{q} a_{ij} X_i X_j + \sum_{i=1}^{q} b_i X_i - \partial_t,$$

in a domain $\Omega_T \subset \mathbf{R}^{n+1}$, where $\{X_i\}_{i=1}^q$, $q < n$, is a set of Hörmander vector fields. Firstly we prove that, under suitable assumptions, there exists a unique strong solution u to the obstacle problem. The method we use is the classical penalization technique. As part of our argument, and this is of independent interest, we prove an embedding type theorem and interior a priori $\mathcal{S}^p$-estimates. Thereafter we study regularity of u. In the interior of the domain we prove that if the obstacle $\varphi \in C^{m,\alpha}$ then $u \in C^{m,\alpha}$ if $m = 0, 1$ and $u \in S^\infty$ if $m = 2$. Near the initial state the boundary data g will also have impact and we prove analogous results but this time assuming that both φ, $g \in C^{m,\alpha}$. To prove regularity we use "blow-ups" and argue by contradiction.

Paper III concerns solutions to $\mathcal{H}u = 0$, with $b_i \equiv 0$. We establish three main results, the first one being a backward Harnack inequality for nonnegative solutions vanishing on the lateral boundary. We also prove that the quotient of two nonnegative solutions which vanish continuously on a portion of the lateral boundary are Hölder continuous and that the parabolic measure associated with the operator $\mathcal{H}$ is doubling. The proof relies on the interior Harnack inequality, the Cauchy problem and the existence of, and Gaussian estimates for, fundamental solutions to the operator $\mathcal{H}$.

Finally, in Paper IV, we study Kolmogorov equations and derive an adaptive method for weak approximation. We demonstrate the method by an example where we price options assuming Hobson-Rogers model.

Sammanfattning

Denna avhandling samlar nya resultat inom teorin för partiella differentialekvationer. De metoder som används kommer från olika grenar av matematiken och spänner från beräkningsmatematik och Malliavinkalkyl till generaliseringar av klassisk teori för partiella differentialekvationer. Gemensamt är att de problem vi studerar har en grundläggande geometrisk struktur som som skiljer sig från den vi är vana vid; här ges inte det kortaste avståndet mellan två punkter av en rät linje. Vi börjar med att redogöra för den geometriska strukturen och den röda tråd som sammanbinder artiklarna.

Artikel I och II behandlar operatorn

$$\mathcal{H} = \sum_{i,j=1}^{q} a_{ij} X_i X_j + \sum_{i=1}^{q} b_i X_i - \partial_t,$$

i ett område $\Omega_T \subset \mathbf{R}^{n+1}$, där $\{X_i\}_{i=1}^{q}$, $q < n$, är en uppsättning Hörmandervektorfält. Speciellt studerar vi hinderproblemet, det vill säga att hitta funktioner u så att

$$\begin{cases} \max\{\mathcal{H}u - f, \varphi - u\} = 0 & \text{i } \Omega_T, \\ u = g & \text{på } \partial_p \Omega_T, \end{cases}$$

för givna funktioner f, g och φ. I Artikel I visar vi att, under vissa antaganden, finns det en entydig stark lösning u till hinderproblemet. När vi vet att det finns en lösning frågar vi oss vilka egenskaper lösningen har. I Artikel II visar vi att i det inre av området kommer hindret, φ, att avgöra hur slät lösningen u är, medan både g och φ avgör hur slät lösningen är nära $t = 0$.

I Artikel III studerar vi lösningar till problemet $\mathcal{H}u = 0$, där $b_i \equiv 0$. Här visar vi tre satser som beskriver hur lösningar uppför sig nära randen av området, det vill säga, vad som händer när vi är på väg ut ur området.

I Artikel IV härleder vi en adaptiv algoritm för svag approximation av stokastiska differentialekvationer, vårt bidrag är att denna metod även fungerar för så kallade Kolmogorovekvationer. Speciellt så använder vi denna metod för att med en given felmarginal kunna uppskatta värdet på en option då vi använder Hobson-Rogers modell för optionsprissättning.

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Notation

$\mathbf{R}^n$	n-dimensional Euclidean space.
Ω	A bounded domain in $\mathbf{R}^n$.
Ω_T	$\Omega \times (0,T)$.
$\partial\Omega,\ \partial_p\Omega_T$	The boundary/parabolic boundary of Ω/Ω_T.
$\overline{\Omega}$	The closure of Ω.
C	Space of continuous functions.
$C^\infty,\ C_b^\infty,\ C_p^\infty$	Space of infinitely differentiable functions which are bounded/polynomially bounded.
$[\cdot,\cdot]$	Lie bracket or commutator.
$\|\cdot\|$	Euclidean norm.
$B_E(x,r)$	Open Euclidean ball (center x and radius r).
$\circ$	Group law.
$\mathbf{G}$	Lie group.
$\mathfrak{g}$	Lie algebra.
δ_λ	Dilations.
$N(s,q)$	Free Lie group of step q with s generators.
$\mathcal{G}(s,q)$	Free Lie algebra of step q with s generators.
$\|\|\cdot\|\|$	Homogeneous norm.
$d_{\mathbb{G}},\ d_{p,\mathbb{G}}$	Homogeneous quasidistance.
$d_X,\ d_{p,X}$	Carnot-Carathéodory distance.
$B_{d_X}(x,t)$	Open Carnot-Carathéodory ball (center x, radius r).
$C_X^{k,\alpha}$	Intrinsic Hölder spaces.
$\mathcal{S}_X^p$	Intrinsic Sobolev spaces.
$C_r(x,t)$	Cylinders, $B_{d_X}(x,r)\times(t-r^2,t+r^2)$.
$\mathcal{C}^{2+\alpha}$	Space of Hölder continuous functions with Lie derivatives of order two.
$C^{2,1}$	Space of functions with two derivatives in space and one derivative in time.
$M,\ r_0,\ A_r(x_0)$	NTA constants.
$\omega^{(x,t)}$	$\mathcal{H}$-parabolic measure.

Chapter 1

Introduction

In this thesis we study subelliptic parabolic equations and obstacle problems. In fact, subelliptic parabolic equations are generalizations of the classical heat equation, which is the prototype example of an elliptic parabolic equation,

$$Hu = \Delta u - \partial_t u = \sum_{i=1}^{n} \partial_{x_i x_i} u - \partial_t u = 0 \quad \text{in } \mathbf{R}^{n+1}. \tag{1.1}$$

This equation typically describes the evolution in time of a density of, for instance, heat or some chemical concentration. The heat equation also appears in the study of Brownian motion and therefore in some option pricing problems in Black-Scholes framework. The classical obstacle problem can be formulated as

$$\begin{cases} \max\{Hu - f, \varphi - u\} = 0, & \text{in } \Omega_T, \\ u = g, & \text{on } \partial_p \Omega_T. \end{cases}$$

Here $\Omega_T = \Omega \times (0, T) \in \mathbf{R}^{n+1}$ is a bounded domain, f, φ and $g : \overline{\Omega_T} \to \mathbf{R}$ are continuous functions and $g \geq \varphi$. Furthermore, $\partial_p \Omega_T$ denotes the parabolic boundary of Ω_T and is defined by $\partial_p \Omega_T := \Omega \times \{t = 0\} \cup (\partial\Omega \times \{t : t \in (0, T)\})$. A frequently used example is to consider a membrane attached to a string at the parabolic boundary, restricted to stay above the obstacle. Other applications of obstacle problems are fluid filtration in porous media, elasto-plasticity, optimal control, pricing of American options and climate research, in particular, glaciology.

We look at generalizations of the classical obstacle problem; instead of having derivatives ∂_{x_i} which define the heat operator in (1.1) we study operators

$$\mathcal{H} = \sum_{i,j=1}^{q} a_{ij} X_i X_j + \sum_{i=1}^{q} b_i X_i - \partial_t, \tag{1.2}$$

which acts on functions in $\mathbf{R}^{n+1}$. Above $\{X_i\}_{i=1}^q$ is a set of smooth vector fields, $X_i = \sum_{j=1}^n c_{ij}(x)\partial_{x_j}$, $c_{ij} \in C^\infty(\mathbf{R}^n)$, which satisfy the so-called Hörmander condition. This condition assures that solutions are hypoelliptic, i.e., they are "nice" functions, in a sense to be described below. Another thing worth noting is that typically $q < n$ which means that $\mathcal{H}$ is not uniformly elliptic and hence classical theory of elliptic parabolic equations does not apply.

In the case of elliptic parabolic operators the first attempts towards developing a rigorous theory was to study operators with constant coefficients. When considering the case of variable coefficients results from the stationary case were used together with perturbation arguments. In the 1970s, Folland noted that stationary elliptic parabolic equations are built by translation invariant operators on the Abelian Lie group $\mathbf{R}^{n+1}$. Since the vector fields X_i are assumed to be arbitrary Hörmander vector fields they are non-commutative which means that $X_iX_j - X_jX_i$ are not identically zero for all $i, j \in \{1, 2, \dots, q\}$. Folland also noted that stationary subelliptic parabolic operators are in fact translation invariant operators on non-Abelian Lie groups where its Lie algebra has a structure reflecting the structure given by the vector fields $\{X_i\}_{i=1}^q$. Hence, the proper geometric setting to study equations structured on Hörmander vector fields is non-Euclidean. Below we will account for the proper setting. Thereafter we give a brief introduction to obstacle problems. We conclude this introductory chapter considering Kolmogorov equations and its connection to subelliptic parabolic equations as well as its stochastic interpretation. In Chapter 2 we present the major results from the appended papers and give some directions of future research.

1.1 Hörmander vector fields and geometry

The starting point for research on subelliptic parabolic equations structured on Hörmander vector fields was the paper [Hör67] by Hörmander. Before we can state the main result in [Hör67] we introduce some notation. Given a set of smooth vector fields we define the commutator of two vector fields by

$$[X_i, X_j] = X_iX_j - X_jX_i.$$

Furthermore, for a given multiindex $\theta = (\theta_1, \dots, \theta_m)$ we say that

$$X^\theta = [X_{\theta_m}, [X_{\theta_{m-1}}, \dots, [X_{\theta_2}, X_{\theta_1}] \dots]] \tag{1.3}$$

is a commutator of order m. Hörmander's condition then reads:

Definition 1.1. *(Hörmander's condition) We say that a set of smooth vector fields $\{X_i\}_{i=1}^q$ on $\mathbf{R}^n$ satisfies Hörmander's condition of order s if there exists a positive integer s such that $\{X_i\}_{i=1}^q$ together with its commutators of order $\leq s$ span $\mathbf{R}^n$ at every point.*

Assuming this condition, Hörmander proved the sum of squares theorem, see Theorem 1.1 in [Hör67].

Theorem 1.2. *(Sum of squares) Assume that the smooth vector fields $\{X_i\}_{i=1}^q$ satisfy Hörmander's condition. Then,*

$$H = \sum_{i=1}^q X_i^2 - \partial_t \tag{1.4}$$

is hypoelliptic. That is, if $Hu = f$ in Ω_T, in distributional sense, and if $f \in C^\infty(U)$ for some set $U \subset \Omega_T$ then $u \in C^\infty(U)$.

What this theorem states, roughly, is that, although the operator in (1.4) is degenerate, it still shares some good properties with classical elliptic parabolic equations as long as the missing directions in the operator are recovered by commutators of the vector fields. In fact, in [Hör67], Hörmander proved that ∂_t can be replaced by X_0 if $\{X_1, \dots, X_q, X_0\}$ are vector fields on $\mathbf{R}^n$ which satisfy Hörmander's condition.

Example 1.3. *(Hörmander's condition) Let $X_1 = \partial_{x_1} + 2x_2\partial_{x_3}$ and $X_2 = \partial_{x_2} - 2x_1\partial_{x_3}$. Obviously, those vector fields can not span $\mathbf{R}^3$, but*

$$[X_1, X_2] = -4\partial_{x_3},$$

so $\{X_1, X_2\}$ satisfies a Hörmander condition of order 2. Moreover,

$$\partial_{x_1x_1} + 4x_2\partial_{x_1x_3} + \partial_{x_2x_2} - 4x_1\partial_{x_2x_3} + 4(x_1^2 + x_2^2)\partial_{x_3x_3} - \partial_t = 0$$

is a hypoelliptic operator although it certainly degenerates at the origin.

A few years later, in [Bon69], Bony proved a weak maximum principle for sum of squares operators. In the proof he used barrier functions, and to assure the existence of such functions he defined what he refers to as

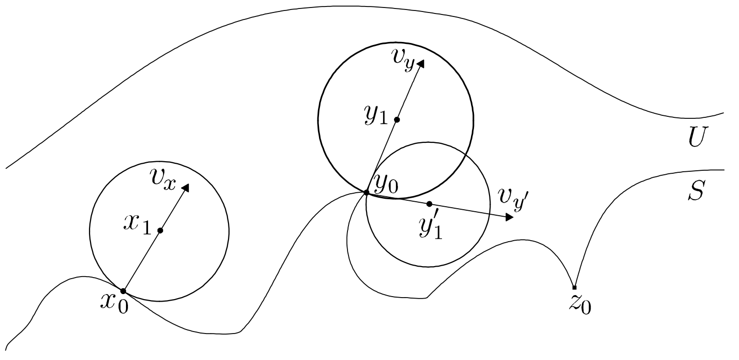

Figure 1.1: At x_0 there is, up to the scaling factor λ, a unique outer normal v_x, while there are infinitely many outer normals at y_0. When the cusp goes inside the domain S we have no outer normal, as illustrated at the point z_0.

an exterior normal. A vector v in $\mathbf{R}^n$ is an exterior normal to a closed set $S \subset \mathbf{R}^n$ relative to an open set U at a point x_0 if there exists an open standard Euclidean ball B_E in $U \backslash S$ centered at x_1 such that $x_0 \in \overline{B_E}$ and $v = \lambda(x_1 - x_0)$ for some $\lambda > 0$. This is illustrated in Figure 1.1. For our purposes the result of Bony can be restated in the following way, although it was slightly more general in Theoreme 5.2 in [Bon69].

Theorem 1.4. *(Bony's maximum principle) Let* $\Omega \times (0,T) = \Omega_T \subset \mathbf{R}^{n+1}$ *be a bounded domain and let* $H := \sum_{i=1}^q X_i^2 - \partial_t = \sum_{i,j=1}^n a_{ij}^* \partial_{x_i x_j} + \sum_{i=1}^n a_i^* \partial_{x_i} - \partial_t$*. Assume that the vector fields* $\{X_1, \ldots, X_q\}$ *satisfy Hörmander´s condition and that* $a_{ij}^*,\ a_i^* \in C^\infty(\Omega_T)$*. In addition, assume that for all* $(x,t) \in \Omega_T$ *and for all* $\xi \in \mathbf{R}^n$ *the quadratic form* $\sum_{i,j=1}^n a_{ij}^*(x,t)\xi_i\xi_j \geq 0$*. Further, assume that* D *is a relatively compact subset of* Ω *and that at every point* $x_0 \in \partial D$ *there exists an exterior normal* v *such that*

$$\sum_{i,j=1}^n a_{ij}^*(x_0,t) v_i v_j > 0, \tag{1.5}$$

for all $t \in [0,T]$*. Then, for all* $g \in C(\partial D_T)$ *and* $f \in C(\overline{D_T})$*, the Dirichlet problem*

$$\begin{cases} Hu = -f, & \text{in } D_T, \\ u = g, & \text{on } \partial_p D_T. \end{cases}$$

has a unique solution $u \in C(\overline{D_T})$*. Furthermore, if* $f \in C^\infty(D_T)$*, then* $u \in C^\infty(D_T)$ *and if* f *and* g *are both positive, then so is* u*.*

It is also fair to mention the work of Oleĭnic and Radkevič, see [OR73][1] and the references therein, where Oleĭnic and Radkevič consider general second order equations with nonnegative characteristic form. Yet another contribution to the theory of subelliptic parabolic equations is the work of Folland, [Fol75]. Inspired by the work carried out for the $\overline{\partial}_b$-complex, see for instance [FK72], [FS74], Folland used similar ideas to develop a regularity theory for subelliptic (parabolic) equations, but with less general assumptions on the vector fields. To explain this further we define;

Definition 1.5. *(Lie group on $\mathbf{R}^n$) Let $\circ$ be a given group law on $\mathbf{R}^n$, and suppose that the map*

$$\mathbf{R}^n \times \mathbf{R}^n \ni (x, y) \to y^{-1} \circ x \in \mathbf{R}^n$$

is smooth. Then $\mathbf{G} = (\mathbf{R}^n, \circ)$ is called a Lie group on $\mathbf{R}^n$.

For $\alpha \in \mathbf{R}^n$ fixed we define $\tau_\alpha(x) := \alpha \circ x$, the left translation of x by α, and we say that a vector field X is left invariant on $\mathbf{G}$ if

$$X(\phi(\tau_\alpha(\cdot)))(x) = (X\phi)(\tau_\alpha(x)),$$

for all test functions $\phi \in C^\infty(\mathbf{R}^n)$. Let $\mathfrak{g}$ denote the set of left invariant vector fields on $\mathbf{G}$. Then $\mathfrak{g}$, viewed as a vector space, together with the commutator operation $[\cdot, \cdot]$, also known as the Lie bracket, is called the Lie algebra of $\mathbf{G}$. It is straightforward to prove that $\mathfrak{g}$ is a Lie algebra, that is, that the Lie bracket is bilinear, anti-commutative and satisfies the Jacobi identity.

Definition 1.6. *(Homogeneous Lie group on $\mathbf{R}^n$) Let $\mathbf{G} = (\mathbf{R}^n, \circ)$ be a Lie group and assume that there exists an n-tuple of real numbers $\sigma = (\sigma_1, \ldots, \sigma_n)$, $1 \leq \sigma_1 \leq \ldots \leq \sigma_n$, such that the dilation δ_λ,*

$$\delta_\lambda(x) := (\lambda^{\sigma_1} x_1, \ldots, \lambda^{\sigma_n} x_n), \tag{1.6}$$

is an automorphism of the group for every $\lambda > 0$. Then $\mathbf{G} = (\mathbf{R}^n, \circ, \delta_\lambda)$ is called a homogeneous group.

Note that it is not restrictive to assume that $\sigma_1 = 1$. Indeed, if the above statement is true for some $\sigma_1 > 1$, then we may consider dilations $\delta_{\lambda^{1/\sigma_1}}$. Moreover, we say that a differential operator D is δ_λ-homogeneous of degree r if $D(\phi(\delta_\lambda(\cdot)))(x) = \lambda^r (D\phi)(\delta_\lambda(x))$ for every test function $\phi \in C^\infty(\mathbf{R}^n)$.

[1] Their work was originally published in Russian in 1971, this is the English translation from 1973.

Definition 1.7. *(Stratified homogeneous Lie group on $\mathbf{R}^n$) Let $\mathfrak{g}$ be the Lie algebra of a homogeneous Lie group $\mathbf{G} = (\mathbf{R}^n, \circ, \delta_\lambda)$. Let $\mathfrak{g}_1$ be the subspace of $\mathfrak{g}$ of left invariant vector fields which are δ_λ-homogeneous of degree σ_1. If $\mathfrak{g}_1$ generates the whole of $\mathfrak{g}$, then $\mathbf{G}$ is a stratified homogeneous group. Moreover, $\mathbf{G}$ has step σ_n/σ_1 and $m = \dim(\mathfrak{g}_1)$ generators.*

Now, assume that we have a set of smooth vector fields $\{X_1, \ldots, X_q\}$, δ_λ-homogeneous of degree 1, which generates the Lie algebra $\mathfrak{g}$ of a homogeneous Lie group $\mathbf{G}$. Then we can find a basis of $\mathfrak{g}$ by considering iterated Lie brackets of those vector fields. In particular, we have the following for $l = \sigma_n/\sigma_1$;

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_l, \text{ where } [\mathfrak{g}_1, \mathfrak{g}_i] = \mathfrak{g}_{i+1} \text{ for } i \in \{1, \ldots, l-1\} \text{ and } [\mathfrak{g}_1, \mathfrak{g}_l] = 0. \tag{1.7}$$

That is, the Lie algebra admits a stratification. Note that this means that

$$\delta_\lambda(x) = (\lambda x^{(1)}, \ldots, \lambda^l x^{(l)}),$$

where $x^{(i)} \in \mathbf{R}^{n_i}$ and $n_1 + \ldots + n_l = n$. The number

$$Q := n_1 + 2n_2 + \ldots + l\ n_l \tag{1.8}$$

is called the homogeneous dimension of $\mathbf{G}$ with respect to δ_λ.

Example 1.8. *(Heisenberg group) Recall the vector fields X_1 and X_2 in Example 1.3. Are these vector fields generators of a homogeneous group? Well, if X_1, X_2 induce a homogeneous Lie group (on $\mathbf{R}^3$), then we can find dilations so that X_1 and X_2 are homogeneous of degree 1, while $X_3 = [X_1, X_2]$ is homogeneous of degree 2. This will be the case if we define*

$$\delta_\lambda(x_1, x_2, x_3) = (\lambda x_1, \lambda x_2, \lambda^2 x_3).$$

Now, δ_λ should also be an automorphism of the group. That is, can we find a group law $\circ$ such that δ_λ preserves the group structure? After some considerations, we see that the group law defined by

$$(x_1, x_2, x_3) \circ (y_1, y_2, y_3) = (x_1 + y_1, x_2 + y_2, x_3 + y_3 - 2(x_1 y_2 - x_2 y_1))$$

will preserve the structure. That is, $(\mathbf{R}^n, \circ, \delta_\lambda)$, with $\circ$ and δ_λ defined as above, is a homogeneous group, the so-called Heisenberg group $\mathbf{H}^1$.

It is in this setting the work of Folland [Fol75] was carried out, that is, on homogeneous stratified Lie groups. Recall that the transpose D^t of a differential operator D is defined so that $\int (D^t u)v = \int u(Dv)$ for all test functions u, v. One of Folland's major achievements was to prove the following, see Theorem 2.1 in [Fol75].

Theorem 1.9. *(Homogeneous fundamental solution) Let $\mathcal{L}$ be a homogeneous differential operator of degree r on $\mathbf{G}$, $0 < r < Q$, such that both $\mathcal{L}$ and $\mathcal{L}^t$ are hypoelliptic. Then there exists a fundamental solution Γ for $\mathcal{L}$ at 0. Moreover, the distribution $\Gamma \in C^\infty(\mathbf{R}^n \backslash \{0\})$ is homogeneous of degree $r - Q$.*

We say that a Lie algebra $\mathfrak{g}$, associated to a homogeneous stratified Lie group $\mathbf{G}$, is free if there are as few relations as possible amongst the generators of $\mathfrak{g}$. That is, the only relations between the generators, and commutators of generators, are the ones forced by anti-commutativity and Jacobi identity. Worth mentioning is that if $\mathcal{G}(s,q)$ is the free Lie algebra of q generators of step s and if $\widehat{\mathcal{G}}(s,q)$ is any other nilpotent Lie algebra of step s with q generators, then there exists a surjective homomorphism of $\mathcal{G}(s,q)$ onto $\widehat{\mathcal{G}}(s,q)$.

Example 1.10. *(A Lie algebra which is not free) Let $X_1 = \partial_{x_1}, X_2 = \partial_{x_2} + x_3\partial_{x_4}$ and $X_3 = \partial_{x_3} - x_2\partial_{x_4}$ be vector fields on $\mathbf{R}^4$. Since $[X_2, X_3] = -2\partial_{x_4}$, the set of vector fields $\{X_1, X_2, X_3\}$ satisfy Hörmander's condition of step 2. We define the group operation $\circ$ by*

$$\begin{aligned}&(x_1, x_2, x_3, x_4) \circ (y_1, y_2, y_3, y_4)\\ &= (x_1 + y_1, x_2 + y_2, x_3 + y_3, x_4 + y_4 - (x_2 y_3 - x_3 y_2)),\end{aligned}$$

and by a direct calculation we find out that X_1, X_2 and X_3 are left invariant with respect to $\circ$. We define dilations by $\delta_\lambda(x) = (\lambda x_1, \lambda x_2, \lambda x_3, \lambda^2 x_4)$ and we note that δ_λ preserves the structure given by $\circ$. Hence, $(\mathbf{R}^4, \circ, \delta_\lambda)$ is a homogeneous group, and $\{X_1, X_2, X_3\}$ are generators of the corresponding Lie algebra. However, $[X_1, X_2] = [X_1, X_3] = 0$, so the Lie algebra is not free. In fact, if $\mathfrak{f}$ is a free Lie algebra with three generators of step 2 then $\dim \mathfrak{f} = 6$.

We continue with some remarks on free Lie algebras. Let $e_1, \dots, e_q$ be the generators of $\mathcal{G}(s,q)$. Then, for all multiindices α, we define e^α in terms of Lie brackets as in (1.3). As a consequence of the Hörmander condition and the fact that $\mathcal{G}(s,q)$ is free, there exists a set A of multiindices α so that $\{e^\alpha\}_{\alpha\in A}$ is, considering $\mathcal{G}(s,q)$ as a vector space, a basis for $\mathcal{G}(s,q)$. Thus $\mathcal{G}(s,q)$ can be identified with $\mathbf{R}^N$, where $N = \dim \mathcal{G}(s,q)$. Next we would like to understand whether there is a group structure on $\mathbf{R}^N$ which allows us to view $\mathbf{R}^N$ as a Lie group. It turns out that the Campbell-Hausdorff series, $X \circ Y = \log(e^X e^Y) = X + Y + \frac{1}{2}[X,Y] + \dots$, defines a group law $\circ$ on $\mathbf{R}^N$ as pointed out in Section 1 in Sanchez-Calle [SC84], for more information see Chapter 15 in the monograph by Bonfiglioli, Lanconelli and Uguzzoni [BLU07]. In particular, we note that the sum will be finite since $[\mathfrak{g}_1, \mathfrak{g}_l] = 0$. What we have discovered so far is that starting with a stratified homogeneous group, we have identified its free Lie algebra with $\mathbf{R}^N$ and in the following we denote the group $\left(\mathbf{R}^N, \circ\right)$ by $N(s,q)$. Then $N(s,q)$ is a simply connected Lie group associated to the Lie algebra $\mathcal{G}(s,q)$ and we refer to $N(s,q)$ as the free Lie group associated to $\mathcal{G}(s,q)$. Since we started with a stratified homogeneous group, $N(s,q)$ can be endowed with a natural family of dilations δ_λ defined as in (1.6) for suitable fixed integers $\sigma_1, \ldots, \sigma_N$. Then $\mathbb{G} := (N(s,q), \delta_\lambda)) = (\mathbf{R}^N, \circ, \delta_\lambda)$ is a homogeneous Lie group, in the sense of Stein, see pages 618-622 in [Ste93], and we define the homogeneous dimension of $\mathbb{G}$ to be the number

$$Q = \sum_{i=1}^{N} \sigma_i, \tag{1.9}$$

analogous to (1.8). It is justified to ask what we have gained through this layout, and the conclusion is that we can identify any homogeneous group $\mathbf{G}$ with q generators of step s with the group $\mathbb{G} := (\mathbf{R}^N, \circ, \delta_\lambda)$. On $\mathbb{G}$ we can define a homogeneous norm $||\cdot||$ through the relation

$$\begin{cases} ||0|| = 0, \\ ||v|| = \rho, \quad \text{iff} \quad \left|\delta_{\rho^{-1}}(v)\right| = 1, \end{cases} \tag{1.10}$$

for $v \in \mathbb{G}$, where $|\cdot|$ denotes the standard Euclidean norm. Using this homogeneous norm we define a quasidistance by

$$d_{\mathbb{G}}(x,y) := ||y^{-1} \circ x||. \tag{1.11}$$

The term quasidistance refers to that it differs from a distance in that

$$d_{\mathbb{G}}(x,y) \leq c_d(d_{\mathbb{G}}(x,z) + d_{\mathbb{G}}(z,y)),$$

for some positive constant c_d. For $d_{\mathbb{G}}$ to be a distance we would require $c_d = 1$. A useful property is that the Lebesgue measure in $\mathbf{R}^N$ is the Haar measure on $\mathbb{G}$, that is the Euclidean volume of $d_{\mathbb{G}}$-balls, $B_{d_{\mathbb{G}}}(x,r) := \{x : ||x|| < r\}$ satisfy

$$|B_{d_{\mathbb{G}}}(x,r)| = r^Q |B_{d_{\mathbb{G}}}(0,1)| \quad \text{for all } x \in \mathbb{G},\ r > 0.$$

The next breakthrough came the following year when Rotschild and Stein in [RS76] proved their celebrated lifting and approximation theorems, see Theorem 4 and Theorem 5 in [RS76]. The idea is that, as demonstrated by Folland, classical tools from analysis are also accessible on homogeneous stratified groups. Hence, if we have a set of smooth vector fields $\{X_1, \ldots, X_q\}$ in

$\mathbf{R}^n$, we would like to work with them on a homogeneous stratified group. The lifting approximation technique provides us with a tool to achieve this. Intuitively, given a set of smooth vector fields $X = \{X_1, \dots, X_q\}$ in $\mathbf{R}^n$ which satisfy Hörmander's condition for some positive integer s, we can locally lift these vector fields into a higher dimensional space $\mathbf{R}^N$. Here N is $\dim \mathcal{G}(s,q)$ and the lifted vector fields $\hat{X} = \{\hat{X}_1, \dots, \hat{X}_q\}$ are free. Moreover, in a certain coordinate system, say ξ, we can locally approximate the free vector fields $\hat{X} = \{\hat{X}_1, \dots, \hat{X}_q\}$ so that $\hat{X}_i = Y_i + R_i$ for $i = 1, \dots, q$. The remarkable thing with this approximation is that Y_i are the unique left invariant vector field, homogeneous of degree 1, which agrees with $\frac{\partial}{\partial \xi_i}$ at the origin on a homogeneous group $\mathbb{G}$, and R_i are differential operators which behaves "nicely" in some sense. In many situations, we can also lift the problem we are facing allowing us to work in a group with a lot of structure, i.e., translations and dilations. In Paper I we use the lifting and approximation technique to prove a priori $\mathcal{S}^p$-estimates (Theorem 1.3) and the lifting technique to prove an embedding theorem (Theorem 1.4). The work of Rothschild and Stein have inspired numerous mathematicians, and we mention a few of them. Folland enlightened the geometric constructions underlying the lifting approximation technique and thereby gave an alternative proof in the case where one starts with homogeneous, left invariant vector fields that are not free in [Fol77]. Other alternative proofs have been provided by Goodman [Goo78] and Hörmander-Melin [HM78]. Generalizations to weighted vector fields was provided by Christ et al in [CNSW99], and non-smooth vector fields was considered by Bramanti, Brandolini and Pedroni in [BBP10].

In view of the results presented it is obvious that the proper setting is that of a non-Abelian Lie group and on this group the geometry differs from the standard Euclidean one. Hence, we need to reconsider proper definitions of function spaces, i.e., analogues to Hölder spaces and Sobolev spaces in the Euclidean setting. In case of a stratified homogeneous Lie group, whose Lie algebra is free, we can work with the distance $d_{\mathbb{G}}$, defined in (1.11). If we start with Hörmander vector fields on $\mathbf{R}^n$ then there is not necessarily a homogeneous structure allowing us to define a distance in a similar fashion. In that case we will use a distance called the control distance or the Carnot-Carathéodory distance. This distance is defined based on X-subunit paths.

Definition 1.11. *(X-subunit) Let $X = \{X_1, \dots, X_q\}$ be a set of vector fields in $\mathbf{R}^n$. We say that a piecewise continuous curve $\gamma : [0,T] \to \mathbf{R}^n, T \geq$*

0, *is X-subunit if there exist measurable functions* $h = (h_1, \ldots, h_q)$ *such that*

$$\dot{\gamma}(t) = \sum_{j=1}^{q} h_j(t) X_j(\gamma(t)) \ \text{a.e.} \quad \text{and} \quad \sum_{j=1}^{q} h_j^2(t) \le 1 \ \text{a.e.}$$

for $t \in [0, T]$.

We remark that if $X = \{X_1, \ldots, X_q\}$ are Hörmander vector fields, then by the Chow-Rashevsky connectivity theorem, see Satz A and Satz B in [Cho40] and [Ras38], we have that for every set of two points $x, y \in \mathbf{R}^n$ there exists $T \ge 0$ and an X-subunit path $\gamma : [0, T] \to \mathbf{R}^n$ such that $\gamma(0) = x, \gamma(T) = y$. Hence, the following definition makes sense.

Definition 1.12. *(Carnot-Carathéodory distance). Let* $X = \{X_1, \ldots, X_q\}$ *be a set of Hörmander vector fields in* $\mathbf{R}^n$*. Then, for every* $x, y \in \mathbf{R}^n$ *we define the Carnot-Carathéodory distance* d_X *by*

$$d_X(x, t) = \inf\{T : \gamma \text{ is } X\text{-subunit}, \gamma(0) = x, \gamma(T) = y\}.$$

Remark 1.13. *(Parabolic distances) The operators we consider are structured on Hörmander vector fields, but we also have the time-derivative* ∂_t *present. The natural extension of the Carnot-Carathéodory distance to this parabolic setting is*

$$d_{p,X}((x, t), (y, s)) = \left(d_X(x, y)^2 + |t - s|\right)^{1/2},$$

the parabolic Carnot-Carathéodory distance. We could extend the distance $d_{\mathbb{G}}$ *in (1.11) analogously by setting*

$$d_{p,\mathbb{G}}((x, t), (y, s)) = \left(||y^{-1} \circ x||^2 + |t - s|\right)^{1/2}.$$

To see that d_X *and* $d_{p,X}$ *in fact are distances we refer to Chapter 5.2 in [BLU07].*

It is not a trivial task to explicitly find the Carnot-Carathéodory distance, if possible at all. However, in the case of the Heisenberg group $\mathbf{H}^1$, see Example 1.3 and Example 1.8, this can be done and we refer to [Mon00], where Monti examines properties of balls in Heisenberg groups. In Figure 1.2 we visualize balls with different radii in $\mathbf{H}^1$, using (2.14) in [Mon00], and we note that different scales are used for different balls. We present another example below, taken from page 1086 in [GN96], which illustrates the fact that the closure of balls in Carnot-Carahtéodory spaces does not need to be compact.

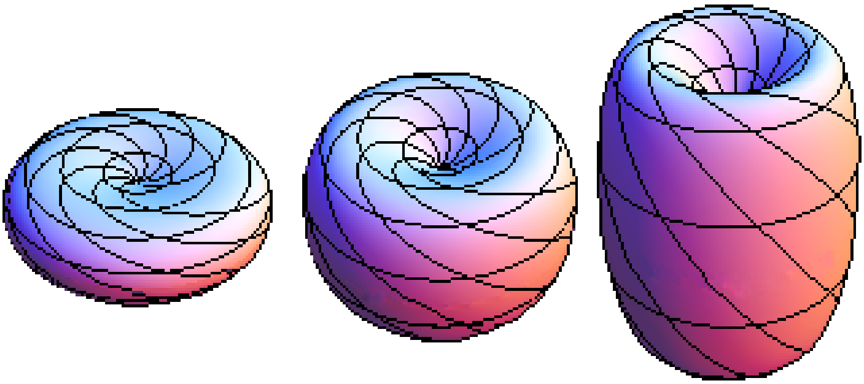

Figure 1.2: Visualization of balls centered at the origin in the Heisenberg group $\mathbf{H}^1$. The leftmost has radii $r = 0.5$, the ball in the middle is the unit sphere while the rightmost has radii $r = 2$. Note that different scales are used for different values of r.

Example 1.14. *(A ball whose closure is not compact) Consider on* $\mathbf{R}$ *the* C^∞*-vector field* $X = (1 + x^2)\partial_x$*. The* X*-subunit curves are given by,*

$$\dot{\gamma}(t) = h(t)(1 + \gamma^2(t)),$$

and to find the infimum over all X*-subunit curves is equivalent to assigning* $h(t) = \pm 1$*. Now assume that* $\gamma(0) = x$ *and* $\gamma(T) = y$*, the Carnot-Carathéodory distance between* x *and* y *is then* T*. Moreover,*

$$T = T - 0 = \gamma^{-1}(y) - \gamma^{-1}(x) = \int_x^y \frac{d}{ds}\gamma^{-1}(s)ds$$
$$= \int_x^y \left(\frac{d}{dt}\gamma(t)\Big|_{t=\gamma^{-1}(s)} \right)^{-1} ds = \int_x^y \frac{1}{1+s^2} ds = \arctan(y) - \arctan(x),$$

that is, $d_X(x, y) = |\arctan(x) - \arctan(y)|$*. Hence, for any radius* $r \geq \pi/2$*,* $B_{d_X}(0, r) = \{y \in \mathbf{R} : d_X(0, y) < r\}$ *is* $\mathbf{R}$*.*

Now we are ready to define proper function spaces. Let U be a bounded domain in $\mathbf{R}^{n+1}$, $\alpha \in (0, 1]$, then we say that $u : U \to \mathbf{R}$ is Hölder continuous with exponent α, $u \in C_X^{0,\alpha}(U)$, if

$$||u||_{C_X^{0,\alpha}(U)} := \sup_U |u| + \sup_{\substack{z,\zeta \in U \\ z \neq \zeta}} \frac{|u(z) - u(\zeta)|}{d_{p,X}(z, \zeta)^\alpha} < \infty.$$

Further, given a positive integer $k \in \mathbf{Z}_+$, $\alpha \in (0,1]$, and a multiindex $I = (i_1, \ldots, i_m)$ with $1 \leq i_j \leq q$, $1 \leq j \leq m$, we define $|I| = m$ and we say that $u \in C_X^{k,\alpha}(U)$ if,

$$||u||_{C_X^{k,\alpha}(U)} := \sum_{|I|+h \leq k} ||X^I \partial_t^h u||_{C_X^{0,\alpha}(U)} < \infty.$$

Moreover, we say that $u \in C_{X,loc}^{k,\alpha}(U)$ if $u \in C_X^{k,\alpha}(V)$ for every compact subset V of U. Sobolev spaces are defined by

$$\mathcal{S}_X^p(U) := \{u \in L^p(U) : X_i u, X_i X_j u, \partial_t u \in L^p(U) \text{ for } i,j = 1, \ldots, q\},$$

and we define

$$||u||_{\mathcal{S}_X^p(U)} = ||u||_{L^p(U)} + \sum_{i=1}^q ||X_i u||_{L^p(U)} + \sum_{i,j=1}^q ||X_i X_j u||_{L^p(U)} + ||\partial_t u||_{L^p(U)}.$$

Above the L^p-norms are taken with respect to the standard Euclidean metric, in particular, we integrate with respect to the Lebesgue measure. If $u \in \mathcal{S}_X^p(V)$ for every compact subset V of U then we say that $u \in \mathcal{S}_{X,loc}^p(U)$. It is in this setting we carry out Paper I-III, with some slight modifications in the definition of Hölder spaces in Paper II.

Another way to approach the Dirichlet problem, different from Theorem 1.4, is by considering non-tangentially accessible domains, NTA$_X$ domains, in the sense of [CG98], Definition 1, and [CGN08], Definition 8.1. Given a bounded open set $\Omega \subset \mathbf{R}^n$, a ball $B_{d_X}(x,r)$ is said to be M-non-tangential in Ω, with respect to the metric d_X, if

$$M^{-1} r < d_X(B_{d_X}(x,r), \partial\Omega) < Mr.$$

Given $x, y \in \Omega$ a sequence of M-non-tangential balls in Ω, $B_{d_X}(x_1, r_1)$, $\ldots$,$B_{d_X}(x_p, r_p)$ is called a Harnack chain of length p joining x and y if: *i)* $x \in B_{d_X}(x_1, r_1)$ and $y \in B_{d_X}(x_p, r_p)$ and *ii)* $B_{d_X}(x_i, r_i) \cap B_{d_X}(x_{i+1}, r_{i+1}) \neq \emptyset$ for $i \in \{1, \ldots, p-1\}$. We explicitly remark that by definition, balls in a Harnack chain have comparable radii.

Definition 1.15. *(NTA$_X$ domain) We say that a connected, bounded open set $\Omega \subset \mathbf{R}^n$ is a NTA$_X$ domain with respect to the set of vector fields $X = \{X_1, \ldots, X_q\}$ if there exist constants $M, r_0 > 0$ such that*

i) (Interior corkscrew condition) For any $x_0 \in \partial\Omega$ and $r \leq r_0$ there exists a point $A_r(x_0) \in \Omega$ such that $M^{-1} r < d_X(A_r(x_0), x_0) \leq r$ and $d_X(A_r(x_0), \partial\Omega) > M^{-1} r$.

ii) *(Exterior corkscrew condition)* $\mathbf{R}^n \setminus \Omega$ *satisfies condition i).*

iii) *(Harnack chain condition) There exists a constant* $c = c(M) > 0$ *such that for any* $\varepsilon > 0$ *and* $x, y \in \Omega$ *such that* $d_X(x, \partial\Omega) > \varepsilon$, $d_X(y, \partial\Omega) > \varepsilon$ *and* $d_X(x, y) < c\varepsilon$, *there exists a Harnack chain joining* x *and* y *whose length depend on* c *but not on* ε.

So far it has been implicitly understood that given a function $u : \mathbf{R}^n \to \mathbf{R}$ and a smooth vector field $X = \sum_{i=1}^n c_i(x)\partial_i$ on $\mathbf{R}^n$ we let $Xu = \sum_{i=1}^n c_i(x)\partial_i u$. Xu is called the Lie derivative of u along the vector field X. Another equivalent definition of Lie derivatives is stated in terms of integral curves, and we account for this through an example.

Example 1.16. *(A different approach to Lie derivatives) On* $\mathbf{R}^4$ *we consider the following set of vector fields;*

$$X_1 = \partial_{x_1} + x_2\partial_{x_3}, \quad X_2 = \partial_{x_2}, \quad \partial_t.$$

Since $[X_1, X_2] = -\partial_{x_3}$ *this set of vector fields satisfy Hörmander's condition. Consider the integral curve* γ *of* X_1, *which passes through* (x, t) *at the origin. We note that* γ *is defined through the ordinary differential equation*

$$\begin{cases} \dot{\gamma}(s) = X_1 I(\gamma(s)), \\ \gamma(0) = (x, t). \end{cases}$$

In this particular case, this reads,

$$\begin{array}{llll} \frac{\partial\gamma_1(s)}{\partial s} = 1, & \frac{\partial\gamma_2(s)}{\partial s} = 0, & \frac{\partial\gamma_2(s)}{\partial s} = \gamma_2(s), & \frac{\partial\gamma_4(s)}{\partial s} = 0, \\ \gamma_1(0) = x_1, & \gamma_2(0) = x_2, & \gamma_3(0) = x_3, & \gamma_4(0) = t. \end{array}$$

Thus, $\gamma(s) = (x_1 + s, x_2, x_3 + x_2 s, t)$ *and the Lie derivative of* u *at* (x, t) *is given by*

$$\begin{aligned} \lim_{s \to 0} \frac{d}{ds} u(\gamma(s)) &= \lim_{s \to 0} \frac{d}{ds} u(x_1 + s, x_2, x_3 + x_2 s, t) \\ &= \partial_{x_1} u(x, t) + x_2 \partial_{x_3} u(x, t) = X_1 u(x, t). \end{aligned}$$

We say that $u \in \mathcal{C}_X^{2+a}(\Omega_T)$ if $u \in C_X^{0,\alpha}(\Omega_T)$ has Lie derivatives up to order two with respect to $\{X_1, \dots, X_q, \partial_t\}$. Here $X_1, \dots, X_q$ are of order one while ∂_t are of order two, which is consistent with the definition of the intrinsic functions spaces. For NTA_X domains we have the following result, which is a consequence of the definition and Theorem 4.1 in [Ugu07].

Theorem 1.17. *(Solvability of the Cauchy-Dirichlet problem) Let* $\Omega \subset \mathbf{R}^n$ *be a* NTA_X *domain and consider the Cauchy-Dirichlet problem*

$$\mathcal{H}u = \sum_{i,j=1}^{q} a_{ij}(x,t) X_i X_j u - \partial_t u = g \text{ in } \Omega_T, \quad u = f \text{ on } \partial_p \Omega_T, \qquad (1.12)$$

where $X = \{X_1, \ldots, X_q\}$ *are smooth vector fields which satisfy Hörmander's condition. Moreover, assume that* $A = \{a_{ij}\}$ *is uniformly elliptic with Hölder continuous elements with exponent* α *(with respect to the vector fields* X*). Then, given* $f \in C(\partial_p \Omega_T)$ *and* $g \in C_X^{0,\beta}(\Omega_T)$*,* $0 < \beta \leq \alpha$*, there exists a unique solution* $u \in \mathcal{C}_X^{2+\beta}(\Omega_T) \cap C(\Omega_T \cup \partial_p \Omega_T)$ *to (1.12).*

Moreover, if Ω_T is a NTA_X domain, then for every $(x,t) \in \Omega_T$ there exists, by Riesz' representation theorem (although not immediate), a unique probability measure $\omega = \omega^{(x,t)}$ with support in $\partial_p \Omega_T$ such that

$$u(x,t) = \int_{\partial_p \Omega_T} f(y,s) d\omega^{(x,t)}(y,s).$$

We will refer to $\omega^{(x,t)}$ as the $\mathcal{H}$-parabolic measure relative to (x,t) and Ω_T. Paper III is carried out in this setting.

1.2 Obstacle problems

To introduce the obstacle problem we will give a simple, yet illustrative example. Consider an elastic string whose endpoints are held fixed. If tightened it will be a line segment. Now, assume that beneath the line we have a rigid object, say a metal wire. As we push the wire upwards the shape of the elastic string will change, see Figure 1.3. Mathematically we can formulate this as: we have a string whose vertical position is given by $u(x)$, where x is the horizontal position, for, say $x \in [0,1]$. The endpoints are fixed so $u(0) = a$, $u(1) = b$ for some fixed $a, b \in \mathbf{R}$. The wire, or obstacle, is given by $\varphi : [0,1] \to \mathbf{R}$ and must satisfy $\varphi(0) \leq a$, $\varphi(1) \leq b$. The obstacle problem then reads: how do we find u? Firstly, u will minimize the tension energy, which will be proportional to the length of the string. That is, we should try to minimize

$$L = \int_0^1 \sqrt{1 + |u'(x)|^2} dx,$$

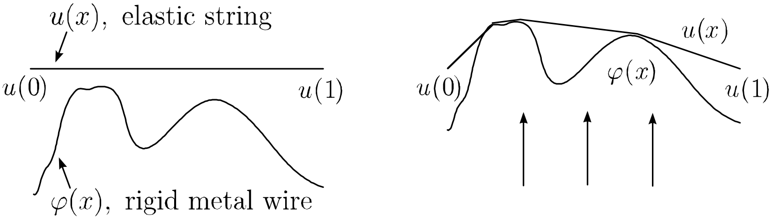

Figure 1.3: A rigid obstacle is pushed upwards affecting the appearance of an elastic string whose endpoints are held fixed.

with the limitation that $u(x) \geq \varphi(x)$. In the one-dimensional case this is equivalent to minimizing

$$I(u) := \int_0^1 |u'(x)|^2 dx.$$

From methods in calculus of variations, see for instance Chapter 1 in the monograph by Friedman [Fri11], it is known that $u''(x) = 0$ whenever $u > \varphi$ and that $u''(x) \leq 0$ everywhere. The problem can thus be formulated as to find the solution u to the following non-linear partial differential equation

$$\begin{cases} \max\{u'', \varphi - u\} = 0 & \text{for } x \in (0,1), \\ u(0) = a, \ u(1) = b. \end{cases}$$

This example account for two different views of obstacle problems, either as minimization problems or as non-linear partial differential equations. Our viewpoint will be the latter one, and the type of problems we consider are

$$\begin{cases} \max\{\mathcal{H}u - f, \varphi - u\} = 0 & \text{in } \Omega_T, \\ u = g & \text{on } \partial_p \Omega_T. \end{cases} \tag{1.13}$$

Above, $\mathcal{H}$ is the operator in (1.2), that is $\mathcal{H}$ is a subelliptic parabolic operator, and $\Omega_T = \Omega \times (0,T)$ is a bounded domain in $\mathbf{R}^{n+1}$. In this particular setting we are not aware of any results other than Paper I and Paper II in this thesis, where we examine basic properties of solutions, i.e., existence and regularity.

1.3 Kolmogorov equations

The prototype example of an operator of Kolmogorov type is the following one in $\mathbf{R}^{2n+1}$,

$$\sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} + \sum_{j=n+1}^{2n} x_{j-n} \frac{\partial}{\partial x_j} - \partial_t.$$

This operator was introduced by Kolmogorov in [Kol34] and describes the density of a system with $2n$ degrees of freedom. The first n variables $(x_1, \ldots, x_n)$ represents the velocity of the system while the following n variables $(x_{n+1}, \ldots, x_{2n})$ represents the position. In Paper IV we study general Kolmogorov equations, in particular, we study operators

$$\mathcal{L} = \frac{1}{2} \sum_{i,j=1}^{q} a_{ij}(x,t) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{q} b_i(x,t) \frac{\partial}{\partial x_i} + \sum_{i=1}^{n} c_{ij} x_i \frac{\partial}{\partial x_j} + \partial_t. \quad (1.14)$$

We assume that $A = \{a_{ij}\}_{i,j=1}^{q}$ is uniformly elliptic, that is, we assume that there exists a constant $\Lambda \in [1, \infty)$ such that

$$\Lambda^{-1} |\xi|^2 \leq \sum_{i,j=1}^{q} a_{ij}(x,t) \xi_i \xi_j \leq \Lambda |\xi|^2,$$

for all $(x,t) \in \mathbf{R}^{n+1}$, $\xi \in \mathbf{R}^q$**.** This means that there exists a unique $q \times q$-matrix $\overline{A} = \{\overline{a}_{ij}\}_{i,j=1}^{q}$ such that $\overline{A}\ \overline{A} = A$. For a moment, assume that $b_i \equiv 0$ for $i = 1, \ldots, q$, and freeze the operator at $(x_0, t_0) \in \mathbf{R}^{n+1}$;

$$\mathcal{L}_{(x_0,t_0)} = \frac{1}{2} \sum_{i,j=1}^{q} a_{ij}(x_0,t_0) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} c_{ij} x_i \frac{\partial}{\partial x_j} + \partial_t.$$

Then we can define new vector fields

$$\begin{aligned} X_0 &= \sum_{i=1}^{n} c_{ij} x_i \frac{\partial}{\partial x_j} + \partial_t, \\ X_i &= \frac{1}{\sqrt{2}} \sum_{j=1}^{q} \overline{a}_{ij}(x_0,t_0) \frac{\partial}{\partial x_j} \quad \text{for } i \in \{1, \ldots, q\}, \end{aligned} \quad (1.15)$$

and rewrite $\mathcal{L}_{(x_0,t_0)}$ in terms of these as,

$$\mathcal{L}_{(x_0,t_0)} = \sum_{i=1}^{q} X_i^2 + X_0.$$

Therefore, a natural assumption is that the Lie algebra generated by the vector fields $\{X_1, \dots, X_q, X_0\}$ span $\mathbf{R}^{n+1}$ for every fixed $(x_0, t_0) \in \mathbf{R}^{n+1}$, or equivalently, that $\{X_1, \dots, X_q, X_0\}$ are Hörmander vector fields. Theorem 1.2 then assures that $\mathcal{L}$ is hypoelliptic. In the case of Kolmogorov equations there is another condition, equivalent to Hörmander's condition, namely, we assume that the matrix $C = \{c_{ij}\}_{i,j=1}^{q}$ has the following block structure

$$\begin{pmatrix} * & C_1 & 0 & \cdots & 0 \\ * & * & C_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & \cdots & C_l \\ * & * & * & \cdots & * \end{pmatrix}$$

where $C_j, j = 1, \dots, q$, is a $q_{j-1} \times q_j$-matrix of rank q_j, $q_0 \geq q_1 \geq \dots \geq q_l \geq 1$, and $q_0 + q_1 + \dots + q_l = n$, while $*$ represents arbitrary matrices with constant entries. The equivalence of these two assumptions are proved in [LP94], Proposition A1. On $\mathbf{R}^{n+1}$ we can define a group law $\circ$ by

$$(x, t) \circ (y, s) = (y + E(s)x, t + s), \quad E(s) = \exp(-sC^T).$$

Moreover, based on the block structure of C, we can define dilations

$$\delta_\lambda(x, t) = (\lambda x^{(1)}, \lambda^3 x^{(2)}, \dots, \lambda^{2l+1} x^{(l)}, \lambda^2 t),$$

where $x^{(i)} \in \mathbf{R}^{q_i}$ for $i = 1, \dots, l$. That is, the induced structure is that of a stratified, homogeneous group. Although not obvious at first, there are a lot of similarities between the operators (1.2) and (1.14).

1.3.1 Stochastic differential equations

Let $\mathcal{L}$ be a Kolmogorov operator and consider the backward in time Cauchy problem

$$\begin{cases} \mathcal{L}u(x, t) = 0 & \text{whenever } (x, t) \in \mathbf{R}^n \times (0, T), \\ u(x, T) = g(x) & \text{whenever } x \in \mathbf{R}^n. \end{cases} \tag{1.16}$$

Although this is a deterministic partial differential equation we may pose the problem as to determine the conditional expected value of a stochastic process $X(t)$ by using the formula of Feynman-Kac, see for instance Subsection 4.4.4 in the monograph [KS88]. We will now give a brief background to this link between partial differential equations and stochastic differential equations. For this matter, let $(\Omega, \mathcal{F}, P)$ be a probability space, where Ω is the space of outcomes, $\mathcal{F}$ is the σ-algebra of events in Ω and $P : \Omega \to [0, 1]$ is a probability measure.

Definition 1.18. *(Wiener process) A stochastic process* $W(t) = W(t, \omega)$, $W : \mathbf{R}_+ \times \Omega \to \mathbf{R}$, *is a one-dimensional Wiener process if the following hold:*

i) $W(0) = 0$ *almost surely,*

ii) W *has independent increments;* $W(t_2) - W(t_1)$ *and* $W(s_2) - W(s_1)$ *are independent for* $0 \leq s_1 < s_2 \leq t_1 < t_2$,

iii) Increments are normally distributed; $W(t) - W(s) \in N(0, t-s)$ *for* $0 \leq s \leq t$.

Moreover, $W(t) = (W_1(t), \ldots, W_m(t))$ *is an* m*-dimensional Wiener process if* $W(t)$ *is a vector of* m *independent one-dimensional Wiener processes* $W_i(t)$, $i = 1, \ldots, m$.

A filtration is an increasing family of σ-algebras and we let $\{\mathcal{F}_t\}_{t \geq 0}$ denote the natural filtration associated with the random variables $\{W(s) : 0 \leq s \leq t\}$. We say that a stochastic process $X(t)$ is adapted to the filtration $\{\mathcal{F}_t\}_{t \geq 0}$ if $X(t)$ is $\mathcal{F}_t$-measurable for all $t \geq 0$. Intuitively this means that if we know $W(s)$ for all $s \in [0, t]$ and if we can determine $X(t)$ by using this information, then $X(t)$ is $\mathcal{F}_t$-measurable.

Definition 1.19. *(Itô integral) Let* $0 = t_0 < t_1 < \ldots < t_n = T$ *be a partition of* $[0, T]$ *and let* $\Delta t_i = t_{i+1} - t_i$. *The Itô integral of a stochastic process* $f(t, \omega) : \mathbf{R}_+ \times \Omega \to \mathbf{R}$, *adapted to the filtration* $\{\mathcal{F}_t\}_{t \geq 0}$, *is defined by*

$$\int_0^T f(t, \omega) dW(s, \omega) := \lim_{n \to \infty} \sum_{i=0}^{n-1} f(t_i, \omega) \left(W(t_{i+1}, \omega) - W(t_i, \omega) \right)$$

where the limit is taken over partitions such that $\max \Delta t_i \to 0$, *provided that the limit exists.*

We emphasize that it takes a lot of effort to extend the ordinary Riemann integral to the stochastic setting since the paths of W has infinite variation. Moreover, unlike the deterministic case, the choice of sample points $f(t_i^*, \omega)$, $t_i^* \in [t_i, t_{i+1}]$, will affect the result. We obtain the Itô integral when we choose the left endpoint t_i. A useful notational convention is to write

$$dX(t) = \mu(X(t), t)dt + \sigma(X(t), t)dW(t), \tag{1.17}$$

meaning that

$$X(t) - X(0) = \int_0^t \mu(X(s), s)ds + \int_0^t \sigma(X(s), s)dW(s).$$

The first integral is an ordinary Lebesgue integral for each $\omega \in \Omega$, while the second one is an Itô integral. If properly defined, then $X(t)$ is called an Itô process, or a solution to the stochastic differential equation (1.17). We have the following results concerning existence and uniqueness for stochastic differential equations, see Theorem 5.2.1 in [Øks00].

Theorem 1.20. *(Solvability of stochastic differential equations) Let $T > 0$ and let μ and σ be measurable functions satisfying the following growth estimates*

$$\begin{aligned} |\mu(x,t)| + |\sigma(x,t)| &\leq C(1+|x|), \\ |\mu(x,t) - \mu(y,t)| + |\sigma(x,t) - \sigma(y,t)| &\leq C|x-y|, \end{aligned}$$

for all $t \in [0, T]$, $x, y \in \mathbf{R}^n$ and for some positive constant C. Then, the stochastic differential equation (1.17) has a unique strong solution $X(t)$ which is adapted to $\mathcal{F}_t$ and

$$E\left[\int_0^T |X(s)|^2 ds\right] < \infty.$$

The term strong solution means that if we have found a solution $X(t)$ to (1.17) given a particular Wiener process, then, should we change the Wiener process and solve (1.17) again, we would obtain the same expression for $X(t)$, but in terms of the new Wiener process. Now, let σ be an $n \times q$ matrix such that $(\sigma\sigma^*)_{ij} = a_{ij}$ for $i, j = 1, \dots, q$, and let

$$\mu_i(x, s) := b_i(x, s) + \sum_{j=1}^{q} c_{ij} x_i,$$

for $i = 1, \dots, n$. Above it is implicitly understood that $b_i \equiv 0$ for $i = q+1, \dots, n$ and that $\sigma_{ij} \equiv 0$ for $i = q+1, \dots, n$. Assume that μ_i and σ_{ij} satisfy the assumptions in Theorem 1.20 and define the n-dimensional process $X(t)$ by

$$X_i(t) = X_i(0) + \int_0^t \mu_i(X(s), s)ds + \sum_{j=1}^{q} \int_0^t \sigma_{ij}(X(s), s)dW_j(s), \quad (1.18)$$

where $W(s) = (W_1(s), \ldots, W_q(s))$ is a q-dimensional Wiener process. Then we have the following result, see for instance Theorem 8.2.1 in [Øks00] or Theorem 4.2 in Chapter 4 of [KS88].

Theorem 1.21. *(Feynman-Kac) Assume that* $g \in C^2(\mathbf{R}^n)$ *and let*

$$u(x,t) = E[g(X(T)) \mid X(t) = x], \tag{1.19}$$

where $X(t)$ *is defined as in (1.18). Then* $u(x,t)$ *is the unique* $C^{2,1}(\mathbf{R}^n \times (0,T))$ *solution to the backward in time Cauchy problem in (1.16).*

In fact, the Feynman-Kac formula is more general, but this version will be sufficient for us. Many practical problems can be formulated as a Kolmogorov backward in time Cauchy problem, or equivalently, as the problem of determining conditional expectations. In Paper IV we focus on problems arising in option pricing theory and in particular we investigate how to approximate (1.19) with a prescribed accuracy. We conclude this section with an example of a problem which can be posed as a Kolmogorov backward in time Cauchy problem.

Example 1.22. *(European Asian options in Black-Scholes model). The simplest example of an option is the European call option. This contract gives the holder the right, but not the obligation, to buy the underlying asset at a pre-specified price* K*, the so-called strike price, at a pre-specified time* T*, the maturity. Assuming that prices of the underlying asset evolves according to a stochastic process* $S(t)$*, the price of the option is given by*

$$Price = e^{-rT} E[\max(S(T) - K, 0)],$$

where E *is the expectation under the so-called risk neutral probability and* r *is the fixed interest rate. The function,* $\max(S(T) - K, 0)$*, which describes the outcome for the holder of the contract, is usually referred to as the pay-off function of the contract. This function indicates that if the underlying asset is cheaper to buy on the market, the contract is worthless. On the contrary, if the underlier is worth more than the strike price* K*, then the holder can buy the asset at a cost of* K *and sell the same asset for* $S(T)$ *resulting in a profit of* $S(T) - K$*. A European Asian call option has pay-off function*

$$\max\left(\frac{1}{T}\int_0^T S(t)dt - K, 0\right).$$

In Black-Scholes model it is assumed that

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t),$$

or equivalently, $S(t) = \exp\left(\left(\mu - \sigma^2/2\right) t + \sigma W(t)\right)$. *Now, define,*

$$A(t) = \frac{1}{t}\int_0^t S(t)dt,$$

then the price P *of the contract satisfies*

$$\frac{1}{2}\sigma^2 S^2 P_{SS} + rSP_S + \frac{1}{t}(S - A)P_A - rP + P_t = 0,$$

which is a Kolmogorov equation. In one dimension we can still solve this problem by means of elliptic parabolic equations, after a change of variables. In higher dimensions no such approach is known.

Worth to be mentioned is that in Paper IV we also use Malliavin calculus. This is an extension of Itô calculus, which we have briefly introduced. In fact, Malliavin calculus was originally developed in [Mal78] to provide a probabilistic proof of Hörmander's sum of squares theorem, see Theorem 1.2.

Chapter 2

Summary of the appended papers

In this chapter we will present the four appended papers and state the main results. We will also explain the ideas behind the proofs. The notation used here might differ from the notation used in the appended papers and the reason is that we choose to be consistent with the introduction.

2.1 Paper I. The obstacle problem for parabolic non-divergence form operators of Hörmander type

In this paper we consider the obstacle problem on a bounded domain $\Omega_T = \Omega \times (0,T) \in \mathbf{R}^{n+1}$, $n \geq 3$,

$$\begin{cases} \max\{\mathcal{H}u - \gamma u - f, \varphi - u\} = 0 & \text{in } \Omega_T, \\ u = g & \text{on } \partial_p \Omega_T. \end{cases} \tag{2.1}$$

We assume that the operator $\mathcal{H}$ is a subelliptic parabolic operator, that is,

$$\mathcal{H} = \sum_{i,j=1}^{q} a_{ij}(x,t) X_i X_j + \sum_{i=1}^{q} b_i(x,t) X_i - \partial_t, \tag{2.2}$$

where $(X_1, \ldots, X_q)$, $X_i = \sum_{j=1}^{n} c_{ij}(x) \partial_{x_j}$, is a set of smooth vector fields in $\mathbf{R}^n$ with $q < n$. Let $C(x)$ denote the $q \times n$-matrix $\{c_{ij}(x)\}$. In particular, we impose the following:

(H1) The smooth vector fields $\{X_1, \dots, X_q\}$ satisfy Hörmander's condition.

(H2) The matrix $A = \{a_{ij}\}$ is real symmetric with bounded and measurable entries and there exists $\Lambda \in [1, \infty)$ such that

$$\Lambda^{-1}|\xi|^2 \leq \sum_{i,j=1}^{q} a_{ij}(x,t)\xi_i\xi_j \leq \Lambda|\xi|^2, \text{ whenever } (x,t) \in \mathbf{R}^{n+1}, \xi \in \mathbf{R}^q.$$

(H3) The coefficients a_{ij} and b_i are measurable, bounded and, in addition, $a_{ij},\, b_i \in C^{0,\alpha}_{X,loc}(\mathbf{R}^{n+1})$ for some constant $\alpha \in (0, 1]$.

(H4) There exists a neighborhood $\widetilde{\Omega}$ of Ω such that; for all points $\varsigma \in \partial\Omega$ there exists an exterior normal v to $\overline{\Omega}$ relative $\widetilde{\Omega}$ such that $C(\varsigma)v \neq \mathbf{0}$.

(H5) The obstacle φ is Lipschitz continuous on $\overline{\Omega_T}$ and there exists a constant $c > 0$ such that

$$\sum_{i,j=1}^{q} \zeta_i\zeta_j \int_{\Omega_T} X_iX_j\psi(z)\varphi(z)dz \geq c|\zeta|^2 \int_{\Omega_T} \psi(z)dz,$$

for all $\zeta \in \mathbf{R}^q$ and for all positive test functions $\psi \in C_0^\infty(\Omega_T)$.

The assumption **(H1)-(H3)** assure that the differential operator is hypoelliptic while **(H4)-(H5)** enable us to use Bony's maximum principle and to construct barriers. We have the following main result on existence of solutions. Recall that we say that u is a strong solution to the obstacle problem (2.1) if $u \in C(\overline{\Omega_T}) \cap \mathcal{S}^1_{X,loc}(\Omega_T)$ satisfy the differential equation (2.1) almost everywhere in Ω_T and the boundary datum is attained at all points of $\partial_p\Omega_T$.

Theorem 2.1. *(Existence of solutions) Consider the obstacle problem (2.1) and assume that $\mathcal{H}$ is as in (2.2). Assume that conditions **(H1)-(H5)** are fulfilled. Let $\gamma, \varphi, f, g : \overline{\Omega_T} \to \mathbf{R}$ be continuous and bounded functions such that $g \geq \varphi$ on $\overline{\Omega_T}$. Then there exists a unique strong solution to the obstacle problem. Furthermore, given p, $1 \leq p < \infty$, and an open subset $U \subset\subset \Omega_T$ there exists a positive constant $c = c(\mathcal{H}, U, \Omega_T, p, \gamma, \varphi, f, g)$ such that*

$$||u||_{\mathcal{S}^p_X(U)} \leq c.$$

Moreover, the constant c only depend on γ, φ, f and g through their L^∞-norm on Ω_T.

The proof of Theorem 2.1, see also Theorem 1.1 in Paper I, is based on the classical penalization technique and in order to complete the proof we establish the following results, which correspond to Theorem 1.3-1.4 in Paper I.

Theorem 2.2. *(A priori $\mathcal{S}^p$ interior estimates) Assume that $\mathcal{H}$ is as in (2.2) and that conditions **(H1)-(H3)** are fulfilled. Let U be a compact subset of Ω_T and let $1 \leq p < \infty$. Then there exists a positive constant $c = c(\mathcal{H}, U, \Omega_T, p)$ such that*

$$||u||_{\mathcal{S}^p_X(U)} \leq c \left(||u||_{L^p_X(\Omega_T)} + ||\mathcal{H}u||_{L^p_X(\Omega_T)} \right),$$

whenever $u \in \mathcal{S}^p_X(\Omega_T)$.

Theorem 2.3. *(Embedding theorem) Assume that $\mathcal{H}$ is as in (2.2) and that conditions **(H1)-(H3)** are fulfilled. Let U be a compact subset of Ω_T and let Q be the homogeneous dimension of the free Lie group associated to $X = \{X_1, \dots, X_q\}$. For $p \in (Q+2, 2(Q+2))$ let $\alpha = (p-(Q+2))/p$. Then, there exists a constant $c = c(\mathcal{H}, U, \Omega_T, p)$ such that*

$$||u||_{C^{1,\alpha}_X(U)} \leq c||u||_{\mathcal{S}^p_X(\Omega_T)}$$

for every $u \in \mathcal{S}^p_X(\Omega_T)$.

In fact, both the a priori S^p interior estimate as well as the embedding theorem were missing in the literature and a substantial part of Paper I is devoted to the proof of these two theorems. In addition, we use Schauder estimates, that is we use that we can estimate the $C^{2,\alpha}_X$-norm of u by using the L^∞-norm of u and the $C^{0,\alpha}_X$-norm of $\mathcal{H}u$. This is a non-trivial result by Bramanti and Brandolini, see Theorem 1.1 in [BB07]. We will now account for the idea behind the proof of Theorem 2.1. Consider a family $(\beta_\varepsilon)_{\varepsilon\in(0,1)}$ of smooth functions with the property that for fixed $\varepsilon \in (0,1)$ β_ε is an increasing function such that

$$\beta_\varepsilon(0) = 0, \quad \beta_\varepsilon(s) \leq \varepsilon, \quad \text{whenever } s > 0,$$

and such that

$$\lim_{\varepsilon \to 0} \beta_\varepsilon(s) = -\infty, \quad \text{whenever } s < 0.$$

Now we can use mollifiers to obtain regularized versions of the coefficients of $\mathcal{H}$ and we obtain a differential operator $\mathcal{H}^\delta$ with smooth coefficients. Then we consider the penalized problem

$$\begin{cases} \mathcal{H}^\delta u_{\varepsilon,\delta} + \gamma^\delta u_{\varepsilon,\delta} = f^\delta - \beta_\varepsilon(u_{\varepsilon,\delta} - \varphi^\delta), & \text{in } \Omega_T, \\ u_{\varepsilon,\delta} = g^\delta & \text{on } \partial_p\Omega_T. \end{cases} \tag{2.3}$$

Above, $u_{\varepsilon,\delta}$ indicates that the solution will depend both on the mollification, through δ, and on the penalization function β_ε. As a first step we prove that there exists a classical solution $u_{\varepsilon,\delta} \in C_X^{2,\alpha}(\Omega_T) \cap C(\overline{\Omega_T})$ to the penalized problem (2.3) which satisfy certain growth estimates. To do this we use a monotone iterative method and consider solutions $u_{\varepsilon,\delta}^j$ to a linear Dirichlet problem. In particular, by Bony's results, there exists a classical solution $u_{\varepsilon,\delta}^j \in C^\infty(\Omega_T)$ to the linearized problem. Then we show that $\{u_{\varepsilon,\delta}^j\}_{j=1}^\infty$ is a decreasing sequence and to do this we use the maximum principle. By an application of the Schauder estimate we find that $\{u_{\varepsilon,\delta}^j\}_{j=1}^\infty$ has a convergent subsequence and by a barrier argument we finally prove that there exists a classical solution $u_{\varepsilon,\delta}$ to (2.3), namely the limit of the convergent subsequence.

Next step is to prove that, as $\varepsilon, \delta \to 0$, the function $u_{\varepsilon,\delta} \to u$ and u is a strong solution to the obstacle problem. We begin to prove that there exist constants c_1 and c_2, independent of both ε and δ, such that $\beta_\varepsilon(u_{\varepsilon,\delta} - \varphi^\delta)$, appearing in (2.3) is bounded from above and below. The boundedness from above follows by definition and to prove boundedness from below we use the maximum principle once again. However, we are only able to establish quite weak bounds on u, if we want the bounds to be independent of both ε and δ. To prove that $u_{\varepsilon,\delta} \to u$ in $\mathcal{S}^p_{X,loc}$ we use the a priori $\mathcal{S}^p$ interior estimate and by the embedding theorem this also holds in $C^{1,\alpha}_{X,loc}$. Using this information we can deduce that u is in fact a strong solution to the obstacle problem.

To prove the embedding theorem we lift the problem, using the lifting theorem of Rothschild and Stein. The reason is that we need a uniform bound on the Euclidean volume of Carnot-Carathéodory balls. Then we represent u and $X_i u$ by means of an integral involving the fundamental solution and the differential operator. By establishing bounds on the fundamental solution and by dividing the integral into sums of integrals over certain Carnot-Carathéodory balls we complete the proof in the lifted setting. However, this carries over directly to the original problem by results of Bramanti and Brandolini in [BB07], Proposition 8.3.

The proof of the a priori $\mathcal{S}^p$ interior estimates heavily relies on the lifting approximation technique. Locally we lift, approximate and freeze the original Hörmander vector fields. The corresponing operator is then left invariant and homogeneous of degree two allowing us to use results of Folland, [Fol75]. In this setting we develop certain local approximation results, in particular, we develop an operator type calculus. This is used to construct a parametrix, which in turn is used to derive L^p estimates in the lifted setting. Getting back to the original vetor fields is then straightforward.

2.2 Paper II. Regularity in the obstacle problem for parabolic non-divergence operators of Hörmander type

This paper, which is a continuation of the study carried out in Paper I, examines regularity properties of solutions to the obstacle problem. Results on regularity in the interior of the domain as well as near the initial state are established. However, we are unable to carry out the study for the wider class of operators considered in Paper I and we restrict ourselves by imposing a stronger version of condition **(H1)**, namely;

(H1') The smooth vector fields $X = \{X_1, \dots, X_q\}$ satisfy Hörmander's condition and there exists a homogeneous Lie group $\mathbf{G} = (\mathbf{R}^n, \circ, \delta_\lambda)$ such that $X = \{X_1, \dots, X_q\}$ Lie generates $\mathbf{G}$. Moreover, we assume that $X = \{X_1, \dots, X_q\}$ are left invariant and homogeneous of degree one on $\mathbf{G}$.

Below we will discuss the necessity of this restriction. In addition, we also assume that $\gamma = 0$ in (2.1). The main result is that for smooth as well as non-smooth obstacles the solution is, up to $\mathcal{S}_X^\infty$-smoothness, as smooth as the obstacle, φ, in the interior of the domain. Near, and up to, the initial state the smoothness of the solution will in addition depend on the smoothness of the boundary data, g. In particular, we prove the following theorems which correspond to Theorem 1.1 and Theorem 1.2 in Paper III respectively.

Theorem 2.4. *(Interior regularity) Let $\mathcal{H}$ be defined as in (2.2) and assume **(H1'), (H2)-(H3)**, let Ω, Ω' be bounded domains in $\mathbf{R}^n$ such that $\Omega' \subset\subset \Omega$ and let $0 < T'' < T' < T$. Set $\Omega_T = \Omega \times (0,T)$, $\Omega'_T = \Omega' \times (T'', T')$. Let $g, f, \varphi : \overline{\Omega_T} \to \mathbf{R}^{n+1}$ be such that $g \geq \varphi$ on $\overline{\Omega_T}$ and assume that g, f, φ are continuous and bounded on $\overline{\Omega_T}$. Let $\alpha \in (0,1)$ and let u be a strong solution to problem (2.1) in Ω_T. Then the following holds:*

i) if $\varphi \in C_X^{0,\alpha}(\Omega_T)$ then $u \in C_X^{0,\alpha}(\Omega' \times (T'', T'))$ and

$$\|u\|_{C_X^{0,\alpha}(\Omega'_T)} \leq c\left(\alpha, \Omega_T, \Omega'_T, \mathcal{H}, \|f\|_{C_X^{0,\alpha}(\Omega_T)}, \|g\|_{L^\infty(\Omega_T)}, \|\varphi\|_{C_X^{0,\alpha}(\Omega_T)}\right);$$

ii) *if* $\varphi \in C^{1,\alpha}_X(\Omega_T)$ *then* $u \in C^{1,\alpha}_X(\Omega' \times (T'', T'))$ *and*

$$\|u\|_{C^{1,\alpha}_X(\Omega'_T)} \leq c\left(\alpha, \Omega_T, \Omega'_T, \mathcal{H}, \|f\|_{C^{0,\alpha}_X(\Omega_T)}, \|g\|_{L^\infty(\Omega_T)}, \|\varphi\|_{C^{1,\alpha}_X(\Omega_T)}\right);$$

iii) *if* $\varphi \in C^{2,\alpha}_X(\Omega_T)$ *then* $u \in \mathcal{S}^\infty_X(\Omega' \times (T'', T'))$ *and*

$$\|u\|_{\mathcal{S}^\infty_X(\Omega'_T)} \leq c\left(\alpha, \Omega_T, \Omega'_T, \mathcal{H}, \|f\|_{C^{0,\alpha}_X(\Omega_T)}, \|g\|_{L^\infty(\Omega_T)}, \|\varphi\|_{C^{2,\alpha}_X(\Omega_T)}\right).$$

Theorem 2.5. *(Regularity up to the initial state) Under the same assumptions as in Theorem 2.4, but for* $\Omega_T = \Omega \times (0, T)$, $\Omega'_T = \Omega' \times (0, T')$, *the following holds:*

i) *if* $g, \varphi \in C^{0,\alpha}_X(\Omega_T)$ *then* $u \in C^{0,\alpha}_X(\Omega'_T)$ *and*

$$\|u\|_{C^{0,\alpha}_X(\Omega'_T)} \leq c\left(\alpha, \Omega_T, \Omega'_T, \mathcal{H}, \|f\|_{C^{0,\alpha}_X(\Omega_T)}, \|g\|_{C^{0,\alpha}_X(\Omega_T)}, \|\varphi\|_{C^{0,\alpha}_X(\Omega_T)}\right);$$

ii) *if* $g, \varphi \in C^{1,\alpha}_X(\Omega_T)$ *then* $u \in C^{1,\alpha}_X(\Omega'_T)$ *and*

$$\|u\|_{C^{1,\alpha}_X(\Omega'_T)} \leq c\left(\alpha, \Omega_T, \Omega'_T, \mathcal{H}, \|f\|_{C^{0,\alpha}_X(\Omega_T)}, \|g\|_{C^{1,\alpha}_X(\Omega_T)}, \|\varphi\|_{C^{1,\alpha}_X(\Omega_T)}\right);$$

iii) *if* $g, \varphi \in C^{2,\alpha}_X(\Omega_T)$ *then* $u \in \mathcal{S}^\infty_X(\Omega'_T)$ *and*

$$\|u\|_{\mathcal{S}^\infty_X(\Omega'_T)} \leq c\left(\alpha, \Omega_T, \Omega'_T, \mathcal{H}, \|f\|_{C^{0,\alpha}_X(\Omega_T)}, \|g\|_{C^{2,\alpha}_X(\Omega_T)}, \|\varphi\|_{C^{2,\alpha}_X(\Omega_T)}\right).$$

The proofs rely on "blow-up" arguments and polynomial approximations and we will shortly account for the proof of Theorem 2.4. The core of the argument uses the function

$$S_k^-(u) = \sup_{C^-_{2^{-k}(0,0)}} |u|,$$

wherethe supremum is taken over the cylinder $C_r^-(0,0) = B_d(0,r) \times (-r^2, 0)$ for $r = 2^{-k}$. Then, depending on the regularity of the obstacle φ we consider functions F and exponents γ,

$$\begin{aligned} &(i) && F = P_0^{(0,0)}\varphi,\ \gamma = \alpha, \\ &(ii) && F = P_1^{(0,0)}\varphi,\ \gamma = 1 + \alpha, \\ &(iii) && F = \varphi,\ \gamma = 2, \end{aligned}$$

where $\alpha \in (0,1)$ is the Hölder exponent of φ. Moreover, $P_0^{(0,0)}\varphi$ and $P_1^{(0,0)}\varphi$ are certain intrinsic Taylor polynomials of φ. The major step in proving Theorem 2.4 is to prove that

$$S_{k+1}^-(u-F) \leq \max\left(c2^{-(k+1)\gamma}, \frac{S_k^-(u-F)}{2^\gamma}, \dots, \frac{S_0^-(u-F)}{2^{(k+1)\gamma}}\right). \tag{2.4}$$

Indeed, if (2.4) holds, then, by an iterative argument, $S_k^-(u-F) \leq c/2^{k\gamma}$. This means that if $u(0,0) = \varphi(0,0)$, then

$$\sup_{C_r^-} |u-\varphi| \leq cr^\gamma, \quad r \in (0,1). \tag{2.5}$$

That is, u cannot differ largely from φ on small cylinders. In fact, we can also prove that the same holds true when the supremum is taken over cylinders $C_r = B_d(0,r) \times (-r^2, r^2)$. To prove this we argue by contradiction, in particular, we assume that for every $j \in \mathbf{N}$ there exist functions $(u_j, f_j, g_j, \varphi_j)$, which belong to a certain function class, such that $u_j(0,0) = \varphi_j(0,0)$ while the analogue of (2.4) does not hold for some integer $k = k_j$. Let (x_j, t_j) be a point in $C_{2^{-k}}^-(0,0)$ where $S_{k_j}^-(u)$ attains its maximum. Then we blow up functions, vector fields and operators near the point (x_j, t_j). Using the regularity assumptions on f_j, g_j and φ_j we define functions v_j and $\widetilde{v}_j$, which bound u from below and above respectively. Moreover, the functions v_j and $\widetilde{v}_j$ are defined as solutions to certain Dirichlet problems, and hence, solvability of the Dirichlet problem is important ingredient of the proof. The contradiction consists in proving that the blow-up version of $u_j(0,0)$ is not equal to the blow-up version of $\varphi_j(0,0)$. This task is completed using v_j and $\widetilde{v}_j$. The final proof of Theorem 2.4, when we have established (2.4), has to be divided into several cases depending both on the regularity of the obstacle φ and on geometrical aspects. By geometrical aspects we mean that we have to take into account how far apart the points in the definition of Hölder regularity are. Using some technical lemmas, results for solutions to the Dirichlet problem, maximum and comparison principles and the estimate in (2.5) the proof is completed. Finally, we remark that condition **(H1')** is used extensively in the proof; it is used to approximate functions with polynomials of a certain degree, to construct blow-ups and to dilate points. Also, it is not clear whether Hölder estimates can be carried through the approximation theorem of Rotschild and Stein, and we are therefore unable to use that machinery at the present time.

2.3 Paper III. Non-divergence form parabolic equations associated with non-commuting vector fields: Boundary behavior of nonnegative solutions

We divide the parabolic boundary so that $\partial_p \Omega_T = S_T \cup (\Omega \times \{t = 0\})$, with $S_T = \partial_p \Omega \times (0, T)$ denoting the lateral boundary. Moreover, we let $A_r(x_0, t_0) = (A_r(x_0), t_0)$, with $A_r(x_0)$ as in the definition of NTA$_X$ domains, and we define cylinders $C_r(x, t) = B_{d_X}(x, r) \times (t - r^2, t + r^2)$. In this paper we consider nonnegative solutions to

$$\mathcal{H}u = \sum_{i,j=1}^{q} a_{ij}(x, t) X_i X_j u - \partial_t u = 0, \quad (x, t) \in \mathbf{R}^n \times \mathbf{R}_+,$$

on bounded cylinders $\Omega_T = \Omega \times (0, T)$, for some bounded domain Ω and for some $T > 0$. Throughout this section we assume that Ω_T is a NTA$_X$ domain with parameters M and r_0 and that the set of vector fields $X = \{X_1, \dots, X_q\}$ satisfy condition **(H1)-(H3)**, see Section 2.1. We will now present our main results, Theorem 1.1-1.3 in Paper III, In Figure 2.1 we include a schematic picture of the situation in Theorem 2.6.

Theorem 2.6. *(Backward Harnack inequality) Let u be a nonnegative solution of $\mathcal{H}u = 0$ in Ω_T with the additional property that $u = 0$ on S_T. Let τ, $0 < \tau \ll \sqrt{T}$ be a fixed constant, let $(x_0, t_0) \in S_T$ be such that $\tau^2 \leq t_0 \leq T - \tau^2$, and assume that $r < \min\{r_0/2, \frac{1}{2}\sqrt{(T - \tau^2 - t_0}, \frac{1}{2}\sqrt{t_0 - \tau^2}\}$. Then there exists a constant $c = c(\mathcal{H}, M, r_0, diam(\Omega), T, \tau) \in [1, \infty)$, such that for every $(x, t) \in C_{r/4}(x_0, t_0) \cap \Omega_T$*

$$u(x, t) \leq c u(A_r(x_0, t_0)).$$

Theorem 2.7. *(Boundary Hölder continuity of quotients of solutions) Let u and v be two nonnegative solutions of $\mathcal{H}u = 0$ in Ω_T. Given $(x_0, t_0) \in S_T$, assume that $r < \min\{r_0/2, \frac{1}{2}\sqrt{T - \tau^2 - t_0}, \frac{1}{2}\sqrt{t_0 - \tau^2}\}$. If $u = v = 0$ on $S_T \cap C_{2r}(x_0, t_0)$ then the quotient u/v is Hölder continuous on the closure of $\Omega_T \cap (C_r(x_0, t_0) \cap \{t < t_0\})$.*

Theorem 2.8. *(Doubling property of the $\mathcal{H}$-parabolic measure) Let $K \geq 100$ and $\nu \in (0, 1)$ be fixed constants. Let $(x_0, t_0) \in S_T$ and assume that*

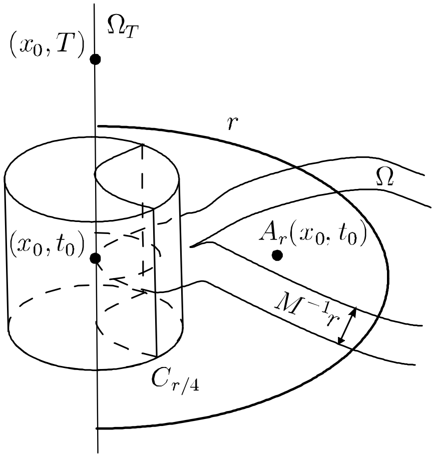

Figure 2.1: This schematic picture represents the situation in the backward Harnack inequality. Given a point (x_0, t_0) on the lateral boundary, the value of u at any point in the cylinder $C_{r/4}(x_0, t_0)$, which also belong to Ω_T, can be bounded from above in terms of the value of u at a certain point $A_r(x_0, t_0)$.

$r < \min\{r_0/2, \frac{1}{2}\sqrt{T - \tau^2 - t_0}, \frac{1}{2}\sqrt{t_0 - \tau^2}\}$. *Then there exists a constant* $c = c(\mathcal{H}, M, r_0, K, \nu) \in [1, \infty)$, *such that for every* $(x, t) \in \Omega_T$, *with* $d_X(x_0, x) \leq K|t - t_0|^{1/2}$ *and* $t - t_0 \geq 16r^2$, *we have that*

$$\omega^{(x,t)}(S_T \cap C_{2r}(x_0, t_0)) \leq c\omega^{(x,t)}(S_T \cap C_r(x_0, t_0)).$$

The corresponding theorems in the case of uniformly parabolic equations have been proved by Fabes, Safonov and Yuan in [FSY99] and [SY99]. In that case the corresponding statements were proved for $\{a_{ij}\}$ being only bounded and measurable. We restrict ourselves to Hölder continuity since estimates on the fundamental solution, Harnack inequalities and so forth are unavailable at present time for $\{a_{ij}\}$ being only measurable and bounded. Instead we rely on the results presented in [BBLU09]. The proofs make use of several technical lemmas, and to finally conclude the proof numerous steps are taken, which makes it a difficult task to simplify. Instead we settle by commenting on the results we use. First of all we make extensive use of *i*) − *iii*) in the definition of NTA_X domains, see Definition 1.15. The Harnack inequality, strong maximum principle, solvability of the Dirichlet problem and, as mentioned above, estimates on the fundamental solution are used repeatedly. At many instances we use iterative arguments.

2.4 Paper IV. Adaptive stochastic weak approximation of degenerate parabolic equations of Kolmogorov type

We consider the problem of pricing European options assuming the model proposed by Hobson and Rogers in [HR98]. This problem could be stated as a backward in time Kolmogorov equation with Cauchy data as in (1.16), but we choose to approach the problem using stochastic differential equations and Malliavin calculus. That is, we assume that

$$X_i(t) = X_i(0) + \int_0^t \mu_i(X(s), s)ds + \sum_{j=1}^{q} \int_0^t \sigma_{ij}(X(s), s)dW_j(s), \qquad (2.6)$$

for $i \in \{1, \ldots, n\}$, and for $X(t) = (X_1(t), \ldots, X_n(t))^T$ we wish to approximate

$$u(x, t) = E[g(X(T) \mid X(t)], \qquad (2.7)$$

with a prescribed accuracy. For notation, we refer to Subsection 1.3.1. We assume that $u(x,t)$ satisfy a Kolmogorov equation according to Section 1.3, that $\mu_i, \sigma_{ij} \in C_b^\infty(\mathbf{R}^n \times \mathbf{R}_+)$ and that $g \in C_p^\infty(\mathbf{R}^n)$. We will use an Euler scheme to approximate (2.7), that is, for a given T we let $\{t_k\}_{k=1}^N$ be a partition of the interval $[0, T]$, $0 = t_0 < t_1 < \ldots < t_N = T$, and we set $\Delta t_k = t_{k+1} - t_k$ and $\Delta W_k = W(t_{k+1}) - W(t_k)$. Then we define $X^\Delta(t)$ by

$$X^\Delta(t_{k+1}) = X^\Delta(t_k) + \mu(X^\Delta(t_k), t_k)\Delta t_k + \sum_{j=1}^{q} \sigma_j(X^\Delta(t_k), t_k)\Delta W_j(t_k),$$

where $\mu = (\mu_1, \ldots, \mu_n)^T$ and $\sigma_j = (\sigma_{1j}, \ldots, \sigma_{nj})^T$. $X^\Delta(t_{k+1})$ is referred to as the discrete Euler approximation of $X(t)$. Moreover, we define the continuous Euler approximation in terms of $\phi(t) = \sup\{t_i : t_i < t\}$, which is defined relative a certain discretization, by

$$\begin{aligned} X^\Delta(t) &= X^\Delta(\phi(t)) + \int_{\phi(t)}^t \mu(X^\Delta(\phi(s)), s)ds \\ &\quad + \sum_{j=1}^{q} \int_{\phi(t)}^t \sigma_j(X^\Delta(\phi(s)), s)dW_j(s). \end{aligned}$$

The continuous Euler approximation will only be used for theoretical purposes, while the discrete Euler approximation will be used in actual computations. In particular, we approximate

$$u(x, t_k) \approx u^\Delta(x, t_k) := E[X^\Delta(T) \mid X^\Delta(t_k) = x],$$

for $k \in \{0, \ldots, N-1\}$. The standard method to approximate $u^{\Delta}(x) = u^{\Delta}(x, 0)$ is to use a Monte Carlo routine,

$$u^{\Delta}(x) \approx u^{\Delta,M}(x) = \frac{1}{M} \sum_{l=1}^{M} g(X^{\Delta}(T, \omega_l)).$$

Above $\{\omega_l\}_{l=1}^{M}$ represent M independent realizations of the discrete Euler scheme. The error in this realization can be split in two parts, which we treat differently, namely

$$u(x) = u(x,0) = u^{\Delta,M}(x) + \underbrace{u(x) - u^{\Delta}(x)}_{E_d^{\Delta}(x)} + \underbrace{u^{\Delta}(x) - u^{\Delta,M}}_{E_s^{\Delta,M}(x)}.$$

The first part, $E_d^{\Delta}(x)$, is the discretization error while the second part, $E_s^{\Delta,M}(x)$, is the statistical error. The statistical error can be estimated by means of the central limit theorem and the Berry-Esséen theorem, and this estimate is valid with a certain probability, to be chosen in the algorithm presented in Paper IV. The discretization error is expanded in a computable a posteriori form in Theorem 3.2 in Paper IV and the proof is divided into five steps. First we use Itô's lemma to represent the discretization error in terms of Itô integrals. Then we use Itô's lemma once again together with assumptions on smoothness and boundedness on μ_i, σ_{ij} and g to be able to represent the error as a sum, not involving integrals. The next step is to prove that the derivatives of u, which is part of terms in the sum, can be approximated using derivatives of u^{Δ}. To do this we use Malliavin calculus. Thereafter we replace derivatives of u^{Δ} with certain dual functions, that is, Malliavin derivatives of certain functions. Along the way we control the error introduced and this lead us to a computable a posteriori expression for the discretization error.

Such algorithms have been available in the uniformly parabolic setting in [STZ01]. Our approach is therefore more general when it comes to which underlying partial differential equations that can be treated, but at a cost of smoothness conditions on $\mu_{i,}\sigma_{ij}$ and g. However, in option pricing, few pay-offs are smooth and to overcome this we used a regularized version of g. In order to fasten computations, various variance reduction techniques can be used, note however that using Quasi Monte Carlo is not one of them.

2.5 Future research

There are still numerous open questions I would like to study and I will give an account for a few of them. For instance, I believe that it is possible to prove existence results for the obstacle problem if we replace ∂_t with the more general term X_0. In that case we should assume that $\{X_0, X_1, \ldots, X_q\}$ Lie generates $\mathbf{R}^{n+1}$. The benefit would be that a lot of real life problems can be described in this setting. Another question which arose when working with both Paper I and Paper II is whether or not we can carry $C_X^{k,\alpha}$-estimates through the lifting approximation machinery. I have no educated guess on this issue, although a partially affirmative answer is provided in [BB07]. Moreover, a natural continuation is to pursue towards free boundary problems. To explain this further we define the sets

$$\begin{aligned}\mathcal{E} &= \{(x,t) \in \Omega_T : u(x,t) = \varphi(x,t)\},\\ \mathcal{C} &= \{(x,t) \in \Omega_T : u(x,t) > \varphi(x,t)\}.\end{aligned}$$

The boundary of $\mathcal{E}$, which we denote $\mathcal{F}$, is called the associated free boundary and I would like to study the behavior of $\mathcal{F}$. Finally, when more is known about obstacle problems in this setting I would like to construct numerical schemes to solve such problems. A starting point would be to construct an adaptive method for estimating the expected value of the supremum of a stochastic process.

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