The Plurisubharmonic Mergelyan Property

Lisa Hed

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Abstract

In this thesis, we study two different kinds of approximation of plurisubharmonic functions.

The first one is a Mergelyan type approximation for plurisubharmonic functions. That is, we study which domains in $\mathbb{C}^n$ have the property that every continuous plurisubharmonic function can be uniformly approximated with continuous and plurisubharmonic functions defined on neighborhoods of the domain. We will improve a result by Fornæss and Wiegerinck and show that domains with C^0-boundary have this property. We will also use the notion of plurisubharmonic functions on compact sets when trying to characterize those continuous and plurisubharmonic functions that can be approximated from outside. Here a new kind of convexity of a domain comes in handy, namely those domains in $\mathbb{C}^n$ that have a negative exhaustion function that is plurisubharmonic on the closure. For these domains, we prove that it is enough to look at the boundary values of a plurisubharmonic function to know whether it can be approximated from outside.

The second type of approximation is the following: we want to approximate functions u that are defined on bounded hyperconvex domains Ω in $\mathbb{C}^n$ and have essentially boundary values zero and bounded Monge-Ampère mass, with increasing sequences of certain functions $\{u_j\}$ that are defined on strictly larger domains. We show that for certain conditions on Ω, this is always possible. We also generalize this to functions with given boundary values. The main tool in the proofs concerning this second approximation is subextension of plurisubharmonic functions.

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Contents

Introduction

In this thesis, we study approximation of plurisubharmonic functions with plurisubharmonic functions defined on strictly larger domains. We shall begin to study a Mergelyan type approximation for plurisubharmonic functions. Let $\Omega \subset \mathbb{C}^n$ be a domain, i.e. an open and connected set, and let $\mathcal{O}(\Omega)$ denote the functions that are holomorphic on Ω. Remember that a domain Ω has the *Mergelyan property* if every $f \in \mathcal{O}(\Omega) \cap C(\bar{\Omega})$ can be approximated uniformly on $\bar{\Omega}$ with functions holomorphic on neighborhoods of $\bar{\Omega}$. By the classical theorem of Mergelyan, [Me], we know that if $K \subset \mathbb{C}$ is a compact set such that $\mathbb{C} \setminus K$ is connected, then every function $f \in \mathcal{O}(K^\circ) \cap C(K)$, where K° denotes the interior of K, can be approximated uniformly on K by polynomials. The problem is more difficult in $\mathbb{C}^n$, when $n > 1$, and has been studied for example by Henkin, Kerzman and Lieb, [Hen, Ker, Li]. Let $\mathcal{PSH}(\Omega)$ denote the class of plurisubharmonic functions defined on a domain Ω in $\mathbb{C}^n$. We say that a bounded domain Ω in $\mathbb{C}^n$ has the *$\mathcal{PSH}$-Mergelyan property* if every function $\varphi \in \mathcal{PSH}(\Omega) \cap C(\bar{\Omega})$ can be approximated uniformly on $\bar{\Omega}$ with continuous and plurisubharmonic functions defined on neighborhoods of $\bar{\Omega}$. The question of which domains have the $\mathcal{PSH}$-Mergelyan property has been studied by, for example, Sibony [Sib, Theorem 2.2] who showed that if Ω is a pseudoconvex domain with smooth boundary, then Ω has the $\mathcal{PSH}$-Mergelyan property. Fornæss and Wiegerinck then proved that this is true even for arbitrary bounded domains with C^1-boundary (see [FW, Theorem 1]). One of the main results in this thesis is Theorem 7.3, where we improve this result and prove that it is enough with C^0-boundary.

In Chapter 8, another type of approximation is discussed. Here we will use Monge-Ampère techniques and we need to consider some classes of negative plurisubharmonic functions where the Monge-Ampère operator is well-defined. For the definition of the classes $\mathcal{F}(\Omega)$, $\mathcal{N}(\Omega)$, $\mathcal{E}(\Omega)$, and $\mathcal{F}(\Omega, F)$, see Chapter 3. Let $\Omega \Subset \Omega_{j+1} \Subset \Omega_j$ be hyperconvex domains. Here $\Subset$ denotes that Ω_{j+1} is relatively compact in Ω_j. We say that a domain Ω has the *$\mathcal{F}$-approximation property* if every function $u \in \mathcal{F}(\Omega)$ can be approximated with an increasing sequence $\{u_j\}$, $u_j \in \mathcal{F}(\Omega_j)$, a.e. on Ω. Just as the $\mathcal{PSH}$-Mergelyan property, this gives us approximation with functions defined on strictly larger domains, but the convergence here is no longer uniform. But, on the other hand, we get control of the approximating sequence. In Theorem

8.2, we show that a hyperconvex domain has the $\mathcal{F}$-approximation property if, and only if, we can find a function $0 > v \in \mathcal{N}(\Omega)$ and a sequence $\{v_j\}$, $v_j \in \mathcal{N}(\Omega_j)$, such that $v_j \to v$ a.e. on Ω. This condition is of course necessary and is needed to make sure that the sequence $\{\Omega_j\}$ converges to Ω in the sense that $\operatorname{cap}(K,\Omega) = \lim_j \operatorname{cap}(K,\Omega_j)$ for every compact subset K of Ω. Here, $\operatorname{cap}(K,\Omega)$ is the relative Monge-Ampère capacity defined by Bedford and Taylor in [BT2]. In Theorem 8.3 we prove that the result in Theorem 8.2 can be generalized to functions in the class $\mathcal{F}(\Omega, H)$, where $H \in \mathcal{E}$ is a maximal function that is continuous up to the boundary.

When discussing these two approximation properties we shall use the notion of plurisubharmonic functions on compact sets, which is discussed in Chapter 5. Plurisubharmonic functions on compact sets have proven to be very useful when, for example, characterizing plurisubharmonic hulls and studying uniform algebras of holomorphic functions on compacts (see [Po4]). Recently, in [PS], Poletsky and Sigurdsson studied plurisubharmonic functions on compact sets from the point of view of several different generalized Dirichlet problems. There are different ways to define plurisubharmonic functions on a compact set $X \subset \mathbb{C}^n$. In Chapter 5, we define them as upper semicontinuous functions on X, that satisfy a submean inequality with respect to a certain class of Jensen measures, $\mathcal{J}_z(X)$. This definition is inspired by Sibony in [Sib]. By Theorem 5.11, it follows that a domain Ω has the $\mathcal{PSH}$-Mergelyan property if, and only if, every $\varphi \in \mathcal{PSH}(\Omega) \cap C(\bar{\Omega})$ is also plurisubharmonic on the compact $\bar{\Omega}$. It is therefore interesting to know when a given function $\varphi \in \mathcal{PSH}(\Omega) \cap C(\bar{\Omega})$ also is plurisubharmonic on $\bar{\Omega}$. To do this, we introduce a new type of convexity that we will call P-hyperconvexity. A bounded domain in $\mathbb{C}^n$ is called *P-hyperconvex* if it has a negative plurisubharmonic exhaustion function that is plurisubharmonic on the closure of the domain. In Chapter 6, this notion is discussed and in Theorem 6.7, we show that if Ω is a P-hyperconvex domain, then a function $\varphi \in \mathcal{PSH}(\Omega) \cap C(\bar{\Omega})$ is plurisubharmonic on $\bar{\Omega}$ if, and only if, φ is plurisubharmonic on $\partial\Omega$. The notion of P-hyperconvexity is also used to characterize those domains in $\mathbb{C}^n$ that have the $\mathcal{F}$-approximation property. In Corollary 8.4 we show that if Ω is P-hyperconvex and has a Stein neighborhood basis, then Ω has the $\mathcal{F}$-approximation property.

In Appendix A, we shall follow the construction of plurisubharmonic functions on compact sets as it is done by Poletsky in [Po4]. He constructs a class of Jensen measures that he calls *holomorphic measures*. These measures are defined as weak*-limits of measures that are push forwards of the arc length measure on the unit disk with analytic disks. He later uses these measures to define plurisubharmonic functions on compact sets in the same way as we do in Chapter 5. In Appendix A, we prove that his holomorphic measures and the class $\mathcal{J}_z(X)$ are the same. This result has been hinted by Poletsky in [Po4] and proven by Nguyễn, Dung and Hung in [NDH]. In Appendix A we give complete proofs of the theorems involved.

The main tool in the proofs concerning the $\mathcal{F}$-approximation property is subextension, which is the main subject of Chapter 4. Let Ω and $\widehat{\Omega}$ be hyperconvex domains such that $\Omega \Subset \widehat{\Omega}$. Given a plurisubharmonic function u on Ω, when can we find a plurisubharmonic function v on $\widehat{\Omega}$, $v \not\equiv -\infty$, such that $v \leq u$ on Ω? The function v is called a *subextension* of u to $\widehat{\Omega}$. In Chapter 4, we discuss the problem of subextending functions with given boundary values without increasing the total Monge-Ampère mass. This has proven to be important in applications, for example when estimating the volume of plurisubharmonic sublevel sets (see [ACKPZ]). The main result regarding subextension is Theorem 4.6 which says the following: Let Ω and $\widehat{\Omega}$ be hyperconvex domains such that $\Omega \subset \widehat{\Omega}$. Given a function $F \in \mathcal{E}(\Omega)$, a maximal plurisubharmonic function $G \in \mathcal{E}(\widehat{\Omega})$, such that $G \leq F$ on Ω, and a function $u \in \mathcal{F}(\Omega, F)$, we show that there exists a subextension v of u such that $v \in \mathcal{F}(\widehat{\Omega}, G)$ and $\int_{\widehat{\Omega}} (dd^c v)^n \leq \int_{\Omega} (dd^c u)^n$. If we add the conditions that $\Omega \Subset \widehat{\Omega}$, $F \in \mathcal{E}(\Omega) \cap C(\bar{\Omega})$, and $\int_{\Omega} (dd^c u)^n < \infty$, then we obtain the stronger result that: if v is the maximal subextension of u to $\widehat{\Omega}$, then we get control of the Monge-Ampère measure of v instead of the Monge-Ampère mass.

In Chapter 9, we study B-regularity. In [Sib], Sibony used the class of Jensen measures, $\mathcal{J}_z(\bar{\Omega})$, to draw conclusions regarding the solvability of the Dirichlet problem. Specifically, he studied compact sets whose only Jensen measures are the Dirac measures, the so-called B-regular sets. He also connects these sets to Catlin's Property (P), [Ca], and the $\bar{\partial}$-Neumann problem. We will recall Sibony's definition of a B-regular compact set and also define a B-regular domain. Here a domain Ω will be called B-regular if one can solve the Dirichlet problem, i.e. if every $f \in C(\partial\Omega)$ can be extended to a function $F \in \mathcal{PSH}(\Omega) \cap C(\bar{\Omega})$. In [Sib], Sibony proved that for domains with C^1-boundary, being B-regular is the same as having B-regular boundary. In

Theorem 9.3, we show that this is actually true for domains with the $\mathcal{PSH}$-Mergelyan property (i.e. especially when Ω has C^0-boundary). We also show that every B-regular domain with C^1-boundary has the $\mathcal{F}$-approximation property. To show this we use results regarding Property (P), and in Theorem 9.6 we prove that, for domains that have the $\mathcal{PSH}$-Mergelyan property, Property (P) is the same as having B-regular boundary. For domains Ω that are not B-regular, it is interesting to characterize which $f \in C(\partial\Omega)$ has a continuous and plurisubharmonic extension to Ω. This is studied in Chapter 10, where we connect plurisubharmonic extension with plurisubharmonic functions on compact sets. We show that if Ω is P-hyperconvex, then $f \in C(\partial\Omega)$ has an extension F that is continuous and plurisubharmonic on $\bar{\Omega}$ if, and only if, $f \in \mathcal{PSH}(\partial\Omega)$.

Chapter 1

Preliminaries

1.1 Basic definitions in pluripotential theory

In this section we state some definitions and well known results from pluripotential theory. We begin with the definition of subharmonic functions. Note that, with a domain we mean an open and connected set

Definition 1.1. Let Ω be a domain in $\mathbb{R}^n$. An upper semicontinuous function $u : \Omega \to \mathbb{R} \cup \{-\infty\}$ is said to be *subharmonic* if, for every relatively compact open subset U of Ω and every continuous function $h : \bar{U} \to \mathbb{R}$ that is harmonic on U, we have the implication

$$u \leq h \text{ on } \partial U \Rightarrow u \leq h \text{ on } U.$$

The study of subharmonic functions is often referred to as the classical potential theory and has been of great importance in the study of holomorphic functions in $\mathbb{C}$. In higher dimension (i.e. in $\mathbb{C}^n \simeq \mathbb{R}^{2n}, n > 1$), the classical potential theory has not had the same impact. One reason for this is that the class of subharmonic functions is to large. In pluripotential theory one therefore studies a smaller class of functions namely those subharmonic functions whose composition with biholomorphic mappings are subharmonic. This class is precisely the class of plurisubharmonic functions that will be properly defined below. For further information about pluripotential theory, see for example the monograph *Pluripotential theory* by Klimek [Kl]. From now on, let $\Omega \subset \mathbb{C}^n$ be a bounded domain. Note that a *complex line* in $\mathbb{C}^n$ is a set of the form $\{a + b\zeta : \zeta \in \mathbb{C}\}$ where $a, b \in \mathbb{C}^n$ are fixed.

Definition 1.2. Let $\varphi : \Omega \to \mathbb{R} \cup \{-\infty\}$ be an upper semicontinuous function which is not identically $-\infty$ on any connected component of Ω. We say that φ is *plurisubharmonic* if, for each complex line $l = \{a + b\zeta\} \subseteq \mathbb{C}^n$, the function

$$\zeta \mapsto \varphi(a + b\zeta)$$

is subharmonic on $\Omega_l = \{\zeta \in \mathbb{C} : a + b\zeta \in \Omega\}$. The class of plurisubharmonic functions on Ω is denoted by $\mathcal{PSH}(\Omega)$ and the subclass of nonpositive plurisubharmonic functions is denoted by $\mathcal{PSH}^-(\Omega)$.

Note that, for $n = 1$, plurisubharmonic is the same as subharmonic. As mentioned above, we can see the plurisubharmonic functions as a counterpart to the subharmonic functions. A natural counterpart to the class of harmonic functions is the class of maximal plurisubharmonic functions that is defined below. Note that harmonic functions are maximal subharmonic functions in the same sense. In the definition below, we use the same terminology as Sadullaev in [Sa].

Definition 1.3. A plurisubharmonic function u on Ω is said to be *maximal* if for each relatively compact open subset ω of Ω and for each upper semicontinuous function v defined on $\bar{\omega}$ such that $v \in \mathcal{PSH}(\omega)$ and $v \leq u$ on $\partial\omega$, we have that $v \leq u$ on ω. The maximal plurisubharmonic functions on a domain Ω is denoted by $\mathcal{MPSH}(\Omega)$.

One important reason why plurisubharmonic functions are interesting to study is that they can be used to define pseudoconvex domains.

Definition 1.4. A domain $\Omega \subseteq \mathbb{C}^n$ is called *pseudoconvex* if there exists a continuous plurisubharmonic function Φ on Ω such that for every $c \in \mathbb{R}$, we have that $\{z \in \Omega : \Phi(z) < c\} \Subset \Omega$.

Such a function Φ is called an *exhaustion function* for Ω. Here $\Subset$ means that the closure of $\{z \in \Omega : \Phi(z) < c\}$ is compact in Ω. If moreover, the domain has a *negative* exhaustion function, then Ω is called hyperconvex.

Definition 1.5. A domain $\Omega \subseteq \mathbb{C}^n$ is called *hyperconvex* if there exists a plurisubharmonic function $\psi : \Omega \to (-\infty, 0)$ such that $\{z \in \Omega : \psi(z) < c\} \Subset \Omega$, for every $c \in (-\infty, 0)$.

Kerzman and Rosay, [KR], showed that if $\Omega \Subset \mathbb{C}^n$ is hyperconvex, then the exhaustion function ψ can be chosen C^∞ smooth and strictly plurisubharmonic, and by Cegrell, [C7], ψ can be chosen from the class $\mathcal{E}_0(\Omega)$ (see

Definition 3.1 for the definition of $\mathcal{E}_0$). Note that every hyperconvex domain is pseudoconvex and Demailly showed in [De] that a bounded pseudoconvex domain with Lipschitz boundary is hyperconvex. Kerzman and Rosay also showed in [KR], that a domain is hyperconvex if, and only if, it has a weak plurisubharmonic barrier at every boundary point. Here a weak plurisubharmonic barrier is defined as the following:

Definition 1.6. A domain $\Omega \Subset \mathbb{C}^n$ has a *weak plurisubharmonic barrier* at a boundary point z_0 if there exists a function $v \in \mathcal{PSH}(\Omega)$ such that $v < 0$ on Ω and $\lim_{w \to z_0} v(w) = 0$.

Another characterization of hyperconvex domains has been obtained by Carlehed, Cegrell and Wikström in [CCW]. They used the class of Jensen measures $\mathcal{J}_z^c(\bar{\Omega})$ that they defined as the following:

Definition 1.7. Let $\Omega \Subset \mathbb{C}^n$ be a domain and for $z \in \bar{\Omega}$, let $\mathcal{J}_z^c(\bar{\Omega})$ be the collection of all probability measures μ defined on $\bar{\Omega}$ such that

$$u(z) \leq \int_{\bar{\Omega}} u \, d\mu$$

for every $u \in \mathcal{PSH}(\Omega) \cap C(\bar{\Omega})$.

The characterization theorem can then be formulated as below, see [CCW, Theorem 2.8] for the proof of this theorem.

Theorem 1.8. *Let $\Omega \Subset \mathbb{C}^n$ be a domain. Then Ω is hyperconvex if, and only if, for every $z \in \partial\Omega$ and every measure $\mu \in \mathcal{J}_z^c(\bar{\Omega})$, μ is supported on $\partial\Omega$.*

In many situations we need to glue two plurisubharmonic functions together and the next theorem helps us with this (for the proof of this theorem see [Kl]). Note that this property is special (i.e. works) for subharmonic functions and does not hold for example for holomorphic functions.

Theorem 1.9. *Let Ω be an open set in $\mathbb{C}^n$, and let ω be a non-empty proper open subset of Ω. If $u \in \mathcal{PSH}(\Omega)$, $v \in \mathcal{PSH}(\omega)$ and $\limsup_{x \to y} v(x) \leq u(y)$ for each $y \in \partial\omega \cap \Omega$, then*

$$\psi = \begin{cases} \max(u, v), & \text{on } \omega \\ u, & \text{on } \Omega \setminus \omega, \end{cases}$$

defines a plurisubharmonic function on Ω.

1.2 Weak*-topology

In this thesis we will consider non-negative measures and the space of non-negative measures will be equipped with the weak*-topology. We will now explain what the weak*-topology is and remind the reader of some of its properties. Let $\mathcal{X}$ be a normed vector space, then the dual of $\mathcal{X}$ consists of all continuous linear functionals on $\mathcal{X}$ and is denoted by $\mathcal{X}^*$. The dual space of $\mathcal{X}^*$, $\mathcal{X}^{**}$, is then the space of continuous linear functionals on $\mathcal{X}^*$. There is a natural imbedding of $\mathcal{X}$ in $\mathcal{X}^{**}$ by the point evaluations. That is, for every $x \in \mathcal{X}$ we can define a function $\widehat{x}$ on $\mathcal{X}^*$ by $\widehat{x}(f) = f(x)$ for all $f \in \mathcal{X}^*$.

Definition 1.10. Let $\mathcal{X}$ be a normed vector space. The *weak**-*topology* on $\mathcal{X}^*$ is the weakest topology on $\mathcal{X}^*$ that makes all the point evaluations continuous.

A sequence $\{\phi_j\}$ in $\mathcal{X}^*$ converges to ϕ in the weak*-topology if, and only if, $\phi_j(x) \to \phi(x)$ for every $x \in \mathcal{X}$.

In the following, by a measure we mean a non-negative regular Borel measure. If Ω is a bounded domain in $\mathbb{C}^n$, then the set of measures on Ω is, because of Riesz representation theorem, equal to $C_0(\Omega)^*$. Here $C_0(\Omega)$ denotes the set of continuous real-valued functions on Ω whose support is compact. This means that a sequence $\{\mu_j\}$ of measures on Ω, converges to a measure μ on Ω, as $j \to \infty$, in the weak*-topology if, and only if,

$$\int_\Omega \varphi\mu_j \to \int_\Omega \varphi\mu,$$

for every $\varphi \in C_0(\Omega)$.

If K is a compact set in $\mathbb{C}^n$, the measures on K only have support on K so they can be seen as measures on $\mathbb{C}^n$. This means that if $\{\mu_j\}$ and μ are measures on K, then μ_j converges weak*- to μ, as $j \to \infty$ if, and only if,

$$\int_K \varphi\mu_j \to \int_K \varphi\mu,$$

for every $\varphi \in C(\mathbb{C}^n)$.

Observe that if $\{\mu_j\}$ converges to μ in the weak*-topology, then it doesn't necessarily imply that $\mu_j(\Omega) \to \mu(\Omega)$, even for $\Omega = \mathbb{C}^n$. But, we have the following theorem that will be used later. For the proof of the theorem and more about measures, see e.g. [Ma].

Theorem 1.11. *Let $\mu_1, \mu_2, \ldots$ be Radon measures on a locally compact metric space X. If $\{\mu_j\}$ converges to μ in the weak*-topology and if $K \subset X$ is compact and $G \subset X$ is open, then*

$$\mu(K) \geq \limsup_{i\to\infty} \mu_j(K)$$

and

$$\mu(G) \leq \liminf_{i\to\infty} \mu_j(G).$$

The next theorem, Banach-Alaoglu theorem, is used in Chapter 5 where we show that a certain class of Jensen measures, $\mathcal{J}_z(X)$, is compact in the weak*-topology. For the proof the Banach-Alaoglu theorem see for example [RS, Theorem IV.21].

Theorem 1.12 (Banach-Alaoglu Theorem)**.** *Let $\mathcal{X}$ be a normed vector space. The closed unit ball B in the dual $\mathcal{X}^*$ is compact in the weak*-topology.*

As said earlier, $\mathcal{X}$ is in a natural way imbedded in $\mathcal{X}^{**}$, but there can be elements in $\mathcal{X}^{**}$ that are not point evaluations. But, in the case when $\mathcal{X}^*$ has the weak*-topology, the dual $\mathcal{X}^{**}$ only consists of point evaluations. For proof see [Co, Theorem 1.3, p 125].

Theorem 1.13. *Let $\mathcal{X}$ be a normed vector space. If $\mathcal{X}^*$ is equipped with the weak*-topology, then $\mathcal{X}^{**} = \mathcal{X}$.*

Chapter 2

Introduction to B-regularity

We will now discuss the definitions and some properties of B-regular domains and B-regular compact sets. These notions will be used throughout the thesis and will also be investigated further in Chapter 9. In Chapter 1 we defined a hyperconvex domain Ω as a domain in $\mathbb{C}^n$ that has a negative plurisubharmonic exhaustion function. We also mentioned that they can be characterized as those domains that admit a weak plurisubharmonic barrier at every boundary point. The domains that admit a *strong* plurisubharmonic barrier at every boundary point are called B-regular domains and were studied by Sibony in [Sib]. For the definition of a strong plurisubharmonic barrier, see Definition 2.4. Sibony showed that a domain has a strong plurisubharmonic barrier at every boundary point if, and only if, every continuous function f on $\partial\Omega$ can be extended to a plurisubharmonic function defined on Ω that is continuous on $\bar{\Omega}$. This will be how we define B-regular domains. Note that, by Bouligand's lemma, see [Bo] or [Hö, Theorem 3.4.16], we have that if $\Omega \subset \mathbb{R}^n$, then given a boundary point x_0, it is equivalent that Ω has a weak subharmonic barrier at x_0 and that Ω has a strong subharmonic barrier at x_0. This means that in $\mathbb{C}$ the notions of B-regularity and hyperconvexity are the same. In [Sib], Sibony also defines B-regular compact sets and because of its connection to approximation from outside, they will play an important role in this thesis.

Definition 2.1. A compact set $X \subset \mathbb{C}^n$ is *B-regular* if every continuous function defined on X can be approximated uniformly on X with continuous plurisubharmonic functions defined on neighborhoods of X.

Proposition 1.3 in [Sib] gives the following characterization of B-regular compact sets.

Theorem 2.2. *For a compact set $X \subset \mathbb{C}^n$ the following assertions are equivalent:*

(1) *X is B-regular.*

(2) *$\mathcal{J}_z(X) = \{\delta_z\}$ for every $z \in X$. For the definition of $\mathcal{J}_z(X)$, see Definition 5.2.*

(3) *For every $M > 0$ there exists a function λ that is plurisubharmonic and smooth in a neighborhood of X, $0 < \lambda < 1$, such that $\langle L\lambda(z)w, w\rangle \geq M|w|^2$ for $z \in X$ and $w \in \mathbb{C}^n$.*

Here we denote the Levi form of a C^2-function ψ by

$$\langle L\psi(z)w, w\rangle = \sum_{i,j} \frac{\partial^2\psi(z)}{\partial z_i \partial \bar{z}_j} w_i \bar{w}_j,$$

where $w \in \mathbb{C}^n$.

Definition 2.3. A domain $\Omega \Subset \mathbb{C}^n$ is called *B-regular* if every continuous function defined on $\partial\Omega$ has a continuous extension to $\bar{\Omega}$ which is plurisubharmonic on Ω.

Before we come to the next theorem, that is due to Sibony, we recall the definition of a strong plurisubharmonic barrier. For the definition of a *weak* plurisubharmonic barrier, see Definition 1.6. Here v^* denotes the upper semicontinuous regularization of v, i.e. $v^*(z) = \limsup_{\Omega \ni w \to z} v(w)$.

Definition 2.4. A domain $\Omega \Subset \mathbb{C}^n$ has a *strong plurisubharmonic barrier* at a boundary point z_0 if there exists a function $v \in \mathcal{PSH}(\Omega)$ such that $\lim_{w \to z_0} v(w) = 0$ and $v^*(z) < 0$ for every $z \in \bar{\Omega} \setminus \{z_0\}$.

The following theorem was showed by Sibony in [Sib, Theorem 2.1].

Theorem 2.5. *Let Ω be a bounded domain in $\mathbb{C}^n$. Then the following assertions are equivalent:*

(1) *Ω is B-regular.*

(2) *Every boundary point z_0 admits a strong plurisubharmonic barrier.*

Since a domain that has a strong plurisubharmonic barrier also has a weak plurisubharmonic barrier (see Definition 1.6), every B-regular domain is also hyperconvex, but there are hyperconvex domains that are not B-regular, for example the polydisc in $\mathbb{C}^2$.

Example 2.6. Let P be the polydisc in $\mathbb{C}^2$ and assume that P is B-regular. Let $f \in C(\partial P)$ be a function such that $f = 0$ where $|z_1| = |z_2| = 1$ but $f(z_0) > 0$ at a point $z_0 \in \partial P$ (see Figure 2.1). Since P is assumed to be B-regular we can find a function $g \in PSH(P) \cap C(\bar{P})$ such that $g|_{\partial\Omega} = f$. Let $\{z_j\}$ be a sequence in P such that $z_j \to z_0$. Then we can, for every z_j in the sequence $\{z_j\}$, find a complex line Π_j through z_j that is parallel to the part of the boundary that contains z_0. It follows from the maximum principle that $g(z_j)$ has to be smaller than the boundary value of Π_j. But, these boundary values tends to zero when $j \to \infty$, since we are approaching the point where $|z_1| = |z_2| = 1$. Thus $f(z_0) = g(z_0) \leq 0$, which gives us a contradiction. Hence P is not B-regular. □

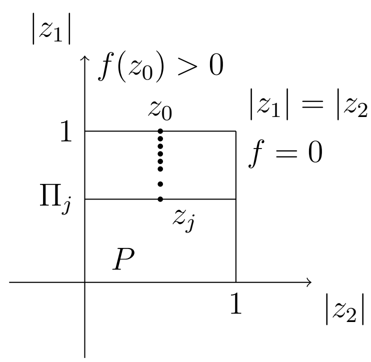


Figure 2.1: The polydisc in $\mathbb{C}^2$ is not B-regular

Remark 1. If Ω is a bounded domain in $\mathbb{C}^n$ and $\mathbb{D}$ is the unit disk in $\mathbb{C}$, we can use the same argument as in the example above to see the following: if there exist holomorphic mappings $\varphi_j : \mathbb{D} \to \mathbb{C}^n$, $\varphi_j(\mathbb{D}) \subset \Omega$ such that φ_j converges locally uniformly on $\mathbb{D}$ to a holomorphic mapping $\varphi : \mathbb{D} \to \mathbb{C}^n$ that satisfies $\varphi(\mathbb{D}) \subset \partial\Omega$, then Ω can not be a B-regular domain.

Chapter 3

The complex Monge-Ampère operator

3.1 The complex Monge-Ampère operator

In this section we define a generalization of the Laplace operator. Note that the Laplace operator applied to a locally integrable function defines a distribution and this distribution is a non-negative measure exactly on the subharmonic functions (and is equal to zero on harmonic functions). The complex Monge-Ampère operator, $(dd^c\cdot)^n$, defines a non-negative measure on certain plurisubharmonic functions and for those functions we have that the operator is equal to zero on maximal function (see Definition 1.3 for the definition of maximal). Let f be a C^1-function defined on a domain Ω in $\mathbb{C}^n$, then the differential of f is $df = \partial f + \bar{\partial} f$, where

$$\partial f = \sum_{j=1}^{n} \frac{\partial f}{\partial z_j} dz_j,$$

$$\bar{\partial} f = \sum_{j=1}^{n} \frac{\partial f}{\partial \bar{z}_j} d\bar{z}_j.$$

Let the operator d^c be defined by

$$d^c = i(\bar{\partial} - \partial),$$

and note that

$$dd^c = 2i\partial\bar{\partial}.$$

If u is a C^2-function defined on a domain Ω in $\mathbb{C}^n$, then the complex Monge-Ampère operator $(dd^c\cdot)^n$ is defined by

$$(dd^cu)^n = (dd^cu) \wedge \ldots \wedge (dd^cu) = 4^n n! \det \left(\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k} \right) dV,$$

where

$$dV = dx_1 \wedge dy_1 \wedge \ldots \wedge dx_n \wedge dy_n = \left(\frac{i}{2} \right)^n dz_1 \wedge d\bar{z}_1 \ldots \wedge dz_n \wedge d\bar{z}_n$$

is the usual volume form in $\mathbb{C}^n$ or $\mathbb{R}^{2n}$. If $n = 1$, then $(dd^cu)^n = \Delta u\, dV$, so u is subharmonic if, and only if, $(dd^cu)^n \geq 0$ and harmonic if, and only if, $(dd^cu)^n = 0$. It was shown by Shiffman and Taylor, [Siu], that this operator can not be extended in a meaningful way to the whole class of plurisubharmonic functions and still have the range contained in the class of non-negative measures, see also [Ki, Example 3.1] or [C3, Example 4]. But, Bedford and Taylor showed in [BT1] that the complex Monge-Ampère operator is well-defined on n-tuples of plurisubharmonic and locally bounded functions. Here locally bounded functions on Ω are denoted by $L^\infty_{\text{loc}}(\Omega)$. If $u \in \mathcal{PSH}(\Omega) \cap L^\infty_{\text{loc}}(\Omega)$, then $(dd^cu)^n$ is a positive measure on Ω.

3.2 The classes $\mathcal{E}_0$, $\mathcal{F}$, $\mathcal{N}$ and $\mathcal{E}$

In [C5], Cegrell defined a subclass, $\mathcal{E}$, of the negative plurisubharmonic functions that is the natural domain of definition for the complex Monge-Ampère operator, in the sense that it is closed under the operation of taking the maximum of two functions and the Monge-Ampère operator has a continuity property on decreasing limits of functions in this class (see [C5, Theorem 4.5]). In this section we define this class and other related classes. In Chapter 4, we will study subextension of functions from these classes and in Chapter 8 we study approximation of functions in the class $\mathcal{F}$, that will be defined below. For further reading about these classes, see [ACCH, C4, C5, C6, Czy1]. In this section, let $\Omega \subset \mathbb{C}^n$ be a bounded hyperconvex domain.

Definition 3.1. Let $\mathcal{E}_0(\Omega)$ be the class of bounded plurisubharmonic functions φ on Ω such that $\lim_{z\to\xi} \varphi(z) = 0$ for every $\xi \in \partial\Omega$ and $\int_\Omega (dd^c\varphi)^n < +\infty$.

By Bedford and Taylor, the complex Monge-Ampère operator is well-defined on $\mathcal{E}_0$. By the maximum principle one can see that if $\varphi \in \mathcal{E}_0$, then either $\varphi < 0$ or $\varphi \equiv 0$ on Ω.

Example 3.2. Let $B(0,1) = B$ be the unit ball in $\mathbb{C}^n$ and let $M < 0$ be a constant. If we define $u(z) = \max(\log|z|, M)$, then $u|_{\partial B} = 0$ and one can show that $\int (dd^c u)^n = (2\pi)^n$, so $u \in \mathcal{E}_0(B)$. □

Definition 3.3. Let $\mathcal{F}(\Omega)$ be the class of negative plurisubharmonic functions φ on Ω such that there exists a decreasing sequence $\{\varphi_j\}$, $\varphi_j \in \mathcal{E}_0(\Omega)$, that converges to φ on Ω and such that $\sup_j \int_\Omega (dd^c\varphi_j)^n < +\infty$.

Example 3.4. Let $B(0,1) = B$ be the unit ball in $\mathbb{C}^n$ and let $u(z) = \log|z|$. Then u can be approximated from above with $u_j(z) = \max(\log|z|, -j)$, which by Example 3.2 are in the class $\mathcal{E}_0(B)$, and $\sup \int (dd^c u_j)^n = (2\pi)^n$, hence $u \in \mathcal{F}(B)$. Note that $u \notin \mathcal{E}_0(B)$. □

Now we come to the definition of the class $\mathcal{E}$.

Definition 3.5. Let $\mathcal{E}(\Omega)$ be the class of all negative plurisubharmonic functions φ on Ω such that for every $z_0 \in \Omega$ there exists a neighborhood ω of z_0 and a decreasing sequence $\{\varphi_j\}$, $\varphi_j \in \mathcal{E}_0(\Omega)$, that converges to φ on ω and such that $\sup_j \int_\Omega (dd^c\varphi_j)^n < +\infty$.

By the definition above we see that $\mathcal{E}(\Omega)$ is the class of all negative plurisubharmonic functions that are locally in $\mathcal{F}(\Omega)$. From [C5], we get that if $u \in \mathcal{E}$ and $\omega \Subset \Omega$, $\omega \neq \emptyset$, is an open set, then there exists a function $u_\omega \in \mathcal{F}$ such that $u_\omega \geq u$ on Ω and $u_\omega = u$ on ω. An example of a function in $\mathcal{E}(B(0,1))$ which is not in $\mathcal{F}(B(0,1))$, is the function $u(z) = -1$ (this function is locally in $\mathcal{F}$, see Example 3.4).

Remark 2. Since we have already mentioned that the class $\mathcal{E}(\Omega)$ is, in some sense, the largest class of negative plurisubharmonic functions where the Monge-Ampère operator is well-defined (see [C5]) and since Bedford and Taylor, [BT1], showed that the Monge-Ampère operator is well-defined on locally bounded plurisubharmonic functions we have that

$$\mathcal{PSH}^-(\Omega) \cap L^\infty_{\text{loc}}(\Omega) \subset \mathcal{E}(\Omega).$$

Remark 3. In Definition 1.3 we defined what it means to be a *maximal* plurisubharmonic function. In pluripotential theory they play an important role since by [Bło1, Bło2] we have that if $u \in \mathcal{E}(\Omega)$, then

$$u \text{ is maximal} \Leftrightarrow (dd^c u)^n = 0 \text{ on } \Omega.$$

In one dimension, the maximal subharmonic functions and the harmonic functions are the same. In several variables, all plurisubharmonic function that does not depend on every variables is maximal. Hence, there are maximal plurisubharmonic functions that are not in the class $\mathcal{E}(\Omega)$.

We now define the class $\mathcal{N}(\Omega)$ that was first defined in [C6]. The functions in $\mathcal{N}(\Omega)$ are those functions in $\mathcal{E}(\Omega)$ that have essentially boundary values zero, in the sense that their smallest maximal plurisubharmonic majorant is identically zero. We define it as follows: (Here $\mathcal{C}\Omega^j$ denotes the complement of Ω^j and a fundamental sequence $\{\Omega^j\}$ of Ω is a sequence of strictly pseudoconvex domains such that $\Omega^j \Subset \Omega^{j+1} \Subset \Omega$ for every j, and $\cup_j \Omega^j = \Omega$).

Definition 3.6. Let $u \in \mathcal{PSH}(\Omega)$, $u \leq 0$, and let $\{\Omega^j\}$ be a fundamental sequence of Ω. Let u^j be the function

$$u^j = \sup\left\{\varphi \in \mathcal{PSH}(\Omega) : \varphi \leq u \text{ on } \mathcal{C}\Omega^j\right\}.$$

and define

$$\tilde{u} = \left(\lim_{j\to\infty} u^j\right)^*,$$

here w^* denotes the upper semicontinuous regularization of w.

Note that the definition of $\tilde{u}$ is independent of the sequence $\{\Omega^j\}$ and if $u \in \mathcal{E}(\Omega)$, then $u \leq u^j \leq u^{j+1}$, so we have that $\tilde{u} = (\lim_{j\to\infty} u^j)^* \in \mathcal{E}$. Also note that since $(dd^c\tilde{u})^n = 0$, $\tilde{u}$ is maximal (see Remark 3) and if $u \in C(\bar{\Omega})$, then $\tilde{u} \in C(\bar{\Omega})$ (see [Wa]).

Definition 3.7. Let $\mathcal{N}(\Omega)$ be the class

$$\mathcal{N}(\Omega) = \{u \in \mathcal{E} : \tilde{u} = 0\}.$$

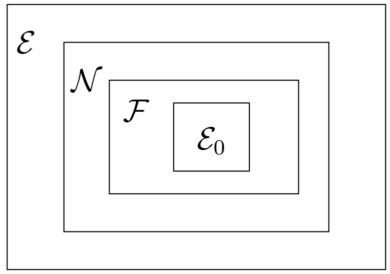


Figure 3.1: Note that $\mathcal{E}_0 \subset \mathcal{F} \subset \mathcal{N} \subset \mathcal{E}$.

By [C5], the complex Monge-Ampère operator is well-defined on $\mathcal{E}$ as the limit measure given by the next theorem.

Theorem 3.8. *For k=1,...,n let $u^k \in \mathcal{E}$ and let $\{u_j^k\}, u_j^k \in \mathcal{E}_0$, be a decreasing sequence such that $\{u_j^k\}$ converges pointwise to u^k as $j \to \infty$. Then*

$$(dd^c u_j^1) \wedge (dd^c u_j^2) \wedge \ldots \wedge (dd^c u_j^n)$$

is weak-convergent and the limit measure is independent of the sequence $\{u_j^k\}$.*

Remark 4. If $u \in \mathcal{F}$, then $\int_\Omega (dd^c u)^n < +\infty$. If $u \in \mathcal{E}$, then $\int_w (dd^c u)^n < +\infty$ for any open set w such that $w \Subset \Omega$.

We now recall some well known results about these classes that are extensively used throughout this thesis. The first theorem is from [C4, C5].

Theorem 3.9. *Let $\mathcal{K} \in \{\mathcal{E}_0, \mathcal{F}, \mathcal{E}\}$, $u \in \mathcal{K}$ and $v \in \mathcal{PSH}^-(\Omega)$. Then*

$$\max(u, v) \in \mathcal{K}.$$

The next theorem explains why we sometimes say that functions in $\mathcal{F}$ have *essentially* boundary values zero. For a proof of the theorem below see [A2, Corollary 1.5].

Theorem 3.10. *Let $u \in \mathcal{F}$, then*

$$\limsup_{z \to \xi} u(z) = 0$$

for every $\xi \in \partial\Omega$.

Observe that a function $u \in \mathcal{F}(\Omega)$ does not have to have boundary values zero in the sense that $\lim_{\Omega \ni w \to z} u(w) = 0$, for every $z \in \partial\Omega$. In Example 1.6 in [A2], Åhag constructed a function $u \in \mathcal{F}$ that has the property that $\liminf_{z \to \xi} u(z) = -\infty$, for every $\xi \in \partial\Omega$. Using that functions in $\mathcal{F}$ have essentially boundary values zero and finite total Monge-Ampère mass Cegrell showed a formula for integration by parts with no boundary terms (see [C5]). This makes the class $\mathcal{F}$ very useful.

Theorem 3.11. *If v, u_1,...,$u_n \in \mathcal{F}$ then*

$$\int_\Omega v(dd^c u_1) \ \wedge \ldots \wedge (dd^c u_n) \ = \int_\Omega u_1(dd^c v) \ \wedge \ldots \wedge (dd^c u_n) \ .$$

A corollary of this theorem is the following comparison theorem, see [C5] and also [A1, Corollary 2.11].

Corollary 3.12. *Let $u, v \in \mathcal{F}$ be such that $u \leq v$ on Ω, then*

$$\int_\Omega \varphi(dd^c u)^n \leq \int_\Omega \varphi(dd^c v)^n,$$

where $\varphi \in \mathcal{PSH}^-(\Omega)$.

In [C5], Cegrell proves a useful approximation theorem for negative plurisubharmonic functions.

Theorem 3.13. *Suppose that $u \in \mathcal{PSH}^-(\Omega)$. Then there is a decreasing sequence of functions $u_j \in \mathcal{PSH}(\Omega) \cap C(\bar{\Omega})$ with $u_j|_{\partial\Omega} = 0$, $\forall\ j \in \mathbb{N}$, $\lim_{j \to +\infty} u_j(z) = u(z)$, $\forall\ z \in \Omega$ and $\int_\Omega (dd^c u_j)^n < +\infty$.*

Remark 5. In the theorem above, the functions u_j can be chosen in $\mathcal{E}_0(\Omega) \cap C(\bar{\Omega})$, $\text{supp}(dd^c u_j)^n \Subset \Omega$.

3.3 The classes $\mathcal{E}_0(\Omega, H)$, $\mathcal{F}(\Omega, H)$ and $\mathcal{N}(\Omega, H)$

When it comes to boundary values of plurisubharmonic functions it is not always clear what we should mean. If u is a plurisubharmonic function defined on a domain Ω that is continuous on $\bar{\Omega}$, then we simply say that $u|_{\partial\Omega}$ is the boundary values of u. If the function is not continuous on the closure of the domain it is not that straightforward. For more about boundary values of plurisubharmonic functions, see for example [Kem]. We will now define some classes of negative plurisubharmonic functions that can be seen as generalizations of the classes defined earlier. These functions have more general boundary values and originate from [C4]. Let $\mathcal{K} \in \{\mathcal{E}_0, \mathcal{F}, \mathcal{N}\}$. We say that a plurisubharmonic function u defined on Ω belongs to the class $\mathcal{K}(\Omega, H)$, $H \in \mathcal{E}$, if there exists a function $\varphi \in \mathcal{K}$ such that

$$H \geq u \geq \varphi + H\,.$$

Note that $\mathcal{K}(\Omega, 0) = \mathcal{K}$ and one can say that the functions in $\mathcal{K}(\Omega, H)$ have boundary values given by the function H (but note that, H does not need to be defined on $\partial\Omega$). Also note that functions belonging to $\mathcal{K}(\Omega, H)$ not necessarily have finite total Monge-Ampère mass. For more details about these classes see [ACCH, C6]. The following theorem was proved in [ACCH, Lemma 3.3] and is extensively used in this thesis.

Theorem 3.14. *Let $u, v \in \mathcal{N}(\Omega, H)$, $H \in \mathcal{E}(\Omega)$, be such that $u \leq v$ and $\int_\Omega \varphi(dd^c u)^n < +\infty$, where $\varphi \in \mathcal{PSH}(\Omega)$, $\varphi \leq 0$. Then*

$$\int_\Omega (-\varphi)(dd^c u)^n \geq \int_\Omega (-\varphi)(dd^c v)^n.$$

Some other well known results are formulated in the proposition below.

Proposition 3.15. *Let $\Omega \subset \mathbb{C}^n$ be a hyperconvex domain and $H \in \mathcal{E}(\Omega)$.*

a) If $\{u_j\}$, $u_j \in \mathcal{N}(\Omega, H)$ is a decreasing sequence that converges pointwise to a function $u \in \mathcal{N}(\Omega, H)$ as $j \to +\infty$, then

$$\lim_{j\to+\infty} \int_\Omega (dd^c u_j)^n = \int_\Omega (dd^c u)^n.$$

b) If $u \in \mathcal{E}(\Omega)$ and $\int_\Omega (dd^c u)^n < +\infty$, then $u \in \mathcal{F}(\Omega, \tilde{u})$ (for the definition of $\tilde{u}$ see Definition 3.6).

Proof. *a*) See [ACCH, Corollary 3.4]
b) See [C6, Theorem 2.1]. □

Chapter 4

Subextension

In this chapter we discuss the problem of subextending plurisubharmonic functions. This will later, in Chapter 8, be used as a very important tool when discussing the $\mathcal{F}$-approximation property. It was shown by Bedford and Burns in [BB] and later by Cegrell in [C1] that any smooth bounded domain that satisfies certain boundary conditions is a domain of existence for plurisubharmonic functions. But, since inequalities are often used when working with plurisubharmonic functions, it is quite natural to study the problem of subextension. The main problem is: given two domains Ω and $\widehat{\Omega}$ in $\mathbb{C}^n$, with $\Omega \subset \widehat{\Omega}$ and given $u \in \mathcal{PSH}(\Omega)$, when can we find $\widehat{u} \in \mathcal{PSH}(\widehat{\Omega})$, $\widehat{u} \not\equiv -\infty$, such that $\widehat{u} \leq u$ on Ω? The function $\widehat{u}$ is called a *subextension* of u to $\widehat{\Omega}$. A version of this problem was studied by El Mir in [El], where he constructed an example of a plurisubharmonic function defined on the unit bidisk in $\mathbb{C}^2$ for which the restriction to any smaller bidisk admits no subextension to $\mathbb{C}^2$. Bedford and Taylor, [BT3], showed that, for any domain in $\mathbb{C}^n$, $n \geq 2$, with C^2-boundary, there exists a smooth negative plurisubharmonic function that does not subextend to any larger domain. This improves an example by Fornæss and Sibony in [FS].

We are interested in subextension without increasing the total Monge-Ampère mass. This has been proven to be important in applications, for example when estimating the volume of plurisubharmonic sublevel sets (see [ACKPZ]). In the case when $\Omega \Subset \widehat{\Omega}$ are bounded hyperconvex domains in $\mathbb{C}^n$, Cegrell and Zeriahi, [CZ, Theorem 2.2], showed that if u is a function in

$\mathcal{F}(\Omega)$, then one can always find a subextension $\widehat{u}$ of u to $\widehat{\Omega}$, such that

$$\int_{\widehat{\Omega}} (dd^c\widehat{u})^n \leq \int_{\Omega} (dd^c u)^n.$$

The same authors, together with Kołodziej, later showed that functions in $\mathcal{F}(\Omega)$ always have a global subextension which is plurisubharmonic of logarithmic growth on $\mathbb{C}^n$ (see [CKZ1, Theorem 5.1]). For results on subextension on compact Kähler manifolds see for example [CKZ2].

It was shown by Wiklund, [Wi, Theorem 3.2], that, for every hyperconvex domain Ω, there exists a function in the class $\mathcal{E}$ which can not be subextended. In Example 4.4 we show that this function actually must be in $\mathcal{N}\backslash\mathcal{F}$. Wiklund also shows that having control of the Monge-Ampère mass does not imply that the function has a subextension. The boundary values of the function seem to play an important role. Wiklund points out that most examples of functions that have no subextension work because the function has to large singularity at a boundary point. But note that, by Example 4.4, we show that even if a function has essentially boundary values zero, it is possible that the function can not be subextended.

In Section 4.2 we discuss subextension of functions in $\mathcal{E}_0(\Omega, H) \cap C(\bar{\Omega})$ and $\mathcal{F}(\Omega, H)$, where $H \in \mathcal{E}(\Omega)$ is a given function. Here we will demand that the subextensions also have certain given boundary values. But, we begin with Section 4.1 where we look at the special case of subextension of functions in $\mathcal{E}_0(\Omega)$ and $\mathcal{F}(\Omega)$ (when $H = 0$).

The results in Section 4.1 are based on the paper [CH], by Cegrell and Hed. Section 4.2 is based on the paper [CzH], by Czyż and Hed, and also the paper [Hed] by Hed. Section 4.3 contains results from [Hed].

4.1 Subextension in $\mathcal{E}_0$ and $\mathcal{F}$

Let $\Omega \Subset \widehat{\Omega}$ be hyperconvex domains in $\mathbb{C}^n$. As mentioned earlier, by [CZ], we know that every function in $\mathcal{F}$ has a subextension to $\widehat{\Omega}$. We want to show that if we have a function u in $\mathcal{E}_0(\Omega) \cap C(\bar{\Omega}), \mathcal{E}_0(\Omega)$ or $\mathcal{F}(\Omega)$, then the largest subextension of u to $\widehat{\Omega}$ will have Monge-Ampère measure with support on Ω and the function will have the same properties as u. Before we come to this theorem we need the follow proposition (see [BT2, Proposition 9.1]).

Proposition 4.1. *Let Ω be a domain in $\mathbb{C}^n$ and $\psi \in \mathcal{PSH}(\Omega) \cap L^\infty_{loc}(\Omega)$. If $D \Subset \Omega$ is strictly pseudoconvex, then there exists a unique function $\Phi \in \mathcal{PSH}(\Omega) \cap L^\infty_{loc}(\Omega)$ such that*

$$(dd^c\Phi)^n = 0 \text{ on } D,$$
$$\Phi = \psi \text{ on } \Omega \backslash D$$

and

$$\Phi \geq \psi.$$

We are now ready for a useful theorem about subextension in the classes $\mathcal{E}_0$ and $\mathcal{F}$. The proof uses an idea of Phạm in Section 4 in [Ph].

Theorem 4.2. *Let Ω and $\widehat{\Omega}$ be hyperconvex domains in $\mathbb{C}^n$ such that $\Omega \Subset \widehat{\Omega}$. If $\mathcal{K}(\Omega) \in \{\mathcal{E}_0(\Omega) \cap C(\bar{\Omega}), \mathcal{E}_0(\Omega), \mathcal{F}(\Omega)\}$ and if $u \in \mathcal{K}(\Omega)$, then*

$$\widehat{u}(z) = \sup\{\varphi(z) \in \mathcal{PSH}^-(\widehat{\Omega}) : \varphi \leq u \text{ on } \Omega\}$$

belongs the class $\mathcal{K}(\widehat{\Omega})$ and $(dd^c\widehat{u})^n \leq \chi_\Omega (dd^c u)^n$.

Proof. Assume that $u \in \mathcal{E}_0(\Omega) \cap C(\bar{\Omega})$. First we observe that $\widehat{u}$ is plurisubharmonic on $\widehat{\Omega}$. This is since if we take $z \in \Omega$, then obviously $\widehat{u}(z) \leq \widehat{u}^*(z)$ and since

$$\widehat{u}^*(z) = \lim_{r\to 0} \sup_{w \in B(z,r)} \widehat{u}(w) \leq \lim_{r \to 0} \sup_{w\in B(z,r)} u(w) \leq u(z),$$

$\widehat{u}^*$ is competing in the construction of $\widehat{u}$, so $\widehat{u} = \widehat{u}^*$. Here we have used that Ω is open and that u is upper semicontinuous. Since $\Omega \Subset \widehat{\Omega}$ we can find $\varphi \in \mathcal{E}_0(\widehat{\Omega})$ such that $\varphi \leq u$ on Ω. Hence $\widehat{u} \geq \varphi$ and by Theorem 3.9 we get that $\widehat{u} \in \mathcal{E}_0(\widehat{\Omega})$. To show that $\operatorname{supp}(dd^c\widehat{u})^n \Subset \Omega$ we begin to prove that $(dd^c\widehat{u})^n = 0$ near $\widehat{\Omega}\backslash\Omega$. Take an open ball $B \subset \widehat{\Omega} \setminus \Omega$. By Proposition 4.1 there is a function $\psi \in \mathcal{PSH}(\widehat{\Omega}) \cap L^\infty_{loc}(\widehat{\Omega})$ such that $(dd^c\psi)^n = 0$ on B, $\psi = \widehat{u}$ on $\widehat{\Omega} \setminus B$ and $\psi \geq \widehat{u}$. By the definition of $\widehat{u}$ we get that $\widehat{u} \geq \psi$, hence $\widehat{u} = \psi$ and $(dd^c\widehat{u})^n = 0$ on B. Now take $z_0 \in \partial\Omega$ and let B be a small ball with center in z_0 such that $\sup\{\widehat{u}(z) : z \in B\} \leq \inf\{u(z) : z \in B \cap \Omega\}$. This is possible since $u \in \mathcal{E}_0(\Omega)$ and $\widehat{u} \in \mathcal{E}_0(\widehat{\Omega})$. By Theorem 4.1 again, there is a function $\psi \in \mathcal{PSH}(\widehat{\Omega}) \cap L^\infty_{loc}(\widehat{\Omega})$ such that $(dd^c\psi)^n = 0$ on B, $\psi = \widehat{u}$ on $\widehat{\Omega} \setminus B$ and $\psi \geq \widehat{u}$. Note that $\psi \leq u$ on $\Omega \setminus B$ and by the choice of B it is also true on $\Omega \cap B$. Hence, $\psi = \widehat{u}$ and $(dd^c\widehat{u})^n = 0$ on B.

We now want to show that $(dd^c\widehat{u})^n \leq (dd^c u)^n$ on Ω. By the same argument as above we know that $(dd^c\widehat{u})^n = 0$ on the open set $\{z \in \Omega : \widehat{u}(z) < u(z)\}$. It remains to show that $(dd^c\widehat{u})^n \leq (dd^c u)^n$ on the set $A = \{u(z) = \widehat{u}(z)\} \cap \Omega$. Take a compact set $K \Subset A$. Then, since $K \subset \{\widehat{u} > u - \varepsilon\}$, we can use [C4, Lemma 5.4] to see that

$$\begin{aligned}\int_K (dd^c\widehat{u})^n &= \int_K \chi_{\{\widehat{u}>u-\varepsilon\}}(dd^c\widehat{u})^n\\ &= \int_K \chi_{\{\widehat{u}>u-\varepsilon\}}\big(dd^c \max(\widehat{u}, u-\varepsilon)\big)^n\\ &\leq \int_K \big(dd^c \max(\widehat{u}, u-\varepsilon)\big)^n.\end{aligned}$$

Since $\max(\widehat{u}, u-\varepsilon) \nearrow u$ when $\varepsilon \to 0$ we know that the measures $(dd^c \max(\widehat{u}, u-\varepsilon))^n$ converges to $(dd^c u)^n$ in the weak*-topology. The characteristic function χ_K is upper semicontinuous so we can approximate χ_K with a decreasing sequence of continuous functions φ_j that are bounded from above. Then, by Lebesgue's dominated convergence theorem, it follows that

$$\begin{aligned}\limsup_{\varepsilon\to 0}\int_\Omega \chi_K\big(dd^c \max(\widehat{u}, u-\varepsilon)\big)^n &= \limsup_{\varepsilon\to 0}\left[\lim_j \int_\Omega \varphi_j\big(dd^c \max(\widehat{u}, u-\varepsilon)\big)^n\right]\\ &\leq \limsup_{\varepsilon\to 0}\int_\Omega \varphi_j\big(dd^c \max(\widehat{u}, u-\varepsilon)\big)^n\\ &= \int_\Omega \varphi_j (dd^c u)^n\end{aligned}$$

for every fixed $j \in \mathbb{N}$. Since $\int_\Omega \varphi_j (dd^c u)^n \searrow \int_\Omega \chi_K (dd^c u)^n = \int_K (dd^c u)^n$ we know that $(dd^c\widehat{u})^n \leq (dd^c u)^n$ on $\{u(z) = \widehat{u}(z)\} \cap \Omega$.

Let us now see that $\widehat{u} \in C(\widehat{\Omega})$. Let

$$v = \sup\{\varphi \in \mathcal{PSH}^-(\widehat{\Omega}) \cap C(\widehat{\Omega}) : \varphi \leq u \text{ on } \Omega\},$$

then v is lower semicontinuous and we want to show that $\widehat{u} = v$. Take $\varepsilon > 0$, then the function $\widehat{u} - \varepsilon$ is negative and plurisubharmonic on $\widehat{\Omega}$. By Theorem 3.13 there exists a sequence $\{\varphi_j\}$, $\varphi_j \in \mathcal{E}_0(\widehat{\Omega}) \cap C(\widehat{\Omega})$, such that $\varphi_j \searrow \widehat{u} - \varepsilon$ on $\widehat{\Omega}$. Let $\varphi_j' = \max(\varphi_j, u-\varepsilon)$, then since $\Omega \Subset \widehat{\Omega}$, $u \in C(\bar{\Omega})$ and $\widehat{u} - \varepsilon \leq u - \varepsilon$

on Ω, we know (by Dini's Theorem) that $\{\varphi_j'\}$ converges uniformly to $u-\varepsilon$ on $\bar{\Omega}$. This means that we can find j_ε such that $\varphi_j < u$ on Ω, for every $j > j_\varepsilon$. Hence φ_j is competing in the construction of v. Since ε is chosen arbitrary, we have that $\widehat{u} = v$.

Now assume that $u \in \mathcal{F}(\Omega)$. From Theorem 3.13 we know that there is a sequence $u_j \in \mathcal{E}_0(\Omega) \cap C(\bar{\Omega})$ such that $u_j \searrow u$. Let

$$\widehat{u}_j = \sup\{v \in \mathcal{PSH}^-(\widehat{\Omega}) : v|_\Omega \leq u_j|_\Omega\},$$

then $\widehat{u}_j \searrow \widehat{u}$. By the argument above we know that $\widehat{u}_j \in \mathcal{E}_0(\widehat{\Omega})$ and $(dd^c\widehat{u}_j)^n \leq \chi_\Omega (dd^c u_j)^n$ on $\widehat{\Omega}$. Since $u_j \geq u$, $u_j, u \in \mathcal{F}(\Omega)$, Corollary 3.12 gives us that

$$\int_\Omega (dd^c u_j)^n \leq \int_\Omega (dd^c u)^n < +\infty. \tag{4.1}$$

Now

$$\sup_j \int_\Omega (dd^c \widehat{u}_j)^n \leq \sup_j \int_\Omega (dd^c u_j)^n < +\infty,$$

hence $\widehat{u} \in \mathcal{F}(\widehat{\Omega})$. To prove that $(dd^c\widehat{u})^n \leq \chi_\Omega(dd^c u)^n$ on $\widehat{\Omega}$ it remains to show that $\chi_\Omega(dd^c u_j)^n$ converges to $\chi_\Omega(dd^c u)^n$ on $\widehat{\Omega}$ in the weak*-topology. Since $(dd^c u_j)^n$ converges weak*- to $(dd^c u)^n$ we know from Theorem 1.11 that $\lim_j \chi_\Omega(dd^c u_j)^n \geq \chi_\Omega(dd^c u)^n$ and by equation (4.1) we see that $\int_\Omega (dd^c u)^n \geq \lim_j \int_\Omega (dd^c u_j)^n$ so

$$(dd^c\widehat{u})^n = \lim (dd^c\widehat{u}_j)^n \leq \lim \chi_\Omega (dd^c u_j)^n = \chi_\Omega (dd^c u)^n,$$

and we are finished.

Finally, assume that $u \in \mathcal{E}_0(\Omega)$. We know from above that $\widehat{u} \in \mathcal{F}(\widehat{\Omega})$ and $(dd^c\widehat{u})^n \leq \chi_\Omega(dd^c u)^n$. Take a function $\psi \in \mathcal{E}_0(\widehat{\Omega})$ that is not identically zero. Since $\Omega \Subset \widehat{\Omega}$ and by multiplying ψ with a constant, we can assume that $\psi < u$ on Ω. This implies that $\widehat{u} \geq \psi$, hence $\widehat{u} \in \mathcal{E}_0(\widehat{\Omega})$.

□

4.2 Subextension with given boundary values

Now we extend the definition of subextensions and look at functions u with boundary values given by a function in the class $\mathcal{E}$. We will let $\widehat{u}$ be the largest subextension of u with certain given boundary values (see Chapter 3 for the definitions of $\mathcal{E}$ and $\mathcal{E}_0(\Omega, F)$). Remember that when $u \in \mathcal{E}_0(\Omega, F)$, $F \in \mathcal{E}(\Omega)$, then the total Monge-Ampère mass of u not necessarily has to be finite. We can use a lot of the same techniques as in the previous section.

Theorem 4.3. *Let Ω and $\widehat{\Omega}$ be hyperconvex domains in $\mathbb{C}^n$ such that $\Omega \Subset \widehat{\Omega}$ and let $F \in \mathcal{E}(\Omega)$ and $G \in \mathcal{MPSH}(\widehat{\Omega}) \cap \mathcal{E}(\widehat{\Omega})$ be such that $G \leq F$ on Ω. If $u \in \mathcal{E}_0(\Omega, F) \cap C(\bar{\Omega})$, then*

$$\widehat{u} = \sup\{\varphi \in \mathcal{PSH}(\widehat{\Omega}) : \varphi \leq G \text{ on } \widehat{\Omega} \text{ and } \varphi \leq u \text{ on } \Omega\}$$

belongs to the class $\mathcal{E}_0(\widehat{\Omega}, G)$ and $(dd^c\widehat{u})^n \leq \chi_\Omega (dd^c u)^n$.

Remark 6. If G moreover is continuous on the closure of $\widehat{\Omega}$, we can use [ACCH, Proposition 2.5] and the same method as in the proof of Theorem 4.2, to show that $\widehat{u}$ is also continuous on the closure of $\widehat{\Omega}$.

Proof. Since $u \in \mathcal{E}_0(\Omega, F)$, there exists a function $\psi \in \mathcal{E}_0(\Omega)$ such that

$$F \geq u \geq F + \psi$$

on Ω. By Theorem 4.2, we know that if $\widehat{\psi} = \sup\{\varphi \in \mathcal{PSH}^-(\widehat{\Omega}) : \varphi \leq \psi \text{ on } \Omega\}$, then $\widehat{\psi} \in \mathcal{E}_0(\widehat{\Omega})$. If we look at the function $G + \widehat{\psi}$, we have that $G + \widehat{\psi} \leq G$ on $\widehat{\Omega}$ and $G + \widehat{\psi} \leq F + \psi \leq u$ on Ω. So we have that $\widehat{u}$ is well-defined, it is plurisubharmonic by the same argument as in the proof of Theorem 4.2 and since $G \geq \widehat{u}$ by the construction

$$G \geq \widehat{u} \geq G + \widehat{\psi},$$

so $\widehat{u} \in \mathcal{E}_0(\widehat{\Omega}, G)$.

We show that $\int_{\widehat{\Omega}\setminus\Omega} (dd^c\widehat{u})^n = 0$ by proving that $\operatorname{supp}(dd^c\widehat{u})^n \Subset \Omega$. Note that, we can without loss of generality, assume that $F, G \leq -1$. By replacing F by $F + \varepsilon$, $\varepsilon > 0$, we can assume that $G + \varepsilon \leq F$ on Ω. Take a small open ball B such that $\bar{B} \subset \widehat{\Omega}\setminus\Omega$. Then, by Proposition 4.1, there is a unique function $\Phi \in \mathcal{PSH}(\widehat{\Omega}) \cap L^\infty_{\text{loc}}(\widehat{\Omega})$ such that $\Phi \geq \widehat{u}$, $\Phi = \widehat{u}$ on $\widehat{\Omega}\setminus B$, and $(dd^c\Phi)^n = 0$ on B. Since G is maximal, $\Phi \leq G$ on $\widehat{\Omega}$ and by the construction of $\widehat{u}$, $\Phi \leq u$ on Ω. So, $\Phi = \widehat{u}$ and $(dd^c\widehat{u})^n = 0$ on B. Near $\partial\Omega$ we can do the same thing if we take a point $z_0 \in \partial\Omega$ and a neighborhood B of z_0 that is small enough, that is such that $\sup\{\widehat{u}(z) : z \in B\} < \inf\{u(z) : z \in B \cap \Omega\}$.

This is possible since $G+\varepsilon \leq F$ on Ω. Use the same argument as above.

To show that $(dd^c\widehat{u})^n \leq (dd^c u)^n$ on Ω we start by using the same argument as above on the set $\{z\in\Omega : \widehat{u}(z) < u(z)\}\cap\Omega$. Note that, since $u\in C(\bar{\Omega})$ we have that the set $\{z\in\Omega : \widehat{u}(z) < u(z)\}\cap\Omega$ is open. It remains to show that $(dd^c\widehat{u})^n \leq (dd^c u)^n$ on $\{z\in\Omega : \widehat{u}(z) = u(z)\}\cap\Omega$. Let $K\subset\{z\in\Omega : \widehat{u}(z) = u(z)\}\cap\Omega$ be a compact set, then especially $K\subset\{z\in\Omega : \widehat{u}(z)+\varepsilon > u(z)\}$. From [NP, Theorem 4.1] (see [BGZ, Theorem 2.2] for an alternative proof) we get the second equality

$$\begin{aligned}\int_K (dd^c\widehat{u})^n &= \int_K \chi_{\{\widehat{u}+\varepsilon>u\}}(dd^c\widehat{u})^n\\ &= \int_K \chi_{\{\widehat{u}+\varepsilon>u\}}\big(dd^c\max(\widehat{u}+\varepsilon,u)\big)^n\\ &\leq \int_K \big(dd^c\max(\widehat{u}+\varepsilon,u)\big)^n.\end{aligned}$$

Note that, since we have assumed that $G+\varepsilon \leq F$ on Ω, we have that $\max(\widehat{u}+\varepsilon,u), \max(\widehat{u},u) \in \mathcal{E}_0(\Omega,F)\subset\mathcal{E}(\Omega)$ and since $\max(\widehat{u}+\varepsilon,u)\searrow \max(\widehat{u},u)$, when ε decreases to zero, Lemma 3.2 in [C6] implies that the measure $(dd^c\max(\widehat{u}+\varepsilon,u))^n$ converges to $(dd^c\max(\widehat{u},u))^n$ in the weak*-topology. We can approximate the characteristic function χ_K with a decreasing sequence of continuous functions φ_j that are bounded from above. Then the Lebesgue's dominated convergence theorem gives us that

$$\begin{aligned}\limsup_{\varepsilon\to 0}\int_\Omega \chi_K\big(dd^c\max(\widehat{u}+\varepsilon,u)\big)^n &= \limsup_{\varepsilon\to 0}\left[\lim_j\int_\Omega \varphi_j\big(dd^c\max(\widehat{u}+\varepsilon,u)\big)^n\right]\\ &\leq \limsup_{\varepsilon\to 0}\int_\Omega \varphi_k\big(dd^c\max(\widehat{u}+\varepsilon,u)\big)^n\\ &= \int_\Omega \varphi_k\big(dd^c\max(\widehat{u},u)\big)^n.\end{aligned}$$

Now, since $\int_\Omega\varphi_j(dd^c\max(\widehat{u},u))^n \searrow \int_\Omega\chi_K(dd^c\max(\widehat{u},u))^n = \int_\Omega\chi_K(dd^c u)^n$ we get that $(dd^c\widehat{u})^n \leq (dd^c u)^n$ on $\{\widehat{u}=u\}\cap\Omega$.

□

To be able to generalize this theorem to the class $\mathcal{F}(\Omega,F)$ we need some more results. If $u\in\mathcal{E}_0(\Omega)$, then it is possible to prove that $(dd^c u)^n$ vanishes on pluripolar sets (see for example [C5, Theorem 5.11]). Next lemma, that follows almost immediately from [ACCH, Lemma 4.12], provides us with the same result for $\mathcal{E}_0(\Omega, H)$, when $H \in \mathcal{MPSH}(\Omega) \cap \mathcal{E}(\Omega)$.

Lemma 4.4. *Let $\Omega \subset \mathbb{C}^n$ be a hyperconvex domain and $H \in \mathcal{MPSH}(\Omega) \cap \mathcal{E}(\Omega)$. If $u \in \mathcal{E}_0(\Omega, H)$, then the measure $(dd^c u)^n$ vanishes on pluripolar sets.*

Proof. Since $u \in \mathcal{E}_0(\Omega, H)$, there exists $\varphi \in \mathcal{E}_0$ such that $H \geq u \geq H + \varphi$. Then $H - \varphi \geq u - \varphi \geq u$ and hence $|u - H| \leq -\varphi$. By [ACCH, Lemma 4.12], $(dd^c u)^n$ vanishes on pluripolar sets. □

The next proposition gives us a characterization of $\mathcal{F}(\Omega, H)$, $H \in \mathcal{E}$. This is a generalization of [AC, Theorem 3.7].

Proposition 4.5. *Let Ω be a hyperconvex domain in $\mathbb{C}^n$ and let $H \in \mathcal{E}(\Omega)$. If $u \in \mathcal{F}(\Omega, H)$ is such that*

$$\int_\Omega (dd^c u)^n < +\infty \,, \tag{4.2}$$

then there exists a decreasing sequence $\{u_j\}$, $u_j \in \mathcal{E}_0(\Omega, H)$, that converges pointwise to u as j tends to $+\infty$, and

$$\sup_j \int_\Omega (dd^c u_j)^n < +\infty \,. \tag{4.3}$$

Moreover if $\{u_j\}$, $u_j \in \mathcal{F}(\Omega, H)$, is a decreasing sequence that converges pointwise to a function u, as j tends to $+\infty$, such that (4.3) is satisfied, then $u \in \mathcal{F}(\Omega, H)$ and (4.2) holds.

Proof. Assume that $u \in \mathcal{F}(\Omega, H)$ is such that (4.2) holds. It follows from [ACCH, Proposition 2.5] that there exists a decreasing sequence $\{u_j\}$, $u_j \in \mathcal{E}_0(\Omega, H)$, that converges pointwise to u on Ω, as $j \to +\infty$. By Proposition 3.15 a) and assumption (4.2) we have that

$$\sup_j \int_\Omega (dd^c u_j)^n < +\infty \,.$$

The outline of the rest of the proof is the following: First we show that if there exists a sequence $\{u_j\}, u_j \in \mathcal{E}_0(\Omega, H)$, that converges pointwise to u and such that (4.3) holds, then $u \in \mathcal{F}(\Omega, H)$ and (4.2) holds. If the functions

u_j are in $\mathcal{F}(\Omega, H)$, then we show that there also exists a sequence of functions in $\mathcal{E}_0(\Omega, H)$ that converges pointwise to u and such that (4.3) holds. The result then follows from the earlier argument.

Now assume that $\{u_j\}$, $u_j \in \mathcal{E}_0(\Omega, H)$, is a decreasing sequence such that (4.3) holds and such that $\{u_j\}$ converges pointwise to a function u, as $j \to \infty$. From (4.3) and Theorem 3.14 we have that $\int_\Omega (dd^c H)^n < +\infty$ (since $u_j, H \in \mathcal{F}(\Omega, H)$ and $u_j \leq H$). Proposition 3.15 *b*) implies that $H \in \mathcal{F}(\Omega, \tilde{H})$, where $\tilde{H}$ is defined as in Definition 3.6. Hence, we can without loss of generality assume that $(dd^c H)^n = 0$. The measure $(dd^c u_j)^n$ has finite total mass and vanishes on pluripolar sets by Lemma 4.4. Therefore Lemma 5.14 in [C5] implies that there exists a uniquely determined function $\varphi_j \in \mathcal{F}(\Omega)$ such that $(dd^c \varphi_j)^n = (dd^c u_j)^n$. Furthermore,

$$(dd^c(\varphi_j + H))^n \geq (dd^c u_j)^n \,.$$

Thus, $u_j \geq \varphi_j + H$, by Corollary 3.2 in [ACCH]. Let φ'_j be the function defined by $\varphi'_j = (\sup_{k \geq j} \varphi_k)^*$. This construction implies that $\{\varphi'_j\}$, $\varphi'_j \in \mathcal{F}(\Omega)$, is a decreasing sequence and

$$\sup_j \int_\Omega \left(dd^c \varphi'_j\right)^n \leq \sup_j \int_\Omega (dd^c \varphi_j)^n < +\infty \,,$$

by (4.3) and the fact that $(dd^c \varphi_j)^n = (dd^c u_j)^n$. Thus, by [Czy2, Lemma 2.1], $\varphi = (\lim_{j \to +\infty} \varphi'_j) \in \mathcal{F}(\Omega)$. For every $k \in \mathbb{N}$ it holds that $u_j \geq u_{(j+k)} \geq \varphi_{(j+k)} + H$. Hence, for every $j \in \mathbb{N}$ we have that $u_j \geq \varphi + H$. By letting $j \to +\infty$ we get that $u \in \mathcal{F}(\Omega, H)$. Now (4.3) and Proposition 3.15 *a*) imply that

$$\int_\Omega (dd^c u)^n = \lim_{j \to +\infty} \int_\Omega (dd^c u_j)^n < +\infty.$$

For the last step in the proof, assume that there exists a decreasing sequence $\{u_j\}$, $u_j \in \mathcal{F}(\Omega, H)$, that converges pointwise to a function u and such that (4.3) holds. Then we can take $\psi \in \mathcal{E}_0(\Omega)$, $\psi \not\equiv 0$ and define the functions u'_j as

$$u'_j = \max(u_j, j\psi + H).$$

Since $j\psi + H \in \mathcal{E}_0(\Omega, H)$, for every fixed j, we know that $u'_j \in \mathcal{E}_0(\Omega, H)$. By the construction, $u'_j \searrow u$ when $j \to \infty$ and then Theorem 3.14 and (4.3) imply that $\int_\Omega \left(dd^c u_j'\right)^n \leq \int_\Omega (dd^c u_j)^n$. It follows from (4.3) that

$$\sup_j \int_\Omega \left(dd^c u_j'\right)^n < \infty,$$

hence it follows from the argument above that $u \in \mathcal{F}(\Omega, H)$ and $\int_\Omega (dd^c u)^n < +\infty$. $\square$

We will now state two different theorems about subextension in the class $\mathcal{F}(\Omega, F)$, where $F \in \mathcal{E}(\Omega)$.

Theorem 4.6. *Let Ω and $\widehat{\Omega}$ be bounded hyperconvex domains in $\mathbb{C}^n$ such that $\Omega \subset \widehat{\Omega}$. Given $F \in \mathcal{E}(\Omega)$ and $G \in \mathcal{MPSH}(\widehat{\Omega}) \cap \mathcal{E}(\widehat{\Omega})$ such that*

$$F \geq G \quad on \quad \Omega\,. \tag{4.4}$$

If $u \in \mathcal{F}(\Omega, F)$, then there exists $\widehat{u} \in \mathcal{F}(\widehat{\Omega}, G)$ such that $\widehat{u} \leq u$ on Ω and

$$\int_{\widehat{\Omega}} (dd^c \widehat{u})^n \leq \int_\Omega (dd^c u)^n\,.$$

Under the assumption that Ω is relatively compact in $\widehat{\Omega}$ and that F and G are the Perron-Bremermann envelope of certain continuous functions f and g that satisfy (4.4), the theorem above has been proved in [AC, Theorem 5.4]. Example 5.5 in [AC] shows that assumption (4.4) is necessary. At this point the author do not know if the assumption that G is a maximal function is necessary, but note that it is necessary that $\int_{\widehat{\Omega}} (dd^c G)^n \leq \int_\Omega (dd^c F)^n$.

If we now assume that F is continuous up to the boundary, that $\Omega \Subset \widehat{\Omega}$ and that $\int_\Omega (dd^c u)^n < +\infty$, we get a stronger result and control of the Monge-Ampère measure of the subextension. Note that the subextension we get in the theorem below is maximal on $\widehat{\Omega} \setminus \Omega$. This differs from the result in Theorem 4.6 where we have no knowledge about maximality of the subextension.

Theorem 4.7. *Let Ω and $\widehat{\Omega}$ be hyperconvex domains in $\mathbb{C}^n$ such that $\Omega \Subset \widehat{\Omega}$. Given $F \in \mathcal{E}(\Omega) \cap C(\bar{\Omega})$ and $G \in \mathcal{MPSH}(\widehat{\Omega}) \cap \mathcal{E}(\widehat{\Omega})$ that satisfy $G \leq F$ on Ω. If $u \in \mathcal{F}(\Omega, F)$ and*

$$\int_\Omega (dd^c u)^n < +\infty, \tag{4.5}$$

then there exists a function $\widehat{u} \in \mathcal{F}(\widehat{\Omega}, G)$ such that $\widehat{u} \leq u$ on Ω, $(dd^c \widehat{u})^n \leq \chi_\Omega (dd^c u)^n$.

Now we come to the proofs of the two theorems. The proof of Theorem 4.7 uses the same techniques as earlier and relies heavily on Theorem 4.3 where we prove the corresponding subextension theorem for the class $\mathcal{E}_0(\Omega, F) \cap C(\bar{\Omega})$, $F \in \mathcal{E}$. The proof of Theorem 4.6 uses solutions of the Dirichlet problem.

Proof of Theorem 4.6. Let $u \in \mathcal{F}(\Omega, F)$. First assume that

$$\int_{\Omega} (dd^c u)^n < +\infty\,. \tag{4.6}$$

This assumption and Theorem 3.14 imply that $\int_{\Omega} (dd^c F)^n < +\infty$, since $u, F \in \mathcal{F}(\Omega, F)$ and $u \leq F$. Proposition 3.15 *b*) implies that $F \in \mathcal{F}(\Omega, \tilde{F})$, where $\tilde{F}$ is defined as in Definition 3.6. Hence, we can without loss of generality assume that $(dd^c F)^n = 0$. Proposition 4.5 implies that there exists a decreasing sequence $\{u_j\}$, $u_j \in \mathcal{E}_0(\Omega, F)$, which converges pointwise to u on Ω, as $j \to +\infty$, and

$$\sup_j \int_{\Omega} (dd^c u_j)^n < +\infty\,. \tag{4.7}$$

Consider the measure $\mu_j = \chi_{\Omega}(dd^c u_j)^n$ defined on $\widehat{\Omega}$, where χ_{Ω} is the characteristic function defined in $\widehat{\Omega}$ for the set Ω. The measure μ_j is a Borel measure in $\widehat{\Omega}$ and it vanishes on pluripolar sets by Lemma 4.4. Moreover, from (4.7) it follows that $\mu_j(\widehat{\Omega}) < +\infty$. Since G is maximal we can use [ACCH, Theorem 3.7] together with Proposition 3.15 *b*) to see that there exists a uniquely determined function $\psi_j \in \mathcal{F}(\widehat{\Omega}, G)$ such that $(dd^c \psi_j)^n = \mu_j$ on $\widehat{\Omega}$. Theorem 5.11 in [C5] implies that there exist functions $w_j \in \mathcal{E}_0(\widehat{\Omega})$ and $\varphi_j \in L^1\big(\widehat{\Omega}, (dd^c w_j)^n\big)$, $\varphi_j \geq 0$, such that $\mu_j = \varphi_j (dd^c w_j)^n$ on $\widehat{\Omega}$. For $k \in \mathbb{N}$ let the measure μ_{jk} be defined on $\widehat{\Omega}$ by

$$\mu_{jk} = \min(\varphi_j, k)(dd^c w_j)^n\,.$$

In the same way as above it follows from [ACCH, Theorem 3.7] and Proposition 3.15 *b*) that there exist decreasing sequences $\{\psi_{jk}\}_{k=1}^{\infty}$, $\psi_{jk} \in \mathcal{F}(\widehat{\Omega}, G)$, $\{\varphi_{jk}\}_{k=1}^{\infty}$, $\varphi_{jk} \in \mathcal{F}(\Omega, F)$ such that

$$(dd^c \psi_{jk})^n = \mu_{jk} \text{ on } \widehat{\Omega} \text{ and } (dd^c \varphi_{jk})^n = \mu_{jk} \text{ on } \Omega\,.$$

Furthermore, the sequence $\{\psi_{jk}\}_{k=1}^{\infty}$ converges pointwise to ψ_j on $\widehat{\Omega}$ and $\{\varphi_{jk}\}_{k=1}^{\infty}$ converges pointwise to u_j on Ω, as $k \to +\infty$. Corollary 3.2 in [ACCH] and (4.4) yield that

$$\psi_{jk} \leq \varphi_{jk} \quad \text{on} \quad \Omega\,.$$

Thus, $\psi_j \leq u_j$ on Ω. For each $j \in \mathbb{N}$ let the function v_j be defined by $v_j = (\sup_{l\geq j} \psi_l)^*$. By construction we have that $v_j \in \mathcal{F}(\widehat{\Omega}, G)$,

$$v_j \leq u_j \;\text{ on } \Omega\ , \tag{4.8}$$

and $v_j \geq \psi_j$ on $\widehat{\Omega}$. Therefore it follows from Theorem 3.14 that

$$\int_{\widehat{\Omega}} (dd^c v_j)^n \leq \int_{\widehat{\Omega}} (dd^c \psi_j)^n = \int_{\Omega} (dd^c u_j)^n\,,$$

hence

$$\sup_j \int_{\widehat{\Omega}} (dd^c v_j)^n \leq \sup_j \int_{\Omega} (dd^c u_j)^n < +\infty\,. \tag{4.9}$$

Thus, $(\lim_{j\to+\infty} v_j) \in \mathcal{F}(\widehat{\Omega}, G)$, by Proposition 4.5. Let $\widehat{u} = (\lim_{j\to+\infty} v_j)$, then it follows from (4.8) that $\widehat{u} \leq u$ on Ω and by (4.9) and Theorem 3.14 we have that

$$\int_{\widehat{\Omega}} (dd^c \widehat{u})^n \leq \int_{\Omega} (dd^c u)^n\,,$$

which completes the proof in this case.

Now assume that $u \in \mathcal{F}(\Omega, F)$ is such that

$$\int_{\Omega} (dd^c u)^n = +\infty\,. \tag{4.10}$$

Then it suffice to construct a function $\widehat{u}$ in $\mathcal{F}(\widehat{\Omega}, G)$ such that $\widehat{u} \leq u$ on Ω. By definition there exists a function $u' \in \mathcal{F}(\Omega)$ such that

$$F \geq u \geq u' + F.$$

From the first part of the proof there exists a function $\widehat{u}' \in \mathcal{F}(\widehat{\Omega})$ such that $\widehat{u}' \leq u'$ on Ω. Now let $\widehat{u} = \widehat{u}' + G$, then $\widehat{u} \in \mathcal{F}(\widehat{\Omega}, G)$ and it follows by assumption (4.4) that

$$u \geq u' + F \geq \widehat{u}' + G = \widehat{u}\,,$$

on Ω. Thus, the proof is completed. □

Proof of Theorem 4.7. Take $u \in \mathcal{F}(\Omega, F)$ and let $\widehat{u} = \sup\{\varphi \in \mathcal{PSH}(\widehat{\Omega}) : \varphi \leq G \text{ on } \widehat{\Omega} \text{ and } \varphi \leq u \text{ on } \Omega\}$. Since $F \in \mathcal{E}(\Omega) \cap C(\bar{\Omega})$, it follows from [ACCH, Proposition 2.5] that there exists a decreasing sequence $\{u_j\}$, $u_j \in \mathcal{E}_0(\Omega, F) \cap C(\bar{\Omega})$ that converges pointwise to u on Ω. Then by (4.5) together with Proposition 3.15 *a*) we have that

$$\lim_j \int_\Omega (dd^c u_j)^n = \int_\Omega (dd^c u)^n < +\infty,$$

so we have that

$$\sup_j \int_\Omega (dd^c u_j)^n < +\infty. \tag{4.11}$$

Now let

$$\widehat{u}_j = \sup\{\varphi \in \mathcal{PSH}(\widehat{\Omega}) : \varphi \leq G \text{ on } \widehat{\Omega} \text{ and } \varphi \leq u_j \text{ on } \Omega\}.$$

Then $\widehat{u}_j \searrow \widehat{u}$ and by Theorem 4.3, $\widehat{u}_j \in \mathcal{E}_0(\widehat{\Omega}, G)$ and

$$(dd^c \widehat{u}_j)^n \leq \chi_\Omega (dd^c u_j)^n. \tag{4.12}$$

Because of (4.11), we know that

$$\sup_j \int_{\widehat{\Omega}} (dd^c \widehat{u}_j)^n < +\infty.$$

Proposition 4.5 then implies that $\widehat{u} \in \mathcal{F}(\widehat{\Omega}, G)$.

It remains to show that $(dd^c \widehat{u})^n \leq \chi_\Omega (dd^c u)^n$. We have that $\widehat{u}_j \searrow \widehat{u}$ and $\widehat{u}_j$, $\widehat{u} \in \mathcal{F}(\widehat{\Omega}, G) \subset \mathcal{E}(\widehat{\Omega})$, so by [C6, Lemma 3.2], we have that $(dd^c \widehat{u}_j)^n$ converges to $(dd^c \widehat{u})^n$ in the weak*-topology, so $(dd^c \widehat{u})^n = \lim_{j\to\infty} (dd^c \widehat{u}_j)^n$. Now we show that $\lim_j \chi_\Omega (dd^c u_j)^n = \chi_\Omega (dd^c u)^n$ on $\widehat{\Omega}$. Since Ω is open and by Theorem 1.11 we know that $\lim_j \chi_\Omega (dd^c u_j)^n \geq \chi_\Omega (dd^c u)^n$, so it is enough to show that $\int_\Omega (dd^c u)^n \geq \lim_j \int_\Omega (dd^c u_j)^n$. But since $u_j \geq u$ and $u_j, u \in \mathcal{F}(\Omega, F)$ we can use Theorem 3.14 to see that $\int_\Omega (dd^c u_j)^n \leq \int_\Omega (dd^c u)^n$ for every j. Therefore we have that $\lim_j \chi_\Omega (dd^c u_j)^n = \chi_\Omega (dd^c u)^n$. Hence, (4.12) gives us that

$$(dd^c \widehat{u})^n = \lim_{j\to\infty} (dd^c \widehat{u}_j)^n \leq \lim_{j\to\infty} \chi_\Omega (dd^c u_j)^n = \chi_\Omega (dd^c u)^n \text{ on } \widehat{\Omega}.$$

□

4.3 An example

In this section we construct an example of a function in $\mathcal{N}\backslash\mathcal{F}$ that can not be subextended (see Example 4.4). We begin with a characterization theorem for the class $\mathcal{N}$. Let Ω be a hyperconvex domain in $\mathbb{C}^n$.

Proposition 4.1. *Let $u \in \mathcal{E}(\Omega)$. Then the following assertions are equivalent:*

(1) $u \in \mathcal{N}(\Omega)$,

(2) *there exists a plurisubharmonic function $\varphi = \sum_{j=1}^{\infty} \varphi_j$, $\varphi_j \in \mathcal{F}(\Omega)$, such that $u \geq \varphi$ on Ω.*

Proof. Assume that $u \in \mathcal{N}(\Omega)$, i.e. $\tilde{u} = 0$. Let $\{\Omega^j\}$ be a fundamental sequence of Ω. The sequence $\{u^j\}$, where u^j is defined as in Definition 3.6, increases pointwise to $\tilde{u}$ on $\Omega\backslash A$, where A is a pluripolar subset of Ω. Hence there exists a point $a \in \Omega$ and a subsequence $\{u^{j_k}\}$ of $\{u^j\}$ with the properties that $u(a) > -\infty$ and $u^{j_k}(a) \geq -\frac{1}{2^{j_k}}$. To simplify the notation $\{u^j\}$ and $\{-\frac{1}{2^j}\}$ will be used instead of $\{u^{j_k}\}$ and $\{-\frac{1}{2^{j_k}}\}$. The original sequence will not be used any more. Let ω_j be defined by

$$\omega_j = \begin{cases} \Omega^2 & \text{if } j = 1 \\ \Omega^{j+1}\backslash\bar{\Omega}^{j-1} & \text{if } j \geq 2\,. \end{cases}$$

This construction implies that ω_j is an open and connected set such that $\bar{\omega}_j \subseteq \Omega$, $\Omega = \bigcup_{j=1}^{+\infty} \omega_j$ and $\omega_j \subseteq \mathcal{C}\Omega^{j-1}$ (remember that $\mathcal{C}\Omega^{j-1}$ denotes the complement of Ω^{j-1}). For each $j \geq 1$ define

$$\varphi_j = \sup\{\psi \in \mathcal{PSH}^-(\Omega) : \psi \leq u \text{ on } \omega_j\}.$$

In particular this construction yields that $\varphi_j \geq u$ on Ω and $\varphi_j = u$ on ω_j. Since $u \in \mathcal{E}(\Omega)$ and $\omega_j \Subset \Omega$ we know from Section 3.2 that there exists a function $u_{\omega_j} \in \mathcal{F}(\Omega)$ such that $u_{\omega_j} \geq u$ on Ω and $u = u_{\omega_j}$ on ω_j. By the construction of φ_j, $\varphi_j \geq u_{\omega_j}$, so $\varphi_j \in \mathcal{F}(\Omega)$ by Theorem 3.9. Furthermore, for $j \geq 2$, note that $\varphi_j \geq u^{j-1}$ on Ω . Then we have that

$$\sum_{j=1}^{\infty} \varphi_j(a) = \varphi_1(a) + \sum_{j=2}^{\infty} \varphi_j(a) \geq u(a) + \sum_{j=2}^{\infty} u^{j-1}(a) \geq u(a) - \sum_{j=2}^{\infty} \frac{1}{2^{j-1}} > -\infty\,,$$

since $u(a) > -\infty$. Thus, the function defined by $\varphi = \sum_{j=1}^{\infty} \varphi_j$ is plurisubharmonic, since $\{\sum_{j=1}^{k} \varphi_j\}_{k=1}^{\infty}$ is a decreasing sequence of plurisubharmonic functions which converges pointwise to a function φ, which is not identically $-\infty$, as $k \to \infty$. To complete this implication we need to prove that $u \geq \varphi$ on Ω. Let $z \in \Omega$, then there exists a j_0, which is not necessarily uniquely determined, such that $z \in \omega_{j_0}$ and therefore we have that

$$u(z) = \varphi_{j_0}(z) \geq \sum_{j=1}^{\infty} \varphi_j(z) = \varphi(z)\,.$$

For the converse assume that $u \in \mathcal{E}(\Omega)$ is such that (2) holds. Let $v_k = \sum_{j=1}^{k} \varphi_j$, then $\{v_k\}$ is a decreasing sequence which converges pointwise to $\varphi \in \mathcal{PSH}(\Omega)$, $\varphi \leq 0$, as $k \to \infty$. The fact that $v_k \in \mathcal{F}(\Omega)$, that $u \geq \varphi$ and the definition of the $\tilde{}$-operator yields that

$$\tilde{u} \geq \tilde{\varphi} \geq \sum_{j=k}^{\infty} \varphi_j$$

for every $k \geq 1$. Let $k \to \infty$, then it follows that $\tilde{u}(z) = 0$, since $\{v_k\}$ converges pointwise to φ. Thus, $u \in \mathcal{N}(\Omega)$, since $u \in \mathcal{E}(\Omega)$ by the assumption. □

Before we come to the main example in this section we remind the reader of the definition of the Lelong number. It is a way to measure the singularity of a plurisubharmonic function and can be seen as a generalization of the multiplicity of a holomorphic function.

Definition 4.2. Let Ω be a domain in $\mathbb{C}^n$ and $u \in \mathcal{PSH}(\Omega)$. *The Lelong number of u at z* is defined by

$$\nu(u, z) = \lim_{r \to 0} \frac{\sup\{u(w) : w \in B(z, r)\}}{\log(r)}.$$

Remark 7. It is well known that whenever u is plurisubharmonic, $u \not\equiv -\infty$, and $z \in \Omega$, then $\nu(u, z)$ is finite. Also note that if u does not have a singularity at z, then $\nu(u, z) = 0$.

Example 4.3. If $u(z) = \log|z|$, then $\nu(u, 0) = 1$, while $v(z) = \log|z|^2$, which has a larger singularity at 0, has $\nu(v, 0) = 2$. □

Remark 8. If $u, v \in \mathcal{PSH}(\Omega)$ are such that $v \le u$, then $\nu(u,z) \le \nu(v,z)$ for all $z \in \Omega$. $\square$

In Example 4.4 we show that subextension is not in general possible in $\mathcal{N}\backslash\mathcal{F}$. This example is based on an idea by Wiklund (see the proof of Theorem 3.2 in [Wi]).

Example 4.4. Let Ω be a bounded hyperconvex domain in $\mathbb{C}^n$ and $g(z,w)$ be the pluricomplex Green function for Ω with pole at $w \in \Omega$. Let the function u be defined by

$$u(z) = \sum_{j=1}^{\infty} j\, g(z, w_j)\,,$$

where the poles $w_j \in \Omega$ are chosen such that $w_j \to z_0$ as $j \to \infty$, for some $z_0 \in \partial\Omega$ and such that $u \in \mathcal{PSH}(\Omega)$ (see the proof of Theorem 3.2 in [Wi]). We now show that u has no subextension by using the Lelong number ν. Note that $\nu(u, w_j) \ge j$. Assume that v is a subextension of u to a larger domain $\Omega' \supset \Omega$ such that $z_0 \in \Omega'$. Then $v \le u$ on Ω so $\nu(v, w_j) \ge \nu(u, w_j) \ge j$ and $\lim_j \nu(v, w_j) = \infty$. This is a contradiction since the Lelong number of a plurisubharmonic function is finite. Hence u has no subextension to any larger domain containing Ω. By the proof of Theorem 3.2 in [Wi] we have that $u \in \mathcal{E}(\Omega)$. Proposition 4.1 implies that $u \in \mathcal{N}$, since $(j\, g(z, w_j)) \in \mathcal{F}(\Omega)$ for all j. Finally, since the pluricomplex Green function is the fundamental solution to the Monge-Ampère operator and u is a sum of infinitely many such functions, the total Monge-Ampère mass of u is infinite and it follows that $u \notin \mathcal{F}$. $\square$

Example 4.5 shows that Proposition 4.1 is not, in general, true if we remove the assumption that $u \in \mathcal{E}$.

Example 4.5. Let $P = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| < 1, |z_2| < 1\}$ be the unit polydisc in $\mathbb{C}^2$ and for each $j \ge 1$ let u_j be the function defined by

$$u_j(z_1, z_2) = \max\left(j^2 \ln|z_1|, \frac{1}{j^2} \ln|z_2| \right) .$$

Then $u_j \in \mathcal{PSH}(P)$, $\lim_{z \to \xi} u_j(z) = 0$ for every $\xi \in \partial P$, and $(dd^c u_j)^2 = (2\pi)^2 \delta_{(0,0)}$, where $\delta_{(0,0)}$ denotes the Dirac measure at $(0,0) \in \mathbb{C}^2$. Hence, $u_j \in \mathcal{F}(P)$. Let $v_k : P \to \mathbb{R} \cup \{-\infty\}$ be defined by $v_k = \sum_{j=1}^{k} u_j$. The sequence $\{v_k\}$ is decreasing and for every point $(z_1, z_2) \in P$, $z_2 \neq 0$, we have that

$$\lim_{k\to\infty} v_k(z_1, z_2) = \sum_{j=1}^{\infty} u_j(z_1, z_2) \geq 2\ln|z_2| > -\infty\,,$$

which implies that $u = (\lim_{k\to\infty} v_k) \in \mathcal{PSH}(P)$. We are now going to show that $\tilde{u} = 0$ and $u \notin \mathcal{E}$, hence $u \notin \mathcal{N}$. For each $k \geq 1$ it holds that

$$0 \geq \tilde{u} \geq \sum_{j=k}^{\infty} u_j$$

since $(\sum_{j=1}^{k} u_k) \in \mathcal{N}$. For the definition of $\tilde{u}$ see Definition 3.6. Since $v_k \to u$ on P we have that $\tilde{u} = 0$ on $P \setminus A$, where A is pluripolar, which implies that $\tilde{u} = 0$ everywhere on P. Assume now that $u \in \mathcal{E}(P)$, then

$$\int_\omega (dd^c u)^n < +\infty$$

for every $\omega \Subset P$. But if $\omega \Subset P$ is an open neighborhood of $(0, 0)$, then

$$\int_\omega (dd^c v_k)^2 = \int_\omega (dd^c(u_1 + \ldots + u_k))^2 \geq \sum_{j=1}^{k} \int_\omega (dd^c u_j)^2 = (2\pi)^2 k\,.$$

Thus, $\lim_{k\to\infty} \int_\omega (dd^c v_k)^2 = \infty$, which is a contradiction to the assumption that $u \in \mathcal{E}$ (see Remark 4 on page 17 or [C5]). $\square$

Chapter 5

Plurisubharmonic functions on compact sets

As we saw in Chapter 1, we usually define plurisubharmonic functions on open sets. In this chapter we will define the notion of plurisubharmonic functions on *compact* sets. This has earlier been studied by, for example Poletsky [Po4, PS], Sibony [Sib] and Gamelin [Ga]. The plurisubharmonic functions on compact sets are closely related to approximation from outside. We will later see that every plurisubharmonic function u on a compact set X can be approximated with a monotone sequence of continuous plurisubharmonic functions defined on neighborhoods of X. If the function u additionaly is continuous, this approximation will be uniform (see Theorem 5.11 and Theorem 5.12). There are different ways to define plurisubharmonic functions on compact sets. In this chapter we consider a construction that follows a Choquet-theoretical point of view (as was done by Sibony in [Sib]). We do this by defining a function cone of continuous functions on X and associating to this a class of representing measures, $\mathcal{J}_z(X)$. Note that this class is different from the class $\mathcal{J}_z^c(X)$ that was defined in Chapter 1. We then say that a plurisubharmonic function on X is an upper semicontinuous function that satisfies the submean inequality with respect to these measures. The same measures can be constructed as the weak*- closure of measures that are push-forwards of the arc length measure on the unit disk with analytic disks. This was done by Poletsky in [Po4]. In Appendix A we discuss this construction and show that these two different approaches are the same. The Choquet theoretical approach is not as technical as the approach by Poletsky which perhaps makes it, at first, easier to understand.

The advantage of the approach by Poletsky is that it gives us more explicit information about how these measures, $\mathcal{J}_z(X)$, really are constructed. The main new results in this chapter are Theorem 5.12 and Theorem 5.16. As mentioned above, Theorem 5.12, says that a function u is plurisubharmonic on X if, and only if, it can be approximated with a decreasing sequence of functions that are plurisubharmonic and continuous on neighborhoods of X. The second theorem, Theorem 5.16, says that it is enough to look at $z \in \partial\Omega$ if we want to know if a function $\varphi \in \mathcal{PSH}(\Omega) \cap C(\bar{\Omega})$ is plurisubharmonic on $\bar{\Omega}$. This chapter is based on results by Hed and Persson in [HP].

Let $X \subset \mathbb{C}^n$ be a compact set. To be able to define $\mathcal{J}_z(X)$, we need the following definition.

Definition 5.1. Define

$$\mathcal{PSH}^o(X) = \{u|_X : u \in \mathcal{PSH}(U) \cap C(U) \text{ for some open } U \supset X\}.$$

Definition 5.2. For given $z \in X$, let $\mathcal{J}_z(X)$ be the set of all probability measures μ defined on X such that

$$u(z) \leq \int u \, d\mu, \quad \forall\, u \in \mathcal{PSH}^o(X).$$

Remark 9. This set is always non-empty, since $\delta_z \in \mathcal{J}_z(X)$.

Remark 10. The class $\mathcal{J}_z(X)$ is compact in the weak*-topology. Too see this, take a sequence $\{\mu_j\} \subset \mathcal{J}_z(X)$ and show that μ_j converges to a measure $\mu \in \mathcal{J}_z(X)$ in the weak*-topology. By the Banach-Alaoglu theorem (see Theorem 1.12 on page 9) we know that the space of probability measures on X is compact when equipped with the weak*-topology of $C(X)$. This means that there is a subsequence $\mu_{j(k)}$ converging to a probability measure μ on X. By the definition of $\mathcal{J}_z(X)$ one can confirm that $\mu \in \mathcal{J}_z(X)$.

Remark 11. It follows immediately that $\mathcal{J}_z^c(\bar{\Omega}) \subseteq \mathcal{J}_z(\bar{\Omega})$ (for the definition of $\mathcal{J}_z^c(\bar{\Omega})$ see Definition 1.7 on page 7). In Example 6.2 we present an example of a domain where $\mathcal{J}_z^c(\bar{\Omega}) \neq \mathcal{J}_z(\bar{\Omega})$.

One important property of $\mathcal{PSH}^o(X)$ is that it is a convex cone of continuous functions that contains the constants and separates points. This means that we can use techniques from Choquet theory when working with this class. In the terminology of Choquet theory the measures in $\mathcal{J}_z(X)$

are called representing measures or $\mathcal{PSH}^o$-measures for z. For more about Choquet theory see the monograph [Ga] by Gamelin. Note that we would have ended up with the same set $\mathcal{J}_z(X)$ if we, in the definition of $\mathcal{PSH}^o(X)$, had let go of the continuity requirement or considered the uniform closure of $\mathcal{PSH}^o(X)$. The later definition was used by Sibony in [Sib]. Now we are ready to define what we mean with a plurisubharmonic function on a compact set.

Definition 5.3. Let u be an upper semicontinuous function defined on X. We say that u is *plurisubharmonic* on X if, for all $z \in X$, it holds that

$$u(z) \leq \int u\, d\mu, \quad \forall\ \mu \in \mathcal{J}_z(X).$$

We denote the set of all plurisubharmonic functions on X by $\mathcal{PSH}(X)$.

Remark 12. By the definition, we see that $\mathcal{PSH}^o(X) \subseteq \mathcal{PSH}(X)$.

The functions in $\mathcal{PSH}(X)$ share a lot of nice properties with ordinary plurisubharmonic functions. Here follows some examples of that.

Example 5.4. Let $u_1, u_2 \in \mathcal{PSH}(X)$. Then it follows from the properties of the integral that for real numbers $s, t \geq 0$, $s\, u_1 + t\, u_2 \in \mathcal{PSH}(X)$. □

Example 5.5. Let $u_1, u_2 \in \mathcal{PSH}(X)$. Then for $z \in X$ and $\mu \in \mathcal{J}_z(X)$,

$$\int \max(u_1, u_2)\, d\mu \geq \int u_i\, d\mu \geq u_i(z), \quad \text{for } i = 1, 2.$$

Hence $\max(u_1, u_2) \in \mathcal{PSH}(X)$. □

Example 5.6. Let u be a plurisubharmonic function defined on X. Suppose also that ϕ is a convex, strictly increasing function defined on the range of u. Then it follows from Jensen's inequality that $\phi \circ u \in \mathcal{PSH}(X)$. □

Example 5.7. Let u be a plurisubharmonic function defined on a neighborhood of X, then $u \in \mathcal{PSH}(X)$. If u is continuous, this follows directly from Definition 5.3. If u is not continuous, we use regularization to approximate u with a decreasing sequence of functions, u_j, that are plurisubharmonic and smooth on neighborhoods of X. Let $z \in X$ and take $\mu \in \mathcal{J}_z(X)$, then (by the monotone convergence theorem)

$$u(z) = \lim_j u_j(z) \leq \lim_j \int_X u_j\, d\mu = \int_X \lim_j u_j\, d\mu = \int_X u\, d\mu.$$

Hence u is plurisubharmonic on X. □

Example 5.8. If $\Omega \Subset \mathbb{C}^n$ is a domain and $u \in \mathcal{PSH}(\bar{\Omega})$, then $u \in \mathcal{PSH}(\Omega)$. This follows since the measure we get when checking the submean inequality on complex lines is in $\mathcal{J}_z(X)$. □

One important theorem in Choquet theory is, the very useful, Edwards' theorem [Ed]. In our setting:

Theorem 5.9 (Edwards' Theorem)**.** *Let φ be a lower semicontinuous function defined on X. Then*

$$\sup\left\{\psi(z) : \psi \in \mathcal{PSH}^o(X), \psi \le \varphi\right\} = \inf\left\{\int \varphi\, d\nu : \nu \in \mathcal{J}_z(X)\right\}.$$

Let us now remind ourselves of Choquet's lemma (for proof see [Kl]). We will use it in the proofs of Theorem 5.11, Theorem 5.16 and Theorem 7.5.

Theorem 5.10 (Choquet's Lemma)**.** *Let X be a compact set in $\mathbb{C}^n$ and let $\{u_\alpha\}_{\alpha \in A}$ be a family of functions on X that are locally bounded from above. Then there exists a countable subset $\{\alpha_j\} \subset A$ such that*

$$(\sup\{u_\alpha : \alpha \in A\})^* = (\sup\{u_{\alpha_j} : j \ge 1\})^*.$$

Moreover, if the functions u_α are lower semicontinuous, then we can choose $\{\alpha_j\}$ such that

$$\sup\{u_\alpha : \alpha \in A\} = \sup\{u_{\alpha_j} : j \ge 1\}.$$

The main reason for us to study plurisubharmonic functions on compact sets in this thesis, is the next theorem that connects them with approximation from outside. This theorem is due to Poletsky, [Po4, Lemma 3.1], but here we present an alternative proof that is based on Choquet's lemma.

Theorem 5.11. *Let $X \subset \mathbb{C}^n$ be a compact set and suppose that u is an upper semicontinuous function defined on X. Then $u \in \mathcal{PSH}(X) \cap C(X)$ if, and only if, there exist functions $u_j \in \mathcal{PSH}^o(X)$ such that $u_j \nearrow u$ on X.*

Remark 13. By Dini's theorem, $u_j \to u$ uniformly on X.

Proof of Theorem 5.11. Let $u \in \mathcal{PSH}(X) \cap C(X)$. Since the Dirac measure $\delta_z \in \mathcal{J}_z(X)$, we have that $u(z) = \inf\{\int u\, d\mu : \mu \in \mathcal{J}_z(X)\}$. By using Edwards' theorem we see that

$$u(z) = \inf\{\int u\, d\mu : \mu \in \mathcal{J}_z(X)\} = \sup\{\varphi(z) : \varphi \in \mathcal{PSH}^o(X), \varphi \le u\}.$$

Since the functions in $\mathcal{PSH}^o(X)$ are continuous, Choquet's lemma (Theorem 5.10) says that there exists a sequence $u_j \in \mathcal{PSH}^o(X)$ such that $u_j \nearrow u$.

Now assume that u is upper semicontinuous on X and that there exists a sequence $u_j \in \mathcal{PSH}^o(X)$ such that $u_j \nearrow u$. Then u can be written as the supremum of continuous functions and u is lower semicontinuous. Since we have assumed that u is upper semicontinuous, u is continuous. Let $z \in X$ and $\mu \in \mathcal{J}_z(X)$, then

$$u(z) = \lim_j u_j(z) \leq \lim_j \int u_j \, d\mu = \int \lim_j u_j \, d\mu = \int u \, d\mu,$$

hence $u \in \mathcal{PSH}(X) \cap C(X)$. □

If we no longer assume continuity on our function u, then we loose the uniform approximation but we still have pointwise approximation. The following theorem gives us a nice characterization of the plurisubharmonic functions on a compact set X.

Theorem 5.12. *Let $X \subset \mathbb{C}^n$ be a compact set and let u be a function defined on X, then $u \in \mathcal{PSH}(X)$ if, and only if, there is a sequence $u_j \in \mathcal{PSH}^o(X)$ such that $u_j \searrow u$ on X.*

Proof. We begin by assuming that u can be approximated as in the statement of the theorem. Then, being the decreasing limit of continuous functions, u must be upper semicontinuous and for $z \in X$ and $\mu \in \mathcal{J}_z(X)$ we have that

$$u(z) = \lim_j u_j(z) \leq \lim_j \int u_j \, d\mu = \int u \, d\mu.$$

Hence $u \in \mathcal{PSH}(X)$. Now suppose that $u \in \mathcal{PSH}(X)$. We begin to show that for every $f \in C(X)$ such that $u < f$ on X, we can find $v \in \mathcal{PSH}^o(X)$ such that $u < v \leq f$. Let

$$F(z) = \sup\{\varphi(z) : \varphi \in \mathcal{PSH}^o(X), \varphi \leq f\} = \inf\{\int f \, d\mu : \mu \in \mathcal{J}_z(X)\}.$$

For each $z \in X$, we can find $\mu_z \in \mathcal{J}_z(X)$ such that $F(z) = \int f \, d\mu_z$. That such a measure exists is because of the fact that $\mathcal{J}_z(X)$ is weak*-compact (see Remark 10). Since then, the continuous point evaluation $\mu \mapsto \int f \, d\mu$ must attain its minimum on $\mathcal{J}_z(X)$. We have that

$$F(z) = \int f \, d\mu_z > \int u \, d\mu_z \geq u(z),$$

so $u < F$. By the construction of F we know that for every given $z \in X$, there exists a function $v_z \in \mathcal{PSH}^o(X)$ such that $v_z \le F$ and $u(z) < v_z(z) \le F(z)$. Since the function $u - v_z$ is upper semicontinuous, the set $U_z = \{w \in X : u(w) - v_z(w) < 0\}$ is open in X. Since X is compact, there are finite many points $z_1, ..., z_k$ with corresponding functions $v_{z_1}, ..., v_{z_k}$ and open sets $U_{z_1}, ..., U_{z_k}$ such that $u \le v_{z_j}$ in U_{z_j} and $X = \bigcup_{j=1}^k U_{z_j}$. The function $v = \max(v_{z_1}, ..., v_{z_k})$ belongs to $\mathcal{PSH}^o(X)$ and $u < v \le f$.

It is now easy to prove that u can be approximated as in the statement of the theorem. Since u is upper semicontinuous, it can be approximated with a decreasing sequence $\{f_j\}$ of continuous functions. We can then find a function $v_1 \in \mathcal{PSH}^o(X)$ such that $u < v_1 \le f_1$. Assuming that we have found a decreasing sequence $\{v_1, v_2, \dots, v_k\}$ such that $v_j \in \mathcal{PSH}^o(X)$ and $u < v_j$ for $j = 1, \dots k$, we find a function $v_{k+1} \in \mathcal{PSH}^o(X)$, such that $u < v_{k+1}$ and $v_{k+1} \le \min\{f_{k+1}, v_k\}$. Now the conclusion of the theorem follows by induction. □

In [FW, Theorem 1], Fornæss and Wiegerinck showed that every arbitrary bounded domain Ω in $\mathbb{C}^n$ with C^1-boundary has the $\mathcal{PSH}$-Mergelyan property, i.e. every $\varphi \in \mathcal{PSH}(\Omega) \cap C(\bar{\Omega})$ can be approximated, uniformly on $\bar{\Omega}$, with functions in $\mathcal{PSH}^o(\bar{\Omega})$. In [FW], they comment that Sibony has made the observation that the proof of this theorem actually says that approximating plurisubharmonic functions in this way is a local property. Gauthier made this statement precise (see [G, Corollary 2]). The localization theorem below is very important in this thesis since it will be used to prove that P-hyperconvexity is a local property (Theorem 6.10) and that every bounded domain in $\mathbb{C}^n$ with C^0-boundary has the $\mathcal{PSH}$-Mergelyan property (Theorem 7.3). For the readers convenience and for the completeness of this thesis we include the proof of the localization theorem from Gauthier in [G].

Theorem 5.13 (Localization theorem)**.** *If $X \subset \mathbb{C}^n$ is a compact set then $u \in \mathcal{PSH}(X) \cap C(X)$ if, and only if, for each $z \in X$, there is a neighborhood $B = B_z$ such that $u|_{X \cap \bar{B}} \in \mathcal{PSH}(X \cap \bar{B}) \cap C(X \cap \bar{B})$.*

Proof. That the restriction of a function $u \in \mathcal{PSH}(X) \cap C(X)$ to $X \cap \bar{B}$ is plurisubharmonic on $X \cap \bar{B}$ follows easily, we show the converse.

Since X is compact there exists a finite open covering $\{B_j\}$ of X such that $u|_{X \cap \bar{B}_j} \in \mathcal{PSH}(X \cap \bar{B}_j) \cap C(X \cap \bar{B}_j)$ for all j. For every j, we can find compact sets $K_{j,k}$ such that $K_{j,k} \subset B_k$ and

$$\partial B_j \cap X \subset \bigcup_{k \neq j} K_{j,k}.$$

Let $K_k = \bigcup_j K_{j,k}$ and note that $K_k \subset B_k$. Set $d_k = \text{dist}(K_k, \partial B_k)$. For every k there exists a function χ_k that is smooth on $\mathbb{C}^n$, $-1 \leq \chi_k \leq 0$, $\chi_k(z) = 0$ when $\text{dist}(z, K_k) \leq \frac{d_k}{2}$ and $\chi_k = -1$ outside of B_k. Choose an arbitrary constant $c > 0$. Since the function $|z|^2$ is strictly plurisubharmonic, there exists a constant $\eta_k^0 > 0$ such that for every $0 < \eta_k < \eta_k^0$, the function $\eta_k \chi_k + c|z|^2$ is plurisubharmonic and continuous on an open set V_k, $B_k \Subset V_k$.

Choose a sequence $\{\varepsilon_j\}$ of positive numbers such that

$$2 \max_{z \in \bar{B}_j} \varepsilon_j < \min_{z \in \bar{B}_j} \eta_j \tag{5.1}$$

for every $z \in X$. The reason for this will be clear later. Since we have assumed that $u|_{X \cap \bar{B}_j} \in \mathcal{PSH}(X \cap \bar{B}_j) \cap C(X \cap \bar{B}_j)$ for every j, Theorem 5.11 says that there exist open sets U_j, $(X \cap B_j) \Subset U_j \Subset V_j$ and functions $u_j \in \mathcal{PSH}(U_j) \cap C(U_j)$ such that

$$|u - u_j| < \varepsilon_j \text{ on } X \cap \bar{B}_j. \tag{5.2}$$

For $z \in (U_j \setminus X) \bigcup (X \cap \bar{B}_j)$ set

$$f_j(z) = u_j(z) + \eta_j \chi_j(z) + c|z|^2$$

and elsewhere, set $f_j = -\infty$. Now define the function

$$v(z) = \max f_j(z).$$

It remains to show that v approximates u uniformly on X and that $v \in \mathcal{PSH}^o(X)$, then the result will follow from Theorem 5.11.

For $z \in X$ we have that

$$\begin{aligned} |u(z) - v(z)| &= |u(z) - \max f_j(z)| \\ &= |u(z) - \max_{z \in X \cap \bar{B}_j} (u_j(z) + \eta_j \chi_j + c|z|^2)| \\ &\leq \max_{z \in X \cap \bar{B}_j} \eta_j + |u(z) - \max_{z \in X \cap \bar{B}_j} u_j(z)| + c|z|^2. \end{aligned}$$

By choosing the constants c, η_j, ε_j in the right order and small enough, the expression above can be made arbitrary small. Hence v approximates u uniformly on X.

To prove that $v \in \mathcal{PSH}^o(X)$, first take $z \in X$ that does not lie on the boundary of any B_j. The functions f_k, that are not $-\infty$ at z, are finitely many and they are continuous and plurisubharmonic in a neighborhood of z. If $z \in \partial B_j \cap X$, then there exists a k such that $z \in (X \cap K_k) \subset (X \cap B_k)$. For this j and k we have that

$$\begin{aligned} f_j(z) &= u_j(z) + \eta_j \chi_j + c|z|^2 \\ &= u_j(z) - \eta_j + c|z|^2 \\ &= \big(u_j(z) - u_k(z)\big) + \big(u_k(z) + \eta_k 0 + c|z|^2\big) - \eta_j \\ &= u_k(z) + \big(u_j(z) - u_k(z)\big) - \eta_j \leq f_k(z) \end{aligned}$$

where the last inequality follows from assumption (5.1) together with (5.2) (that makes sure that $|u_j(z) - u_k(z)| < \varepsilon_j + \varepsilon_k$). This means that locally, near z, we can assume that the function v is the maximum of functions f_k, $k \neq j$, where the functions f_k are continuous and plurisubharmonic in a neighborhood of z. This concludes the proof. $\square$

A consequence of the localization theorem is that we can glue plurisubharmonic functions on compacts together and still have a plurisubharmonic function defined on a compact set.

Theorem 5.14. *Let $\omega \subset \Omega \subset \mathbb{C}^n$ be open sets and let $u \in \mathcal{PSH}(\bar{\omega}) \cap C(\bar{\omega})$, $v \in \mathcal{PSH}(\bar{\Omega}) \cap C(\bar{\Omega})$ and $u(z) \leq v(z)$ for all $z \in \partial\omega$. Define the function ψ by*

$$\psi(z) = \begin{cases} \max\big(u(z), v(z)\big), & z \in \omega \\ v(z), & z \in \bar{\Omega} \setminus \omega. \end{cases}$$

Then $\psi \in \mathcal{PSH}(\bar{\Omega}) \cap C(\bar{\Omega})$.

Proof. Let

$$\psi_\varepsilon(z) = \begin{cases} \max\big(u(z), v(z) + \varepsilon\big), & z \in \omega, \\ v(z) + \varepsilon, & z \in \bar{\Omega} \setminus \omega. \end{cases}$$

Then, by Theorem 5.13, it holds that $\psi_\varepsilon \in \mathcal{PSH}(\bar{\Omega}) \cap C(\bar{\Omega})$ and $\psi_\varepsilon \searrow \psi$ when $\varepsilon \to 0$. Since $\psi \in C(\bar{\Omega})$ it follows from Theorem 5.11 that $\psi \in \mathcal{PSH}(\bar{\Omega})$. $\square$

As said earlier, the motivation for us to study plurisubharmonic functions on compact sets is to learn more about which domains have the $\mathcal{PSH}$-Mergelyan property. A direct consequence of Theorem 5.11 is the following:

Corollary 5.15. *A domain $\Omega \Subset \mathbb{C}^n$ has the $\mathcal{PSH}$-Mergelyan property if, and only if, $\mathcal{PSH}(\Omega) \cap C(\bar{\Omega}) = \mathcal{PSH}(\bar{\Omega}) \cap C(\bar{\Omega})$.*

Remark 14. Note that we always have that $\mathcal{PSH}(\bar{\Omega}) \cap C(\bar{\Omega}) \subseteq \mathcal{PSH}(\Omega) \cap C(\bar{\Omega})$.

So, it is interesting to know when a function $\varphi \in \mathcal{PSH}(\Omega) \cap C(\bar{\Omega})$ also is plurisubharmonic on $\bar{\Omega}$. By the definition of plurisubharmonicity on compact sets, we know that $\varphi \in \mathcal{PSH}(\bar{\Omega}) \cap C(\bar{\Omega})$ if, for every $z \in \bar{\Omega}$ and every $\mu \in \mathcal{J}_z(\bar{\Omega})$, it holds that $\varphi(z) \leq \int \varphi \, d\mu$. The next theorem shows that it is enough to check this for $z \in \partial\Omega$. In Chapter 6, where we study P-hyperconvex domains, we will investigate this question further (see Corollary 6.7).

Theorem 5.16. *Let $\Omega \Subset \mathbb{C}^n$ be a domain. If $\varphi \in \mathcal{PSH}(\Omega) \cap C(\bar{\Omega})$ and for every $z \in \partial\Omega$ we have that*

$$\varphi(z) \leq \int \varphi \, d\mu \quad \forall \; \mu \in \mathcal{J}_z(\bar{\Omega}), \tag{5.3}$$

then $\varphi \in \mathcal{PSH}(\bar{\Omega}) \cap C(\bar{\Omega})$.

Proof. Assume that $\varphi \in \mathcal{PSH}(\Omega) \cap C(\bar{\Omega})$ and let

$$v = \sup\{u : u \in \mathcal{PSH}^o(\bar{\Omega}), u \leq \varphi\}.$$

By Edwards' Theorem (Theorem 5.9) it follows that

$$v(z) = \inf\{\int_{\bar{\Omega}} \varphi \, d\mu : \mu \in \mathcal{J}_z(\bar{\Omega})\},$$

so by (5.3) we see that $v|_{\partial\Omega} = \varphi|_{\partial\Omega}$ and especially that $v \in C(\partial\Omega)$. By using Choquet's lemma (Theorem 5.10) we can find a sequence $v_j \in \mathcal{PSH}^o(\bar{\Omega})$ such that $v_j \leq \varphi$ and $v_j \nearrow v$. Since $v \in C(\partial\Omega)$ and by using Dini's Theorem we see that $v_j \to v$ uniformly on $\partial\Omega$. This means that for every $\varepsilon > 0$ we can find $\widehat{v} \in \{v_j\}$ such that $v - \widehat{v} < \varepsilon$ on $\partial\Omega$. This also means that $\varphi - \varepsilon < \widehat{v}$ on $\partial\Omega$. Since $\widehat{v} \in \mathcal{PSH}^o(\bar{\Omega})$, we can assume that $\widehat{v}$ is continuous and plurisubharmonic in a neighborhood of $\bar{\Omega}$. Now let

$$v_\varepsilon(z) = \begin{cases} \max\big(\widehat{v}(z), \varphi(z) - \varepsilon\big), & z \in \Omega \\ \widehat{v}(z), & \text{otherwise.} \end{cases}$$

Then $v_\varepsilon|_{\bar{\Omega}} \in \mathcal{PSH}^o(\bar{\Omega})$ and since $v \leq \varphi$ on $\bar{\Omega}$ we have that

$$v - \varepsilon \leq \varphi - \varepsilon \leq v_\varepsilon \leq v \leq \varphi \text{ on } \bar{\Omega}.$$

Hence $v_\varepsilon \to \varphi$ uniformly on $\bar{\Omega}$ and by Theorem 5.11 we know that $\varphi \in \mathcal{PSH}(\bar{\Omega}) \cap C(\bar{\Omega})$. $\square$

Chapter 6

P-hyperconvexity

We shall now introduce a new notion of convexity called P-hyperconvexity. This notion is quite natural when working with approximation of plurisubharmonic functions and it uses plurisubharmonic functions on compact sets that were defined in Chapter 5. Remember that a domain is hyperconvex if it has a negative exhaustion function in $\mathcal{PSH}(\Omega)\cap C(\bar{\Omega})$ (see Definition 1.5). A domain will be called P-hyperconvex if it has a negative plurisubharmonic exhaustion function that is plurisubharmonic on $\bar{\Omega}$. The main results in this chapter are that P-hyperconvex domains can be fully characterized using the support of the Jensen measures $\mathcal{J}_z(\bar{\Omega})$ in the boundary points and that, if $z \in \partial\Omega$, then $\mathcal{J}_z(\bar{\Omega}) = \mathcal{J}_z(\partial\Omega)$ for every P-hyperconvex domain Ω. Finally, we see that for a P-hyperconvex domain Ω, a function $\varphi \in \mathcal{PSH}(\Omega) \cap C(\bar{\Omega})$ is in $\mathcal{PSH}(\bar{\Omega})\cap C(\bar{\Omega})$ if, and only if, $\varphi|_{\partial\Omega} \in \mathcal{PSH}(\partial\Omega)$. This chapter is based on results by Hed and Persson in [HP].

Definition 6.1. A domain $\Omega \Subset \mathbb{C}^n$ is called *P-hyperconvex* if it has a negative exhaustion function in $\mathcal{PSH}(\bar{\Omega}) \cap C(\bar{\Omega})$.

Remark 15. If Ω is a P-hyperconvex domain, then Ω is fat, i.e. $\Omega = (\bar{\Omega})^\circ$. To see this, assume that Ω is P-hyperconvex but not fat. Then Ω has an exhaustion function $\psi \in \mathcal{PSH}(\bar{\Omega}) \cap C(\bar{\Omega})$, $\psi|_{\partial\Omega} = 0$. Assume that $z \in \partial\Omega \cap (\bar{\Omega})^\circ$, then $\psi(z) = 0$. But, by Example 5.8, $\psi \in \mathcal{PSH}(\Omega)$ so ψ attains a maximum in an interior point, hence Ω is not P-hyperconvex.

Every P-hyperconvex domain is hyperconvex (by Example 5.8). The converse is not true. This follows from Example 6.2 below. It is still unknown to the author if there are hyperconvex domains that are fat but not P-hyperconvex.

Example 6.2. Let $\Omega = \mathbb{D} \setminus [-\frac{1}{2}, \frac{1}{2}]$, where $\mathbb{D}$ is the unit disk in $\mathbb{C}$ (see Figure 6.1). This domain is hyperconvex since it is regular with respect to the Laplace equation (see [Ra]) but it is not fat so it can not be P-hyperconvex. Observe that this is also an example of a domain where $\mathcal{J}_z(\bar{\Omega}) \neq \mathcal{J}_z^c(\bar{\Omega})$. To see this, let μ be the measure we get when integrating the Lebesgue measure on $\partial B(0, \frac{3}{4})$. We claim that this is a measure in $\mathcal{J}_0(\bar{\Omega}) \setminus \mathcal{J}_0^c(\bar{\Omega})$. Let $\varphi \in \mathcal{PSH}^o(\bar{\Omega})$, then

$$\varphi(0) \leq \int \varphi \, d\mu,$$

hence $\mu \in \mathcal{J}_0(\bar{\Omega})$. Since Ω is hyperconvex, it has a negative plurisubharmonic exhaustion function ψ. Then

$$\psi(0) > \int \psi \, d\mu$$

which shows that $\mu \notin \mathcal{J}_0^c(\bar{\Omega})$. By taking the cross product of Ω and some hyperconvex domain, we get an example in higher dimension of a domain that is hyperconvex but not P-hyperconvex. □

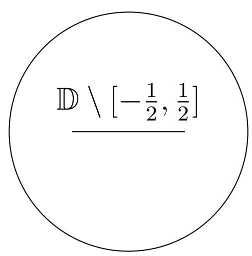


Figure 6.1: An example of a domain that is hyperconvex but not P-hyperconvex.

Let us now look at some examples of domains that are P-hyperconvex.

Example 6.3. Every hyperconvex domain Ω with the $\mathcal{PSH}$-Mergelyan property is P-hyperconvex. In Theorem 7.3 we will see that every hyperconvex domain with C^0-boundary has the $\mathcal{PSH}$-Mergelyan property, so they are P-hyperconvex. □

Example 6.4. Every strongly hyperconvex domain is P-hyperconvex. With strongly hyperconvex we mean that Ω has a negative plurisubharmonic exhaustion function $\psi \in \mathcal{PSH}^o(\bar{\Omega})$. By Example 5.7, we see that $\psi \in \mathcal{PSH}(\bar{\Omega}) \cap C(\bar{\Omega})$, hence Ω is P-hyperconvex. An analytic polyhedron is an example of a strongly hyperconvex domain, hence an analytic polyhedron is P-hyperconvex. $\square$

Remark 16. It follows by the definition of P-hyperconvexity that if Ω_1 and Ω_2 are P-hyperconvex domains, then so are $\Omega_1 \cap \Omega_2$ and $\Omega_1 \times \Omega_2$.

In the same way as hyperconvexity can be fully characterized using the Jensen measures $\mathcal{J}_z^c(\bar{\Omega})$ (see Theorem 1.8 on page 7), the notion of P-hyperconvexity has a corresponding characterization using the measures $\mathcal{J}_z(\bar{\Omega})$, for $z \in \partial\Omega$.

Theorem 6.5. *Let $\Omega \Subset \mathbb{C}^n$ be a domain. The following assertions are then equivalent:*

(1) Ω is P-hyperconvex,

(2) for every $z \in \partial\Omega$ and every $\mu \in \mathcal{J}_z(\bar{\Omega})$, we have that $\operatorname{supp}(\mu) \subset \partial\Omega$,

(3) for every $z \in \partial\Omega$ there exists a function $\varphi \in \mathcal{PSH}(\bar{\Omega}) \cap C(\bar{\Omega})$, $\varphi \not\equiv 0$ such that $\varphi \leq 0$ and $\varphi(z) = 0$.

Proof. (1) $\Rightarrow$ (2) Assume that Ω is P-hyperconvex, then there exists a negative exhaustion function $\psi \in \mathcal{PSH}(\bar{\Omega}) \cap C(\bar{\Omega})$. Take $z \in \partial\Omega$ and let $\mu \in \mathcal{J}_z(\bar{\Omega})$, then

$$0 = \psi(z) \leq \int_{\bar{\Omega}} \psi \, d\mu \leq 0.$$

Hence $\int_{\bar{\Omega}} \psi \, d\mu = 0$ and since $\psi < 0$ on Ω we have that $\operatorname{supp}(\mu) \subset \partial\Omega$.

(2) $\Rightarrow$ (1) Now assume that for $z \in \partial\Omega$, every measure $\mu \in \mathcal{J}_z(\bar{\Omega})$ has support on $\partial\Omega$. Since $\mathcal{J}_z^c(\bar{\Omega}) \subseteq \mathcal{J}_z(\bar{\Omega})$, it follows from Theorem 1.8 that Ω is hyperconvex. Then Ω has a negative exhaustion function $\psi \in \mathcal{PSH}(\Omega) \cap C(\bar{\Omega})$. We want to show that $\psi \in \mathcal{PSH}(\bar{\Omega})$. By Theorem 5.16 it is enough to show that for every $z \in \partial\Omega$ we have that

$$\psi(z) \leq \int \psi \, d\mu \quad \forall \, \mu \in \mathcal{J}_z(\bar{\Omega}).$$

By the assumption every $\mu \in \mathcal{J}_z(\bar{\Omega})$ has support on $\partial\Omega$ when $z \in \partial\Omega$ and we know that $\psi|_{\partial\Omega} = 0$. Hence $\psi \in \mathcal{PSH}(\bar{\Omega})$ and Ω is P-hyperconvex.

(1) $\Rightarrow$ (3) Assume that Ω is P-hyperconvex, then Ω has an exhaustion function $\psi \in \mathcal{PSH}(\bar{\Omega}) \cap C(\bar{\Omega})$. This function satisfies the condition in (3) for every $z \in \partial\Omega$.

(3) $\Rightarrow$ (2) Let $z \in \partial\Omega$ and assume that there exists a function $\varphi \in \mathcal{PSH}(\bar{\Omega}) \cap C(\bar{\Omega})$, $\varphi \not\equiv 0$ such that $\varphi \leq 0$ and $\varphi(z) = 0$. Let $\mu \in \mathcal{J}_z(\bar{\Omega})$, then

$$0 = \varphi(z) \leq \int_{\bar{\Omega}} \varphi \, d\mu \leq 0.$$

Hence $\operatorname{supp}(\mu) \subset \partial\Omega$ by the same argument as above. $\square$

If we want to know when a function $\varphi \in \mathcal{PSH}(\Omega) \cap C(\bar{\Omega})$ also is plurisubharmonic on $\bar{\Omega}$, then we showed in Theorem 5.16 that it is enough to check the submean inequality for measures in $\mathcal{J}_z(\bar{\Omega})$ when $z \in \partial\Omega$. The next theorem gives us the additional advantage that if Ω is P-hyperconvex, then it is not only enough to look at points in the boundary, it is also enough to look at measures in $\mathcal{J}_z(\partial\Omega)$.

Theorem 6.6. *If Ω is P-hyperconvex, then $\mathcal{J}_z(\partial\Omega) = \mathcal{J}_z(\bar{\Omega})$ for every $z \in \partial\Omega$.*

Before we prove this theorem we formulate two useful corollaries that will be used to prove that P-hyperconvexity is a local property (see Theorem 6.10).

Corollary 6.7. *Let $\Omega \Subset \mathbb{C}^n$ be a P-hyperconvex domain. A function φ is in $\mathcal{PSH}(\bar{\Omega}) \cap C(\bar{\Omega})$ if, and only if, $\varphi \in \mathcal{PSH}(\Omega) \cap C(\bar{\Omega})$ and $\varphi|_{\partial\Omega} \in \mathcal{PSH}(\partial\Omega)$.*

Note that the next corollary says that if we can approximate one plurisubharmonic and continuous function from outside, then we can approximate every plurisubharmonic and continuous function, that has the same boundary values, from outside.

Corollary 6.8. *Let $\Omega \Subset \mathbb{C}^n$ be a P-hyperconvex domain. If $\varphi \in \mathcal{PSH}(\Omega) \cap C(\bar{\Omega})$ and if there exists a function $\psi \in \mathcal{PSH}(\bar{\Omega}) \cap C(\bar{\Omega})$ such that $\varphi|_{\partial\Omega} = \psi|_{\partial\Omega}$, then $\varphi \in \mathcal{PSH}(\bar{\Omega})$.*

We are now ready to prove Theorem 6.6.

Proof of Theorem 6.6. First note that for $z \in \partial\Omega$, we always have that $\mathcal{J}_z(\partial\Omega) \subseteq \mathcal{J}_z(\bar{\Omega})$, so it is enough to show the reverse inclusion. Let $f \in \mathcal{PSH}^o(\partial\Omega)$. We can assume that f is smooth in a neighborhood of $\partial\Omega$ (otherwise use regularization). Since Ω is P-hyperconvex, it has an exhaustion function $\psi \in \mathcal{PSH}(\bar{\Omega}) \cap C(\bar{\Omega})$. By Theorem 5.11 there are functions $\psi_j \in \mathcal{PSH}(\Omega_j) \cap C(\Omega_j)$, $\bar{\Omega} \subset \Omega_j$, such that $\psi_j \to \psi$ uniformly on $\bar{\Omega}$. Note that Ω especially is hyperconvex, so it also has a strictly plurisubharmonic exhaustion function ϕ (see [KR]).

Let U be an open set such that $\partial\Omega \subset U$ and $f \in \mathcal{PSH}(U) \cap C^\infty(U)$. Finally, let K be a compact set such that $\bar{\Omega} \subset K \cup U$ and $\partial K \subset U$.

Choose M large enough so that

$$\phi(z) - 1 > M\psi(z) \quad \forall\, z \in K.$$

Since $\psi_j \to \psi$ uniformly on $\bar{\Omega}$, we can assume that $|\psi(z) - \psi_j(z)| < \frac{1}{jM}$ for $z \in \bar{\Omega}$. Let

$$\widehat{\psi_j} = \begin{cases} \max\left(\phi - \frac{1}{j}, M\psi_j\right) & z \in \Omega \\ M\psi_j & z \in \Omega_j \setminus \Omega. \end{cases}$$

Then, by Theorem 1.9 on page 7, $\widehat{\psi_j} \in \mathcal{PSH}(\Omega_j)$ and $\widehat{\psi_j} = \phi - \frac{1}{j}$ on K.

Now, let θ be a smooth function identically 1 in a neighborhood of $\partial\Omega$ and such that $\operatorname{supp}(\theta) \subset U$. By choosing C large enough the function

$$F_j = C\widehat{\psi_j} + \theta f,$$

will belong to $\mathcal{PSH}^o(\bar{\Omega})$. Furthermore, since

$$|F_j(z) - f(z)| = |C\widehat{\psi_j}(z)| = |CM\psi_j| < \frac{C}{j}, \quad \forall\, z \in \partial\Omega,$$

it holds that F_j converges to f uniformly on $\partial\Omega$. Hence, for $z \in \partial\Omega$ and $\mu \in \mathcal{J}_z(\bar{\Omega})$, we have that

$$f(z) = \lim_j F_j(z) \le \lim_j \int F_j \, d\mu = \int \lim_j F_j \, d\mu = \int f \, d\mu.$$

Here we have used Theorem 6.5 which gives us that $\operatorname{supp}(\mu) \subset \partial\Omega$, since Ω is P-hyperconvex. Hence $\mu \in \mathcal{J}_z(\partial\Omega)$. $\square$

Remark 17. For a P-hyperconvex domain Ω, the exhaustion function $\psi \in \mathcal{PSH}(\bar{\Omega}) \cap C(\bar{\Omega})$ can be chosen such that ψ is strictly plurisubharmonic on Ω and $\psi \in \mathcal{E}_0(\Omega) \cap C^\infty(\Omega)$. This follows from the fact that every P-hyperconvex domain Ω is hyperconvex, and by Cegrell, [C7, Corollary 1.3], Ω has a strictly plurisubharmonic exhaustion function $\varphi \in \mathcal{E}_0(\Omega) \cap C^\infty(\Omega)$. Then $\varphi|_{\partial\Omega} = 0$ so by Corollary 6.7, $\varphi \in \mathcal{PSH}(\bar{\Omega})$.

In the same way as hyperconvexity is a local property (see [KR, Proposition 1.1]), so is P-hyperconvexity. Before we prove this we need another result. Remember that condition (3) in Theorem 6.5 says that if Ω is a P-hyperconvex domain, then every boundary point admits a *weak* barrier that is plurisubharmonic on $\bar{\Omega}$. If moreover the boundary is B-regular, this barrier will be a *strong* barrier. For more about B-regularity see Chapter 2 and Chapter 9. In the proof of Theorem 6.9 below, we use a result from Chapter 10 that says that for a P-hyperconvex domain Ω, every function $f \in \mathcal{PSH}(\partial\Omega) \cap C(\partial\Omega)$ has an extension $F \in \mathcal{PSH}(\bar{\Omega}) \cap C(\bar{\Omega})$ (see Corollary 10.6).

Theorem 6.9. *Let $\Omega \Subset \mathbb{C}^n$ be a P-hyperconvex domain such that $\partial\Omega$ is B-regular. Then for every $z \in \partial\Omega$ there exists a function $\varphi \in \mathcal{PSH}(\bar{\Omega}) \cap C(\bar{\Omega})$ such that $\varphi(z) = 0$ and $\varphi(\xi) < 0$ for every $\xi \neq z$.*

Proof. Let $z \in \partial\Omega$ and let $f \in C(\partial\Omega)$ be a function such that $f(z) = 0$ and $f < 0$ otherwise. Since $\partial\Omega$ is B-regular, we know that $\mathcal{J}_z(\partial\Omega) = \{\delta_z\}$ (Theorem 2.2 on page 11), so $f \in \mathcal{PSH}(\partial\Omega)$. By using this, together with the fact that Ω is P-hyperconvex and Corollary 10.6, we know that there exists $F \in \mathcal{PSH}(\bar{\Omega}) \cap C(\bar{\Omega})$ such that $F|_{\partial\Omega} = f$. Then F will be a strong barrier in z. $\square$

We are now ready to prove that a domain that is locally P-hyperconvex, also is globally P-hyperconvex.

Theorem 6.10. *Let $\Omega \Subset \mathbb{C}^n$ be a domain such that for every $a \in \partial\Omega$ there exists a neighborhood V_a such that $\Omega \cap V_a$ is P-hyperconvex, then Ω is P-hyperconvex.*

Proof. By the assumption, Ω is locally P-hyperconvex, then it is also locally hyperconvex. By Kerzman and Rosay, [KR, Proposition 1.1], we know that Ω must be hyperconvex. Hence, there exists $\psi \in \mathcal{PSH}(\Omega) \cap C(\bar{\Omega})$, $\psi \not\equiv 0$, such that $\psi|_{\partial\Omega} = 0$. We want to show that $\psi \in \mathcal{PSH}(\bar{\Omega})$. By Theorem

5.13 it is enough to show that for every $z \in \bar{\Omega}$ there is a ball B_z such that $\psi|_{\bar{\Omega}\cap\bar{B}_z} \in \mathcal{PSH}(\bar{\Omega} \cap \bar{B}_z)$. For $z \in \Omega$ this is obviously true so it is enough to look at $z \in \partial\Omega$. Fix a point $z_0 \in \partial\Omega$ and a small ball B_{z_0} around z_0. We can assume that $B_{z_0} \Subset V_{z_0}$ so $\Omega \cap B_{z_0}$ is P-hyperconvex. Then, by Corollary 6.7, it is enough to show that $\psi \in \mathcal{PSH}\big(\partial(\Omega \cap B_{z_0})\big)$, i.e. that for every $z \in \partial(\Omega \cap B_{z_0})$ and every $\mu \in \mathcal{J}_z(\partial(\Omega \cap B_{z_0}))$

$$\psi(z) \leq \int \psi \, d\mu. \tag{6.1}$$

Suppose that $z \in \partial\Omega \cap B_{z_0} \setminus \partial B_{z_0}$. We want to show that $\mu \in \mathcal{J}_z(\partial(\Omega \cap B_{z_0}))$ only has support on $\partial\Omega$ since then (6.1) holds. Since $\Omega \cap V_{z_0}$ is P-hyperconvex, it has an exhaustion function $\varphi \in \mathcal{PSH}(\bar{\Omega} \cap \bar{V}_{z_0})$ and especially $\varphi \in \mathcal{PSH}\left(\partial(\bar{\Omega} \cap \bar{B}_{z_0})\right)$. Let $\mu \in \mathcal{J}_z\left(\partial(\Omega \cap B_{z_0})\right)$, then

$$0 = \varphi(z) \leq \int \varphi \, d\mu \leq 0.$$

Hence, μ has support where $\varphi = 0$, i.e. on $\partial\Omega$.

Now suppose that $z \in \Omega \cap \partial B_{z_0}$. We claim that $\mathcal{J}_z\left(\partial(\Omega \cap B_{z_0})\right) = \{\delta_z\}$ and this makes that (6.1) holds. Since B_{z_0} is P-hyperconvex and ∂B_{z_0} is B-regular, we know by Theorem 6.9 that every boundary point admits a strong barrier that lies in $\mathcal{PSH}(\bar{B}_{z_0})$. This means that for every $z \in \Omega \cap \partial B_{z_0}$ there exists a function $\varphi \in \mathcal{PSH}\left(\partial(\Omega \cap B_{z_0})\right)$ such that $\varphi(z) = 0$ and $\varphi(\xi) < 0$ for every $\xi \neq z$. By the same argument as above, we see that $\mathcal{J}_z\left(\partial(\Omega \cap B_{z_0})\right) = \{\delta_z\}$. The same argument holds for $z \in \partial\Omega \cap \partial B_{z_0}$ and this finishes the proof. □

Since the measures in $\mathcal{J}_z(\bar{\Omega})$ can be constructed using analytic disks (see the appendix) and since they only have support on $\partial\Omega$ when Ω is P-hyperconvex and $z \in \partial\Omega$, the next corollary follows immediately.

Corollary 6.11. *Let Ω be a P-hyperconvex domain and let f be a holomorphic function from the unit disk $\mathbb{D}$ to $\bar{\Omega}$ such that $f(0) \in \partial\Omega$, then $f(\mathbb{D}) \subset \partial\Omega$.*

Chapter 7

$\mathcal{PSH}$-Mergelyan property

Let $\Omega \subset \mathbb{C}^n$ be a domain and let $\mathcal{O}(\Omega)$ denote the functions that are holomorphic on Ω. We say that a domain Ω has the *Mergelyan property* if every $f \in \mathcal{O}(\Omega) \cap C(\bar{\Omega})$ can be approximated uniformly on $\bar{\Omega}$ with functions holomorphic on neighborhoods of $\bar{\Omega}$. By the classical theorem by Mergelyan, [Me], we know that if $K \subset \mathbb{C}$ is a compact set with an interior such that $\mathbb{C} \setminus K$ is connected, then every function $f \in \mathcal{O}(K^\circ) \cap C(K)$ can be approximated uniformly on K with polynomials. In the case when $n > 1$, there are still many open problems. Henkin, Kerzman and Lieb, [Hen, Ker, Li], proved that every smoothly bounded, strictly pseudoconvex domain has the Mergelyan property. They proved this by showing the deep result that one can solve the $\bar{\partial}$ with uniform estimates. There are examples of smooth pseudoconvex domains that does not have the Mergelyan property. An example of this is the worm domain defined by Diederich and Fornæss (see [DF, Theorem 2]). The worm domain is also an example of a smooth pseudoconvex domain that does not have a Stein neighborhood basis and it is conjectured that a smooth domain Ω has the Mergelyan property if, and only if, Ω has a Stein neighborhood basis. For more about the Mergelyan property see [BF] and [FN].

In this chapter we discuss the corresponding Mergelyan property for plurisubharmonic functions. A domain $\Omega \Subset \mathbb{C}^n$ has the *$\mathcal{PSH}$-Mergelyan property* if every function $\varphi \in \mathcal{PSH}(\Omega) \cap C(\bar{\Omega})$ can be approximated uniformly on $\bar{\Omega}$ with functions that are plurisubharmonic and continuous on neighborhoods of $\bar{\Omega}$. The question of investigating which domains has this property has been studied by, for example, Sibony ([Sib, Theorem 2.2]) who showed that if Ω is a pseudoconvex domain with smooth boundary, then Ω has the $\mathcal{PSH}$-Mergelyan property. Fornæss and Wiegerinck then showed that this is true even for arbitrary bounded domains with C^1-boundary (see [FW, Theorem 1]). Note that the worm domain, discussed above, does not have the Mergelyan property but, since it has smooth boundary, it has the $\mathcal{PSH}$-Mergelyan property. We will begin by improving the result by Fornæss and Wiegerinck and show that it is enough with C^0-boundary for a domain to have the $\mathcal{PSH}$-Mergelyan property. Later we use the notion of plurisubharmonic functions on compact sets and P-hyperconvex domains when studying this property. As we will see in Theorem 7.2, C^0-boundary is the same as having the segment property, so we begin by defining this property.

Definition 7.1. We say that a domain $\Omega \Subset \mathbb{C}^n$ has the *segment property* if there, for every $z \in \partial\Omega$, exists a neighborhood U of z and a vector $w \in \mathbb{C}^n$ such that

$$U \cap \bar{\Omega} + tw \subset \Omega, \quad \forall\, 0 < t < 1.$$

A domain that has the segment property can not at the same time lie on both sides of any given part of its boundary (see Figure 7.1), but we can allow cusps (see Figure 7.2). Observe that domains that have cusps does not have Lipschitz boundary. The segment property has been studied by, for example, Birtel in [Bi], who studied its connection to the Mergelyan property, and by Kerzman and Rosay in [KR], who called it condition (*).

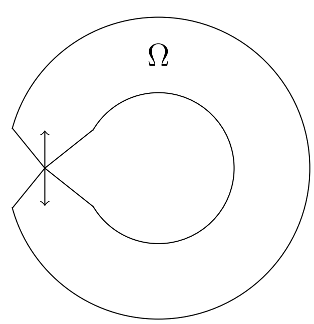


Figure 7.1: Ω does not have the segment property.

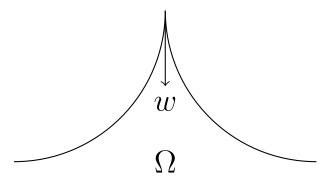


Figure 7.2: A part of the boundary that has a cusp

The segment property is very useful when it comes to approximation from outside and it is actually equivalent to having C^0-boundary. Here C^0-boundary means that the boundary locally can be seen as the graph of a continuous function.

Theorem 7.2. *A domain $\Omega \subset \mathbb{C}^n$ has the segment property if, and only if, Ω has C^0-boundary.*

Proof. For a proof see [DZ, Theorem 7.3] or [Fr, Theorem 3.3]. The idea of the proof is to, first of all, point out that if the boundary is locally the graph of a continuous function, then there has to be a direction into the domain, hence the domain has the segment property. The proof of the converse is a bit more work. The main idea is to rotate and translate the domain and then define the continuous function that locally defines the boundary, as the least distance from some plane to the boundary. Since the domain has the segment property, this functions will be continuous. □

Next theorem is one of the main theorems in this thesis. It improves the result by Fornæss and Wiegerinck in [FW] that says that a domain in $\mathbb{C}^n$ with C^1-boundary has the $\mathcal{PSH}$-Mergelyan property. We show that it is enough with C^0-boundary. The main idea in the proof by Fornæss and Wiegerinck is that if $\varphi \in \mathcal{PSH}(\Omega) \cap C(\bar{\Omega})$, then the C^1-boundary gives us locally a normal vector that makes it possible to extend φ locally to a small neighborhood. Theorem 7.3 below can be proved in a similar way. But, here we have chosen to use techniques more similar to the rest of this thesis. Remember that in Chapter 5 we noted that a domain $\Omega \Subset \mathbb{C}^n$ has the $\mathcal{PSH}$-Mergelyan property if, and only if, $\mathcal{PSH}(\Omega)\cap C(\bar{\Omega}) = \mathcal{PSH}(\bar{\Omega})\cap C(\bar{\Omega})$. This follows directly from Theorem 5.11. Hence, we want to show that our function φ is in $\mathcal{PSH}(\bar{\Omega})$. The proof relies heavily on Theorem 5.13 that says that $\varphi \in \mathcal{PSH}(\bar{\Omega})\cap C(\bar{\Omega})$ if, and only if, it is true locally.

Theorem 7.3. *Let $\Omega \Subset \mathbb{C}^n$ be a domain with C^0-boundary, then Ω has the $\mathcal{PSH}$-Mergelyan property.*

Proof. Take $\varphi \in \mathcal{PSH}(\Omega) \cap C(\bar{\Omega})$, we want to show that $\varphi \in \mathcal{PSH}(\bar{\Omega})$. By Theorem 5.13 it is enough to show that for every $z_0 \in \bar{\Omega}$ there exists a neighborhood B such that $\varphi|_{\bar{\Omega}\cap\bar{B}} \in \mathcal{PSH}(\bar{\Omega}\cap\bar{B})$. If $z_0 \in \Omega$, this is obviously true so let $z_0 \in \partial\Omega$. Since Ω has C^0-boundary, Theorem 7.2 says that Ω has the segment property. Then there exists a neighborhood U of z_0 and a vector $w \in \mathbb{C}^n$ such that $U \cap \bar{\Omega} + tw \subset \Omega$ for every $0 < t < 1$. Let $B = B(z_0, \frac{|w|}{2})$. To show that $\varphi|_{\bar{\Omega}\cap\bar{B}} \in \mathcal{PSH}(\bar{\Omega}\cap\bar{B})$ we can use Theorem 5.16 to see that it is enough to show that for every $z \in \partial(\Omega \cap B)$ we have that

$$\varphi(z) \leq \int \varphi\, d\mu \quad \forall\ \mu \in \mathcal{J}_z(\bar{\Omega}\cap\bar{B}). \tag{7.1}$$

Let

$$\varphi_j(z) = \varphi(z + \frac{1}{j}w),$$

then $\varphi_j \in \mathcal{PSH}^o(\bar{\Omega}\cap\bar{B})$, $\varphi_j \to \varphi$ and the functions φ_j are bounded. By Lebesgue's dominated convergence theorem we have that (7.1) is satisfied, hence $\varphi \in \mathcal{PSH}(\bar{\Omega}\cap\bar{B})$. □

Remark 18. A consequence of Theorem 7.3 is that every hyperconvex domain with C^0-boundary is P-hyperconvex.

Next proposition uses the fact that on a B-regular compact set, every continuous function is also plurisubharmonic.

Proposition 7.4. *Let Ω be a bounded hyperconvex domain in $\mathbb{C}^n$ such that $\partial\Omega$ is B-regular. Then Ω has the $\mathcal{PSH}$-Mergelyan property if, and only if, Ω is P-hyperconvex.*

Proof. That a hyperconvex domain that has the $\mathcal{PSH}$-Mergelyan property is P-hyperconvex follows directly from the definition of P-hyperconvexity. To show the converse, let $\varphi \in \mathcal{PSH}(\Omega) \cap C(\bar{\Omega})$ and show that $\varphi \in \mathcal{PSH}(\bar{\Omega})$.

Since $\partial\Omega$ is B-regular, we know that $\mathcal{J}_z(\partial\Omega) = \{\delta_z\}$, for every $z \in \partial\Omega$ (see Theorem 2.2), hence $\varphi \in \mathcal{PSH}(\partial\Omega)$. The result then follows from Corollary 6.7. □

We can also use Jensen measures to characterize the $\mathcal{PSH}$-Mergelyan property. It was proven by Nguyễn, [N, Proposition 3.1], that a domain $\Omega \Subset \mathbb{C}^n$ has the $\mathcal{PSH}$-Mergelyan property if, and only if, $\mathcal{J}_z(\bar{\Omega}) = \mathcal{J}_z^c(\bar{\Omega})$ for all $z \in \bar{\Omega}$. We can now improve this result for P-hyperconvex domains. Remember that when Ω is hyperconvex and $z \in \partial\Omega$, then the measures in $\mathcal{J}_z^c(\bar{\Omega})$ only have support on $\partial\Omega$ (see Theorem 1.8 on page 7). We can then see these measures as measures on $\partial\Omega$ and we can prove the following result:

Theorem 7.5. *A P-hyperconvex domain $\Omega \Subset \mathbb{C}^n$ has the $\mathcal{PSH}$-Mergelyan property if, and only if, $\mathcal{J}_z(\partial\Omega) = \mathcal{J}_z^c(\bar{\Omega})$ for all $z \in \partial\Omega$.*

Before we prove this theorem we show that for a hyperconvex domain Ω we have that $\mathcal{J}_z^c(\bar{\Omega}) \subseteq \mathcal{J}_z(\partial\Omega)$ for all $z \in \partial\Omega$. We will use the following result that can be found in [NDH, Lemma 2.9].

Theorem 7.6. *Let $\Omega \Subset \mathbb{C}^n$ be a hyperconvex domain and u be a $\mathcal{C}^2$ smooth plurisubharmonic function defined on a neighborhood U of $\partial\Omega$. Then there exists $u' \in \mathcal{PSH}(\Omega) \cap C(\bar{\Omega})$ such that $u' = u$ on $\partial\Omega$.*

Lemma 7.7. *Let $\Omega \Subset \mathbb{C}^n$ be a hyperconvex domain, then $\mathcal{J}_z^c(\bar{\Omega}) \subseteq \mathcal{J}_z(\partial\Omega)$, for every $z \in \partial\Omega$.*

Proof. Take $\mu \in \mathcal{J}_z^c(\bar{\Omega})$ and $\varphi \in \mathcal{PSH}^o(\partial\Omega)$. By regularization there is a sequence $\{\varphi_j\}$ of plurisubharmonic and smooth functions defined on neighborhoods of $\partial\Omega$ such that $\varphi_j \searrow \varphi$. Hence, by Theorem 7.6 there are functions $\varphi_j' \in \mathcal{PSH}(\Omega) \cap C(\bar{\Omega})$ such that $\varphi_j'|_{\partial\Omega} = \varphi_j$ and $\varphi_j' \searrow \varphi$ on $\partial\Omega$. If $z \in \partial\Omega$, we have that

$$\varphi(z) = \lim_j \varphi_j'(z) \leq \lim_j \int_{\partial\Omega} \varphi_j' d\mu = \int_{\partial\Omega} \varphi d\mu$$

and hence $\mu \in \mathcal{J}_z(\partial\Omega)$. □

Proof of Theorem 7.5. Assume that Ω is P-hyperconvex and has the $\mathcal{PSH}$-Mergelyan property. By Lemma 7.7 it is enough to show that $\mathcal{J}_z(\partial\Omega) = \mathcal{J}_z(\bar{\Omega}) \subseteq \mathcal{J}_z^c(\bar{\Omega})$ for every $z \in \partial\Omega$. Let $\varphi \in \mathcal{PSH}(\Omega) \cap C(\bar{\Omega})$. Since Ω has the $\mathcal{PSH}$-Mergelyan property, φ can be uniformly approximated on $\bar{\Omega}$ with functions $\varphi_j \in \mathcal{PSH}^o(\bar{\Omega})$. Take $z \in \partial\Omega$ and let $\mu \in \mathcal{J}_z(\bar{\Omega})$.

$$\varphi(z) = \lim_j \varphi_j(z) \le \lim_j \int_{\bar{\Omega}} \varphi_j d\mu = \int_{\bar{\Omega}} \varphi d\mu \tag{7.2}$$

hence $\mu \in \mathcal{J}_z^c(\bar{\Omega})$.

Now assume that Ω is P-hyperconvex and that $\mathcal{J}_z(\partial\Omega) = \mathcal{J}_z^c(\bar{\Omega})$ for all $z \in \partial\Omega$. It is enough to show that $\mathcal{PSH}(\Omega) \cap C(\bar{\Omega}) \subseteq \mathcal{PSH}(\bar{\Omega}) \cap C(\bar{\Omega})$, then Ω has the $\mathcal{PSH}$-Mergelyan property by Corollary 5.15. Take $z \in \partial\Omega$ and $\varphi \in \mathcal{PSH}(\Omega) \cap C(\bar{\Omega})$. Let

$$D_1\varphi(z) = \sup\{u(z) : u \in \mathcal{PSH}(\Omega) \cap C(\bar{\Omega}), u \le \varphi\},$$
$$D_2\varphi(z) = \sup\{u(z) : u \in \mathcal{PSH}^o(\bar{\Omega}), u \le \varphi\}.$$

By Edwards' theorem (Theorem 5.9) together with the corresponding version for $\mathcal{J}_z^c(\bar{\Omega})$ (see for example [W, Corollary 2.2]) we get that

$$D_1\varphi(z) = \inf\{\int_{\bar{\Omega}} u(z) d\mu : \mu \in \mathcal{J}_z^c(\bar{\Omega})\},$$
$$D_2\varphi(z) = \inf\{\int_{\bar{\Omega}} u(z) d\mu : \mu \in \mathcal{J}_z(\bar{\Omega})\}.$$

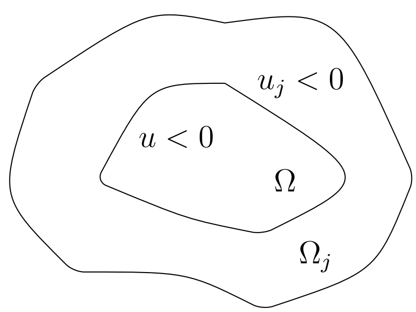
Since, for $z \in \partial\Omega$, we have assumed that $\mathcal{J}_z(\partial\Omega) = \mathcal{J}_z^c(\bar{\Omega})$ and since $\mathcal{J}_z(\partial\Omega) = \mathcal{J}_z(\bar{\Omega})$ (see Theorem 6.6) we have that $D_1\varphi(z) = D_2\varphi(z)$ for all $z \in \partial\Omega$. But $\varphi = D_1\varphi$ and Choquet's lemma (Theorem 5.10) gives us that there is a sequence in $\mathcal{PSH}^o(\bar{\Omega})$ that increases to φ on $\partial\Omega$, hence $\varphi \in \mathcal{PSH}(\partial\Omega) \cap C(\partial\Omega)$. Now, by Corollary 6.7, we know that $\varphi \in \mathcal{PSH}(\bar{\Omega}) \cap C(\bar{\Omega})$. $\square$

Chapter 8

$\mathcal{F}$-approximation property

Another approximation property that we study in this thesis is, what we call, the $\mathcal{F}$-approximation property. Here we study approximation from outside of functions in the class $\mathcal{F}(\Omega)$ (see Section 3.2 for the definition of $\mathcal{F}$).

Definition 8.1. A hyperconvex domain $\Omega \Subset \mathbb{C}^n$ has the *$\mathcal{F}$-approximation property* if there are hyperconvex domains $\{\Omega_j\}$, $\Omega \Subset \Omega_{j+1} \Subset \Omega_j$, such that we can approximate each function $u \in \mathcal{F}(\Omega)$ with an increasing sequence of functions $u_j \in \mathcal{F}(\Omega_j)$ a.e. on Ω.



The difference between the $\mathcal{PSH}$-Mergelyan property and this approximation property is that, in the $\mathcal{F}$-approximation property, we do not have uniform convergence, the functions in the approximating sequence are not necessarily continuous and we only have convergence inside Ω. The main advantages of the $\mathcal{F}$-approximation property are that we are able to approximate functions that are not continuous and we have control of the functions in the approximating sequence. The worm domain, in Example 8.12, is an example of a domain that has the $\mathcal{PSH}$-Mergelyan property but not the $\mathcal{F}$-approximation property. By the proof of Corollary 8.4, we will see that a domain that has the $\mathcal{PSH}$-Mergelyan property and a Stein neighborhood basis has the $\mathcal{F}$-approximation property. It is still unknown to the author if there are domains that have the $\mathcal{F}$-approximation property but not the $\mathcal{PSH}$-Mergelyan property.

A similar approximation as the $\mathcal{F}$-approximation property was studied by Benelkourchi in [Be] but he considers functions in $\mathcal{F}^a$, i.e. functions in $\mathcal{F}$ whose Monge-Ampère measure vanishes on pluripolar subsets of Ω. Another difference is that he assumes that his domain is strongly hyperconvex and that $\{\Omega_j\}$ is a Stein neighborhood basis. We will see that a sufficient condition for Ω to have the $\mathcal{F}$-approximation property is that one single function ($\not\equiv 0$) in the class $\mathcal{N}(\Omega)$ can be approximated with functions in $\mathcal{N}(\Omega_j)$. To show this, we will use subextensions that were discussed in Chapter 4. The results in Section 8.1 are by Cegrell and Hed in [CH], and by Hed in [Hed]. In Section 8.2 we show that if Ω has the $\mathcal{F}$-approximation property, then Ω has a weak Stein neighborhood basis.

8.1 $\mathcal{F}$-approximation property

The two main theorems in this section are the following:

Theorem 8.2. *Let $\Omega \Subset \Omega_{j+1} \Subset \Omega_j$ be hyperconvex domains such that there exists a function $0 > v \in \mathcal{N}(\Omega)$ and a sequence $v_j \in \mathcal{N}(\Omega_j)$ such that $v_j \to v$ a.e. on Ω, then Ω has the $\mathcal{F}$-approximation property.*

The next theorem says that if we can approximate functions in $\mathcal{F}(\Omega)$, then we can also approximate functions in $\mathcal{F}(\Omega, G)$, for certain boundary values G.

Theorem 8.3. *Let $\Omega \Subset \Omega_{j+1} \Subset \Omega_j$ be hyperconvex domains and assume that Ω has the $\mathcal{F}$-approximation property. Let $G \in \mathcal{MPSH}(\Omega_1) \cap C(\bar{\Omega})$, $G \leq 0$. Then, to every function $u \in \mathcal{F}(\Omega, G|_\Omega)$, such that*

$$\int_\Omega (dd^c u)^n < +\infty,$$

there exists an increasing sequence of functions $u_j \in \mathcal{F}(\Omega_j, G|_{\Omega_j})$ such that $u_j \to u$ a.e. on Ω.

Before we prove these theorems we have some work to do. First we formulate a corollary that also will be proven in the end of this section. For this corollary we have to assume that the sequence $\{\Omega_j\}$ is a Stein neighborhood basis, i.e. that the domains Ω_j are pseudoconvex, $\Omega \Subset \Omega_j$ and $\bar{\Omega} = \cap \Omega_j$.

Corollary 8.4. *Let Ω be a P-hyperconvex domain with a Stein neighborhood basis, then Ω has the $\mathcal{F}$-approximation property.*

Remark 19. Note that Corollary 8.4 and Theorem 7.3 imply that every hyperconvex domain Ω with C^0-boundary and a Stein neighborhood basis has the $\mathcal{F}$-approximation property. See also Corollary 9.9 for a result that connects the $\mathcal{F}$-approximation property with B-regular domains.

Remark 20. Note that a strictly pseudoconvex domain with C^2-boundary has a Stein neighborhood basis. Then, by Corollary 8.4, it has the $\mathcal{F}$-approximation property.

That polydiscs are examples of non-smooth domains that have the $\mathcal{F}$-approximation property follows either from Example 6.4, which points out that polydiscs are P-hyperconvex, or by observing that we easily can approximate one function from $\mathcal{E}_0$ from outside. This is shown in the following example:

Example 8.5. Let $\Omega = \{(z_1, z_2) : |z_1| < 1, |z_2| < 1\}$ be the unit polydisc in $\mathbb{C}^2$. Note that Ω is hyperconvex. Let $\{r_j\}$ be a sequence of real numbers such that $r_j \searrow 1$ and let $\Omega_j = \{(z_1, z_2) : |z_1| < r_j, |z_2| < r_j\}$. Look at the function

$$v = \max(\log |z_1|, \log |z_2|, -1) \in \mathcal{E}_0(\Omega).$$

Then the functions

$$v_j = \max(\log |\frac{z_1}{r_j}|, \log |\frac{z_2}{r_j}|, -1)$$

are in $\mathcal{E}_0(\Omega_j)$ and $v_j \to v$ on Ω. □

To be able to prove the theorems above we need some other results. We start by defining the relative Monge-Ampère capacity defined by Bedford and Taylor in [BT2]. If $K \subset \Omega$ is a compact set, then the Monge-Ampère capacity of K relative Ω is defined as

$$\operatorname{cap}(K,\Omega) = \sup\left\{ \int_K (dd^c v)^n : v \in \mathcal{PSH}(\Omega), -1 \leq v \leq 0 \right\}.$$

If $h_{K,\Omega}$ is the relative extremal function defined by

$$h_{K,\Omega}(z) = \sup\{v(z) : v \in \mathcal{PSH}(\Omega), v|_K \leq -1, v|_\Omega < 0\},$$

then Bedford and Taylor, [BT2], proved that

$$\operatorname{cap}(K,\Omega) = \int_\Omega (dd^c h^*_{K,\Omega})^n = \int_K (dd^c h^*_{K,\Omega})^n,$$

where $h^*_{K,\Omega}$ is the upper semicontinuous regularization of $h_{K,\Omega}$.

In [Be, Proposition 2.1], Benelkourchi gave a new characterization of the class $\mathcal{F}(\Omega)$ in terms of the relative Monge-Ampère capacity. For the readers convenience we include the proof.

Theorem 8.6. *Let Ω be a hyperconvex domain in $\mathbb{C}^n$. A function $\varphi \in \mathcal{PSH}^-(\Omega)$ is in $\mathcal{F}(\Omega)$ if, and only if,*

$$\limsup_{s\to 0} s^n \operatorname{cap}(\{z \in \Omega; \varphi \leq -s\}, \Omega) < +\infty.$$

Proof. Let $\varphi \in \mathcal{F}(\Omega)$, then there is a decreasing sequence of functions $\varphi_j \in \mathcal{E}_0(\Omega)$ such that $\varphi_j \searrow \varphi$ on Ω and $\sup_j \int_\Omega (dd^c\varphi_j)^n < +\infty$. For a fixed j we have that $h^*_{\{\varphi_j \leq -s\},\Omega} \geq \frac{\varphi_j}{s}$, where $h_{\{\varphi_j \leq -s\},\Omega}$ is the relative extremal function. Since both functions are in $\mathcal{E}_0(\Omega)$, integration by parts (see Theorem 3.12) gives us that

$$\int_\Omega (dd^c h^*_{\{\varphi_j \leq -s\},\Omega})^n \leq \int_\Omega (dd^c \frac{\varphi_j}{s})^n$$

and hence

$$s^n \operatorname{cap}(\{\varphi_j \leq -s\}, \Omega) \leq \int_\Omega (dd^c\varphi_j)^n.$$

Since $\sup_j \int_\Omega (dd^c\varphi_j)^n < +\infty$ we get that

$$\limsup_{s\to 0} s^n \operatorname{cap}(\{\varphi(z) \leq -s\}, \Omega) < +\infty.$$

For the reverse, suppose that $\varphi \in \mathcal{PSH}^-(\Omega)$ and that

$$\limsup_{s\to 0} s^n \operatorname{cap}(\{z \in \Omega : \varphi(z) \leq -s\}, \Omega) < +\infty.$$

By Theorem 3.13 (see also [C5, Theorem 2.1]) there is a decreasing sequence of functions $\varphi_j \in \mathcal{E}_0(\Omega) \cap C(\bar{\Omega})$ such that $\varphi_j \searrow \varphi$ when $j \to \infty$. It remains to show that $\sup_j \int_\Omega (dd^c\varphi_j)^n < +\infty$. Take $s > 0$ fixed, then

$$\begin{aligned}
\frac{1}{s^n}\int_{\{\varphi_j \leq -s\}} (dd^c\varphi_j)^n &= \int_{\{\frac{\varphi_j}{s} \leq -1\}} (dd^c \frac{\varphi_j}{s})^n \\
&= \int_\Omega (dd^c \frac{\varphi_j}{s})^n - \int_{\{\frac{\varphi_j}{s} > -1\}} (dd^c \frac{\varphi_j}{s})^n \\
&= \int_\Omega \big(dd^c \max(\frac{\varphi_j}{s}, -1)\big)^n - \int_{\{\frac{\varphi_j}{s} > -1\}} \big(dd^c \max(\frac{\varphi_j}{s}, -1)\big)^n \\
&= \int_{\{\frac{\varphi_j}{s} \leq -1\}} \big(dd^c \max(\frac{\varphi_j}{s}, -1)\big)^n \\
&\leq \operatorname{cap}(\{\varphi_j \leq -s\}, \Omega)\,.
\end{aligned}$$

Here we have used [C4, Lemma 4.1] together with the fact that both $\frac{\varphi_j}{s}$ and $\max(\frac{\varphi_j}{s}, -1)$ are in $\mathcal{E}_0(\Omega)$. Hence we have that

$$\int_{\{\varphi_j \leq -s\}} (dd^c\varphi_j)^n \leq s^n \operatorname{cap}(\{\varphi_j \leq -s\}, \Omega) \;\; \forall\; s > 0$$

and then

$$\int_\Omega (dd^c\varphi_j)^n \leq \limsup_{s\to 0} s^n \operatorname{cap}(\{\varphi \leq -s\}, \Omega) < +\infty$$

for all j. This gives us that $\varphi \in \mathcal{F}(\Omega)$. □

We also need an identity principle for functions in $\mathcal{F}(\Omega, H)$. The following theorem is from [ACCH, Theorem 3.6].

Theorem 8.7. *Let $H \in \mathcal{E}$. If $u, v \in \mathcal{N}(\Omega, H)$ is such that $u \leq v$, $(dd^cu)^n = (dd^cv)^n$ and $\int_\Omega (-\omega)(dd^cu)^n < +\infty$ for some $\omega \in \mathcal{E}(\Omega)$ which is not identically 0, then $u = v$ on Ω.*

The assumption in Theorem 8.2, that there exists a function $0 > v \in \mathcal{N}(\Omega)$ and a sequence of functions $v_j \in \mathcal{N}(\Omega_j)$ such that $\lim v_j = v$ a.e. on Ω will make sure that Ω_j tends to Ω in the following way:

Theorem 8.8. *Assume that $\Omega \Subset \Omega_{j+1} \Subset \Omega_j$ are hyperconvex domains and that there exists a function $0 > v \in \mathcal{N}(\Omega)$ and a sequence of functions $v_j \in \mathcal{N}(\Omega_j)$ such that $\lim v_j = v$ a.e. on Ω, then $\operatorname{cap}(K, \Omega) = \lim_{j \to +\infty} \operatorname{cap}(K, \Omega_j)$ for every compact subset K of Ω.*

Remark 21. By the proof of Theorem 8.2 we get that the two properties above actually are equivalent.

Before we prove this theorem we observe that if we have a sequence $v_j \in \mathcal{N}(\Omega_j)$ that converges to some $v \in \mathcal{N}(\Omega)$ $(v \not\equiv 0)$ a.e. on Ω, then we can assume that our sequence $\{v_j\}$ is increasing. We can create functions $v^j = (\sup_{j \leq k} v_k)^*$ which will be in $\mathcal{N}(\Omega)$ (since $v^j \geq v$) and $v^j \searrow v$ a.e. on Ω. Observe that $(\sup_{k \geq j} v_k)^* = (\sup_{k \geq j} v_k)$ a.e. on Ω. Choose $j_0 \in \mathbb{N}$ such that $v_j \not\equiv 0 \ \forall \ j > j_0$. Now let $v_s^{'} = \sup_{j_0 \leq p \leq s} v_p$ then $v_s^{'} \in \mathcal{N}(\Omega_s)$ since $v_s^{'} \geq v_s$. We see that $v_s^{'} \nearrow v^{j_0} = (\sup_{j_0 \leq k} v_k)^*$ a.e. on Ω and $v^{j_0} < 0$.

Proof of Theorem 8.8. Assume that there exists a function $0 > v \in \mathcal{N}(\Omega)$ and an increasing sequence of functions $v_j \in \mathcal{N}(\Omega_j)$ such that $\lim v_j = v$ a.e on Ω. Let $K \subset \Omega$ be a compact set and let $h_{K,\Omega}$ be the relative extremal function for K in Ω. Then $h^*_{K,\Omega} \in \mathcal{E}_0(\Omega) \cap C(\bar{\Omega})$, $-1 \leq h^*_{K,\Omega} \leq 0$, $\operatorname{supp}(dd^c h^*_{K,\Omega})^n \subset K$. Let

$$h_j(z) = \sup\{\varphi(z); \varphi \in \mathcal{PSH}^-(\Omega_j); \varphi|_\Omega \leq h^*_{K,\Omega}\}.$$

By Theorem 4.2, $h_j \in \mathcal{E}_0(\Omega_j)$ and $(dd^c h_j)^n \leq \chi_\Omega (dd^c h^*_{K,\Omega})^n$ on Ω_j. By multiplying v and the v_j:s with a positive constant, we can assume that $v < -1$ near K so that $v_j \leq h^*_{K,\Omega}$ on Ω. Then $v_j \leq h_j$ so if we define $f = (\lim h_j)^*$, $v \leq f$ and $f \in \mathcal{N}(\Omega)$. Because of the construction, $f \leq h^*_{K,\Omega}$ and $(dd^c f)^n \leq (dd^c h^*_{K,\Omega})^n$. It follows that $\int (dd^c f)^n \leq \int (dd^c h^*_{K,\Omega})^n < +\infty$ and by Proposition 3.15 *b*) on page 19, $f \in \mathcal{F}$. But, since $f \leq h^*_{K,\Omega}$, it follows from Theorem 3.12 that $\int (dd^c f)^n \geq \int (dd^c h^*_{K,\Omega})^n$ so we get that $\int (dd^c f)^n = \int (dd^c h^*_{K,\Omega})^n$. Therefore, $(dd^c f)^n = (dd^c h^*_{K,\Omega})^n$, so by Theorem 8.7, $f = h^*_{K,\Omega}$. Then, since h_j is an increasing sequence, we know that the measure $(dd^c h_j)^n$ converges to $(dd^c h^*_{K,\Omega})^n$ in the weak*-topology. But

$\text{supp}(dd^c h_j)^n \subset K$ and $\text{supp}(dd^c h^*_{K,\Omega})^n \subset K$ so

$$\int_K (dd^c h_j)^n \to \int_K (dd^c h^*_{K,\Omega})^n.$$

By the definition of the capacity, $\text{cap}(K, \Omega_j) \geq \int_K (dd^c h_j)^n$, so the result follows. □

Remark 22. Example 4.4 shows that Theorem 8.2 is not possible to generalize to the whole class $\mathcal{N}$. This is since the example shows that not every function in $\mathcal{N}$ can be subextended.

We are now ready to prove Theorem 8.2, Theorem 8.3 and Corollary 8.4.

Proof of Theorem 8.2. Let $\{\Omega_j\}$ be a sequence of bounded hyperconvex domains such that $\Omega \Subset \Omega_{j+1} \Subset \Omega_j$. Let $u \in \mathcal{F}(\Omega)$ and let

$$u_j = \sup\{\varphi \in \mathcal{PSH}^-(\Omega_j) : \varphi|_\Omega \leq u\}.$$

Then $\{u_j\}$ will be an increasing sequence and Theorem 4.2 gives us that $u_j \in \mathcal{F}(\Omega_j)$ and $(dd^c u_j)^n \leq \chi_\Omega (dd^c u)^n$ on Ω_j. It remains to show that $f = (\lim u_j)^* = u$ and we will do that by proving that $f \in \mathcal{F}(\Omega)$ and use Theorem 8.7 to see that $f = u$. Suppose that $s > 0$ and that K is a compact subset of $\{z \in \Omega : f(z) \leq -s\}$. By Theorem 8.8 and the proof of Theorem 8.6 we get that

$$s^n \,\text{cap}(K, \Omega) = s^n \lim_{j\to\infty} \text{cap}(K, \Omega_j) \leq s^n \lim_{j\to\infty} \text{cap}(\{z \in \Omega : f(z) \leq -s\}, \Omega_j) \leq$$

$$s^n \lim_{j\to\infty} \text{cap}(\{z \in \Omega_j : u_j(z) \leq -s\}, \Omega_j) \leq \lim_{j\to\infty} \int_{\Omega_j} (dd^c u_j)^n \leq \int_\Omega (dd^c u)^n.$$

Hence $s^n \,\text{cap}(\{f \leq -s\}, \Omega) \leq \int_\Omega (dd^c u)^n < +\infty$ for every $s > 0$, so by Theorem 8.6, $f = (\lim u_j)^* \in \mathcal{F}(\Omega)$. We know by the construction that $f \leq u$ so by Theorem 3.12, $\int_\Omega (dd^c u)^n \leq \int_\Omega (dd^c f)^n$. But, Theorem 4.2 makes sure that $(dd^c f)^n \leq (dd^c u)^n$ and hence, $(dd^c f)^n = (dd^c u)^n$. Now it follows from Theorem 8.7 that $f = u$. □

Proof of Theorem 8.3. Let $G \in \mathcal{MPSH}(\Omega_1) \cap C(\bar{\Omega})$, $G \leq 0$, and let $u \in \mathcal{F}(\Omega, G|_\Omega)$ be such that

$$\int_\Omega (dd^c u)^n < +\infty\,. \tag{8.1}$$

Let $H = G|_\Omega$ and $H_j = G|_{\Omega_j}$. Since $u \in \mathcal{F}(\Omega, H)$ there exists $\psi \in \mathcal{F}(\Omega)$ such that

$$H \geq u \geq \psi + H.$$

Now set

$$\begin{aligned} \psi_j &= \sup\{\varphi \in \mathcal{PSH}^-(\Omega_j) : \varphi \leq \psi \text{ on } \Omega\}\,, \\ u_j &= \sup\{\varphi \in \mathcal{PSH}(\Omega_j) : \varphi \leq H_j \text{ on } \Omega_j, \text{ and } \varphi \leq u \text{ on } \Omega\}\,. \end{aligned}$$

By Theorem 4.2 we have that $\psi_j \in \mathcal{F}(\Omega_j)$ and by the proof of Theorem 8.2 we know that $\psi_j \nearrow \psi$ a.e. on Ω. By the assumption that $G \in C(\bar{\Omega})$ we can use Theorem 4.7 to get that $u_j \in \mathcal{F}(\Omega_j, H_j)$ and that

$$(dd^c u_j)^n \leq \chi_\Omega (dd^c u)^n. \tag{8.2}$$

Since $u_j,\ \psi_j + H_j \in \mathcal{F}(\Omega_j, H_j)$ we get from the construction of the u_j's that

$$H_j \geq u_j \geq \psi_j + H_j \text{ for every } j. \tag{8.3}$$

Let $f = (\lim_j u_j)^*$, then by (8.3), $f \in \mathcal{F}(\Omega, H)$ and by (8.2) and since $\{u_j\}$ is increasing, we get that $(dd^c f)^n \leq (dd^c u)^n$. Moreover, since $u_j \leq u$, we know that $f \leq u$ and Theorem 3.14 implies that $\int_\Omega (dd^c u)^n \leq \int_\Omega (dd^c f)^n$. So we have that $(dd^c f)^n = (dd^c u)^n$, $f \leq u$, and by Theorem 8.7 and (8.1), we have that $u = f$. □

Proof of Corollary 8.4. Let $\{\Omega_j\}$ be a Stein neighborhood basis to $\bar{\Omega}$, then we can assume that Ω_j are hyperconvex for every j. Take a closed ball $B \subset \Omega$. Then the relative extremal function $h_{B,\Omega} \in \mathcal{E}_0(\Omega) \cap C(\bar{\Omega})$. Since Ω is P-hyperconvex, we know from Corollary 6.7 that $h_{B,\Omega}$ can be approximated with functions $u_i \in \mathcal{PSH}^o(\bar{\Omega})$ uniformly on $\bar{\Omega}$. Take $\varepsilon > 0$, then there exists $N > 0$ such that $\sup_{\bar{\Omega}} |h_{B,\Omega} - u_i| < \varepsilon$ if $i > N$. Since Ω_j is a Stein neighborhood, we can take a large j so that $u_i \in \mathcal{PSH}(\Omega_j) \cap C(\Omega_j)$. Let

$$h_k(z) = \sup\{\varphi(z); \varphi \in \mathcal{PSH}^-(\Omega_k); \varphi|_\Omega \leq h_{B,\Omega}\}.$$

Then $\{h_k\}$ is an increasing sequence and $h_k \in \mathcal{E}_0(\Omega_k)$ by Theorem 4.2. We know that $u_i - \varepsilon < h_{B,\Omega}$ on Ω, so $h_k \geq u_i - \varepsilon$ for $k > j$. Thus, $\lim h_k = h_{B,\Omega}$ and by Theorem 8.2, Ω has the $\mathcal{F}$-approximation property. □

8.2 Weak Stein neighborhood basis

A hyperconvex domain Ω has the $\mathcal{F}$-approximation property if there are hyperconvex domains $\{\Omega_j\}$, $\Omega \Subset \Omega_{j+1} \Subset \Omega_j$ and if every function $u \in \mathcal{F}(\Omega)$ can be approximated a.e. on Ω with an increasing sequence $u_j \in \mathcal{F}(\Omega_j)$. A natural question is whether these two assumptions imply that $\bar{\Omega} = (\cap \Omega_j)$, i.e. that $\{\Omega_j\}$ is a Stein neighborhood basis to $\bar{\Omega}$. The answer is still unknown to the author but in this section we will see that $\{\Omega_j\}$ has to be a *weak* Stein neighborhood basis.

Definition 8.9. Let $\Omega \subset \mathbb{C}^n$ be a domain. The *Nebenhülle*, $\mathrm{NH}(\Omega)$, of Ω is the connected component of the set

$$\Big(\bigcap \{\widehat{\Omega} : \Omega \Subset \widehat{\Omega}, \widehat{\Omega} \text{ pseudoconvex}\}\Big)^{\circ}$$

that contains Ω.

By this definition we see that the Nebenhülle of Ω is the smallest pseudoconvex domain that contains Ω.

Definition 8.10. A domain $\Omega \subset \mathbb{C}^n$ has a *weak Stein neighborhood basis* if $\mathrm{NH}(\Omega) = \Omega$.

Note that this is not the same as having a Stein neighborhood basis. In [Ste, Example 2], Stensønes gives an example of a bounded, pseudoconvex domain in $\mathbb{C}^2$ that has a weak Stein neighborhood basis but not a Stein neighborhood basis.

Theorem 8.11. *Let Ω be a hyperconvex domain in $\mathbb{C}^n$. If Ω has the $\mathcal{F}$-approximation property, then Ω has a weak Stein neighborhood basis.*

Proof. Since Ω has the $\mathcal{F}$-approximation property, we know that there are hyperconvex domains $\{\Omega_j\}$ such that $\Omega \Subset \Omega_{j+1} \Subset \Omega_j$. If $\psi \in \mathcal{E}_0(\Omega) \cap C(\bar{\Omega})$ is an exhaustion function for Ω, then we can approximate ψ with an increasing sequence of functions $\psi_j \in \mathcal{E}_0(\Omega_j) \cap C(\bar{\Omega}_j)$ a.e. on Ω (it follows from Theorem 4.2 that ψ_j are continuous). By Dini's theorem this approximation is uniform on Ω. Now assume that Ω does not have a weak Stein neighborhood basis, i.e. that $\mathrm{NH}(\Omega) \setminus \Omega$ contains interior points. Then $\widehat{\psi} = (\lim_j \psi_j)^*$ is a plurisubharmonic function defined on $\mathrm{NH}(\Omega) \subset \cap \Omega_j$ and it must be equal to ψ on $\bar{\Omega}$. This means that $\widehat{\psi}|_{\partial\Omega} = 0$ so $\widehat{\psi}$ attains a global maximum on an interior point and we get a contradiction. □

Example 8.12. The worm domain, defined by Diederich and Fornæss in [DF], is an example of a smooth pseudoconvex domain in $\mathbb{C}^2$ that does not have a weak Stein neighborhood basis. This implies that the worm domain does not have the $\mathcal{F}$-approximation property. □

Theorem 8.13. *Let Ω be a domain in $\mathbb{C}^n$ that has a weak Stein neighborhood basis, then Ω is fat, i.e. $(\bar{\Omega})^\circ = \Omega$.*

Proof. This follows from the fact that $(\bar{\Omega})^\circ \subset \widehat{\Omega}$ for every pseudoconvex domain $\widehat{\Omega}$ such that $\bar{\Omega} \subset \widehat{\Omega}$. Since Ω has a weak Stein neighborhood basis we know that $\mathrm{NH}(\Omega) = \left(\bigcap\{\widehat{\Omega} : \Omega \Subset \widehat{\Omega}, \widehat{\Omega} \text{ pseudoconvex}\}\right)^\circ = \Omega$. Hence $(\bar{\Omega})^\circ = \Omega$. □

By using Theorem 8.11 and Theorem 8.13 we get the next corollary. Note that this result is quite natural since the $\mathcal{F}$-approximation property especially says that every function in $\mathcal{E}_0$, i.e. with boundary values zero, can be approximated from outside with functions that have zero boundary values on their strictly larger domains of definition. If the domain is not fat, this should contradict the maximum principle.

Corollary 8.14. *Let $\Omega \subset \mathbb{C}^n$ be a domain that has the $\mathcal{F}$-approximation property, then Ω is fat.*

Example 8.15. The domain $\Omega = \mathbb{D} \setminus [-\frac{1}{2}, \frac{1}{2}]$, where $\mathbb{D}$ is the unit disk in $\mathbb{C}$, (see Example 6.2) is hyperconvex but not fat so it can not have the $\mathcal{F}$-approximation property. □

When reading about the Nebenhülle in the literature the definition differs a bit. Note that we say that it is defined as the *connected component* of the set

$$(\cap\{\widehat{\Omega} : \Omega \Subset \widehat{\Omega}, \widehat{\Omega} \text{ pseudoconvex}\})^\circ$$

that contains Ω. At this point, the author do not know whether the Nebenhülle always is connected. The following example shows that, when Ω is the unit ball in $\mathbb{C}^n$, then there can exist hyperconvex domains Ω_j, $\Omega \Subset \Omega_j$ such that $(\cap\Omega_j)^\circ$ is not connected. But of course, in this example, there exists a much better choice of neighborhoods Ω_j.

Example 8.16. Let $\Omega = B(0,1) \subset \mathbb{C}^n$. The idea is to construct our neighborhood basis $\{\Omega_j\}$ by using the Barbell lemma (see [Sh] or [JP, Lemma

4.1.44]). Take a sequence $\{\varepsilon_j\}$ such that $\varepsilon_j \searrow 0$. Create neighborhoods $\{B^0_{\varepsilon_j}\}$ to Ω by letting $B^0_{\varepsilon_j} = B(0, 1 + \varepsilon_j)$. Do the same thing for $B(3, 1)$ and call the neighborhoods $\{B^3_{\varepsilon_j}\}$. By the Barbell Lemma we can glue $B^0_{\varepsilon_j}$ and $B^3_{\varepsilon_j}$ together and get the domain Ω_j which will be hyperconvex. Then we have that $\Omega \Subset \Omega_{j+1} \Subset \Omega_j$ and $(\cap \Omega_j)^\circ = B(0, 1) \cup B(3, 1)$. □

Chapter 9

B-regularity

In Chapter 1, we defined B-regular compact sets and B-regular domains in $\mathbb{C}^n$. Remember that a compact set $X \subset \mathbb{C}^n$ is B-regular if every continuous function defined on X can be approximated uniformly on X with continuous plurisubharmonic functions defined on neighborhoods of X. In [Sib], Sibony showed that X is B-regular if, and only if, $\mathcal{J}_z(X) = \{\delta_z\}$ for every $z \in X$ (see Theorem 2.2). This makes B-regular compact sets very special. For instance, this implies that for a B-regular compact set X, every upper semicontinuous function f on X is also plurisubharmonic on X. A bounded open set Ω is called B-regular if every continuous function defined on $\partial\Omega$ has a continuous extension to $\bar{\Omega}$ which is plurisubharmonic on Ω. In [Sib], Sibony studies domains with C^1-boundary and for those domains he shows that being B-regular and having B-regular boundary is the same. In Theorem 9.3 we will show that this is true when the domain has the $\mathcal{PSH}$-Mergelyan property (for example when Ω has C^0-boundary). We also show that every B-regular domain with C^1-boundary has the $\mathcal{F}$-approximation property (see Theorem 9.9). To show this, we use results regarding Property (P). This notion was introduced by Catlin in [Ca] and in Theorem 9.6 we show that for hyperconvex domains with the $\mathcal{PSH}$-Mergelyan property, Property (P) is the same as having B-regular boundary.

We begin to show that if Ω is a hyperconvex domain with B-regular boundary, then Ω is B-regular. This was shown in [Sib] but we prove it by using the following theorem by Wikström [W, Corollary 3.8]: (For the definition of $\mathcal{J}_z^c(\bar{\Omega})$ see Definition 1.7).

Theorem 9.1. *A hyperconvex domain $\Omega \Subset \mathbb{C}^n$ is B-regular if, and only if, $\mathcal{J}_z^c(\bar{\Omega}) = \{\delta_z\}$ for every $z \in \partial\Omega$.*

Theorem 9.2. *Let Ω be a hyperconvex domain in $\mathbb{C}^n$ and assume that $\partial\Omega$ is B-regular. Then Ω is B-regular.*

Proof. Since $\partial\Omega$ is B-regular we know that $\mathcal{J}_z(\partial\Omega) = \{\delta_z\}$ for every $z \in \partial\Omega$ (see Theorem 2.2 or [Sib, Proposition 1.3]). Remember that, when Ω is hyperconvex and $z \in \partial\Omega$, Theorem 1.8 says that the measures in $\mathcal{J}_z^c(\bar{\Omega})$ only have support on $\partial\Omega$, so then we can see the measures in $\mathcal{J}_z^c(\bar{\Omega})$ as measures on $\partial\Omega$. This is needed since we now use Lemma 7.7 which says that if Ω is hyperconvex, then $\mathcal{J}_z^c(\bar{\Omega}) \subseteq \mathcal{J}_z(\partial\Omega)$ for every $z \in \partial\Omega$. Hence, $\mathcal{J}_z^c(\bar{\Omega}) = \{\delta_z\}$ for every $z \in \partial\Omega$ and the result follows from Theorem 9.1. □

Now we come to one of the main results in this chapter. As said earlier, Sibony showed the result below for domains with C^1-boundary (see [Sib]). Recall that, by Theorem 7.3, the theorem below is especially true for bounded B-regular domains with C^0-boundary.

Theorem 9.3. *Let Ω be a bounded B-regular domain with the $\mathcal{PSH}$-Mergelyan property. Then $\partial\Omega$ is B-regular.*

Proof. Let $f \in C(\partial\Omega)$, then we can extend this function to $\psi \in \mathcal{PSH}(\Omega) \cap C(\bar{\Omega})$. By the $\mathcal{PSH}$-Mergelyan property, we know that we can approximate this function, uniformly on $\bar{\Omega}$ with functions in $\mathcal{PSH}^o(\bar{\Omega})$, hence $\partial\Omega$ is B-regular (see Definition 2.1). □

Wikström showed, [W, Theorem 4.1], that on bounded B-regular domains, every upper bounded plurisubharmonic function u on Ω has a decreasing sequence $u_j \in \mathcal{PSH}(\Omega) \cap C(\bar{\Omega})$ such that $u_j \searrow u^*$ on $\bar{\Omega}$. But, what do we know about approximation from outside on B-regular domains? In Proposition 7.4 we proved that if Ω is a hyperconvex domain with B-regular boundary then Ω has the $\mathcal{PSH}$-Mergelyan property if, and only if, Ω is P-hyperconvex. When it comes to the $\mathcal{F}$-approximation property we saw in Corollary 8.4 that every P-hyperconvex domain with a Stein neighborhood basis has the $\mathcal{F}$-approximation property. We can now use B-regularity to make sure that $\bar{\Omega}$ has a Stein neighborhood basis. We do this by using the notion of Property (P) which was introduced by Catlin in [Ca]. This was originally introduced as a sufficient condition for compactness of the $\bar{\partial}$-Neumann operator.

Definition 9.4. A pseudoconvex domain Ω in $\mathbb{C}^n$ is said to satisfy *Property (P)* if, for every $M > 0$, there exists a function ψ that is plurisubharmonic and smooth in a neighborhood of $\bar{\Omega}$ such that $0 \leq \psi \leq 1$ on $\bar{\Omega}$ and $\langle L\psi(z)w, w\rangle \geq M|w|^2$ for $z \in \partial\Omega$ and $w \in \mathbb{C}^n$.

It follows from the definition that if a pseudoconvex domain Ω has Property (P), then $\partial\Omega$ is B-regular. Sibony proved, in [Sib], that if the domain moreover has smooth boundary then these notions are the same. We will now improve this and show that this is true if the domain is hyperconvex and has the $\mathcal{PSH}$-Mergelyan property (i.e. especially when Ω has C^0-boundary). To do this we need the following theorem by Cegrell, [C7, Corollary 2.3], that allows us to glue smooth plurisubharmonic functions together in a smooth way.

Theorem 9.5. *If $\varphi, \psi \in \mathcal{PSH}(\Omega) \cap C^\infty(\Omega)$, and if ω is a neighborhood of $\{\varphi = \psi\}$, then there is a function $\widetilde{\max}(\varphi, \psi) \in \mathcal{PSH}(\Omega) \cap C^\infty(\Omega)$ such that $\widetilde{\max}(\varphi, \psi) \geq \max(\varphi, \psi)$ on Ω and $\widetilde{\max}(\varphi, \psi) = \max(\varphi, \psi)$ on $\Omega \setminus \omega$.*

Theorem 9.6. *Let $\Omega \Subset \mathbb{C}^n$ be a hyperconvex domain with the $\mathcal{PSH}$-Mergelyan property. Then Ω has Property (P) if, and only if, $\partial\Omega$ is B-regular.*

Proof. If Ω has Property (P), then it follows from the definition that $\partial\Omega$ is B-regular. Now assume that $\partial\Omega$ is B-regular. Fix a constant $M > 0$. By Theorem 2.2 there exists a function λ that is plurisubharmonic and smooth in a neighborhood V of $\partial\Omega$, $0 < \lambda < 1$ and

$$\langle L\lambda(z)w, w\rangle \geq M|w|^2 \tag{9.1}$$

for $z \in \partial\Omega$ and $w \in \mathbb{C}^n$. Since Ω is hyperconvex, it has a smooth strictly plurisubharmonic exhaustion function ψ (see [KR]). Let θ be a smooth function with compact support in V such that $\theta \equiv 1$ in a neighborhood U of $\partial\Omega$. We want to find a function φ that is plurisubharmonic and smooth in a neighborhood of $\bar{\Omega}$, $0 \leq \varphi \leq 1$ and such that φ satisfies (9.1) for $z \in \partial\Omega$.

Choose a constant $D > 0$ such that $(|z|^2 - D) < -1$ on some ball B, $\bar{\Omega} \subset B$ and choose a constant $C > 0$ such that

$$\lambda\theta + C(|z|^2 - D)$$

is plurisubharmonic on B. By using the $\mathcal{PSH}$-Mergelyan property together with the technique of regularization and Dini's theorem, we can find functions

ψ_j that are plurisubharmonic and smooth in neighborhoods of $\bar{\Omega}$ such that $\psi_j \searrow \psi$ uniformly on $\bar{\Omega}$. For a fixed j, choose $N > 0$ big enough such that

$$N\psi_j < C(|z|^2 - D) \text{ on } \Omega \setminus U.$$

Observe that $N\psi_j > C(|z|^2 - D)$ on a neighborhood of $\partial\Omega$. Set

$$\varphi = \lambda\theta + \widetilde{\max}\big(N\psi_j, C(|z|^2 - D)\big),$$

where $\widetilde{\max}$ is the smooth maximum from Theorem 9.5. Now φ is plurisubharmonic and smooth in a neighborhood of $\bar{\Omega}$, φ satisfies (9.1) for $z \in \partial\Omega$ and by adjusting φ with suitable constants, we can make sure that $0 \leq \varphi \leq 1$.
□

In [Sib, Theorem 4.1], Sibony shows that, if Ω has C^3-boundary and $\partial\Omega$ is B-regular, then $\bar{\Omega}$ has a Stein neighborhood basis. We can now use results about Property (P) to improve this result. Harrington showed the following: (see [Ha, Theorem 1.1])

Theorem 9.7. *Let Ω be a pseudoconvex domain in $\mathbb{C}^n$ with Property (P) and C^1-boundary, then $\bar{\Omega}$ has a Stein neighborhood basis.*

This result together with Theorem 9.6 gives us that:

Theorem 9.8. *Let Ω be a B-regular domain in $\mathbb{C}^n$ with C^1-boundary, then $\bar{\Omega}$ has a Stein neighborhood basis.*

In Corollary 8.4 we saw that every P-hyperconvex domain with a Stein neighborhood basis has the $\mathcal{F}$-approximation property. Every B-regular domain with C^1-boundary is of course P-hyperconvex so the next corollary follows immediately.

Corollary 9.9. *Let Ω be a B-regular domain in $\mathbb{C}^n$ with C^1-boundary, then Ω has the $\mathcal{F}$-approximation property.*

Remark 23. Note that the $\mathcal{F}$-approximation property together with a Stein neighborhood basis does not imply B-regularity. The polydisc is an example of that (see Example 2.6).

Chapter 10

Plurisubharmonic extension

Let Ω be a bounded domain in $\mathbb{C}^n$. One important problem in pluripotential theory is whether every function $f \in C(\partial\Omega)$ can be extended to a plurisubharmonic function F on Ω that is continuous on $\bar{\Omega}$. If such a function exists, Bremermann shows in [Br], that it can be written as

$$F(z) = \sup\{v(z) : v \in \mathcal{PSH}(\Omega) : v^*|_{\partial\Omega} \leq f\}.$$

Here v^* denotes the upper semicontinuous regularization of v. The function above is called the Perron-Bremermann envelope of f. Observe that this function may not be upper semicontinuous, but, by a result by Walsh in [Wa], F is continuous on $\bar{\Omega}$ if $\lim_{w\to z} F(w) = f(z)$ for every $z \in \partial\Omega$. The class of domains where the Perron Bremermann envelope solves the above problem are called *B-regular* domains and were discussed in Chapter 9. For domains that are not B-regular it is interesting to characterize those functions $f \in C(\partial\Omega)$ that has a continuous extension to $\bar{\Omega}$ that is plurisubharmonic on Ω. In this chapter we look at the connection between plurisubharmonic functions on compact sets and plurisubharmonic extension. We will also discuss when the extension F can be chosen in $\mathcal{PSH}(\bar{\Omega}) \cap C(\bar{\Omega})$. In Corollary 10.6 we see that if Ω is P-hyperconvex then a function $f \in C(\partial\Omega)$ has an extension $F \in \mathcal{PSH}(\bar{\Omega}) \cap C(\bar{\Omega})$ if, and only if, $f \in \mathcal{PSH}(\partial\Omega)$. This chapter is based on results by Hed and Persson in [HP].

We will begin to show that if Ω is a hyperconvex domain, then every function $f \in \mathcal{PSH}(\partial\Omega) \cap C(\partial\Omega)$ is a boundary value of a function $F \in \mathcal{PSH}(\Omega) \cap C(\bar{\Omega})$. To do this, we need a result by Wikström who used the measures $\mathcal{J}_z^c(\bar{\Omega})$ to characterize those continuous functions on $\partial\Omega$ that has a plurisubharmonic extension to Ω (for the definition of $\mathcal{J}_z^c(\bar{\Omega})$, see Definition 1.7 on page 7). Remember that, if Ω is hyperconvex and if $z \in \partial\Omega$, then the measures in $\mathcal{J}_z^c(\bar{\Omega})$ only have support on $\partial\Omega$ (see Theorem 1.8). Wikström showed the following in [W, Theorem 3.5]:

Theorem 10.1. *Let Ω be a hyperconvex domain in $\mathbb{C}^n$ and let $f \in C(\partial\Omega)$. There exists a function $F \in \mathcal{PSH}(\Omega) \cap C(\bar{\Omega})$ such that $F|_{\partial\Omega} = f$ if, and only if, for every $z \in \partial\Omega$ and every measure $\mu \in \mathcal{J}_z^c(\bar{\Omega})$ it holds that*

$$f(z) \leq \int_{\partial\Omega} f \, d\mu.$$

By using this theorem, and the fact that, if we see the measures in $\mathcal{J}_z^c(\bar{\Omega})$ as measures on $\partial\Omega$, then $\mathcal{J}_z^c(\bar{\Omega}) \subseteq \mathcal{J}_z(\partial\Omega)$ for all $z \in \partial\Omega$ (see Lemma 7.7 on page 58), we get the following:

Theorem 10.2. *Let Ω be a bounded hyperconvex domain in $\mathbb{C}^n$ and $f \in \mathcal{PSH}(\partial\Omega) \cap C(\partial\Omega)$. Then there exists a function $u \in \mathcal{PSH}(\Omega) \cap C(\bar{\Omega})$ such that $u|_{\partial\Omega} = f$.*

A natural question is: for which domains Ω do we have that $\mathcal{J}_z(\partial\Omega) \subseteq \mathcal{J}_z^c(\bar{\Omega})$ when $z \in \partial\Omega$? For hyperconvex domains Ω where this holds, we have a full characterization of which continuous functions f on $\partial\Omega$ can be extended to a continuous and plurisubharmonic function defined on Ω.

Theorem 10.3. *Let Ω be a bounded hyperconvex domain in $\mathbb{C}^n$ where $\mathcal{J}_z^c(\bar{\Omega}) = \mathcal{J}_z(\partial\Omega)$ for every $z \in \partial\Omega$. Let $f : \partial\Omega \to \mathbb{R}$ be a continuous function, then the following assertions are equivalent:*

(1) there exists a function $u \in \mathcal{PSH}(\Omega) \cap C(\bar{\Omega})$ such that $u|_{\partial\Omega} = f$,

(2) $f \in \mathcal{PSH}(\partial\Omega)$.

Here are some examples of hyperconvex domains where $\mathcal{J}_z^c(\bar{\Omega}) = \mathcal{J}_z(\partial\Omega)$ for every $z \in \partial\Omega$.

Example 10.4. If Ω has the $\mathcal{PSH}$-Mergelyan property (for example if Ω has C^0-boundary, see Theorem 7.3) then $\mathcal{J}_z^c(\bar{\Omega}) = \mathcal{J}_z(\partial\Omega)$. Take $z \in \partial\Omega$, $\mu \in \mathcal{J}_z(\partial\Omega)$ and let $\varphi \in \mathcal{PSH}(\Omega) \cap C(\bar{\Omega})$. Since Ω has the $\mathcal{PSH}$-Mergelyan property, we can approximate φ uniformly on $\bar{\Omega}$ with continuous plurisubharmonic functions φ_j defined on neighborhoods of $\bar{\Omega}$. Then

$$\varphi(z) = \lim_j \varphi_j(z) \leq \lim_j \int_{\bar{\Omega}} \varphi_j \, d\mu = \int_{\bar{\Omega}} \varphi \, d\mu,$$

and $\mu \in \mathcal{J}_z^c(\bar{\Omega})$. □

Example 10.5. If Ω is a hyperconvex domain such that $\partial\Omega$ is B-regular, then (by Theorem 2.2 on page 11) we have that the Jensen measures $\mathcal{J}_z(\partial\Omega) = \{\delta_z\}$ for every $z \in \partial\Omega$. Using Theorem 9.2 we know that Ω is B-regular, so Theorem 9.1 gives us that $\mathcal{J}_z^c(\bar{\Omega}) = \{\delta_z\}$ for every $z \in \partial\Omega$. Hence $\mathcal{J}_z^c(\bar{\Omega}) = \mathcal{J}_z(\partial\Omega)$ for every $z \in \partial\Omega$. □

Now we want to study plurisubharmonic extension of a function $f \in C(\partial\Omega)$ to a function that is plurisubharmonic and continuous on $\bar{\Omega}$. Note that, when Ω is P-hyperconvex, we know that every $\mu \in \mathcal{J}_z(\bar{\Omega})$ has support on $\partial\Omega$ when $z \in \partial\Omega$ (see Theorem 6.5), so the integral below makes sense.

Corollary 10.6. *Let $\Omega \Subset \mathbb{C}^n$ be a P-hyperconvex domain and let $f \in C(\partial\Omega)$. Then the following assertions are equivalent:*

(1) *there exists $F \in \mathcal{PSH}(\bar{\Omega}) \cap C(\bar{\Omega})$ such that $F|_{\partial\Omega} = f$,*

(2) *for every $z \in \partial\Omega$ and every $\mu \in \mathcal{J}_z(\bar{\Omega})$ we have that*

$$f(z) \le \int_{\partial\Omega} f \, d\mu,$$

(3) $f \in \mathcal{PSH}(\partial\Omega)$.

Proof. (1) $\Rightarrow$ (2) Follows directly from the definition of $\mathcal{PSH}(\bar{\Omega})$.
(2) $\Rightarrow$ (1) Since Ω is P-hyperconvex, it is also hyperconvex and since $\mathcal{J}_z^c(\bar{\Omega}) \subseteq \mathcal{J}_z(\bar{\Omega})$, it follows from Theorem 10.1 that there exists $F \in \mathcal{PSH}(\Omega) \cap C(\bar{\Omega})$ such that $F|_{\partial\Omega} = f$. The result then follows from Theorem 5.16 on page 45.
(2) $\Leftrightarrow$ (3) Follows directly from Theorem 6.6 on page 50. □

Remark 24. An analytic polyhedron $\mathcal{P}$ is an example of a strongly hyperconvex domain, so it is especially P-hyperconvex. In [ACLW], they characterize when a function $f \in C(\partial\mathcal{P})$ has a plurisubharmonic extension to $\mathcal{P}$.

Remark 25. If Ω is P-hyperconvex and $\partial\Omega$ is B-regular, then every function $f \in C(\partial\Omega)$ is plurisubharmonic on $\partial\Omega$ and hence has an extension $F \in \mathcal{PSH}(\bar{\Omega}) \cap C(\bar{\Omega})$.

Appendix A

Approximation of Jensen measures

Let $X \subset \mathbb{C}^n$ be a compact set. The main tool in our construction of $\mathcal{PSH}(X)$ in Chapter 5 is the measures $\mathcal{J}_z(X)$, but we have very little knowledge of them other than how they act on the functions in $\mathcal{PSH}^o(X)$. It is therefore desirable to find a more explicit description of these measures. Fortunately, using Poletsky's theory of analytic disks, such a description has been obtained. In [Po4], Poletsky introduces a class, $\mathcal{M}_z(X)$, of Jensen measures on X that are defined using analytic disks. The measures in $\mathcal{M}_z(X)$ are called holomorphic measures. Poletsky also defines the notion of plurisubharmonic functions on a compact set X as upper semicontinuous functions on X that satisfy a submean inequality with respect to $\mathcal{M}_z(X)$. This definition is exactly the same as the definition of plurisubharmonic functions on compact sets that we saw in Chapter 5, but with the class $\mathcal{M}_z(X)$ instead of $\mathcal{J}_z(X)$. The purpose of this appendix is to follow Poletsky's construction and show that $\mathcal{M}_z(X) = \mathcal{J}_z(X)$ for every $z \in X$. This means that the notion of plurisubharmonic functions on compacts defined by Poletsky and by Sibony are the same and the measures in $\mathcal{J}_z(X)$ can be seen as weak*-limits of push-forwards of the arc length measures with analytic disks. The fact that $\mathcal{M}_z(X) = \mathcal{J}_z(X)$ has been hinted by Poletsky and proved in an appendix to a paper by Nguyễn, Dung and Hung, [NDH], but it is the authors opinion that it has not been given the proper emphasis and detailed discussion as it deserves. Because of this, we dedicate this appendix to a more thorough study of this subject. This appendix is based on the work by Czyż, Hed and Persson in [CHP].

We recall that an upper semicontinuous function u on an open set $\Omega \subset \mathbb{C}^n$ is plurisubharmonic on Ω if its restriction to complex lines are subharmonic. It is a quite remarkable fact that an upper semicontinuous function is plurisubharmonic if, and only if, its compositions with analytic mappings $f : \mathbb{D} \to \Omega$ are subharmonic. Here $\mathbb{D}$ is the unit disk in $\mathbb{C}$. If we want to generalize this to a compact set X, we get a problem because it is not certain that a compact set has an analytic structure, i.e. there might not exist a holomorphic mapping $f : \mathbb{D} \to X$. An example of such a compact set is the examples by Stolzenberg [Sto] and Wermer [We] of a compact set X with non-trivial polynomial hull $\widehat{X}$, such that $\widehat{X} \setminus X$ has no analytic structure. To overcome this problem, Poletsky used approximation of analytic disks. We will discuss the main ideas of his construction but try to avoid most of the technical refinements. For more about this see [Po4].

A.1 Holomorphic measures

In the following, a bounded holomorphic mapping $f : \mathbb{D} \to \mathbb{C}^n$ will be called an analytic disk. We will consider the continuation of an analytic disk f in radial directions as a mapping from $\overline{\mathbb{D}}$ to $\mathbb{C}^n$. By σ, we mean the normalized arc length measure on $\mathbb{T} = \partial\mathbb{D}$. As is common in the literature we will also allow ourself to confuse an analytic disk with its image. We start by noting that every analytic disk f induces a measure on $\mathbb{C}^n$ by pushing forward the arc length measure on the unit disk. To be more precise:

Definition A.1. For $E \subset \mathbb{C}^n$ and f an analytic disk in $\mathbb{C}^n$, let

$$\mu_f(E) = f_*\sigma(E) = \int\limits_{\mathbb{T} \cap f^{-1}(E)} d\sigma.$$

The idea is that it should be possible to weak*-approximate any $\mu \in \mathcal{J}_z(X)$ with a sequence of push forward measures μ_f. To be able to show this we need some more notation and results.

Definition A.2. Let $L = \{f_j\}$ be a uniformly bounded sequence of analytic disks. We say that L is *weak**- *convergent* if the measures μ_{f_j} weak*-converges on $\mathbb{C}^n$, that is if there is a measure μ_L such that

$$\int \varphi(z)\, d\mu_L = \lim_{j\to\infty} \int \varphi(z)\, d\mu_{f_j}, \quad \forall\, \varphi \in C(\mathbb{C}^n).$$

Given such a sequence, we will denote its limit measure by μ_L and we let $\lim_j f_j(0) = z_L$.

Definition A.3. Let $L = \{f_j\}$ be a sequence of analytic disks. *The cluster of* L, denoted $\mathcal{K}(L)$, is the set of all points $z \in \mathbb{C}^n$ such that for every $r > 0$ the set

$$\{j \in \mathbb{N} : f_j(\mathbb{D}) \cap B(z,r) \neq \emptyset\},$$

is infinite.

Remark 26. Since the complement of the cluster is easily seen to be open, the cluster itself will always be a closed set.

The set $\mathcal{K}(L)$ should be understood as the points where the limit measure of L potentially could have mass. The following easy example shows that the cluster of a sequence of analytic disks can be disjoint from the range of the individual disks.

Example A.4. Let L be the sequence

$$f_j(z) = \frac{z+4}{4^j}.$$

The point $z_L = 0$ does not belong to the closure of the set $f_j(\bar{\mathbb{D}})$ for any $f_j \in L$, but $\text{supp}(\mu_L) = \{0\} = \mathcal{K}(L)$. □

We now recall Poletsky's definition of a holomorphic measure (see [Po4]).

Definition A.5. Suppose that L is a uniformly bounded sequence of analytic disks. If $\mathcal{K}(L) \subset X$ and L is weak*-convergent, we say that the limit measure μ_L is a *holomorphic measure* for X at the point $z = z_L$. We denote the set of all such measures by $\mathcal{M}_z(X)$.

Remark 27. Note that, by the definition of weak*-convergence and by testing against a constant function, we see that all holomorphic measures are probability measures.

Remark 28. Suppose that $z \in X$. Then, by letting $f_j(\zeta) = z$, we get the Dirac measure $\delta_z = \lim \mu_{f_j}$, which consequently belongs to $\mathcal{M}_z(X)$.

Since we later will show that $\mathcal{M}_z(X) = \mathcal{J}_z(X)$ for every $z \in X$, the plurisubharmonic functions on compact sets in the sense of Poletsky are exactly the same as the plurisubharmonic functions on compact sets that were discussed in Chapter 5. Because of this, they will not be discussed further. But, in the proofs of Theorem A.9 and Theorem A.14 we need the following lemma that actually says that a plurisubharmonic function defined on a neighborhood of X is plurisubharmonic on X in Poletsky's sense (see [Po4]).

Lemma A.6. *Let u be a plurisubharmonic function defined on a neighborhood of X. Then, for $z \in X$, we have that*

$$u(z) \leq \int_X u\, d\mu$$

for every $\mu \in \mathcal{M}_z(X)$.

Proof. Suppose that $\mu \in \mathcal{M}_z(X)$, then there are analytic disks f_j, $\lim_j f_j(0) = z$, such that

$$\int \varphi\, d\mu = \lim_{j\to\infty} \int_{\mathbb{T}} \varphi \circ f_j\, d\sigma,$$

for all $\varphi \in C(\mathbb{C}^n)$. By regularization, the function u can be approximated with a decreasing sequence $\{u_k\}$ of smooth plurisubharmonic functions defined on slightly smaller neighborhoods of X. By looking at the functions $u_k \circ f_j$, we can conclude that they will be subharmonic on $\mathbb{D}$. From this it follows that

$$\int u\, d\mu = \lim_{j,k\to\infty} \int_{\mathbb{T}} u_k \circ f_j\, d\sigma \geq \lim_{j,k\to\infty} u_k \circ f_j(0) = u(z).$$

□

One useful property of holomorphic measures is that they have a rather strong compactness property in the weak*-topology.

Theorem A.7. *Suppose that $X \subset X_{j+1} \subset X_j$ is a decreasing sequence of compact sets in $\mathbb{C}^n$ such that $X = \cap X_j$. Suppose also that $\mu_j \in \mathcal{M}_{z_j}(X_j)$. Then there is a subsequence $\mu_{j(k)}$ converging to a measure $\mu \in \mathcal{M}_z(X)$, with $z = \lim z_j$.*

Proof. Let B be the closure of a ball sufficiently large such that $X_1 \subset B$. By the same argument as in Remark 10 (on page 38) we get that there is a subsequence $\mu_{j(k)}$ converging to a probability measure μ on B. It remains to show that $\mu \in \mathcal{M}_z(X)$. For notational comfort, we begin by renaming our subsequence μ_k. By definition, there exist sequences $\{f_{k\ell}\}$ of analytic disks such that $\mu_{f_{k\ell}}$ weak*-converges to μ_k and such that $\mathcal{K}(\{f_{k\ell}\}) \subset X_k$. Now let $g_k = f_{kk}$. Obviously $\mathcal{K}(\{g_k\}) \subset X$, $\lim g_k(0) = z$ and since μ_{g_k} converges weak*- to μ, we are finished. □

Corollary A.8. *The set $\mathcal{M}_z(X)$ is weak*- compact.*

Proof. Since the probability measures on X are compact in the weak*-topology of $C(X)$ (see Remark 10), it is enough to show that if a sequence $\{\mu_j\}$ of measures in $\mathcal{M}_z(X)$, weak*-converges to a measure μ on X, then $\mu \in \mathcal{M}_z(X)$. To do this, set $X_j = X$ and apply the previous theorem. □

A.2 Equality of holomorphic measures and Jensen measures

The key tool in proving that $\mathcal{M}_z(X) = \mathcal{J}_z(X)$ is an Edwards type duality theorem for plurisubharmonic functions and holomorphic measures. A very similar theorem was proved by Poletsky in [Po4], whose proof we will follow.

Theorem A.9. *Suppose that $\varphi \in C(X)$ and define*

$$u(z) = \inf\left\{\int \varphi\, d\mu : \mu \in \mathcal{M}_z(X)\right\},$$
$$v(z) = \sup\left\{\psi(z) : \psi \in \mathcal{PSH}^o(X), \psi \le \varphi\right\}.$$

Then $u = v$ on X and there is an increasing sequence $u_j \in \mathcal{PSH}^o(X)$ converging pointwise to this function.

Remark 29. Note that we always have that $v \le u$, since if $\psi \in \mathcal{PSH}^o(X)$ and $\psi \le \varphi$, then for given $\mu \in \mathcal{M}_z(X)$ we have that $\psi(z) \le \int \psi\, d\mu \le \varphi\, d\mu$.

Remark 30. Poletsky noted that a nice consequence of Lemma 3.1 in [Po4] is that every continuous and plurisubharmonic function on X can be approximated with an increasing sequence of functions in $\mathcal{PSH}^o(X)$. In Theorem 5.11 we proved the same thing using Edwards' theorem and Choquet's lemma.

Before we prove Theorem A.9 we need to define a notion of holomorphic measures on open sets. This definition should be compared to the holomorphic current Φ_1 of Example 3.1 in [Po3].

Definition A.10. Let $\Omega \Subset \mathbb{C}^n$ be open and define $\mathcal{H}_z(\Omega)$ to be the class of holomorphic mappings f from a neighborhood of $\overline{\mathbb{D}}$ to Ω such that $f(0) = z$.

Definition A.11. Suppose that $\Omega \Subset \mathbb{C}^n$ is open. Then we define $\mathcal{M}_z(\Omega)$ to be the weak*-closure of the set $\{\mu_f : f \in \mathcal{H}_z(\Omega)\}$.

Remark 31. Since $\mathcal{M}_z(\Omega)$ is a closed subset of the set of probability measures, and since the set of probability measures is weak*-compact, the same holds true for $\mathcal{M}_z(\Omega)$.

Remark 32. Suppose that $\mu \in \mathcal{M}_z(\Omega)$. Then μ is the weak*- limit of some sequence μ_{f_j} for $f_j \in \mathcal{H}_z(\Omega)$. The sequence $L = \{f_j\}$ is uniformly bounded and $\mathcal{K}(L) \subset \overline{\Omega}$. Hence $\mathcal{M}_z(\Omega) \subset \mathcal{M}_z(\overline{\Omega})$.

We are now ready to prove Theorem A.9.

Proof of Theorem A.9. Since $\delta_z \in \mathcal{M}_z(X)$, it follows that $u \leq \varphi$. If $\psi \in \mathcal{PSH}^o(X)$, $\psi \leq \varphi$ and $\mu \in \mathcal{M}_z(X)$, we can extend ψ to a plurisubharmonic function defined on a neighborhood of X. Then we can use Lemma A.6 to see that

$$\psi(z) \leq \int \psi \, d\mu \leq \int \varphi \, d\mu.$$

Taking the supremum of all such ψ we get that

$$v(z) = \sup\{\psi(z) : \psi \in \mathcal{PSH}^o(X), \psi \leq \varphi\} \leq \int \varphi \, d\mu.$$

Now we take the infimum of all $\mu \in \mathcal{M}_z(X)$ and get that $v \leq u \leq \varphi$ on X. The idea is now to approximate u with functions smaller than v, to see that u and v have to be equal. Let $\{\Omega_j\}$ be a sequence of bounded open domains such that $X \subset \Omega_{j+1} \Subset \Omega_j$ and $\cap_{j=1}^{\infty} \Omega_j = X$. Let $\widehat{\varphi}$ be any continuous extension of φ to Ω_1. Since the set $\mathcal{M}_z(\Omega_j)$ is weak*-compact (see Remark 31), the point-evaluation functional defined by $\mu \mapsto \int_{\Omega_j} \widehat{\varphi} \, d\mu$, must attain its minimum on $\mathcal{M}_z(\Omega_j)$ (since it is continuous in the weak*-topology). This means that there exists a measure $\mu_j^z \in \mathcal{M}_z(\Omega_j)$ such that

$$\int \widehat{\varphi} \, d\mu_j^z = \inf\left\{\int \widehat{\varphi} \, d\mu_f : f \in \mathcal{H}_z(\Omega_j)\right\}.$$

Remember that, by Remark 32, we know that $\mu_j^z \in \mathcal{M}_z(\overline{\Omega}_j)$. Also note that $\int \widehat{\varphi}\, d\mu_j^z$ increases with increasing j (since $\Omega_{j+1} \Subset \Omega_j$). By letting $X_j = \bar{\Omega}_j$ we are in the situation of Lemma A.7, and hence we get that the sequence $\{\mu_j^z\}$ has a convergent subsequence μ_k^z which weak*-converge to $\mu^z \in \mathcal{M}_z(X)$. Define the functions

$$u_k(z) = \int \widehat{\varphi}\, d\mu_k^z - \frac{1}{k}.$$

By [Po1, Theorem 1], $u_k \in \mathcal{PSH}(\Omega_k)$ and by the observation above $u_{k+1} > u_k$. We also have that

$$\lim_{k\to\infty} u_k = \int \varphi\, d\mu^z.$$

Since $\mu^z \in \mathcal{M}_z(X)$, it follows that $\lim_{k\to\infty} u_k(z) \geq u(z)$. By the definition of u_k we know that $u_k \in \mathcal{PSH}^o(X)$ and since $\delta_z \in \mathcal{M}_z(\Omega_k)$ we know that $u_k \leq \varphi$. Hence $u_k \leq v$. Because we have already noted that $v \leq u$, this implies that $v = u$. □

As mentioned earlier, one important element in proving that $\mathcal{M}_z(X) = \mathcal{J}_z(X)$ is that we have similar Edwards type theorems for both of the classes. Remember that Theorem 5.9 says that if φ is a lower semicontinuous function defined on X, then

$$\sup\left\{\psi(z) : \psi \in \mathcal{PSH}^o(X), \psi \leq \varphi\right\} = \inf\left\{\int \varphi\, d\nu : \nu \in \mathcal{J}_z(X)\right\}.$$

As the supremum above is equal to the supremum in Theorem A.9, this gives us the necessary link between the class $\mathcal{J}_z(X)$ and Poletsky's holomorphic measures $\mathcal{M}_z(X)$. We now want to use the Hahn-Banach separation theorem. We remind the reader of the content of that theorem (see [Co, page 111]).

Theorem A.12. *Let V be a real locally convex space and A and B be two disjoint closed convex subsets of V. If B is compact, then there exists a continuous linear functional φ on V and a number α such that*

$$\sup\{\varphi(b) : b \in B\} < \alpha < \inf\{\varphi(a) : a \in A\}.$$

In our case, V will be the set of Radon measures on X equipped with the weak*-topology. Then every continuous linear functional φ on V can be assumed to be a point evaluation (see Theorem 1.13). In Corollary A.8

we saw that $\mathcal{M}_z(X)$ is weak*-compact so, to be able to use Hahn-Banach separation theorem, it remains to show that $\mathcal{M}_z(X)$ is convex. The theorem below is a variant of a theorem by Bu and Schachermayer [BS] but the proof presented here is due to Poletsky [Po2, Theorem 2.1].

Theorem A.13. *The set $\mathcal{M}_z(X)$ is convex.*

Proof. Without loss of generality, we may assume that $z = 0$. We want to show that for $\mu, \nu \in \mathcal{M}_z(X)$ and $\lambda \in [0,1]$, it holds that $\lambda\mu + (1-\lambda)\nu \in \mathcal{M}_z(X)$. Because of the definition of $\mathcal{M}_z(X)$, we know that μ and ν are weak*-limits of some sequences, $\{\mu_{f_j}\}$ and $\{\mu_{g_j}\}$ respectively, where the disks $\{f_j\}$ and $\{g_j\}$ all lie in $B(0, M)$ for some big $M > 0$ and $\mathcal{K}(\{f_j\}), \mathcal{K}(\{g_j\}) \subset X$. Let us begin by supposing that $0 < r_j < 1$ and consider the mappings

$$h_j(\zeta) = f_j(\zeta) + g_j(\frac{r_j}{\zeta}),$$

defined on the annuli $R_j = \{\zeta : r_j \leq \zeta \leq 1\}$.

For every $\zeta \in \mathbb{D}$, either $|\zeta|$ or $r_j/|\zeta|$ is less than $\sqrt{r_j}$. Hence choosing r_j small enough will guarantee that the set $h_j(R_j)$ belongs to an arbitrarily small neighborhood of $f_j(\mathbb{D}) \cup g_j(\mathbb{D})$. Let us from now on suppose that $\{r_j\}$ is chosen such that $r_j \to 0$ when $j \to \infty$. Now let ψ_j be the conformal mapping from $\mathbb{D}$ to $\{\zeta : \log r_j < \operatorname{Re}\zeta < 0\}$ defined by

$$\psi_j(\zeta) = \frac{i \log r_j}{\pi} \log\left(e^{-\lambda i\pi}\frac{\zeta - e^{\lambda i\pi}}{\zeta - e^{-\lambda i\pi}}\right) + \log r_j.$$

Note that $\psi_j(0) = (1-\lambda)\log r_j$, $\psi_j(1) = 0$ and $\psi_j(-1) = \log r_j$. From this it follows that the mapping $\omega_j = e^{\psi_j}$ maps the arc $\gamma_1 = \{e^{i\theta} : |\theta| < \lambda\pi\}$ to $\mathbb{T}$ and the arc $\gamma_2 = \mathbb{T} \setminus \gamma_1$ to $r_j\mathbb{T}$. Now the mappings $p_j = h_j \circ \omega_j$ are analytic disks, and we claim that their corresponding measures $\{\mu_{p_j}\}$ weak*-converges to $\lambda\mu + (1-\lambda)\nu$. To prove this claim we start by observing that $\mathcal{K}(\{p_j\}) \subset \mathcal{K}(\{f_j\}) \cup \mathcal{K}(\{g_j\}) \subset X$. Next, we let φ be an arbitrary continuous function defined on $\mathbb{C}^n$ and consider the integral

$$\int \varphi\, d\mu_{p_j} = \int_{\gamma_1} \varphi\big(f_j(\omega_j) + g_j(r_j/\omega_j)\big)\, d\sigma + \int_{\gamma_2} \varphi\big(f_j(\omega_j) + g_j(r_j/\omega_j)\big)\, d\sigma. \quad \text{(A.1)}$$

We begin by studying the first integral on the right-hand side. Since ω_j maps γ_1 onto $\mathbb{T}$ and by the construction of r_j, it follows from the Schwarz lemma that $|g_j\big(r_j/\omega_j(\zeta)\big)| \to 0$ for $\zeta \in \gamma_1$ when $j \to \infty$. By the continuity of φ it follows that

$$\int_{\gamma_1} \varphi\big(f_j(\omega_j) + g_j(r_j/\omega_j)\big)\, d\sigma = \int_{\gamma_1} \varphi\big(f_j(\omega_j)\big)\, d\sigma + \delta_j, \tag{A.2}$$

for some numbers δ_j tending to zero as j tends to infinity. Now let u_j be the harmonic function defined on $\mathbb{D}$ with boundary values $u_j(\zeta) = \varphi\big(f_j(\zeta)\big)$. Since every u_j is bounded by

$$K = \sup_{B(0,M)} \varphi(z),$$

it once again follows by the Schwarz lemma that

$$|u_j(\zeta) - u_j(0)| < K|\zeta|, \quad \forall\, j \in \mathbb{N}. \tag{A.3}$$

Since $u_j \circ \omega_j$ also is a harmonic function it follows that

$$u_j\big(\omega_j(0)\big) = \int_{\mathbb{T}} u_j(\omega_j)\, d\sigma = \int_{\gamma_1} \varphi\big(f_j(\omega_j)\big)\, d\sigma + \int_{\gamma_2} u_j(\omega_j)\, d\sigma. \tag{A.4}$$

Using (A.3) we know that

$$|u_j\big(\omega_j(\zeta)\big) - u_j(0)| < K r_j$$

for $\zeta \in \gamma_2 \cup \{0\}$. Putting this together with (A.4) we see that

$$u_j(0) = \int_{\gamma_1} \varphi\big(f_j(\omega_j)\big)\, d\sigma + (1-\lambda) u_j(0) + \delta_j'.$$

where $\{\delta_j'\}$ is a sequence of numbers converging to zero. Rearranging and combining (A.2) and the definition of u we arrive at

$$\int_{\gamma_1} \varphi\big(f_j(\omega_j) + g_j(r_j/\omega_j)\big)\, d\sigma = \lambda \int_{\mathbb{T}} \varphi\big(f_j(\zeta)\big)\, d\sigma + \delta_j'',$$

for some sequence $\{\delta_j''\} \to 0$. Doing a similar estimation for the second integral on the right-hand side of (A.1), we can conclude that

$$\lim_{j\to\infty} \int \varphi\, d\mu_{p_j} = \lambda \int \varphi\, d\mu + (1-\lambda) \int \varphi\, d\nu,$$

which concludes the proof. □

After doing all this work, the main theorem in this appendix is now easy to prove.

Theorem A.14. *For every $z \in X$, $\mathcal{M}_z(X) = \mathcal{J}_z(X)$.*

Proof. We begin by proving that $\mathcal{M}_z(X) \subset \mathcal{J}_z(X)$. For this, suppose that $\mu \in \mathcal{M}_z(X)$ and that $u \in \mathcal{PSH}^o(X)$. This means that u can be extended to a plurisubharmonic function defined on a neighborhood of X and then it follows from Lemma A.6 that

$$u(z) \leq \int u \, d\mu, \tag{A.5}$$

which shows that $\mu \in \mathcal{J}_z(X)$.

We will show the reverse inclusion by contradiction. For this, suppose that there is a $\mu \in \mathcal{J}_z(X) \setminus \mathcal{M}_z(X)$. Since we have shown that $\mathcal{M}_z(X)$ is convex and weak*-compact, Theorem A.12 says that there exists a function $\varphi \in C(X)$ such that

$$\begin{aligned} \int \varphi \, d\mu &< \inf \left\{ \int \varphi \, d\nu : \nu \in \mathcal{M}_z(X) \right\} \\ &= \sup \left\{ \psi(z) : \psi \in \mathcal{PSH}^o(X), \psi \leq \varphi \right\}. \end{aligned}$$

By Theorem 5.9, we know that this supremum equals

$$\inf \left\{ \int \varphi \, d\nu : \nu \in \mathcal{J}_z(X) \right\} \leq \int \varphi \, d\mu,$$

and this is a contradiction. $\square$

References

[A1] Åhag P., *The complex Monge-Ampère operator on bounded hyperconvex domains*, Ph.D. thesis, Umeå Univ., 2002.

[A2] Åhag P., *A Dirichlet problem for the complex Monge-Ampère operator in $\mathcal{F}(f)$*, Michigan Math. J. 55 (2007), 123-138.

[ACCH] Åhag P., Cegrell U., Czyż R. & Phạm H.H., *Monge-Ampère measures on pluripolar sets*, J. Math. Pures Appl. (9) 92 (2009), no. 6, 613-627.

[ACKPZ] Åhag P., Cegrell U., Kołodziej S, Phạm H.H. & Zeriahi A., *Partial pluricomplex energy and integrability exponents of plurisubharmonic functions*, Adv. Math. 222 (2009), no. 6, 2036-2058.

[AC] Åhag P. & Czyż R., *On the Cegrell classes*, Math. Z. 256 (2007), 243-264.

[ACLW] Åhag P., Czyż R., Lodin S., Wikström F., *Plurisubharmonic extension in non-degenerate analytic polyhedra*, Univ. Iagel. Acta Math., no. 45 (2007), 139-145.

[BB] Bedford E. & Burns D., *Domains of existence for plurisubharmonic functions*, Math. Ann. 238 (1978), no. 1, 67-69.

[BF] Bedford E. & Fornæss J. E., *Approximation on pseudoconvex domains*, Complex approximation (Proc. Conf., Quebec, 1978), pp. 18-31, Progr. Math., 4, Birkhäuser, Boston, Mass., 1980.

[BT1] Bedford E. & Taylor B.A., *The Dirichlet problem for a complex Monge-Ampère equation*, Invent. Math. 37 (1976), 1-44.

[BT2] Bedford E. & Taylor B.A., *A new capacity for plurisubharmonic functions*, Acta Math. 149 (1982), no. 1-2, 1-40.

[BT3] Bedford E. & Taylor B.A., *Smooth plurisubharmonic functions without subextension*, Math. Z. 198 (1988), 331-337.

[Be] Benelkourchi S., *A note on the approximation of plurisubharmonic functions*, C. R. Acad. Sci. Paris, Ser. I 342 (2006), 647-650.

[BGZ] Benelkourchi S., Guedj V. & Zeriahi A., *Plurisubharmonic functions with weak singularities*, Complex analysis and digital geometry, pp. 57-74, Proceedings of the conference in honour of C. Kiselman (Kiselmanfest, Uppsala, 2006). Acta Univ. Upsaliensis Skr. Uppsala Univ. C Organ. Hist., 86, Uppsala Universitet, Uppsala, 2009.

[Bi] Birtel F.T. *Holomorphic approximation to boundary value algebras*, Bull. Amer. Math. Soc. 84 (1978), no. 3, 406-416.

[Bło1] Błocki Z., *On the definition of the Monge-Ampère operator in* $\mathbb{C}^2$, Math. Ann. 328 (2004), 415-423.

[Bło2] Błocki Z., *The domain of definition of the complex Monge-Ampère operator*, Amer. J. Math. 128 (2006), 519-530.

[Bo] Bouligand G., *Sur le problème de Dirichlet*, Ann. Soc. Polon. Math. 4 (1926), 59-112.

[Br] Bremermann H.J., *On a generalized Dirichlet problem for plurisubharmonic functions and pseudo-convex domains. Characterization of Šilov boundaries*, Trans. Amer. Math. Soc. **91** (1959), 246-276.

[BS] Bu S.Q. & Schachermayer W., *Approximation of Jensen measures by image measures under holomorphic functions and applications,* Trans. Amer. Math. Soc. 331 (1992), no. 2, 585-608.

[CCW] Carlehed M., Cegrell U. & Wikström F. *Jensen measures, hyperconvexity and boundary behaviour of the pluricomplex Green function*, Ann. Polon. Math. 71 (1999), no. 1, 87-103.

[Ca] Catlin D., *Global regularity of the* $\bar{\partial}$*-Neumann problem*, Complex analysis of several variables (Madison, Wis., 1982), pp. 39-49, Proc. Sympos. Pure Math., 41, Amer. Math. Soc., Providence, RI, 1984.

[C1] Cegrell U., *On the domains of existence for plurisubharmonic functions*, Complex analysis (Warsaw, 1979), pp. 33-37, Banach Center Publ., 11, PWN, Warsaw, 1983.

[C2] Cegrell U., *Discontinuité de l'opérateur de Monge-Ampère complexe*, C. R. Acad. Sci. Paris Sér. I Math. 296 (1983), no. 21, 869-871.

[C3] Cegrell U., *Sums of continuous plurisubharmonic functions and the complex Monge-Ampère operator in* $\mathbb{C}^n$, Math. Z. 193 (1986), no. 3, 373-380.

[C4] Cegrell U., *Pluricomplex energy*, Acta Math. 180 (1998), 187-217.

[C5] Cegrell U., *The general definition of the complex Monge-Ampère operator*, Ann. Inst. Fourier (Grenoble) 54 (2004), 159-179.

[C6] Cegrell U., *A general Dirichlet problem for the complex Monge-Ampère operator*, Ann. Polon. Math. 94 (2008), no. 2, 131-147.

[C7] Cegrell U., *Approximation of plurisubharmonic functions in hyperconvex domains*, Complex analysis and digital geometry, pp. 125-129, Proceedings of the conference in honour of C. Kiselman (Kiselmanfest, Uppsala, 2006). Acta Univ. Upsaliensis Skr. Uppsala Univ. C Organ. Hist., 86, Uppsala Universitet, Uppsala, 2009.

[CH] Cegrell U. & Hed L., *Subextension and approximation of negative plurisubharmonic functions*, Michigan Math. J. 56 (2008), no. 3, 593-601.

[CKZ1] Cegrell U., Kołodziej S. & Zeriahi A., *Subextension of plurisubharmonic functions with weak singularities*, Math. Z. 250 (2005), no. 1, 7-22.

[CKZ2] Cegrell U., Kołodziej S. & Zeriahi A., *Maximal subextensions of plurisubharmonic functions* , Ann. Fac. Sci. Toulouse Math. (6) 20 (2011), Fascicule Special, 101-122.

[CZ] Cegrell U. & Zeriahi A., *Subextension of plurisubharmonic functions with bounded Monge-Ampère mass*, C. R. Math. Acad. Sci. Paris 336 (2003), no. 4, 305-308.

[Co] Conway J.B., *A course in functional analysis*, Second edition. Graduate Texts in Mathematics, 96. Springer-Verlag, New York, 1990.

[Czy1] Czyż R., *The complex Monge-Ampère operator in the Cegrell classes*, Dissertationes Math. (Rozprawy Mat.) 466 (2009).

[Czy2] Czyż R., *On a Monge-Ampère type equation in the Cegrell class $\mathcal{E}_\chi$*, Ann. Polon. Math. 99 (2010), no. 1, 89-97.

[CzH] Czyż R. & Hed L., *Subextension of plurisubharmonic functions without increasing the total Monge-Ampère mass*, Ann. Polon. Math. 94 (2008), no. 3, 275-281.

[CHP] Czyż R., Hed L. & Persson H., *Plurisubharmonic functions on compact sets*, Ann. Polon. Math. (to appear).

[DZ] Delfour M.C. & Zolésio J.-P., *Shapes and geometries. Analysis, differential calculus, and optimization*, Advances in Design and Control, 4. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2001.

[De] Demailly J.P., *Mesures de Monge-Ampère et mesures pluriharmoniques*, Math. Z. 194 (1987), 519-564.

[DF] Diederich K. & Fornæss J.E., *Pseudoconvex domains: an example with nontrivial Nebenhülle*, Math. Ann. 225 (1977), no. 3, 275-292.

[Ed] Edwards D.A., *Choquet boundary theory for certain spaces of lower semicontinuous functions*, Function Algebras (Proc. Internat. Sympos. on Function Algebras, Tulane Univ., 1965), pp. 300-309, Scott-Foresman, Chicago, 1966.

[El] El Mir H., *Fonctions plurisousharmoniques et ensembles polaires*, Séminaire Pierre Lelong-Henri Skoda (Analyse). Années 1978/79, pp. 61-76, Lecture Notes in Math. 822, Springer, Berlin, 1980.

[FN] Fornæss J.E. & Narasimhan R., *The Levi problem on complex spaces with singularities*, Math. Ann. 248 (1980), no. 1, 47-72.

[FS] Fornæss J.E. & Sibony N., *Plurisubharmonic functions on ring domains*, Complex analysis (University Park, Pa., 1986), pp. 111-120, Lecture Notes in Math. 1268, Springer, Berlin, 1987.

[FW] Fornæss J.E. & Wiegerinck J., *Approximation of plurisubharmonic functions*, Ark. Mat. 27 (1989), no. 2, 257-272.

[Fr] Fraenkel L.E., *On regularity of the boundary in the theory of Sobolev spaces*, Proc. London Math. Soc. (3) 39 (1979), no. 3, 385-427.

[Ga] Gamelin T.W., *Uniform algebras and Jensen measures*, London Mathematical Society Lecture Note Series, 32. Cambridge University Press, Cambridge-New York, 1978.

[G] Gauthier P.M., *Approximation by (pluri)subharmonic functions: fusion and localization*, Canad. J. Math. 44 (1992), no. 5, 941-950.

[Ha] Harrington P.S., *Property* (P) *and Stein neighborhood bases on* C^1 *domains*, Illinois J. Math. 52 (2008), no. 1, 145-151.

[Hed] Hed L., *Approximation of negative plurisubharmonic functions with given boundary values*, Internat. J. Math. 21 (2010), no. 9, 1135-1145.

[HP] Hed L. & Persson H., *On a Mergelyan type approximation for plurisubharmonic functions*, manuscript (2012).

[Hen] Henkin G.M., *Integral representation of functions which are holomorphic in strictly pseudoconvex regions, and some applications*, Mat. Sb. (N.S.) 78 (120) 1969 611-632.

[Hö] Hörmander L., *Notions of convexity*, Reprint of the 1994 edition. Modern Birkhäuser Classics. Birkhäuser Boston, Inc., Boston, MA, 2007.

[JP] Jarnicki M. & Pflug P., *Extension of holomorphic functions*, Walter de Gruyter & Co, 2000.

[Kem] Kemppe B., *Boundary values of plurisubharmonic functions and related topics*, Ph.D. thesis, Umeå Univ., 2009.

[Ker] Kerzman N., *Hölder and* L^p *estimates for solutions of* $\bar{\partial}u = f$ *in strongly pseudoconvex domains*, Comm. Pure Appl. Math. 24 (1971) 301-379.

[KR] Kerzman N. & Rosay J., *Fonctions plurisousharmoniques d'exhaustion bornées et domaines taut*, Math. Ann. 257 (1981), no. 2, 171-184.

[Ki] Kiselman C.O., *Sur la définition de l'opérateur de Monge-Ampère complexe*, Complex Analysis (Toulouse, 1983), pp. 139-150, Lecture Notes in Math. 1094, Springer, Berlin, 1984.

[Kl] Klimek M., *Pluripotential theory*, London Mathematical Society Monographs. New Series, 6. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1991.

[Le] Lelong P, *Discontinuité et annulation de l'opérateur de Monge-Ampère complexe*, P. Lelong-P. Dolbeault-H. Skoda analysis seminar, 1981/1983, pp. 219-224, Lecture Notes in Math. 1028, Springer, Berlin, 1983.

[Li] Lieb I., *Ein Approximationssatz auf streng pseudokonvexen Gebieten*, Math. Ann. 184 (1969) 56-60.

[Ma] Mattila P., *Geometry of sets and measures in Euclidean spaces*, Cambridge Studies in Advanced Mathematics, 44. Cambridge University Press, Cambridge, 1995.

[Me] Mergelyan S.N., *Uniform approximations to functions of a complex variable*, Uspehi Matem. Nauk (N.S.) 7, (1952). no. 2(48), 31-122.

[NP] Nguyễn V.K. & Phạm H.H., *A comparison principle for the complex Monge-Ampère operator in Cegrell's classes and applications*, Trans. Amer. Math. Soc. 361 (2009), 5539-5554.

[N] Nguyễn Q.D., *Approximation of plurisubharmonic functions on bounded domains in* $\mathbb{C}^n$, Michigan Math. J. 54 (2006), no. 3, 697-711.

[NDH] Nguyễn Q.D., Dung N.T. & Hung D.H., *B-regularity of certain domains in* $\mathbb{C}^n$, Ann. Polon. Math. 86 (2005), no. 2, 137-152.

[Ph] Phạm H.H., *Pluripolar sets and the subextension in Cegrell's classes*, Complex Var. Elliptic Equ. 53 (2008), no. 7, 675-684.

[Po1] Poletsky E.A., *Plurisubharmonic functions as solutions of variational problems*, Several complex variables and complex geometry, Part 1 (Santa Cruz, CA, 1989), pp. 163-171, Proc. Sympos. Pure Math., 52, Part 1, Amer. Math. Soc., Providence, RI, 1991.

[Po2] Poletsky E.A., *Sequences of analytic disks*, arXiv:math/9207202v1.

[Po3] Poletsky E.A., *Holomorphic currents*, Indiana Univ. Math. J. 42 (1993), no. 1, 85-144.

[Po4] Poletsky E.A., *Analytic geometry on compacta in* $\mathbb{C}^n$, Math. Z. 222 (1996), 407-424.

[PS] Poletsky E.A. & Sigurdsson R., *Dirichlet problems for plurisubharmonic functions on compact sets*, Math. Z. (to appear).

[Ra] Ransford T., *Potential theory in the complex plane*, London Mathematical Society Student Texts, 28. Cambridge University Press, Cambridge, 1995.

[RS] Reed M. & Simon B., *Methods of modern mathematical physics. I. Functional analysis*, Academic Press, New York-London, 1972.

[Sa] Sadullaev A., *Plurisubharmonic measures and capacities on complex manifolds*, Uspekhi Mat. Nauk, 36 (1981), no. 4 (220), 53-105, 247.

[Sh] Shcherbina N.V., *On fibering into analytic curves of the common boundary of two domains of holomorphy*, Izv. Akad. Nauk SSSR Ser. Mat., 46:5 (1982), 1106-1123.

[Sib] Sibony N., *Une classe de domaines pseudoconvexes*, Duke Math J. 55 (1987), 299-319.

[Siu] Siu Y.T., *Extension of meromorphic maps into Kähler manifolds*, Ann. of Math. (2) 102 (1975), no. 3, 421-462.

[Ste] Stensønes B., *Stein neighborhoods,* Math. Z. 195 (1987), no. 3, 433-436.

[Sto] Stolzenberg G., *A hull with no analytic structure,* J. Math. Mech. 12 (1963) 103-111.

[Wa] Walsh J.B., *Continuity of envelopes of plurisubharmonic functions*, J. Math. Mech., 18 (1968), 143-148.

[We] Wermer J., *Polynomially convex hulls and analyticity,* Ark. Mat. 20 (1982), no. 1, 129-135.

[Wi] Wiklund J., *On subextension on pluriharmonic and plurisubharmonic functions*, Ark. Mat. 44 (2006), 182-190.

[W] Wikström F., *Jensen measures and boundary values of plurisubharmonic functions,* Ark. Mat. 39 (2001), no. 1, 181-200.