Commodities or not commodities?

Portfolio optimization with robust Mean-Variance and Mean-Conditional Value at Risk strategies.

Paul Kushch
870106-0512

Umea University
Department of Economics
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Supervisor: Carl Lönnbark
Abstract
This paper evaluates features of commodities when they are included in the portfolio allocation decision. Two portfolios are created whereof only one includes commodities besides the traditional asset classes as stocks and bonds. Due to the fallacy of multivariate normality assumption the portfolios in this paper are optimized with two strategies, which aim to mitigate the frailty of assuming normality in optimization settings. The first strategy is the Markowitz Mean-Variance optimization algorithm that is robustly estimated by, first, applying the multivariate Student-t distribution and, second, using the Fast Minimum Covariance Determinant estimator and the Orthogonalized Gnanadesikan-Kettenring estimator. The second strategy is the Mean-Conditional Value at Risk optimization algorithm that uses the historical cumulative density function of the losses. These two strategies aim to find the portfolio weights that minimize the portfolios risk.

The results show that including commodities in the portfolio allocation leads to a higher portfolio return and a lower risk. When the portfolios are later on backtested against a constructed Benchmark index consisting of Swedish bonds and stocks it is seen that Mean-Conditional Value at Risk strategy outperforms robustly estimated Mean-Variance strategy. That fact is explicated when the portfolios are compared by Information Ratio and Sortino Ratio.

The results do also show that all of the portfolios are less risky than Benchmark index. Moreover, if the portfolios are backtested against the OMX Stockholm index both strategies show superior results in terms of risk and return.

Keywords: Mean-Variance (M-V) framework, Mean-Conditional Value at Risk (M-CVaR) framework, Value at Risk (VaR) measure, Conditional Value at Risk (CVaR) measure, Student-t distribution, Fast Minimum Covariance Determinant (FMCD) estimator, Orthogonalized Gnanadesikan-Kettenring (OGK) estimator, efficient frontier, portfolio optimization, backtesting, probability density function (pdf), cumulative density function (cdf), Sortino Ratio, Information Ratio, Central Limit Theorem (CLT), coherent risk measure.

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1. Introduction

1.1 Theoretical background

The year was 1952 when Harry Markowitz published an article concerning portfolio optimization called Mean-Variance (M-V) framework. Markowitz argued that if investors are risk-averse and prefer return before risk they should hold a portfolio offering the highest expected return for a given level of risk. He proposed a way to find such a portfolio by analyzing mean-variance structure of the assets returns. Time went by and still, 60 years after, his idea is the central framework in the portfolio allocation theory. The popularity of Markowitz approach takes place in its simplicity – it requires only three parameters. They are the assets expected mean returns, standard deviations and covariances. The parameters are gathered from the historical data and together create an efficient frontier in two-dimensional space. Each point on the frontier represents an optimal portfolio with an amount of risk for a given level of portfolio return. The investors can then tailor the portfolio allocation to achieve a desired level of risk or return.

The recent years have lead to a huge revision of the portfolio optimization framework. Development in statistical models, computational power and the ease of accessing information on internet resulted in an abundance of theoretically optimal asset allocation strategies. Development has also lead to criticism of the core in the Markowitz allocation decision, namely the covariance matrix. The covariance matrix shows the interaction between the assets and requires only mean and standard deviation of each variable in its computations. Hence, the matrix is highly sensitive to the assumptions about the input variables.

Markowitz M-V model assumes that the assets returns are normally distributed. If they are not, the covariance matrix might be a faulty statistic to base the allocation decision upon. Hence, when normality is not a plausible assumption investors will also care about higher moments of distribution as skewness and kurtosis. But M-V model fails to incorporate these measures (Benson et al., 2008).

1.2 Features of commodities

Disregarding its shortcomings, Markowitz Mean-Variance (M-V) strategy is one of the most frequently used approaches in the financial industry. M-V framework favors the assets with the lowest correlation in absolute terms towards the possible set to choose from. The strategy dictates that such assets are the most desirable in terms of diversification and should be included in the portfolio allocation framework. Traditionally the allocation has been focused on the choice between tree asset classes - stocks, bonds and cash. However, time has lead to a quest of better diversification and improvement of return-risk characteristics by also including other asset classes. One of such a “new” asset classes is commodities. Though, three decades ago Greer (1978) gave commodities attributes of desirable correlation to other asset classes and discussed why they should be included in the portfolio allocation.

From theoretical perspective, the advantage of investing in commodities stem from the fact that they are expected to show small or even negative correlation to other traditional asset
classes such as bonds or stocks. That creates diversification effects which can be explained by the different sources of impact among the asset categories. Erb & Harvey (2006) highlights that commodity prices are driven by a much larger amount of variables (i.e. weather, supply and demand in physical production) compared to the traditional asset classes (i.e. dividend yields and changes in valuation levels).

In the present context the investments in commodities have escalated by the means of today’s financial products. Investors can buy commodities directly by taking the physical delivery, invest in commodity related companies or place their money in commodity- indices and futures. The last two alternatives are also the most common. Futures contracts are liquid exchange-traded financial instruments that obligate the investor to buy or sell a contracted amount of a commodity at a predetermined date and price. Futures contracts can be used for different purposes, whereas the most common and origin purpose is to hedge against price fluctuations in a commodity value. Commodity indices are constructed to track a basket of commodity futures contracts.

The main benefit of investing in a commodity index is that the investor achieves a more diversified portfolio because the index is a basket of futures contracts. From the historical perspective commodity indices had an average annual return of slightly above 10 percent under a 15 year’s period. Although the recent financial crisis did lead to a sharp fall in investments inflow into commodity sectors (in 2011 the estimated value was $15 billion compared to $50 - $60 billion in the three preceding years) mainly due to the recent times highly volatile commodity markets and unusually high correlations to other asset classes, commodities are still seen as a long-term rationale for investing and the inflows are projected to rebound in 2012.

Gorton & Rouwenhorst (2006) investigated the long-term properties of an investment in a fully collateralized commodity futures contracts by constructing an equally weighted commodity index. Authors reached a conclusion that commodity futures demonstrate superior returns compared to an investment in stocks of commodity companies. In addition, they did found that the historical risk of an investment in commodity futures have been small, especially compared to the amount of risk added by stocks and bonds. On its own, a diversified investment in commodity futures has somewhat lower risk than an investment in equities and performs better in periods of unexpected inflation. That can mitigate cyclical variation in returns of stocks and bonds but also create a more diversified portfolio.

Nevertheless, disregarding the features mentioned above, commodities are usually excluded from the portfolio allocation decision. Buttell (2011) gives a possible explanation to that by

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3 Erb & Harvey (2006) explains that the usual way to compare the total return of commodity futures with the return of other assets is to study the performance of a fully collateralized, unlevered, long-only diversified commodity futures portfolio. In example, an investor desiring exposure of $1 to a fully collateralized futures contract will typically go long $1 of a commodity futures contract and at the same time invest $1 in some risk-free asset as U.S. T-bill. Nevertheless, the excess return of commodity futures investment will not include the return of the risk-free asset.
mentioning that even if commodities are negatively correlated with other asset classes that relationship does not hold in the case of extreme events. The recent financial crises lead to an almost perfect correlation between commodities and other asset classes, so diversification benefits did no longer apply. Buttell discusses further that commodities are more difficult to evaluate. Recalling the factorial dependencies mentioned above, commodities risk-premium cannot be estimated as easily as in the case of stocks or bonds. Difficulties also arise due to the huge amount of different ways for commodity investing. Futures are frequently used in trading strategies in order to profit if the price goes in the predicted direction. The strategies may be constructed in different ways, whereas the futures contract can be traded on its own or be involved in more advanced trading technique. Success in such techniques is crucially dependent on the investor’s skills (i.e. commodity derivatives can perform differently than the underlying asset).

Collins (2007) further discusses that commodity indices have become quiet complex instruments. Today’s indices are offering a smorgasbord of products with varying weightings in several sectors. Hence, it can be difficult for an investor to scale down the risk to different risk sources.

Some authors reach a conclusion that exclusion of commodities from portfolio allocation could be due to the fact that commodities show differences in return structure compared with equity returns. In example, Erb & Harvey (2006) discuss that an average commodity futures contract does not have an equity-like return. Historically, the averaged annualized excess return of the average individual commodity futures contract has normally been indistinguishable from zero. In the case of a portfolio of commodity futures the authors argue that such a portfolio can have an equity-like return by benefiting allocation to the commodity futures that are expected to achieve positive returns in the future.

Other studies (e.g. Conover et al., 2010; Bodie & Rosansky, 1980) attribute desirable features to commodities. They are in general discussed to incorporate the property of reducing risk without affecting the portfolio return. Buttell (2011) argues further that commodities also serve as a hedge against inflation by contributing to the portfolio performance when inflation is higher than expected.

Nevertheless, commodities are frequently excluded in portfolio allocation decision and in spite of considerable academic research no define conclusion has been reach about the role of commodities in portfolio settings. The exclusion of commodities could be a result of relatively short histories for major commodity indices, traditions by the means of what can be regarded as an asset class in an investment strategy and the lack of investors experience in the commodity market.

Willenbrock (2011) points out that the main source of diversification in the portfolio context is rebalancing. In order to maintain desired weighting the outperforming assets, that gained weight, are sold and underperforming assets are bought. Hence, rebalancing insures that

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investor keeps a constant risk profile. In comparison, a buy and hold strategy portfolio does not feature diversification effect and risk profile is not held constant because no rebalancing takes place. That is therefore surprising why commodities are not an usual component in portfolio allocation framework, especially when they incorporate the desired properties discussed above.

1.4 Robust covariance estimation
Short-term financial data is often leptokurtic (have fatter tails), skewed or includes other asymmetric properties. Financial time series can display volatility clustering, exhibit long-memory, sometimes non-stationary processes or be depending on other factors (Bade et al., 2009). Welsch & Zhou (2007) highlight that if the researcher simply uses the normality assumption in M-V approach, he must have in mind that the results might be biased because of high estimation errors. It is therefore important to incorporate such features to correctly estimate the covariance matrix. Still, normality is the most commonly used assumption.

The popularity of the assumption rests on several reasons. One of them is that normal density function incorporates attractive analytical properties and a departure from that assumption may lead to considerable unreliability about the statistical conclusion (Liang, 1998). Moreover, when returns are not normally distributed, the estimation of the covariance matrix might be erroneous because of the outlying observations in the data set. Therefore, there are several ways to get a more correct estimate of the covariance matrix.

One commonly used covariance matrix estimation method is to fit the returns to a specific distribution that, among its features, can incorporate fatter tails in the computations of the covariance matrix. One of such commonly used distributions, applied in this paper, is the Student-t distribution.

The other method, instead of assuming a specific distribution for the returns, is to use robust covariance estimators. As well that method takes into account the presence of outliers in the data set. The goal of it is to give larger weights in computations to observations that can be regarded as not outlying, to reduce the impact of outlying observations. Robust covariance estimators used in this paper are the Fast Minimum Covariance Determinant (FMCD) estimator and the Orthogonalized Gnanadesikan-Kettenring (OGK) estimator.

1.5 Mean-Conditional Value at Risk
M-V strategy measures the risk in terms of the standard deviation. In the case of a portfolio of assets M-V optimization problem focuses on the multivariate probability density function (pdf) of the returns and aims to minimize the risk for a given level of return (or maximize return for a given level of risk). If returns are not multivariate normally distributed uncertainty can arise about the true parameters (i.e. the mean and standard deviation for the joint distribution) for the multivariate pdf. The methods used in this paper try to mitigate the issue of fallacy of normality assumption in M-V strategy since the incorrect specification of the parameters can understate or overestimate Value at Risk (VaR) and Conditional Value at Risk (CVaR) measures.
VaR aims to quantify the potential loss in the value of an asset portfolio for a chosen level of confidence. CVaR measures the expected value of the loss given that VaR is exceeded. Rockafellar & Uryasev (2000) and Rockafellar & Uryasev (2002) explain that when the returns are normally distributed CVaR and VaR are the multiples of the standard deviation, so it won’t matter what measure we choose. That is not the case if the returns are not multivariate normally distributed.

Hence, in the case of uncertainty of the multivariate pdf parameters, one could instead focus on the historical cumulative density function (cdf) of the losses - an approach that does not restrict the set of assets to be multivariate normally distributed. Therefore, one departure from the traditional M-V approach and uses instead Mean-Conditional Value at Risk (M-CVaR) strategy. Conclusively, the risk is now measured by CVaR of the historical cdf of the losses instead of the standard deviation.

1.6 Research purpose
The purpose of this paper is to investigate if inclusion of commodities in portfolio allocation framework can give superior results in terms of diversification and return. The question is answered by optimization of two portfolios with two different optimization techniques. Portfolios are different in the sense that only one portfolio includes commodities, besides the traditional asset classes as stocks and bonds, which are also included in the other portfolio. The optimization techniques are Markowitz Mean-Variance approach and Mean-Conditional Value at Risk strategy, whereof both approaches aim to find the portfolio allocation that minimizes risk. Mean-Variance approach is robustly estimated by, first, fitting the data to the Student-t distribution, second, by estimating the covariance matrix with the Minimum Covariance Determinant estimator and, third, by estimating the covariance matrix with the Orthogonalized Gnanadesikan-Kettenring estimator. Mean-Conditional Value at Risk approach uses instead the historical probability density function of the portfolio losses.

Furthermore, in order to access the performance of each strategy, the optimized portfolios are backtested against a benchmark index. The purpose of backtesting is to investigate which strategy is the most desirable in terms of, among all, risk and return; and if an active investing decision that uses a specific strategy can outperform a passive index investing approach. That question is further investigated by evaluation of the optimized portfolios with Information Ratio and Sortino Ratio.
2. Theoretical review

2.1 Markowitz Mean-Variance model

Harry Markowitz was not the first to reflect on the desirability of diversification. The subject was touched by William Shakespeare in “The Merchant of Venice” and was also discussed by the famous mathematician Daniel Bernoulli. Bernoulli argued that investors with risk-averse preferences can benefit from diversification, which Bernoulli summarized as “it is advisable to divide goods which are sensitive to some risks into smaller proportions, rather than to risk them all together”. Still, Markowitz was the first to describe the idea of diversification mathematically by showing how an investor can maximize the portfolio target return while minimizing the portfolio risk measured by the portfolio variance (Rubinstein, 2002).

The brilliance of Markowitz M-V framework concerns the insight that diversification can reduce risk but not completely eliminate it. The model implies that investors are risk-averse, that is they prefer return before risk. M-V approach considers a portfolio consisting of \( n \) securities. The portfolio variance in given by:

\[
\sigma_p^2 = \sum_{i=1}^{n} x_i^2 \sigma_i^2 + \sum_{i=1}^{n} \sum_{j<i} x_i x_j \text{Cov}(i, j),
\]

where in (2.1) \( x_i^2 \) defines the squared weight of \( i \)th security, \( \sigma_i^2 \) shows the variance of \( i \)th security and \( \text{Cov}(i, j) \) stands for the covariance between securities \( i \) and \( j \). Simaan (1997) shows that equation (2.1) can alternatively be written in the variance-covariance notation. Let \( \hat{R}=(r_1, r_2, ..., r_n)^T \) be the vector of assets returns and \( e=(1, ..., 1)^T \) a vector of ones, so that the vector of portfolio weights \( w=(w_1, w_2, ..., w_n)^T \) sums to 1; namely \( w^T e = 1 \). Weights can be positive or negative, depending on if short sales are allowed or not. The expected returns are shown by the vector \( \mu = E(\hat{R}) \) and the estimated covariance matrix by \( \Sigma = \text{Var}(\hat{R}) \). Therefore the portfolio expected return is \( E(r_p) = w^T \mu \) and the portfolio variance is:

\[
\sigma_p^2 = w^T \Sigma w.
\]

Next, the interest is to find the portfolio weights that minimize the portfolio variance. Recalling equation (2.2) we define the portfolio optimization problem in M-V approach as:

\[
\min \sigma_p^2 = w^T \Sigma w
\]

Subject to

\[
w^T \mu \geq E(r_p)
\]

\[
w^T e = 1.
\]
Expression (2.3) states that we minimize the portfolio variance given the constraints expressed in (2.4) and (2.5). The portfolio target or expected return constraint is expressed by equation (2.4). Equation (2.5) states that the available capital is fully invested (Huang & Litzenberger, 1988).

Solving equation (2.3) with respect to (2.4) and (2.5) gives portfolio weights that minimize portfolio variance. Generally, by varying the portfolio weights, the model treats any portfolio as a point in two-dimensional space \{\sigma, r\}. Huang & Litzenberger (1988) and Vanini & Vignola (2001) show that Markowitz portfolio model has a unique solution:

\[ w^* = \mu w_0^* + w_1^* \]  

(2.6)

where

\[ w_0^* = \frac{1}{\Delta} (B \Sigma^{-1} \mu - C \Sigma^{-1} \mathbf{1}) \]

\[ w_1^* = \frac{1}{\Delta} (C \Sigma^{-1} \mu - A \Sigma^{-1} \mathbf{1}) \]

\[ \Delta = AB - C^2 \]

and

\[ A = \mu^T \Sigma^{-1} \mu \]

\[ B = \mathbf{1}^T \Sigma^{-1} \mathbf{1} \]

\[ C = \mathbf{1}^T \Sigma^{-1} \mu^T \]

The corresponding standard deviation for the portfolio in equation (2.6) with weights \(w^*\) is

\[ \sigma_p = \sqrt{\frac{1}{\Delta} (\mu B - 2 \mu C + A)} \]

(2.7)

and the expected portfolio return is

\[ w^T \mu = E(r_p). \]

(2.8)

If the parameters in equations (2.7) and (2.8) are varied and the solutions are plotted in the same graph, they form a hyperbola in the two-space \(\{\sigma, r\}\). The hyperbolas equation is:

\[ \frac{\sigma^2}{1/B} - \frac{(\mu - C/B)}{\Delta/B^2} = 1. \]

(2.9)

Equation (2.9) has an apex, the separation point, with the equation:

\[ (\sqrt{1/B}, \frac{C}{B}). \]

(2.10)

The border above the point in equation (2.10) is called the efficient frontier, and the border below is called the minimum variance locus. The set inside the frontier is the feasible set of

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\( ^6 \) This optimization problem requires a quadratic programming solver. A brief explanation can be found in Appendix D1.
mean/standard deviation portfolios. Any investor holding a portfolio stated on the efficient frontier can keep a constant standard deviation while increasing the portfolio return by moving up to the efficient frontier (Huang & Litzenberger, 1988).

Huang & Litzenberger (1988) also explain that the separation point between the efficient frontier and the minimum variance locus is situated at the minimum variance portfolio. This minimum variance portfolio, the solution to equation (2.3) - (2.5), has the set of weights shown in equation (2.11):

$$w_* = \frac{\Sigma^{-1}1}{1' \Sigma^{-1}1}. \quad (2.11)$$

Different combinations of a risk-free asset (i.e. the risk-free governmental bond) and a risky portfolio form a line with the intercept on the y-axis equal to the risk-free rate. That line is called the capital market line and the point where the line is the tangent to the efficient frontier is known as the optimal risky portfolio (also called tangency portfolio). That portfolio can be shown as the portfolio that maximizes the ratio:

$$\max h(w) = \frac{\mu^T w - r_f}{w' \Sigma w}$$

subject to (2.4) and (2.5). The ratio in (2.12) is called the Sharpe ratio (Sharpe, 1994). Figure 1 summarizes the discussion above.

![Figure 1. Efficient frontier and minimum variance locus form together the hyperbola for the Mean-Variance model. Minimum variance portfolio separates the efficient frontier from minimum variance locus. The plotted line is the capital market line if the risk-free rate is set to 0.7 percent. The dotted line shows Sharpe ratios for different portfolios. The tangency portfolio has the highest Sharpe ratio.](image)

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Weaknesses of Markowitz Mean-Variance model

Sharpe ratio in (2.12) is used for ranking the portfolio performance by sorting the portfolios by their Sharpe value in descending order. The origins of the measure are deeply rooted in Markowitz M-V model. Investors are assumed to only care about the assets expected return and variance. Therefore, optimizing a portfolio using M-V approach has its shortcomings.

While the model offers a great tool for controlling the portfolio exposure to various sources of risk (i.e. the assets volatility or changes in expected return) and is able to quantify large amount of information quickly, the major weakness arises when the assets are not multivariate normally or more generally, as Embrechts et al. (2001) explains, elliptically contoured\(^7\). The authors discuss that empirical studies have clearly showed the inappropriateness of elliptically contoured assumption in the M-V framework. They discuss the fallacy of the correlation measure and hence the covariance matrix in a non-elliptical case. In such a case, investors do also need to consider the shape of distribution measured by skewness and kurtosis (Benson et al., 2008). Ignoring such features and simply using M-V model may lead to results that are inferior to returns of an equally weighted portfolio (Michaud, 1989). Jorion (2007) mentions further that non-normality features might be accounted for by using a symmetric and unimodal distribution\(^8\), such as the Student-t distribution, which belongs to the class of elliptical densities.

Even if skewness and kurtosis are taken into account, the M-V model involves a large number of variables in the computations and suffers from the problem of high estimation errors (Welsch & Zhou, 2007). Konno (2003) highlights that the difficulties in estimation arise when the number of assets increases above a few hundred.

Fisher & Statman (1997) discuss that M-V model does not take in consideration market capitalization weights for the assets. Securities with a low level of capitalization and high expected returns, while having relatively low correlation to other assets, do often attain high weights after optimization. Michaud (1989) tells that such securities are precisely the subject to largest estimation errors, but M-V model does not distinct between levels of uncertainty associated with the models inputs. When the portfolio weights are unrestricted (i.e. short-sales are allowed) it might lead to a problem of high transaction costs if an investor wants to purchase the assets in the market place. Therefore, Fisher & Statman (1997) concludes that M-V model optimization can lead to an allocation with unreasonably large weights in some assets and zero weights in other.

Fisher & Statman (1997) do also highlight that Markowitz model can generate very unstable portfolios with regard to changes in inputs, namely that small variation in the expected returns or covariances will drastically alter the portfolio allocation. At the same time, the investors do

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\(^7\) Riquelme et al. (2009) explains the properties of elliptically contoured distributions.
\(^8\) Unimodal distribution implies that the pdf of the distribution has only one peak. Hence, mode of a continuous pdf is a value at which the pdf has its maximum value. Controversy, a bimodal distribution has two peaks. A symmetric distribution is a distribution that has a skewness parameter near zero, or as in the case of normal distribution skewness is zero. Jorion (2007) explains that unimodal and symmetric distribution gives the familiar bell shaped curve.
not only care about the expected returns and variance - they also care about the intuition behind the investment. If the optimization leads to unintuitive allocation, even if the historical data can be seen as a good estimator for future returns, investors will care about this abnormal allocation and may instead use some behavioral approach for the portfolio choice.

2.2 Value at Risk measure

The term Value at Risk (VaR) appeared for the first time in the financial community at the beginning of 1990’s but the history of VaR goes further back. VaR was already present in the beginning of the 20th century and used in the capital requirements of US security firms. The measure was theoretically discussed in the context of emerging portfolio theory, at a time when the most portfolios consisted solely of equity (Zikovic et al., 2011). In recent years VaR did receive a lot of attention and has become a widely used measure in the calculations of financial risk. VaR is used by financial institutions in order to measure the risks in financial positions and by regulatory committees, such as Basel committee of the Bank for International Settlements, to set marginal requirements. According to the Basel Committee, VaR is applied to determine the capital requirements necessary for the market risk of a financial position\(^9\). Hence, VaR is used to ensure that the financial institution can handle a catastrophic financial event without going bankrupt.

Modern financial markets incorporate several types of risk. The three main categories are credit risk, operational risk and market risk. The consideration of VaR is mainly the market risk. With other words, VaR shows the sensitivity of the financial position to changes in stochastic variables as the equity prices, interest rates or different macroeconomic variables. VaR can therefore be described as the maximal loss of a financial position for a chosen level of confidence. The confidence level is usually set to 99 or 95 percent, meaning that with 99 (or 95) percent confidence the losses should be smaller or equal to VaR (Tsay, 2010).

We can define VaR of a financial position by choosing the confidence level, “the tail probability” \(\alpha\) :

\[
\alpha = \Pr[L(\xi) \geq VaR] = 1 - \Pr[L(\xi) < VaR],
\]

where \(L(\xi)\) in (2.13) is the portfolio loss of a random loss variable of value \(\xi \in R\). Equation (2.13) can be interpreted as the probability that the loss of the position will be equal or greater than VaR. Alternatively VaR is the probability \((1 - \alpha)\) that the loss of the financial position is less then VaR (Tsay, 2010).

VaR is concerned with the upper tail behavior of the cumulative density function (cdf) of \(L(\xi)\), denoted by \(F(\xi)\). For an univariate \(F(\xi)\) and probability \(q\) \((0 < q < 1)\) the quantity

\[
\zeta_q = \inf\{\xi | F(\xi) \geq q\}
\]

\(^9\)The mission of the Bank for International Settlements is to serve central banks in their pursuit of monetary and financial stability, to foster international cooperation in those areas and to act as a bank for central banks (http://www.bis.org/about/index.htm).
is the $q$-th quantile of $F(\zeta)$. Denotation $\inf$ in (2.14) characterizes the smallest real number $\zeta$ that satisfies $F(\zeta) \geq q$. It implies that if the random variable $L(\zeta)$ of $F(\zeta)$ is continuous, then:

$$\begin{align*}
q &= \Pr[L(\zeta) \leq \zeta_q].
\end{align*}$$

(2.15)

Equation (2.15) shows the quantile of $F(\zeta)$ that satisfies the equation. If the $F(\zeta)$ is known, and recalling equation (2.13)-(2.14), VaR is simply the $(1-\alpha)$-th quantile of $F(\zeta)$. That is why VaR is sometimes referred to as the upper $\alpha$-th quantile, since $\alpha$ is the upper tail probability of the loss distribution (Tsay, 2010).

Rockafellar & Uryasev (2000), Rockafellar & Uryasev (2002) and Krokhmal et al. (2001) show that VaR can in general be expressed by considering a portfolio of assets with random returns. Let $f(x, y)$ be the loss function envisioned by the decision vector $x \in X \in \mathbb{R}^n$ representing decision constraints, and a stochastic vector $y \in Y \in \mathbb{R}^m$ incorporating market uncertainties (i.e. the future returns of the assets). Also, denote the cdf of the loss associated with the decision vector $x$ by $\psi(x, \zeta)$, where $\zeta \in R$ is again the value of a random loss variable. In example, the probability of the loss, $f(x, y)$, not exceeding a given value $\zeta$ is given in (2.16):

$$\begin{align*}
\psi(x, \zeta) &= P\{f(x, y) \leq \zeta\}.
\end{align*}$$

(2.16)

Or, equivalently, in integral form:

$$\begin{align*}
\psi(x, \zeta) &= \int_{f(x, y) \leq \zeta} p(y)dy.
\end{align*}$$

(2.17)

In (2.17) $p(y)$ is the pdf of $y$. Thus, for a given confidence level $\alpha$, the $VaR_\alpha$ associated with the chosen vector of portfolio weights $x$, is:

$$\begin{align*}
VaR_\alpha(x) &= \zeta_\alpha(x) = \min\{\zeta \in R : \psi(x, \zeta) \geq \alpha\}.
\end{align*}$$

(2.18)

Applying equation (2.18), the portfolio optimization problem under VaR constraint is then:

$$\begin{align*}
\min VaR_\alpha(x)
\end{align*}$$

(2.19)

subject to (2.4) and (2.5).

**The method for the computation of Value at Risk measure**

Three most widely used methods for the computations of VaR are historical simulation, RiskMetrics (also called variance-covariance method) and Monte Carlo simulations (Teker & Akcay, 2004).

Historical simulation approach, used in this paper, is also called the nonparametric approach. It is a strategy that does not make any specific distributional assumption about the factors affecting the financial position. It implies that the covariance matrix does not need to be estimated (Teker & Akcay, 2004). Instead, one assumes that the distribution of the assets returns is constant over the sample period and that the historical trends will continue into the
future\textsuperscript{10}. Hull & White (1998) discusses that those two assumptions do result in a model that correctly reflects the historical multivariate distribution of the assets, but is less sensitive to shifting market conditions. The approach does moreover rely on the historical data as the predictor for the future observations. Hence, VaR calculations use the returns within the sample period and view them as possible scenarios for the future returns. That is accomplished by ranking the historical returns for each day in the sample by their size, in ascending order. That results in an empirical distribution (i.e. the loss distribution of the returns), which is in the next step used for VaR (and CVaR) calculations\textsuperscript{11}.

**Weaknesses of Value at Risk measure**

Value at Risk can be thought of as the best of worst case scenario. Zikovic et al. (2011) discuss that the biggest shortcoming of VaR is that it underestimates the potential portfolio loss with a specified level of confidence. A portfolio that earns small returns with a high level of probability and suffers substantial amount of loss with a very small level of probability will have a negative VaR value. This anomaly is due to the fact that the chance of accruing a loss is smaller than the chosen confidence level. For such kind of portfolios VaR is unable to disclose any risk.

Kaplanski & Kroll (2002) discuss that VaR is a good measure if the distribution of returns is normal. If it is not, Artzner et al. (1999) state that quantile-based VaR is not a coherent risk measure, meaning that VaR does not satisfy the sub-additivity property given in Table 1. A risk measure that satisfies the four properties in Table 1 is said to be coherent:

<table>
<thead>
<tr>
<th>Property</th>
<th>Expression</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Monotonicity</td>
<td>$W_1 \leq W_2 \rightarrow p(W_1) \geq p(W_2)$</td>
<td>If portfolio 1 always accrue lower returns than portfolio 2, the risk of the first portfolio must be greater.</td>
</tr>
<tr>
<td>Translation invariance</td>
<td>$p(W + c) = p(W) - c$</td>
<td>Adding an amount of cash ($c$) to a portfolio should reduce its risk by $c$.</td>
</tr>
<tr>
<td>Homogenity</td>
<td>$p(kW) = kp(W)$</td>
<td>Increasing the size of a portfolio by $k$ should scale its risk by the same factor.</td>
</tr>
<tr>
<td>Sub-additivity</td>
<td>$p(W_1 + W_2) \leq p(W_1) + p(W_2)$</td>
<td>Merging portfolios cannot increase risk.</td>
</tr>
</tbody>
</table>

*Table 1. Properties of a coherent risk measure. Symbol $W$ denotes the portfolio return. $p(W)$ denotes the risk of a portfolio.*

Failure of satisfying the sub-additivity property might lead to a VaR measure that will be greater than the sum of VaR measures for the single unities, when a number of assets or portfolios are merged. It is not intuitive since the diversification effect should reduce risk, not

\textsuperscript{10} http://www.oenb.at/en/img/value_at_risk_e_tcm16-11203.pdf

\textsuperscript{11} The historical simulation approach is further on used for calculations of Conditional Value at Risk measure. The optimization problem for the historical approach is discussed in chapter 2.3 and in Appendix D2. RiskMetrics approach is discussed in Tsay (2010) and Monte Carlo simulations approach is discussed in Jorion (2007).
increase it. Zikovic et al. (2011) argue that if regulators (i.e. the Basel committee) use non-sub-additive risk measures to set capital requirements, a financial institution would be tempted to break up its financial accounts to decrease the regulatory capital requirements. The sum of the capital requirements of the smaller units will simply be less than the overall requirement for the company as a whole. Generally, all the participants of the financial society (i.e. members of organized exchange or hedge funds) would as well prefer to reduce their requirements by using the same strategy. It is a matter of serious concern since the combined risk of the accounts will not cover the risk comprising the overall account.

Further on, VaR does not completely describe the upper tail behavior of the loss function if the returns are not normally distributed. However, while assuming non-normality of returns Jorion (2007) states that if the portfolio consists of a large amount of securities, as the aggregated portfolios at the highest level of a financial institution, then it can benefit from the Central Limit Theorem (CLT). CLT states that the sum of independent random variables tends to converge to a normal distribution.12

Moreover, non-normality of the returns might also lead to a realization of different losses for the assets or portfolios that have the same VaR value, if VaR is exceeded. The losses will also be greater than VaR. Hence, in order to get a more correct estimate of the potential loss if VaR is exceeded one must take into consideration the expected value of the loss function beyond VaR (Krause, 2003). A measure that does it is Conditional Value at Risk (CVaR).

### 2.3 Conditional Value at Risk measure

Rockafellar & Uryasev (2000) and Rockafellar & Uryasev (2002) explain that CVaR measures the expected value of the loss given that VaR is exceeded. That is illustrated in Figure 2. If returns are normally (or more generally elliptically) distributed than VaR and CVaR are simply the multiples of the standard deviation. Therefore we don’t face the problem of tail risk. For example, at the 99 percent level of confidence, VaR is the standard deviation multiplied by 2.33 and CVaR is the standard deviation multiplied by 2.67. If the returns distribution does have fat tails or other aspects that fail to satisfy the normality assumption - the standard deviation and VaR will not correctly describe the distributional properties. VaR will simply fail to capture the downside risk.

---

12 See Greene (2011) for an explanation of the Central Limit Theorem. Merton (1980) does however argue that CLT might not solve the fallacy of non-normality, see page 16.
Rockafellar & Uryasev (2000) and Rockafellar & Uryasev (2002) show that CVaR can be expressed by once again assuming that \( f(x, y) \) is the loss function envisioned by the decision vector \( x \in X \in R^n \) representing decision constraints, and a stochastic vector \( y \in Y \in R^m \) incorporating market uncertainties. Recall that \( \psi(x, \zeta) = P[f(x, y) \leq \zeta] \) shows the cdf of the loss associated with the decision vector \( x \) up to a predetermined value of the random loss variable \( \zeta \in R \). Assume further that \( \psi_a(x, \zeta) \) is the cdf of the loss for a chosen level of the tail probability \( \alpha \). It is zero for the loss below VaR, and equals \([\psi(x, y) - \alpha]/[1 - \alpha]\) for losses equal to or exceeding VaR. In summary:

\[
\psi_a(x, \zeta) = P[f(x, y) \leq \zeta] = \begin{cases} 
0, & \zeta < \zeta_a(x) \\
[\psi(x, \zeta) - \alpha]/[1 - \alpha], & \zeta \geq \zeta_a(x)
\end{cases}
\] (2.20)

Equation (2.20) shows that CVaR is the mean of \( \alpha \)-tail cdf of the loss. CVaR is equal to:

\[
CVaR = \lambda VaR + (1 - \lambda) CVaR^+.
\] (2.21)

Equation (2.21) states that \( \lambda = [\psi(x, y) - \alpha]/[1 - \alpha] \), a value between \( 0 \leq \lambda \leq 1 \). \( CVaR^+ \), usually called “upper CVaR” or “Expected Shortfall”, denotes the expected loss strictly exceeding VaR. If the distribution of returns is continuous CVaR usually coincides with \( CVaR^+ \).

Rockafellar & Uryasev (2000) show that by using the results above and recalling the probability of the loss not exceeding a given value \( \zeta \) given in equation (2.16) or (2.17), along with the definition of \( VaR_a(x) \) associated with the chosen vector of portfolio weights \( x \) in equation (2.18), \( CVaR_a(x) \) can be defined as:

\[
CVaR_a(x) = \frac{1}{1 - \alpha} \int_{f(x,y) \geq \zeta_a(x)} f(x, y) p(y)dy
\] (2.22)

Using equation (2.22) we can express the problem of portfolio optimization using Mean-Conditional Value at Risk (M-CVaR) strategy as:

\[
\min CVaR_a(x)
\] (2.23)

subject to the constraints in (2.4) and (2.5).

Generally, minimizing \( CVaR_a \) and \( VaR_a \) are not equivalent. The definition of \( CVaR_a \) involves the function of \( VaR_a \) which makes it difficult to optimize the function of \( CVaR_a \). Instead, one considers the following function:

\[
F_a(x, \zeta) = \zeta + (1 - \alpha)^{-1} \int_{y \in R^m} [f(x, y) - \zeta]^+ p(y)dy.
\] (2.24)

In (2.24) \([f(x, y) - \zeta]^+ = \max\{[f(x, y) - \zeta], 0\} \). (2.24) is a function of \( \zeta \) that incorporates the following useful properties for computations of \( CVaR_a \) and \( VaR_a \):

- \( F_a(x, \zeta) \) is a convex function of \( \zeta \).
- $VaR_\alpha(x)$ minimizes $F_\alpha(x, \zeta)$.
- The minimum value of the function $F_\alpha(x, \zeta)$ is $CVaR_\alpha(x)$.

Hence, $CVaR_\alpha$ can be optimized by minimizing the function $F_\alpha(x, \zeta)$ with respect to the weights $x$ and VaR value $\zeta$.$^{13}$

*Figure 3* illustrates the positioning of the minimum-variance portfolio, minimum VaR portfolio and minimum-Expected Shortfall portfolio (if the distribution of returns is continuous CVaR usually coincides with Expected Shortfall) on the efficient mean-variance frontier. As seen in the figure, the set of efficient portfolios is reduced with respect to standard deviation.

![Figure 3. Mean-variance efficient frontier showing minimum-variance portfolio minimum VaR portfolio and minimum Expected Shortfall portfolio (De Giorgi, 2002).](image)

**Weaknesses of Conditional Value at Risk measure**

The common practice today is to use VaR to establish the marginal capital requirements. The biggest weakness of CVaR is hence that it is not widely accepted among regulators (Acerbi & Scandolo, 2008).

Moreover, CVaR is a sub-additive measure that gives a more realistic assessment of the loss, a property that financial institutions do not enjoy because they face larger restriction on marginal requirements. Even if a more realistic measure of the losses would make their business safer in case of a catastrophic event (Zikovic et al., 2011).

---

$^{13}$ It is often not possible to compute the joint density function $p(y)$. Instead one uses a number of scenarios that might represent historical values of the returns. Hence, the function in (2.33) needs to be adjusted (i.e. linearized) to then be optimized with a linear programming solver (Wuertz et al., 2009). See Appendix D2 for an explanation.
The choice of CVaR before VaR is however not a clear decision. Whether VaR is valid as a risk measure depends on the aspects relevant to the manager. A single risk measure cannot cover the whole spectrum of aspects and the risk manager must personally choose the most relevant of them. Therefore, one should not directly conclude that VaR is a bad measure only because it is not sub-additive since that aspect might be irrelevant for a risk manager. Even if in the most of the cases the manager does need to consider the property of sub-additivity because the tail risk is a concern about institutional insolvency caused by unfavorable market conditions (Y. Yamai & Yoshina, 2002).

There are also other approaches that aim to mitigate the fallacy of normality assumption besides M-CVaR strategy. Some of them use specific distributional assumptions about the distribution of returns, other intend to estimate the covariance matrix robustly.

2.4 Robust covariance estimation
Dispersion matrices (i.e. covariance matrices) are very important measures in methods of multivariate statistics and are the central part in different statistical frameworks. They are on their own quantities of interest because they measure the interdependency between several statistics (Ma & Genton, 2001). Ma & Genton (2001) and Maronna & Zamar (2002) explains that a covariance matrix can be regarded as an ellipsoid of the data cloud in multidimensional space, an ellipsoid that is very sensitive to outlying observations. Still, the majority of portfolio optimization approaches rely on rather simple assumption of normally of the assets returns distribution. As explained earlier, one could argue that in the long run, for a long investments horizon, CLT could solve those problems (Bade et al., 2009). Merton (1980) did however show that increasing the sampling frequency won’t generally increase the accuracy of estimation of expected returns. And decreasing the frequency will lead to the loss of relevant information about variances and covariances.

There are a number of different approaches to makes the results of M-V strategy more plausible. One such approach is to fit the assets returns to a specific multivariate pdf. Jorion (2007) explains that one of such commonly used distributions is the Student-t distribution.

In the multivariate case, Hu & Kercheval (2008) explains the Student-t distribution by assuming that \( \mu \in \mathbb{R}^d \), \( \Sigma \in \mathbb{R}^{d \times d} \) (a real positive semi-definite matrix) and that the degrees of freedom is larger than 2 (\( n > 2 \)). The \( d \) dimensional jointly Student-t distributed random vector \( X \) is denoted by:

\[ X \sim \text{Student} - t_d(n, \mu, \Sigma). \quad (2.25) \]

The distribution of \( X \) in (2.25) is given by:

\[ X = \mu + \sqrt{WZ}. \quad (2.26) \]

---

\(^{14}\) For a discussion about what the degrees of freedom are, see Pandey & Bright (2008).
In (2.26) \( Z \sim N(0, \Sigma) \) is the multivariate normal distribution with mean 0 and covariance matrix \( \Sigma \) and \( W \sim IG(n/2, n/2) \) is the inverse gamma distribution\(^{15}\) which is independent of \( Z \).

Now let \( X \in \mathbb{R}^{d \times d} \) be jointly Student-\( t \) distributed. The joint pdf of \( X \) is then denoted as:

\[
f(x|n, \mu, \Sigma) = \frac{\Gamma(n + d/2)}{\Gamma(n/2)(\pi n)^{d/2} |\Sigma|^{1/2}} \left(1 + \frac{p(x)}{n}\right)^{-(n+d)/2},
\]

(2.27)

where, in (2.27), \( p(x) = (x - \mu)^T \Sigma^{-1} (x - \mu) \) and \( n \) is again the shape-defining parameter called the degrees of freedom. When \( n \) is very large the Student-\( t \) distribution approaches pdf of the normal distribution. With other words, for larger values of \( n \) the tails of Student-\( t \) distribution become smaller. Jorion (2007) explains that the Student-\( t \) distribution belongs to the class of elliptical densities because its probability density functions is symmetric and unimodal.

The mean and the covariance of the Student-\( t \) distributed random vector \( X \) are:

\[
E(X) = \mu
\]

\[
COV(X) = \frac{n}{n-2} \Sigma.
\]

Therefore, to provide a degree of robustness to outliers in the M-V approach the data is fitted to Student-\( t \) distribution which uses the mean and covariance measures above.

Even if the data is fitted to a distribution that can incorporate the effect of outliers, the outliers in a multivariate data set might not be as easily observable as in the lower dimensional settings. The traditional approach to find the outliers, which relies on the normality of the data, is called Mahalanobis distances. It aims to detect outliers in a multivariate case by computing the distance from each observation to a center of the data by taking into account the shape of the data. Rousseeuw & van Zomeren (1990) define Mahalanobis distances as the distance from a vector \( x \in \mathbb{R}^p \) to a location vector \( \mu \in \mathbb{R}^p \) (i.e. the mean of each variable) given a symmetric and positive definite covariance matrix \( \Sigma \) as:

\[
D_{M_i} = \sqrt{(x - \mu)^T \Sigma^{-1} (x - \mu)}.
\]

(2.28)

Fauconnier & Haesbroeck (2009) explain that the method in (2.28) defines outliers as any observation with a squared Mahalonobis distance lying above a predefined quantile of the \( \chi^2 \) distribution with \( n \) degrees of freedom\(^{16}\). The use of \( \chi^2 \) distribution is due to the assumption of normality, namely when \( X \sim N_p(\mu, \Sigma) \) the Mahalanobis squared distance is distributed as a \( \chi^2 \) random variable with \( n \) degrees of freedom. Hence, in order to get a reliable estimate of the distance the approach in (2.28) relies on correct estimates of the location vector \( \mu \) and covariance matrix \( \Sigma \). If they are not correct, because of the influence of outlying observations, it seems reasonable to replace \( \mu \) and \( \Sigma \) in equation (2.28) by their robust counterparts

\(^{15}\)For an explanation of the inverse gamma distribution see Hu & Kercheval (2008).

\(^{16}\)\( \chi^2 \) (Chi-squared) distribution is explained in Greene (2011).
Two approaches for robust estimation of $\mu$ and $\Sigma$ are discussed below.

**Minimum Covariance Determinant Estimator**

Minimum Covariance Determinant (MCD) estimator is closely related to the traditionally used covariance matrix but it does not use the outlying observations in its computation. For a sample of $n$ points one aims to find a subsample of $h$ observations, such that $n/2 < h < n$, with the minimum possible determinant of the covariance matrix. The mean of that subsample is then used as $\mu$ in (2.28) and the covariance matrix of the subsample as $\Sigma$ in (2.28) (Fauconnier & Haesbroeck, 2009).

In geometrical terms, the chosen covariance matrix specifies an ellipsoid with minimum volume encompassing some percentage of the subset of $n$ points. In spite of its advantages (i.e. deleting the outliers from the computation of the covariance matrix) MCD estimator is rarely applied. In the case of a multidimensional data set the MCD algorithm is very inefficient since it needs to evaluate a sharply increasing number of possible subsets of data. Therefore, Rousseeuw & Driessen (1999) did construct an algorithm that is much faster and can deal with a sample size of tens of thousands. The algorithm is called Fast Minimum Covariance Determinant (FMCD) estimator.

The authors explain that the first step in the computations of FMCD is to choose a subset of $h$ observations of a total of $n$ by iteration. Next, the mean vector and the covariance matrix for each subset are computed. If determinant for each specific covariance matrix is not equal to zero the next step is to compute the Mahalanobis distances for each observation in the subset. After, one takes $h$ smallest distances and creates a new subset in order to calculate the covariance matrix for this new subset. Then, the above steps are repeated a number of times to find and choose the covariance matrix with the smallest determinant.

**Orthogonalized Gnanadesikan-Kettenring estimator**

FMCD estimator has its highest breakdown value when the number of outlying observations is $h = [(n + p + 1)/2]$ (Rousseeuw & Driessen, 1999). In the case of high dimensional data set, which may contain a substantial number of outlying observations, one might need a very large amount of observations. As mentioned above, Rousseeuw & Driessen (1999) solved that problem by FMCD estimator which seems to give a good solution without requiring large number of observations. Nevertheless, FMCD estimator still requires a substantial computational time for large dimensions (Maronna & Zamar, 2002).

Gnanadesikan & Kettenring (1972) proposed an estimator that was later on used as the central assumption in Orthogonalized Gnanadesikan-Kettenring (OGK) estimator. Maronna &

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17 For a visualization, see http://tr8dr.wordpress.com/2010/09/24/minimum-covariance-determination/
18 The breakdown value is of large importance for reliability of an estimator. It indicates the largest proportion of a data set that can be replaced by arbitrary values to restrict the estimator to be contained between the boundaries in parameter space. It is hence the maximum proportion of outlying observations that the estimator can tolerate before giving unreasonable results (Ma & Genton, 2001).
Zamar (2002) explain that it is a very efficient method for large covariance matrices. The estimate defined by Gnanadesikan & Kettenring (1972) is based on the identity:

\[
\text{Cov}(X, Y) = \frac{1}{4} \left( \sigma(X + Y)^2 - \sigma(X - Y)^2 \right)
\]  

(2.29)

In equation (2.29), the authors proposed to substitute the standard deviation for a more robust measure by replacing it by the trimmed standard deviation\(^\text{19}\). (Maronna & Zamar, 2002) base then the OGK estimato other on the observation that the eigenvalues of the resulting covariance matrix are the variances along the directions given by the respective eigenvectors.

Maronna & Zamar (2002) explain that the first step in the computation of OGK estimator is to use the covariance identity in (2.29) for the computations of the covariance matrix. Then, compute the eigenvalues and eigenvectors of the covariance matrix. Next, use matrices \(E\) (which columns are the eigenvectors of the variance-covariance matrix), and \(\Lambda\) (which is a matrix with eigenvalues corresponding to each eigenvector on diagonal and zeros elsewhere). Subsequently, compute the eigenvector decomposition:

\[
\Sigma = E\Lambda E^T
\]

Then, project the data onto the basis eigenvectors and estimate the variances in the direction of coordinates. The OGK estimator is then given by:

\[
\Sigma_{OGK} = E\Gamma E^T.
\]  

(2.30)

\(\Gamma\) in (2.30) is a diagonal matrix of the variances (i.e. eigenvalues) in the direction of eigenvectors, after projection onto the basis.

2.5 Portfolio performance measures

Quantitative risk management plays a key role in quantitative finance. The fund managers are typically expected to outperform target benchmark indices by active investment strategies, with an acceptable level of risk. A manager’s primary objective is hence to provide quality control of the investment above a required minimum acceptable return. It is therefore necessary to be able to compare the manager’s performance across the fund industry, for which task there exists a large number of different comparison measures (Chaudhry & Johnson, 2008). A common practice is to measure the risks and rewards of an investment portfolio in computable terms in order to verify the performance of the managed portfolio (Chen et al., 2011).

The use of Capital Asset Pricing Model (CAPM) or the related multi-factor models may be inappropriate because they don’t explicitly account for the manager’s performance relative to a benchmark index. Therefore, there exist a number of measures that are able to quantify the manager’s performance. The most well-known measure is the Sharpe ratio described in equation (2.12) (Chaudhry & Johnson, 2008). The Sharpe ratio is a measure of efficiency, in terms of reward, given the taken risk. Hence, a higher Sharpe ratio is desirable but the

\(^{19}\)For example, 90% trimmed standard deviation is the standard deviation of the values after truncating (i.e. cutting off) the highest and lowest 5% of the values.
calculations of the measure depend critically on the reliability of the mean and variance measures. Hence, the fallacy of simply assuming normality can lead to questionable correctness of the Sharpe ratio. For this reason there exist a number of different measures developed because of the increased focus on the managerial performance in the recent years.

The main trend has been to criticize the normality assumption and instead use the measures that are less dependent on that assumption. Markowitz proposed a semi-variance measure called Sortino Ratio which focuses only on the negative returns since he argued that the investors do care more about losing money than making them. Semi-variance is therefore discussed to be more consistent with the investor’s preferences about risk (Chen et al., 2011).

**Information Ratio**

Goodwin (1998) explains that the Information ratio is based on Markowitz mean-variance paradigm that takes mean and variance as the only measures needed to characterize an investment portfolio. The measure aims to describe the properties of an active managed portfolio by a single number. Let \( R_t \) be the return of an actively managed portfolio at period \( t \) and \( B_t \) be the return of a benchmark index at period \( t \). Also let \( ER_t \) be the excess return by taking the difference between the portfolio and benchmark, namely \( ER_t = R_t - B_t \). Calculate the standard deviation of the excess return (also called the “tracking error”) and denote it by \( \sigma_{ER} \). Then, the Information Ratio is equal to:

\[
IR = \frac{ER}{\sigma_{ER}}
\]  

(2.31)

The ratio in (2.31) measures the ability of the manager to generate excess returns relative to a benchmark index and at the same time attain the consistency of the manager. Thus, a high ratio can be achieved by a large excess return (which is also the case if the benchmarks return is very low) or a low tracking error.

**Sortino Ratio**

Sortino Ratio is a modification of Sharpe Ratio which measures the excess return of an asset over a unit of risk in an asset. Sharpe Ratio is given in equation (2.12), but can also be written as:

\[
Sharpe = \frac{R_t - R_f}{\sigma_R}
\]  

(2.32)

In (2.32) \( R_t \) is the return of an actively managed portfolio at period \( t \), \( R_f \) is the risk-free rate of return and \( \sigma_R \) is the standard deviation of return. The Sharpe Ratio think of the risk as both positive and negative returns, while Sortino Ratio only considers negative returns in the risk calculations. To calculate Sortino Ratio we must first calculate the semivariance, given in equation (2.33) and then the actual Sortino Ratio, given in (2.34):
\[ Semi\ variance = \frac{1}{n_s} \sum_{R<R_{MAR}}^\infty (R - R_{MAR})^2 \]  \hspace{1cm} (2.33)

\[ Sortino\ ratio = \frac{R_t - R_{MAR}}{\sqrt{Semi\ variance}} \]  \hspace{1cm} (2.34)

In equation (2.33) and (2.34) \( R_{MAR} \) is the minimum acceptable return and \( n_s \) is the total number of returns when the return is below the minimum acceptable return (Chaudhry & Johnson, 2008). \( R_{MAR} \) is set to 0 percent, which is a common practice in the calculations of the Sortino ratio and is the value that Chaudhry & Johnson (2008) set in their paper.
3. Data

3.1 Variables
The data sample consists of 7 variables\textsuperscript{20}. Those are MSCI.EAFE, MSCI.USA, General.SIX, BarCap.US, SV.Obl.Index, ACCI and a constructed Benchmark index. Each variable has 430 monthly observations starting from 1976-01-30 and ending at 2011-10-31.

\textit{MSCI.EAFE} index is a free-float adjusted\textsuperscript{21} market capitalization index that is designed to measure the equity market performance of developed markets, excluding USA and Canada. MSCI.EAFE Index includes 22 developed country indices: Australia, Austria, Belgium, Denmark, Finland, France, Germany, Greece, Hong Kong, Ireland, Israel, Italy, Japan, the Netherlands, New Zealand, Norway, Portugal, Singapore, Spain, Sweden, Switzerland, and the United Kingdom.

\textit{MSCI.USA} index is a free float adjusted market capitalization index that is designed to measure large cap and mid cap US equity market performance\textsuperscript{22}.

\textit{General.SIX} is an index for Swedish stock returns. From year 1919 to year 2005 this index is computed by Frennberg et al. (1992). From year 2006 General SIX is the SIX RX Index. SIX RX is the new general index.

\textit{BarCap.US} denotes the Barclays Capital US Aggregate Bond Index which offers exposure to USD denominated investment grade fixed rate bonds including Treasuries, government-related, securitized and corporate securities. Only bonds with a minimum remaining time to maturity of one year are included in this index\textsuperscript{23}.

\textit{SV.Obl.Index} is the Swedish bond index based on monthly holding period return, calculated as the sum of the monthly percentage price change on the Stockholm Stock Exchange. Namely the quoted price of Swedish long-term government bonds and a twelfth of its coupon rate, except for the period after 1986 when the yield is treated as coming from a zero coupon bond (Frennberg et al., 1992)\textsuperscript{24}.

\textit{ACCI} is an abbreviation for Allba Commodity Composite Index constructed for this paper by Allba Asset Management. ACCI includes five different total return (it is assumed that any cash distributions, such as dividends, are reinvested back into the index\textsuperscript{25} commodity indices. \textit{First} is the Standard & Poor’s global share commodity index (SPGCCITR) calculated primarily on a world production weighted basis and comprised of the principal physical commodities that are the subject of active, liquid futures markets. The weight of each commodity in the index is determined by the average quantity of production as per the last five years of available data. The production weights are designed to reflect the relative significance of each of the constituent commodities in the world economy while preserving

\textsuperscript{20} The variables were chosen by consultation with Allba Asset Management.
\textsuperscript{21} Shares not available in the marketplace are excluded.
\textsuperscript{22} http://www.msci.com/products/indices
\textsuperscript{23} http://uk.ishares.com/en/rc/products/SUAG
\textsuperscript{24} The data for General SIX index and Swedish bond index is taken from www.riksbank.se
\textsuperscript{25} http://www.investopedia.com/terms/t/total_return_index.asp#axzz1oRo1VDIV
the tradability of the index. Second is the Dow Jones-UBS Commodity Index (DJUBSTR) which reflects the returns that are potentially available through an unleveraged investment in the futures contracts on physical commodities comprising the index plus the rate of interest that could be earned on cash collateral invested in specified Treasury Bills. Third is the Thomson Reuters/Jefferies Crb Index (CRYTR) which maintains broad diversification through 19 commodities representing all commodity sectors. Commodities are equitably distributed whenever feasible, though exposure to selected markets, in particular those within the petroleum sector, are modified to create a liquid and rational index. Forth is the Summerhaven Dynamic Commodity Index (SDCITR) which is comprised of 14 Futures Contracts that are selected on a monthly basis from a list of 27 possible futures contracts. The index is rules-based and rebalanced monthly based on observable price signals. In this context, the term rules based is meant to indicate that the composition of the index in any given month will be determined by quantitative formulas relating to the prices of the futures contracts that relate to the commodities which are suitable to be included in the index. The fifth and the last index is the UBS Bloomberg Constant Maturity Commodity Index (CMCITR). That index is the first benchmark commodity index to diversify across both commodities and maturities. The index measures the collateralized returns from a basket of 26 commodity futures contracts representing the energy, precious metals, industrial metals, agricultural and livestock sectors. In addition, the commodity futures contracts are diversified across five constant maturities from three months up to three years.

ACCI has the starting value of 100 by the date 1970-01-30 and a value of 375.2 at 1976-01-30 (the sample starting date). ACCI is constructed by first transforming the above five variables into simple returns. In the next step the mean of the indices returns present at a specific point in time are calculated with start from the sample starting date. It means that the indices contained in ACCI have different starting dates which imply that if for example only one variable is present at a specific date – only its return will be used for the calculation of ACCI. If several variables are present then the mean of the variables returns will be used. For example, at the start of the data sample 1976-01-30, only Standard & Poor’s global share commodity index is present. Dow Jones-UBS Commodity Index and Summerhaven Dynamic Commodity Index enter the calculation at date 1991-01-31. Thomson Reuters/Jefferies Crb Index starts at 1994-01-31. UBS Bloomberg Constant Maturity Commodity Index starts at 1997-10-31. The discussion is summarized in Table 1.

---


Table 1. The composition of Allba Composite Commodity Index (ACCI)

<table>
<thead>
<tr>
<th></th>
<th>SPGCCITR</th>
<th>DJUBSTR</th>
<th>CRYTR</th>
<th>SDCITR</th>
<th>CMCITR</th>
</tr>
</thead>
<tbody>
<tr>
<td>Jan 76 to Dec 90</td>
<td>100 %</td>
<td>--</td>
<td>--</td>
<td>--</td>
<td>--</td>
</tr>
<tr>
<td>Jan 91 to Dec 93</td>
<td>33.3 %</td>
<td>33.3 %</td>
<td>--</td>
<td>33.3 %</td>
<td>--</td>
</tr>
<tr>
<td>Jan 94 to Sep 97</td>
<td>25 %</td>
<td>25 %</td>
<td>25 %</td>
<td>25 %</td>
<td>--</td>
</tr>
<tr>
<td>Oct 97 to Oct 11</td>
<td>20 %</td>
<td>20 %</td>
<td>20 %</td>
<td>20 %</td>
<td>20 %</td>
</tr>
</tbody>
</table>

Benchmark index is constructed of an equally weighted portfolio consisting of General.SIX and SV.Obl.Index assets. It is done in order to create an equally weighted exposure index that is sensitive to the changes in Swedish stock index value and Swedish bond index value.

This and the next two chapters apply Open-Source statistical software R. The packages used in R are in example fPortfolio (Wurtz et al., 2009), stats, pastecs, quantmod, PerformanceAnalytics, ggplot2, robustbase.

3.2 Descriptive statistics

In studies of time-series data a common approach is to convert the data into return series. I use simple (discrete) returns because the data is collected on the monthly basis which implies that the logarithmic returns would not be a good approximation to discrete returns. Using simple returns is also a common praxis in the fund industry.

The first observation in the data sample is at date 1976-01-30, but in order to exclude NA’s (not available) values after converting the data into simple returns, I exclude the first observation (1976-01-30). Hence, the data sample of the simple returns will start at the next date (1976-02-27).

The plot of returns distributions in Figure 4 shows the returns for the six variables and Benchmark index. Appendix A1 shows the more traditional plots of the returns distributions for the variables and Benchmark index. As seen in the plots, all of the variables have been quiet volatile in the sample period, where SV.Obl.Index and BarCap.Us are the least volatile. Hence, Figure 4 implies that many of the variables do have rather similar returns structures and namely the risk in terms of volatility. The other way to investigate the riskiness of the assets is to perform K-means clustering, which is an algorithm that attempts to find groups in the data, in order to group the assets in different risk categories (different groups) (Kanungo et al., 2002). K-means clustering takes a given set of \( n \) data points for which one tries to determine a set of \( k \) points (\( k \leq n \)) called clusters, so as to minimize the mean squared distance from each data point to its nearest center:

\[
\min_{\{\mu_1, \ldots, \mu_k\}} \sum_{h=1}^{k} \sum_{x \in S_h} \|x - \mu_h\|^2
\]

(3.1)

31 Simple return is calculated by \( R_{st} = P_t / P_{t-1} - 1 \), where \( R_{st} \) is the simple return at period \( t \) and \( P_t \) is the price of an asset at time \( t \) divided by the price of a previous period \( t-1 \), \( P_{t-1} \). A good discussion about when to use simple or log returns can be found at http://www.r-bloggers.com/a-tale-of-two-returns/
There exist several algorithms to find a locally minimum solution to the problem in (3.1). One such algorithm is the Lloyd’s algorithm. It is based on an iterative scheme for finding a locally minimum solution. Applying Lloyd’s algorithm to the variables gives the following results:

<table>
<thead>
<tr>
<th>MSCI.EAFE</th>
<th>MSCI.USA</th>
<th>General.SIX</th>
<th>BarCap.US</th>
<th>SV.Obl.Index</th>
<th>Benchmark</th>
<th>ACCI</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
</tr>
</tbody>
</table>

Table 2. The results of k-means clustering with Lloyd's algorithm.

Table 2 shows that Lloyd’s algorithm sorts the data in to 2 groups, where the assets in group 2 (ACCI) is the higher risk assets. Appendix A2 shows a number of statistics for the distributional properties of the assets. As seen there, the assets have different skewness and kurtosis values. As seen in the last two rows in the table in Appendix A2, all of the assets except BarCap.US have negative skewness and all of the variables have positive excess kurtosis. Appendix A3 explicates that fact by

---

32 For an explanation of the Lloyd’s k-means clustering algorithm see Wilkin & Xiuzhen (2008) and Kanungo et al. (2002).
showing the distributional plots of the returns. As shown, the distributions of the returns more or less do not follow a normal distribution. However, it is important to test the hypothesis of the assets multivariate normality.

To test whether the assets are multivariate normally distributed or not I use the multivariate Shapiro-Wilk test. Shapiro-Wilk test gives the following result:

<table>
<thead>
<tr>
<th>Statistic</th>
<th>P Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>W 0.9493</td>
<td>5.813e-11</td>
</tr>
</tbody>
</table>

Figure 5. The results of Shapiro-Wilk test.

The printout from the Shapiro test in Figure 5 shows that the hypothesis (i.e. $H_0$ = the assets follow a multivariate normal distribution) of a multivariate normal distribution is rejected because of a very low p-value. Hence, when the returns are not multivariate normally distributed assuming normality would be inadequate.

To better access the dependencies between the assets the correlations plot is shown in Table 3. ACCI has a low or even slightly negative correlation to other assets. It is also the case for SV.Obl.Index. The MSCI indices are relatively highly correlated with each other as well as with General.SIX. SV.Obl.Index is on the other hand negatively correlated with MSCI indices and ACCI.

<table>
<thead>
<tr>
<th></th>
<th>MSCI.EAFE</th>
<th>MSCI.USA</th>
<th>General.SIX</th>
<th>BarCap.US</th>
<th>SV.Obl.Index</th>
<th>ACCI</th>
<th>Benchmark</th>
</tr>
</thead>
<tbody>
<tr>
<td>MSCI.EAFE</td>
<td>1.0000</td>
<td>0.6233</td>
<td>0.5170</td>
<td>0.1537</td>
<td>-0.0075</td>
<td>0.2799</td>
<td>0.4799</td>
</tr>
<tr>
<td>MSCI.USA</td>
<td>0.6233</td>
<td>1.0000</td>
<td>0.5458</td>
<td>0.2293</td>
<td>-0.0585</td>
<td>0.1856</td>
<td>0.4908</td>
</tr>
<tr>
<td>General.SIX</td>
<td>0.5170</td>
<td>0.5458</td>
<td>1.0000</td>
<td>0.0058</td>
<td>0.0531</td>
<td>0.0496</td>
<td>0.9496</td>
</tr>
<tr>
<td>BarCap.US</td>
<td>0.1537</td>
<td>0.2293</td>
<td>0.0058</td>
<td>1.0000</td>
<td>0.2346</td>
<td>-0.0399</td>
<td>0.0791</td>
</tr>
<tr>
<td>SV.Obl.Index</td>
<td>-0.0076</td>
<td>-0.0585</td>
<td>0.0531</td>
<td>0.2346</td>
<td>1.0000</td>
<td>-0.1521</td>
<td>0.3635</td>
</tr>
<tr>
<td>ACCI</td>
<td>0.2796</td>
<td>0.1856</td>
<td>0.0496</td>
<td>-0.0399</td>
<td>-0.1521</td>
<td>1.0000</td>
<td>-0.0015</td>
</tr>
<tr>
<td>Benchmark</td>
<td>0.4799</td>
<td>0.4908</td>
<td>0.9496</td>
<td>0.0791</td>
<td>0.3635</td>
<td>-0.0015</td>
<td>1.0000</td>
</tr>
</tbody>
</table>

Table 3. The assets correlation structure. The correlation of the asset with its self is shown on the main diagonal. Off diagonal elements show the correlations between the assets.

For a discussion about the properties of Shapiro-Wilk test see Yap & Sim (2011).
4. Portfolio optimization

4.1 Optimization settings
The aim of portfolio optimization in this paper is to find the portfolios with the lowest possible risk. M-V strategy minimum risk portfolio is a portfolio that minimizes the portfolio risk measured by the standard deviation. M-CVaR minimum risk portfolio is the portfolio that minimizes risk measured by CVaR.

A comparison will be made between two portfolios – one with commodities (ACCI) and the other without. Portfolio weights are restricted by long-only constraints, which implies that the minimum allowable weight for an asset in the portfolio is zero and maximum weight is one. In the case of M-CVaR strategy, the VaR confidence level is set to 5 percent.

Symbols for the variables are abbreviated in the tables in this chapter. MS.EA stands for MSCI.EAFE, MS.US for MSCI.USA, Gen.SIX for General.SIX, BC.US for BarCap.US, S.O.O for SV.Obl.Index and ACCI for ACCI.

The portfolio weights in this chapter are in absolute terms. In example, a weight of 0.6406 is actually 64.06 percent of the total portfolio allocation. The measures (i.e. return and risk) in this chapter are in the percentage terms. In example a VaR value of 1.2956 implies a VaR value of 1.2956 percent. Or, a return of 0.7658 implies a return of 0.7658 percent.

4.2 Mean-Variance strategy with Student-t distribution
The optimal weights for M-V optimized portfolio including/excluding commodities (ACCI) are shown in Table 4. As seen in the table, when ACCI is included MSCI.EAFE and MSCI.USA assets are given zero weights in the portfolio allocation. Portfolio without ACCI allocates a weight of 0.0225 to MSCI.USA. The biggest weight is given to BarCap.US asset. This minimum variance portfolio has the mean return of 0.7658 percent and a standard deviation that is 1.0496 percent. Portfolio not including commodities has a slightly lower return, 0.7456, and a larger standard deviation, 1.0979.

<table>
<thead>
<tr>
<th>Portfolio Weights:</th>
<th>Student-t portfolio with commodities</th>
<th>Portfolio Weights:</th>
</tr>
</thead>
<tbody>
<tr>
<td>MS.EA. MS.US. Gen.SIX BC.US S.O.I. ACCI</td>
<td>0.0000 0.0000 0.0393 0.6406 0.2207 0.0994</td>
<td>MS.EA. MS.US. Gen.SIX BC.US S.O.I</td>
</tr>
<tr>
<td>Target Return and Risks:</td>
<td>Mean Sigma CVaR VaR</td>
<td>Mean Sigma CVaR VaR</td>
</tr>
<tr>
<td>0.7658 1.0496 2.2561 1.2956</td>
<td>0.7456 1.0979 2.3274 1.3236</td>
<td></td>
</tr>
</tbody>
</table>

Table 4. The results of portfolio optimization with M-V strategy and Student-t distribution.

Figure 6 shows the weight composition depending on the target risk and target return. The thick vertical black separates the efficient frontier from the minimum variance locus. If the investor wants to increase the portfolio return by simultaneously increasing the portfolio risk
(i.e. moving to the right of the thick line) he needs to constantly increase the portfolio allocation in General.SIX asset. It is also seen in the figure that the allocation along the efficient frontier is concentrated in General.SIX, BarCap.US, SV.Obl.Index, and ACCI when commodities are included.

Figure 6. The weights composition along the minimum variance locus and the efficient frontier.

Figure 7 shows the efficient frontiers with- and without commodities. The two fatter dots show the minimum variance portfolios position on the frontier. The efficient frontier with commodities lies slightly to the left of the frontier without commodities.
4.3 Mean-Conditional Value at Risk strategy
The optimal weights for M-CVaR optimized portfolios including/excluding commodities are shown in Table 5. M-CVaR portfolio including ACCI gives MSCI.EAFE and MSCI.USA assets zero weights in the portfolio allocation. BarCap.US gets the highest weight of 0.5732 in that portfolio. Excluding ACCI gives MSCI.USA a weight of 0.0050. The biggest weight is again given to BarCap.US, 0.6351. This minimum variance portfolio with commodities has a mean return of 0.7821 percent and a standard deviation that is 1.0736. Portfolio not including commodities has a slightly lower return, 0.7690 percent, and a larger standard deviation of 1.1128 percent.

<table>
<thead>
<tr>
<th>Portfolio Weights:</th>
<th>M-CVaR portfolio with commodities</th>
<th>M-CVaR portfolio without commodities</th>
</tr>
</thead>
<tbody>
<tr>
<td>MS.EA. MS.US. Gen.SIX BC.US S.O.I. ACCI</td>
<td>0.0000 0.0000 0.0552 0.5732 0.3071 0.0645</td>
<td>0.0000 0.0050 0.0601 0.6351 0.2998</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Target Return and Risks:</th>
<th>Portfolio Weights:</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>Sigma</td>
</tr>
<tr>
<td>0.7821</td>
<td>1.0736</td>
</tr>
<tr>
<td>0.7690</td>
<td>1.1128</td>
</tr>
</tbody>
</table>

Table 5. The results of portfolio optimization with M-CVaR strategy.

Figure 1 in Appendix B1 shows the weight composition depending on the target risk and target return. The thick vertical black separates the efficient frontier from the minimum variance locus. If the investor wants to increase his return by simultaneously increasing risk, he needs to constantly increase the portfolio allocation in General.SIX asset. Allocation along
the efficient frontier is concentrated in General.SIX, BarCap.US, SV.Obl.Index and ACCI (when commodities are included).

*Figure 2 in Appendix B1* shows the efficient frontiers for the portfolios with- and without commodities. The two fatter dots show the minimum variance portfolios position on the frontier. The efficient frontier with commodities lies slightly to the left of the frontier without commodities. The distance between frontiers grows slowly if we move up the efficient frontier.

### 4.4 Mean-Variance strategy with FMCD estimator

The optimal weights for M-V optimized portfolios are shown in *Table 6*. The portfolio including ACCI gives MSCI.EAFE and MSCI.USA assets zero weights in the portfolio allocation. Excluding ACCI gives MSCI.EAFE a weight of 0.0103. BarCap.US gets the highest weight in both portfolios, 0.7612 in the portfolio with commodities and 0.8151 without commodities. The portfolio without commodities has a mean return of 0.7492 percent and a standard deviation that is 1.1409. Portfolio not including commodities has a slightly lower return, 0.7361, and a larger standard deviation, 1.2021.

<table>
<thead>
<tr>
<th>Portfolio Weights:</th>
<th>Portfolio Weights:</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0000 0.0000 0.0490 0.7612 0.1077 0.0821</td>
<td>0.0103 0.0000 0.0532 0.8151 0.1213</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Target Return and Risks:</th>
<th>Target Return and Risks:</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean Sigma CVaR VaR</td>
<td>Mean Sigma CVaR VaR</td>
</tr>
<tr>
<td>0.7492 1.1409 2.4171 1.3284</td>
<td>0.7361 1.2021 2.4475 1.5087</td>
</tr>
</tbody>
</table>

*Table 6. The results of portfolio optimization with M-V strategy and FMCD estimator.*

*Figure 1 in Appendix B2* shows the weight composition depending on the target risk and target return. The thick vertical black separates the efficient frontier from the minimum variance locus. If the investor wants to increase the portfolio return by simultaneously increasing the portfolio risk, he needs to constantly increase the portfolio allocation in General.SIX asset. Allocation is concentrated in General.SIX, BarCap.US, SV.Obl.Index and ACCI (when commodities are included).

*Figure 2 in Appendix B2* shows FMCD efficient frontiers with- and without commodities. The two fatter dots show the minimum variance portfolios position on the frontier. The efficient frontier with commodities lies slightly to the left of the frontier without commodities.

*Figure 3 in Appendix B2* shows a comparison between efficient frontiers estimated for M-V strategy with Student-t distribution and FMCD estimator. ACCI is included in the figure on the top and excluded in the figure below. As the figures shows the efficient frontier estimated by Student-t distribution lies to the left of FMCD estimated frontier.
4.5 Mean-Variance strategy with OGK estimator

The optimal weights for M-V optimized portfolios including/excluding commodities are shown in Table 7. The portfolio including ACCI gives MSCI.EAFE and MSCI.USA assets zero weights in the portfolio allocation. Excluding ACCI gives MSCI.EAFE a weight of 0.0277. The biggest weight in both portfolios is given to BarCap.US, 0.5939 when commodities are included and 0.6677 excluding commodities. The portfolio with commodities has a mean return of 0.7768 percent and a standard deviation that is 1.1640 percent. Portfolio not including commodities has a slightly lower return, 0.7583, and a larger standard deviation, 1.2687 percent.

<table>
<thead>
<tr>
<th>OGK Portfolio with commodities</th>
<th>OGK Portfolio without commodities</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Portfolio Weights:</strong></td>
<td></td>
</tr>
<tr>
<td>MS.EA.</td>
<td>MS.US.</td>
</tr>
<tr>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td><strong>Target Return and Risks:</strong></td>
<td></td>
</tr>
<tr>
<td>Mean</td>
<td>Sigma</td>
</tr>
<tr>
<td>0.7768</td>
<td>1.1640</td>
</tr>
</tbody>
</table>

Table 7. The results of portfolio optimization with M-V strategy and OGK estimator.

Figure 1 in Appendix B3 shows the weight composition depending on the target risk and target return. The thick vertical black line separates the efficient frontier from the minimum variance locus. If the investor wants to increase his return by simultaneously increasing risk, he needs to constantly increase the portfolio allocation in General.SIX asset. Allocation is concentrated in General.SIX, BarCap.US, SV.Obl.Index, and ACCI (when commodities are included).

Figure 2 in Appendix B3 shows FMCD efficient frontiers with- and without commodities. The two fatter dots show the minimum variance portfolios position on the frontier. The efficient frontier with commodities lies slightly to the left of the frontier without commodities.

Figure 3 in Appendix B3 shows a comparison between efficient frontiers estimated with Student-t distribution and OGK estimator. ACCI is included in the figure on the top and excluded in the figure below. As the figures show the efficient frontier estimated by Student-t distribution lies to the left of OGK estimated frontier. The difference between frontiers is slightly larger when ACCI is excluded.
5. Portfolio backtesting

5.1 Backtesting settings

Backtesting can be seen as one of the most important components in portfolio management and is done in order to evaluate the differences in the results between the statistical models. Backtesting procedure reconstructs the trades that would take place historically. The idea of backtesting is that a strategy, which showed desired results in the past, will also work in the future.

The backtesting period in this paper is starting at date 1998-01-30 and ends at date 2011-10-31. Three years (horizon: 36 month) are used as the data set to solve for the minimum risk portfolios. The rebalancing takes place one month after the portfolios are optimized (startup: 1m) and the portfolios are further rebalanced on a monthly basis (shift: 1m). It means that the the last observation in each 3 years window is substituted on a monthly basis to achieve the weights matrix for the whole backtesting period. The weights constraints are once again long-only.

After the backtesting procedure is completed and the optimal portfolio weights matrix is proposed, the portfolio weights are smoothed in order to lower the weights fluctuations and hence the transaction costs. Namely, the weights matrix is smoothed by an exponentially weighted moving average (EWMA) with a chosen smoothing parameter lambda, $\lambda$. 34 Lambda in this paper is set to 6 month (smoothing: 6m) which is a commonly chosen length. The smoothing is applied on the proposed by optimization weights matrix for the whole backtesting period. Smoothing is done in 2 steps. Step 1 is explained in the previous part. As mentioned, step 1 results in the optimal weights matrix for the whole backtesting period. In step 2, the weights matrix is smoothed by EWMA.

Backtesting results also show a comparison of maximum drawdowns for the optimized portfolios and the Benchmark index. A drawdown is a measure of the decline from some historical peak in a variable, and a maximum drawdown is the largest historical drawdown for a variable. Hence, a drawdown measures how much a variable did loose in value before the value is restored to the initial value, also called par value.

The optimized portfolios consist of MSCI.EAFE, MSCI.USA, General.SIX, BarCap.US, Sv.Obl.Index and ACCI (commodities are once again included in only one of the two portfolios) – and are tested against the Benchmark index.

34 In the case of the return series, Exponential Moving Average (EMA) (also called Exponentially Weighted Moving Average (EWMA)) can be written as $EMA_t = \lambda Y_{t-1} + (1-\lambda)EMA_{t-1}$. EMA is defined for time period $t \geq 2$, where $t$ stands for the current period. The return at a time period $t$ is denoted as $Y_t$, the value of EMA at any time period is denoted as $EMA_t$. $EMA_t$ is set to $Y_t$. Lambda parameter, $\lambda$, is calculated by $\lambda = 2/(N + 1)$, where $N$ is equal to the number of periods chosen for smoothing - in this paper 6 month (Lopez & Walter, 2001; Sonesson, 2003).

35 Shortening the lambda means that the portfolio weights are more responsive to changes in the market conditions. That also does imply that transaction costs gets higher as the weights fluctuate more.
5.2 Mean-Variance strategy with Student-t distribution

Mean-Variance portfolio with commodities

*Table 8* shows the backtesting results for M-V portfolio with commodities. As seen in the table, the optimized portfolio is much less volatile than the Benchmark index. During the beginning of the recent financial crisis, year 2008, the portfolio lost 0.33 percent compared to Benchmark index that lost 10.89 percent. The ability to minimize losses when the market conditions are undesirable seems to be the main feature of the minimum variance strategy. The portfolios lower volatility does however imply that the portfolio tends to gain less return than the Benchmark, for example in year 2009 or 2010. The portfolio has a slightly lower per annum (p.a.) 3 years return but a somewhat higher 5 years p.a. return.

<table>
<thead>
<tr>
<th>Net Performance % to 2011-10-31:</th>
</tr>
</thead>
<tbody>
<tr>
<td>1mth 3mths 6mths 1yr 3yrs 5yr 3yrs(p.a.) 5yrs(p.a.)</td>
</tr>
<tr>
<td>Portfolio</td>
</tr>
<tr>
<td>Benchmark</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Net Performance % Calendar Year:</th>
</tr>
</thead>
<tbody>
<tr>
<td>2001 2002 2003 2004 2005 2006 2007 2008 2009 2010</td>
</tr>
<tr>
<td>Portfolio</td>
</tr>
<tr>
<td>Benchmark</td>
</tr>
</tbody>
</table>

| Portfolio | YTD36  Total |
|-----------|
| YTD       |
| 7.57      |
| 68.66     |
| Benchmark |
| 2.98      |
| 72.85     |

*Table 8. Performance of M-V portfolio with Student-t distribution. Commodites are included.*

*Figure 8* shows the graphs of backtesting results and optimization settings. The graph to the upper left shows the returns series for the six assets and Benchmark portfolio. Figure to the upper right shows the weights recommendations over time in percentage points of portfolio allocation to each asset (BarCap.US is given the highest weight). The weights are rebalanced by the weights recommendation shown in the graph in the middle to the left. This graph suggests that portfolio weights’ rebalancing was at maximum of slightly above 6 percent. The picture in the middle to the right shows the cumulative return of the optimized portfolio compared to Benchmark. The cumulative return of the portfolio is calculated to be 68.66 percent, compared to Benchmarks return of 72.85 percent. The lower plot to the left shows drawdowns for the optimized portfolio compared to Benchmark index. The portfolio standard deviation of return is lower than Benchmarks, 1.07 compared to 2.96. The same applies for maximum daily loss, 4.21 percent versus 7.45 percent.

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36 YTD is an abbreviation for Year To Date.
Figure 8. Results for M-V strategy with Student-t distribution. Commodities are included.
Mean-Variance portfolio without commodities

Table 9 shows the backtesting performance results for Markowitz M-V minimum variance portfolio without commodities. As seen in the table, the optimized portfolio is much less volatile than Benchmark index. In year 2008 the portfolio did gain 0.06 percent compared to Benchmark index that lost 10.89 percent. The portfolio is a good strategy when market conditions are poor as year 2007 and 2008. When market conditions are good the portfolio low risk profile results in lower gains compared to Benchmark. The portfolio has a lower per annum (p.a.) 3 years return but almost identical 5 years p.a. return.

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<td>0.36</td>
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<td>5.15</td>
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<td>12.70</td>
<td>-1.40</td>
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<td>20.21</td>
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</table>

Appendix C1 shows the graphs of backtesting results when commodities are excluded. The graph to the upper left shows the returns series for the six assets and Benchmark portfolio. Figure to the upper right shows the weights recommendations over time in percentage points of portfolio allocation to each asset (BarCap.US is given the highest weight). The weights are rebalanced by the weights recommendation shown in the graph in the middle to the left. This graph suggests that portfolio weights’ rebalancing was at maximum around 5 percent. The picture in the middle to the right shows the development of returns of the optimized portfolio compared to Benchmark. The cumulative return of the portfolio is calculated to be 64.42. The lower plot to the left shows drawdowns for the optimized portfolio compared to Benchmark index. Portfolios standard deviation is 1.06 and maximum loss is 3.73 percent.

5.3 Mean-Conditional Value at Risk strategy

Mean-Conditional Value at Risk portfolio with commodities

Table 10 shows the backtesting performance results. As seen in the table, the optimized portfolio is much less volatile than Benchmark index and did also achieve positive returns each year. For example, in 2007 the return was 8.3 percent and in 2008 3.46 percent. The low
risk profile of the portfolio results in smaller returns when market conditions are desirable and give higher returns to Benchmark. For example, in year 2010 the Benchmark outperforms the portfolio by 8.25 percent. The portfolio has a slightly lower per annum (p.a.) 3 years return but a somewhat higher 5 years p.a. return.

**Net Performance % to 2011-10-31:**

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**Net Performance % Calendar Year:**

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Table 10. Performance of M-CVaR portfolio. Commodities are included.

Figure 9 shows the graphs of backtesting results when commodities are included. The graph to the upper left shows the returns series for the six assets and Benchmark. Figure to the upper right shows the weights recommendations over time in percentage points of portfolio allocation to each asset (BarCap.US is given the highest weight). The weights are rebalanced by the weights recommendation shown in the graph in the middle to the left. That graph suggests that portfolio weights’ rebalancing was at maximum around 20 percent. The picture in the middle to the right shows the development of returns of the optimized portfolio compared to Benchmark. The cumulative return of the portfolio is calculated to be 75.34 percent. The lower plot to the left shows drawdowns for the optimized portfolio and Benchmark index. Portfolios standard deviation is 1.14 and maximum loss is 3.73 percent.
Figure 9. Results for M-CVaR strategy. Commodities are included.
Mean-Conditional Value at Risk portfolio without commodities

Table 11 shows the backtesting performance results. As seen in the table, the optimized portfolio is much less volatile than Benchmark and achieves positive returns each year. For example, in year 2007 the return was 6.66 percent and in year 2008 3.47 percent. The low risk profile of the portfolio is again giving lower returns compared to Benchmark when market conditions are desirable. For example, in year 2010 the Benchmark outperforms the portfolio by 8.25 percent. The portfolio has a slightly lower per annum (p.a.) 3 years return but a somewhat higher 5 years p.a. return.

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Table 11. Performance of M-CVaR portfolio. Commodities are excluded.

Appendix C2 shows the graphs of backtesting results when commodities are excluded. The graph to the upper left shows the returns series for the six assets and Benchmark portfolio. Figure to the upper right shows the weights recommendations over time in percentage points of portfolio allocation to each asset (BarCap.US is given the highest weight). The weights are rebalanced by the weights recommendation shown in the graph in the middle to the left. This graph suggests that portfolio weights’ rebalancing was at maximum around 20 percent. The picture in the middle to the right shows the development of returns of the optimized portfolio compared to Benchmark. The cumulative return of the portfolio is calculated to be 72.49 percent. The lower plot to the left shows drawdowns for the optimized portfolio and Benchmark. Portfolios standard deviation is 1.15 and maximum loss is 3.31 percent.

5.4 Mean-Variance strategy with FMCD estimator

Mean-Variance portfolio with commodities

Table 12 shows the backtesting performance results. As seen in the table, the optimized portfolio is much less volatile than Benchmark and achieves positive returns each year. For example, in year 2007 the return was 6.62 percent and in year 2008 0.23 percent. As earlier,
when market conditions are good the portfolio low risk profile results in lower gains compared to Benchmark. For example, in year 2010 the Benchmark outperforms the portfolio by 7.15 percent. The portfolio has a slightly lower per annum (p.a.) 3 years return but a higher 5 years p.a. return.

### Net Performance % to 2011-10-31:

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Table 12. Performance of M-V portfolio with FMCD estimator. Commodities are included.

Figure 10 shows the graphs of backtesting results when commodities are included. The graph to the upper left shows the returns series for the six assets and Benchmark portfolio. Figure to the upper right shows the weights recommendations over time in percentage points of portfolio allocation to each asset (BarCap.US is given the highest weight). The weights are rebalanced by the weights recommendation shown in the graph in the middle to the left. This graph suggests that portfolio weights’ rebalancing was at maximum around 20 percent but in the most of cases about 10 percent. The picture in the middle to the right shows the development of returns of the optimized portfolio compared to Benchmark. The lower plot to the left shows drawdowns for the optimized portfolio compared to benchmark index. The cumulative return of the portfolio is calculated to be 70.22 percent which is somewhat lower than Benchmarks return. Portfolios standard deviation is 1.07 and maximum loss is 3.74 percent.
Figure 10. Results for M-V strategy with FMCD estimator. Commodities are included.
Mean-Variance portfolio without commodities

Table 13 shows the backtesting performance results. As seen in the table, the optimized portfolio is less volatile than Benchmark index but does not achieve positive returns each year. In year 2008 the return was negative, 1.22 percent, which is however still much lower than Benchmarks loss that year. Also, when market conditions are desirable the portfolio achieves lower returns than Benchmark. The portfolio has a slightly lower per annum (p.a.) 3 years return as well as 5 years p.a. return.

<table>
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<tr>
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Net Performance % Calendar Year:

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Table 13. Performance of M-V portfolio with FMCD estimator. Commodities are excluded.

Appendix C3 shows the graphs of backtesting results when commodities are excluded. The graph to the upper left shows the returns series for the six assets and Benchmark portfolio. Figure to the upper right shows the weights recommendations over time in percentage points of portfolio allocation to each asset (BarCap.US is given the highest weight). The weights are rebalanced by the weights recommendation shown in the graph in the middle to the left. This graph suggests that portfolio weights’ rebalancing peaked at about 20 percent, but was in the most of cases lower. The picture in the middle to the right shows the development of returns of the optimized portfolio compared to Benchmark. The lower plot to the left shows rolling drawdowns for the optimized portfolio compared to benchmark index. The cumulative return of the portfolio is calculated to be 63.18 percent which is lower than Benchmarks return. Portfolios standard deviation is 1.10 and maximum loss is 3.79 percent.

5.5 Mean-Variance strategy with OGK estimator

Mean-Variance portfolio with commodities

Table 14 shows the backtesting performance results. As seen in the table, the optimized portfolio is less volatile than the Benchmark index but does not achieve positive returns each year. In year 2008 the return was negative, 0.72 percent, which is still however much lower
than Benchmarks loss that year. Also, when market conditions are desirable the portfolio achieves lower returns than Benchmark. The portfolio has a slightly lower per annum (p.a.) 3 years return but a higher 5 years p.a. return.

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<th>3yrs</th>
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*Table 14. Performance of M-V portfolio with OGK estimator. Commodities are included.*

*Figure 11* shows the graphs of backtesting results when commodities are included. The graph to the upper left shows the returns series for the six assets and Benchmark portfolio. Figure to the upper right shows the weights recommendations over time in percentage points of portfolio allocation to each asset (BarCap.US is given the highest weight). The weights are rebalanced by the weights recommendation shown in the graph in the middle to the left. This graph suggests that portfolio weights’ rebalancing peaked at about 10 percent, but was in the most of cases lower. The picture in the middle to the right shows the development of returns of the optimized portfolio compared to Benchmark. The lower plot to the left shows drawdowns for the optimized portfolio compared to Benchmark. The cumulative return of the portfolio is calculated to be 67.43 percent which is lower than Benchmarks return. Portfolios standard deviation is 1.07 and maximum loss is 4.37 percent.
Figure 11. Results for M-V strategy with OGK estimator. Commodities are included.
Mean-Variance portfolio without commodities

*Table 15* shows the backtesting performance results. As seen in the table, the optimized portfolio is less volatile than the Benchmark index but does not achieve positive returns each year. In year 2008 the return was negative, 0.06 percent, which is however still lower than Benchmarks loss that year. Also, when market conditions are desirable the portfolio achieves lower returns than Benchmark. The portfolio has a slightly lower per annum (p.a.) 3 years return but a higher 5 years p.a. return.

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<td>21.01</td>
<td>12.70</td>
<td>-1.40</td>
<td>-10.89</td>
<td>20.21</td>
<td>14.25</td>
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<td>YTD</td>
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<td>Portfolio</td>
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<td>Benchmark</td>
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*Table 15. Performance of M-V portfolio with OGK estimator. Commodities are excluded.*

Appendix C3 shows the graphs of backtesting results when commodities are excluded. The graph to the upper left shows the returns series for the six assets and Benchmark portfolio. Figure to the upper right shows the weights recommendations over time in percentage points of portfolio allocation to each asset (BarCap.US is given the highest weight). The weights are rebalanced by the weights recommendation shown in the graph in the middle to the left. This graph suggests that portfolio weights’ rebalancing peaked at about 10 percent. The picture in the middle to the right shows the development of returns of the optimized portfolio compared to Benchmark. The lower plot to the left shows drawdowns for the optimized portfolio compared to Benchmark. The cumulative return of the portfolio is calculated to be 64.16 percent which is lower than Benchmarks return. Portfolios standard deviation is 1.06 and maximum loss is 3.84 percent.

### 5.6 Portfolio performance measures for the optimized portfolios

The portfolio strategies are compared with two performance ratios, Information ratio and Sortino ratio. Benchmark index is used as the benchmark. The monthly returns for the portfolios are computed in chapter 5.2 – 5.5. Number 1 (for example S-t1) indicates that commodities are included in optimization; number 2 (for example S-t2) indicates that commodities are excluded from optimization. S-t stands for Student-t distribution. In
example, FMCD is the M-V portfolio optimized with FMCD estimator - FMCD1 implies that commodities are included and FMCD2 that commodities are excluded. And so on. The results are shown in Table 16.

<table>
<thead>
<tr>
<th>Information Ratio</th>
<th>S-t1</th>
<th>S-t2</th>
<th>MCVaR1</th>
<th>MCVaR2</th>
<th>FMCD1</th>
<th>FMCD2</th>
<th>OGK1</th>
<th>OGK2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.1635</td>
<td>0.1026</td>
<td>0.2113</td>
<td>0.2428</td>
<td>0.1635</td>
<td>0.1206</td>
<td>0.1168</td>
<td>0.0675</td>
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</table>

<table>
<thead>
<tr>
<th>Sortino Ratio</th>
<th>S-t1</th>
<th>S-t2</th>
<th>MCVaR1</th>
<th>MCVaR2</th>
<th>FMCD1</th>
<th>FMCD2</th>
<th>OGK1</th>
<th>OGK2</th>
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<tbody>
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<td></td>
<td>0.4285</td>
<td>0.4175</td>
<td>0.4964</td>
<td>0.5222</td>
<td>0.4285</td>
<td>0.4261</td>
<td>0.3781</td>
<td>0.3821</td>
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Table 16. Portfolio performance measures for the backtested strategies for the optimized portfolios.

The upper part in Table 16 shows the Information Ratio for the portfolios. M-CVaR strategy has the highest Information Ratio of 0.2428 when commodities are excluded, followed by the next highest ratio of 0.2113 including commodities. When M-V framework is used the Student-t distribution gives a ratio of 0.1635 including commodities and a ratio of 0.1026 excluding commodities. FMCD estimator achieves the same ratio as the Student-t distribution when commodities are included and a ratio of 0.1206 excluding commodities. OGK estimator gives a ratio of 0.1168 including commodities, respectively a ratio of 0.0675 excluding them.

The lower part in Table 16 shows the Sortino Ratio for the portfolios. M-CVaR strategy achieves the highest ratio of 0.5222 excluding commodities followed by the ratio of 0.4964 including commodities. Using Student-t distribution or FMCD estimator gives the same ratio of 0.4285 when commodities are included. Excluding commodities gives a ratio of 0.4175 with Student-t distribution and a ratio of 0.4261 with FMCD estimator. OGK estimator gives a ratio of 0.3781 when commodities are included and a ratio of 0.3821 excluding them.
6. Discussion

6.1 Optimization
The weights allocation states that both strategies give the highest weight to BarCap.US. When Student-t distribution is used BarCap.US gets roughly 64 percent including commodities and 72 percent excluding them. Using FMCD estimator gives BarCap.US the weight of 76 percent when commodities are included and 81.5 percent otherwise. OGK estimator gives BarCap.US a weight of about 59 percent, and 67 percent when commodities are excluded. M-CVaR strategy gives BarCap.US about 57 percent, respective 63 percent excluding ACCI. Hence, BarCap.US always gets the largest weight in the portfolio, which weight is also relatively larger if commodities are excluded. On the other hand, the assets that always get the smallest weights in portfolio allocation are MSCI.EAFE and MSCI.USA. When commodities are included they are always given zero weights regardless of the strategy used. Not including ACCI and using Student-t distribution gives a small weight of 2.25 percent to MSCI.EAFE. Using FMCD estimator gives MSCI.EAFE a weight of 1.03 percent and applying OGK estimator gives MSCI.EAFE a weight of 2.77 percent. M-CVaR strategy gives MSCI.USA a weight of 0.5 percent. Therefore there are similarities between M-V and M-CVaR strategies in terms of weights allocation.

The similarities in the weights allocation among the portfolios optimized with M-V strategy could be due to the risk features of the assets and the correlation structure.

BarCap.US and SV.Obl.Index get the highest, respective next highest weights in portfolio allocation among the portfolios. BarCap.US is a bond index that offers exposure to USD denominated bonds (i.e. treasuries, government or corporate related issues, which are relatively low risk investments even if it is possible to argue that American government securities can no longer be seen as risk free. Sv.Obl.Index is an index for the Swedish stock returns, which is only slightly correlated with BarCap.US, and might be seen as a lower risk asset. As seen in Appendix A2, BarCap.US has the lowest variance (2.635) among the assets, followed by Sv.Obl.Index with the next lowest variance (4.31). Therefore, M-V strategy tends to pick the least risky assets in the portfolio allocation in order to construct minimum variance portfolios.

Table 3 on page 26 shows that BarCap.US is relatively highly correlated with both MSCI indices, which in their turn are negatively correlated with Sv.Obl.Index, while BarCap.US is negatively correlated with ACCI. The fact that BarCap.US is relatively highly correlated with MSCI indices (and also that the indices are relative highly volatile) might be the cause to why the indices are totally excluded, when commodities are included; and only get small weights when commodities are included in the portfolio allocation. A possible argument to such a pattern can again be that optimization algorithms pick the least risky assets; that algorithms pick commodities (because of their desirable features) and those assets that are negatively correlated with the least risky assets. Sv.Obl.Index is on the other hand negatively correlated with both MSCI indices. However it is hard to tell if the negative correlation between these assets impacts on the portfolio allocation.
Moreover, concerning ACCI, the results of the k-means clustering in Table 2 on page 25 tells that ACCI is the riskier asset among all. It might exactly be why ACCI is given in the most of the cases a weight of 10 percent or less in the portfolio allocation. On the other hand, ACCI’s negative correlation to BarCap.US and Sv.Obl.Index could boost the proposed weight of ACCI in the portfolio. So, even if the recent financial crisis have lead to almost perfect correlation of commodities to other asset classes, the inclusion of commodities might still create diversification, especially in M-V strategy. ACCI index is constructed in such a way that from year 1997 it is equally allocated to 5 different commodity indices, as seen in Table 1 on page 24. The main idea of using ACCI is therefore to access the well-diversified commodity exposure on a global basis. Diversification can be explained by the fact that negative events (i.e. political turbulence or natural catastrophes) would lead to falling market prices for bonds and equities because of higher uncertainty of the future cash flows. Commodities do however use to gain in value during such situations. The reason to this is that investors base their valuation of equities and bonds on their future cash flow, while commodity prices are depending on factors such as supply/demand. In example, if a natural catastrophe or a political event occurs it might lead to higher uncertainty of the future supply of a commodity and will push up its price. Commodities do also serve as a hedge against inflation for the portfolios in both strategies. Hence, the hedging feature of commodities is due to the fact that resources of commodities are finite, while inflation inertia might not be.

It must however be mentioned that Table 3 on page 26 shows the correlation structure for the whole data sample, starting at year 1976, while the correlation do vary across periods. Hence, the years of recent financial turbulence is only a small part of the data sample. The idea of using such a long data sample is exactly to mitigate the impact of recent time’s turbulence and investigate the historically desirable weights for the commodities. But one must be cautious about the interpretation of the correlation structure and be basing the analysis of the assets on the correlation structure of the whole optimization period. Therefore, a proper idea could be to divide the optimization period into different samples by some rule and then optimize the portfolios. In example, one could incorporate the impact of changing market conditions on to the correlation structure or even model the time varying correlation (if it is desirable to forecast future values of the assets).

Correlation structure might also lead to similarities between M-V and M-CVaR strategies in terms of proposed weights if one argues that the cdf of the historical losses is affected by it. However, M-CVaR approach used in this paper aims to minimize the risk measured in terms of CVaR. Hence, M-CVaR strategy is not “directly” affected by the choice of the covariance estimator or the assets distribution since M-CVaR optimization algorithm does not need these measures to calculate the minimum CVaR portfolio composition.

Besides the similarities among the optimal weights proposed by the models, similarities are also found in the results of the optimization process. M-V strategy aims to find a portfolio that minimizes portfolio risk in terms of variance per unit of return. The model will therefore get slightly different estimates of the variance depending on the distribution used, or the choice of the robust covariance estimator. The differences are seen in the expected returns, standard
deviations and consequently in VaR and CVaR measures. The smallest CVaR among M-V portfolios is achieved when OGK estimator is used - 2.2416 percent when ACCI is included and 2.3093 percent excluding ACCI. The differences compared to M-V optimization with Student-t distribution and FMCD estimator are rather small and it is not very striking that these small differences lead to similarities in the assets weights allocation. The smallest CVaR value among the portfolios is obtained when M-CVaR framework. When commodities are included the CVaR for the portfolio is 2.2001, respective 2.2830 when ACCI is excluded. Even if CVaR measures are close to each other between the two strategies, we cannot assume normality and CVaR cannot be expressed as a function of the standard deviation. Moreover, VaR measure is not sub-additive and the combined portfolio VaR may be an overestimation of the true portfolio VaR. That could be the reason to why the smallest CVaR value is actually obtained by using M-CVaR framework.

Generally, when M-V model is used, one must keep in mind that the model is known for giving unreasonably high or low weights to some assets, or especially if long-only constraints are added, zero weights to others. An explanation to this phenomenon is that M-V model does not account for the assets market capitalization and can therefore give large weights to securities that are low market capitalization assets while having small or negative correlation to other asset classes. Therefore, it might be appropriate to set weights restrictions depending on the liquidity of a specific asset. On the other hand, if all of the assets are liquid, than the investor could restrict the weights by the minimum or maximum desired portfolio allocation in a specific asset, or a group of assets.

When commodities are included in the portfolio it results in the leftward shift of the efficient frontier. This shift gives a higher expected return and lowers the standard deviation for the optimized portfolios. Applying M-V strategy with Student-t distribution gives an expected return of 0.7658 percent and a standard deviation of 1.0496, while excluding ACCI implies an expected return of 0.7456 percent and a standard deviation of 1.0979. If FMCD estimator is used the expected return is 0.7492 percent and standard deviation is 1.1409, respectively a return of 0.7361 percent and standard deviation of 1.2021 percent excluding ACCI. OGK estimator gives an expected return of 0.7768 and standard deviation of 1.1640 when commodities are included, and a return of 0.7583 and standard deviation of 1.2687 when ACCI is excluded. In summary, the efficient frontier shifts slightly to the left when commodities are included in the portfolio. Also, if Student-t distribution is used, it results in an efficient frontier that lies slightly to the left of the frontiers estimated with FMCD or OGK estimators, as seen in Appendix B2 and Appendix B3. M-CVaR optimization gives a return of 0.7821 percent and standard deviation is 1.0736 including commodities, a return of 0.7690 percent and standard deviation of 1.1128 when ACCI is excluded. Hence, M-CVaR strategy gives the highest expected return for the optimized portfolio.

The expected returns for the optimized portfolios are nevertheless small which could be explained by the minimum risk characteristics of the portfolios. The differences (as well as the expected return) between strategies could nevertheless be larger if the portfolios minimum acceptable return was restricted to a specified value. Moreover, in the case of M-V strategy,
the differences across portfolios could grow if the portfolio risk, the standard deviation, was restricted to a specified quantity and we would maximize the return for that quantity. The optimization problem would in such a case become different and require a different optimization procedure (i.e. a linear objective function with quadratic constraints instead of quadratic objective function with linear constraints).

If an investor wants to get a higher expected return by raising the standard deviation for the portfolio he needs to adjust the portfolio weights. For example, Figure 6 on page 28 shows the optimal portfolio weights composition along the efficient frontier when M-V strategy is used with the Student-t distribution. The thick vertical black separates the efficient frontier from the minimum variance locus. If the investor moves to the right in Figure 6, he moves up the efficient frontier (moving in the left direction would place the investor on the minimum variance locus which is not a desirable alternative) and needs to constantly increase the holdings in General.SIX asset. That pattern applies for both M-V and M-CVaR strategies. The main differences between the strategies concern the optimal weights for the other assets. When commodities are included both strategies dictate that the investor should after a point, when the expected return is slightly below 1 percent, use General.SIX, SV.Obl.Index and ACCI if commodities are included. If commodities are excluded and investor moves up the efficient frontier, the portfolio will be divided between allocation in General.SIX and SV.Obl.Index.

Also worth mentioning is that there are other strategies besides those used in this paper to optimize a portfolio, especially if one aims to construct an out of sample forecast for the portfolio returns. Two of such techniques are RiskMetrics model and Monte Carlo simulations. With other words, this paper constructed an in sample forecast while an out of sample forecast would aim to find the best model/approach for the estimation forecast of future assets returns. Returns that then would used to find the most desirable optimization strategy.

6.2 Backtesting
The backtesting period in this paper is set to 10 years, or more formally speaking, 13 years if the optimization period of 3 years is taken into consideration. The reason to the choice of such an interval is that it is in primary interest to evaluate the backtested strategies during the last decade’s turbulent financial markets. On the other hand, the portfolio optimization in chapter 4 uses the whole data period, starting at year 1976, in order to mitigate the impact of the recent financial crisis on the correlation structure of the assets (i.e. almost perfect correlation of commodities to other asset classes).

The backtesting results show that M-CVaR minimum risk strategy is the superior approach. M-CVaR strategy achieves a cumulative return of 75.34 percent compared to Benchmarks return of 72.85 percent when commodities are included, and 72.49 percent excluding ACCI. M-CVaR strategy is the only approach that succeeded to beat the Benchmark index but the percentage gap to other strategies is rather small. M-V strategy with Student-t distribution achieved a cumulative return of 68.66 percent by including ACCI respective 64.42 percent.
excluding it. FMCD estimator gave a return as large as 70.22 percent and 63.18 percent excluding ACCI. OGK estimator showed a return of 67.43 percent including commodities respective 64.16 percent excluding them. Therefore, exclusion of commodities lowers the portfolio return exactly in the same manner as when the portfolios were optimized in chapter 4.

It must also be mentioned that the constructed, in this paper, Benchmark index is much less volatile than the otherwise widely used benchmark index - OMX Stockholm (OMXS) index. Benchmark index does furthermore achieve a much higher cumulative return than OMX Stockholm index. As seen in Appendix C5, the best performing M-CVaR strategy is backtested against OMXS index. M-CVaR strategy clearly outperforms OMXS index disregard if commodities are included or not. In general, all of the portfolios would outperform OMXS index because of its low cumulative return of 39.89 percent. Moreover, all of the portfolios are much less risky than the OMXS index as shown by the measures in Appendix C5.

Focusing again on the results of portfolio backtesting against the Benchmark it is clear that all of the strategies incorporate the characteristics of minimum risk portfolios. The Benchmark index has a standard deviation of 2.96 percent. All the optimized portfolios have a standard deviation close to 1 percent. Hence, the main feature of the optimized portfolios is that they are less volatile than Benchmark and do cut the losses substantially when market conditions are undesirable. However, that does come at a price of smaller returns compared to Benchmark when the market conditions are “better”. M-CVaR strategy is the approach that achieves the best results during the recent financial crisis. In year 2008, the portfolio gained 3.46 (3.47 percent if commodities are excluded) percent in value. The Benchmark index lost at the same time 10.89 percent. M-V strategies did nevertheless perform more poorly in year 2008. Using Student-t distribution gave either a small negative return of 0.33 percent or a return of 0.06 percent excluding commodities. FMCD estimator gave a return of 0.23 percent and a negative return of 1.22 percent excluding commodities. OGK estimator achieved a negative return of 0.72 percent respectively a negative return of 0.06 percent. Therefore, M-CVaR strategy seems to the best strategy in adverse market conditions. On the other hand, concerning the maximum daily losses the differences between the portfolios are small.

However, the strategies might perform better in the periods of desirable market conditions, namely when Benchmark index gained superior returns (such as year 2009 and 2010). During such periods the investor could probably achieve a higher return by setting a maximum allowable CVaR risk or a standard deviation risk, instead of minimizing them. The investor could also maximize the portfolios Sharpe ratio as the goal of optimization. If a higher portfolio risk is undesirable the investor might even combine the portfolio with Benchmark index by first investigating the market trends and then, in an uptrend climate, sell an amount of the portfolio value and invest that amount in Benchmark index. The difficulty here is to correctly investigate when the market is in an uptrend. In example, the investor could follow a customized rule as the trend following. In the simple case, the investor chooses to go long an asset if the close price is greater than the N period simple moving average (the mean of the
prices for a given time interval), and go short if the close price is lower than the N period simple moving average. In a more advanced case the investor could also incorporate the role of the market’s volatility. The rule is the same besides that the investor will go long (short) if, additionally, the market volatility is less (larger) than its mean (or median) over the last N periods. Of course, there are a lot of other tailored rules (i.e. spreadtrading with option contracts or leveraged fund strategies as the 130/30 strategy) that an investor could follow.

The backtesting results are depending on the choice of the used parameters. The portfolios are rebalanced on a monthly basis which is a common approach in backtesting. Each time the rebalancing took place the proposed weights were smoothed by exponentially weighted moving average with smoothing parameter of 6 month. Such number of month is commonly chosen, increasing that number does lead to a more sluggish changes in the weights, thus moving away from the weights proposed by the optimization. It also lowers transaction costs and the weights sensitivity to changing market conditions. On the other hand, if it is crucial that the weights are rebalances more frequently (i.e. in the case of trading in highly volatile markets, using derivatives- or leveraged trading strategies), the rebalancing should be done more often. In such scenarios the investor might be partly or fully dependant on the markets intraday movements. The focus in this paper is long run asset allocation, but one might still reason that rebalancing could for example be done on a weekly basis to incorporate the turbulent market environment.

The same smoothing parameter is applied to both strategies but the weights fluctuations are different. M-CVaR strategy gives the highest weights changes that are at maximum 20 percent (the combined change in weights for the assets add up to that number) per month regardless if commodities are included or not. The same applies for FMCD estimator. Using Student-t distribution gives about 6 percent changes in weights; OGK changes are close to 10 percent. Therefore, in terms of transaction costs the Student-t distribution is the best alternative. Even if Student-t together with M-V strategy gives the lowest transaction costs the return of the strategy fails to beat the Benchmark index. Both strategies create also portfolios with lower maximum drawdowns values compared to Benchmark. That is not surprising when the strategies aim to minimize the portfolio risk which in the next turn gives lower drawdowns.

Information Ratio and Sortino Ratio for the strategies are shown in Table 16 on page 45. As shown in the figure, the highest ratios are achieved by using M-CVaR strategy and excluding commodities. As discussed above M-CVaR strategy is the best in terms of beating the Benchmark index and actually the only strategy that succeeded to accrue a higher cumulative return than Benchmark. That is probably the main reason to why M-CVaR strategy gives the highest Information Ratio.

Sortino Ratio results show that M-CVaR strategy is again the best performing one. However, all of the Sortino Ratio measures are rather similar among portfolios and one should be cautious to be basing the choice of the strategy solely by the results of Sortino Ratio.
7. Conclusions

7.1 Results of the paper
This paper investigated if inclusion of commodities in portfolio allocation decision can improve the portfolios risk-return characteristics. Two portfolios were constructed whereof only one of the portfolios included commodities besides the traditional asset classes as stocks and bonds. The portfolio weights were restricted to only be positive. In order to find the assets weights allocation that minimizes the overall portfolio risk, the portfolio weights were optimized with Markowitz Mean-Variance strategy (using Student-t distribution and robust covariance estimation techniques) and Mean-Conditional Value at Risk strategy (using the historical distribution of the assets losses).

The optimization results did show that commodities do incorporate desirable properties by giving a higher expected portfolio return and a lower portfolio risk. Moreover, the two optimization strategies did propose optimal assets weights that were rather similar across the portfolios. The highest weight was always given to the BarCap.US asset and the smallest weights were given to the two MSCI indices. In the case of the portfolio not including commodities MSCI indices were always excluded, and only given small weights when commodities were present in the portfolio allocation.

The portfolios were later on backtested against Benchmark index over a 10 years period. Benchmark index was constructed as an equally weighted portfolio of General.SIX and SV.Obl.Index assets. Such an allocation was chosen in order to create an exposure index that is equally sensitive to the changes in Swedish stock index value and Swedish bond index value. Once again, the portfolios with commodities outperformed the portfolios not including commodities. Both strategies succeeded to create portfolios that were less risky than the Benchmark in terms of, among all, the maximum portfolio loss, the standard deviation and maximum drawdowns. However, the only strategy that succeeded to outperform the Benchmark index was Mean-Conditional Value at Risk strategy. On the other hand, when Mean-Variance strategy was used the results were quiet similar across the optimized portfolios. The highest return was achieved by using Fast Minimum Covariance Determinant estimator. Moreover, M-CVaR strategy was also backtested against OMX Stockholm index. The results did show that M-CVaR strategy did clearly outperform OMX Stockholm index disregarding if commodities were included in the portfolio or not.

The optimized portfolio strategies were further on compared with Information Ratio and Sortino Ratio. Mean-Conditional Value at Risk strategy did once again show to be the best performing and achieved the highest ratio values.

7.2 Further studies
This paper concluded that commodities are desirable in portfolio allocation and that the best optimization technique seems to be Mean-Conditional Value at Risk. The technique uses historical loss distribution of the assets returns. Since the assets correlation structure varies through time it might also be of interest to incorporate the impact of such features on the
historical distribution of the losses and consequently the portfolio allocation. Furthermore, portfolio optimization can be done by using some approach that can among all account for the effects of time varying volatility, investigate the features of different distributional assumptions about the assets returns, or incorporate other decision rules in portfolio trading. In example, as investors might prefer to incorporate their specific views about the future assets returns in the portfolio optimization framework one might apply the Black-Litterman model.

The portfolio weights in this paper were restricted to only allow for positive assets weights. Hence, no constraints were imposed on the weights for the single assets or a group of assets. In reality the fund manager needs to consider several factors in the market before he can construct the desired portfolio. In example, the transaction costs or the assets liquidity in the market place might dictate the allowed maximum or minimum weights for an asset, or a group of assets. It might therefore be desirable to evaluate and backtest the portfolios that are restricted with respect to maximum transaction costs allowance, liquidity ratios, equal risk contributions for the assets or other aspects relevant to a manager.

The portfolios in this paper do always consist of the same assets. Changing market conditions might on the other hand dictate to incorporate changing portfolio allocation as well. It is hence motivating to investigate what asset classes should be included in the portfolio allocation in the different phases of the financial cycle.
References

Books


Articles


Internet


Investopedia, http://www.investopedia.com/terms/t/total_return_index.asp#axzz1oRo1VDIV (2012-02-24)


Appendix A1 – The assets returns series

Plots of the returns series for the six variables.

Plot of the returns series for the Benchmark index.
### Appendix A2 – Distributional properties of the assets

<table>
<thead>
<tr>
<th></th>
<th>MSCI.EAFE</th>
<th>MSCI.USA</th>
<th>General.SIX</th>
<th>BarCap.US</th>
<th>Sv.Obl.Index</th>
<th>ACI</th>
<th>Benchmark</th>
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<td>0.686</td>
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<td>2.635</td>
<td>4.31</td>
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*The distributional properties of the assets.*
Appendix A3 – Distributional plots for the assets and Benchmark index

The blue bins shows the probability for the returns, the orange line the mean value, the red curve a normal density estimation with the same mean and variance as the empirical returns.
Appendix B1 – Mean-Conditional VaR at Risk optimal weights and efficient frontiers

Figure 1

Figure 2
Appendix B2 – Mean-Variance FMCD optimal weights and efficient frontiers

**Figure 1**

**Figure 2**
Appendix B3 – Mean-Variance OGK optimal weights and efficient frontiers
Eff. frontiers with (black) and without (grey) commodities. OGK estimator

Figure 2

Eff. front. with commodities by using covt est. (black) & OGK est. (grey)

Figure 3
Appendix C1 – Mean-Variance Student-t distribution backtesting. Without commodities.
Appendix C2 - Mean-Conditional Value at Risk strategy backtesting. Without commodities.
Appendix C3 - Mean-Variance strategy FMCD estimator backtesting. Without commodities.
Appendix C4 - Mean-Variance strategy OGK estimator backtesting. Without commodities.
Appendix C5 - Mean-Conditional Value at Risk strategy backtested against the OMX Stockholm index.

M-CVaR strategy is backtested against OMX Stockholm index. The figure to the left with the corresponding measures below is when the portfolio includes commodities/the figure to the right excludes commodities.
Appendix D1 – Solving Mean-Variance optimization problem
Goldfarb & Idnani (1983) explains that quadratic programming (QP) is concerned with the strictly convex (positive definite) problem:

\[
\min f(x) = \frac{1}{2} x^T G x,
\]

(1.1a)

Subject to

\[
s(x) = C^T x - b \geq 0
\]

(1.1b)

where \( x \) and \( a \) are n-vectors. \( G \) is a \( n \times n \) symmetric positive definite matrix, \( C \) is a \( n \times m \) matrix, \( b \) is a \( m \)–vector. The authors proposed a dual method for the strictly convex QP problem where the first step is to provide a dual feasible point (optimal point for a subproblem of the original problem) by relaxing all of the constraints in (1.1b) and finding the unconstrained minimum in (1.1a). A dual algorithm then iterates until primal feasibility (i.e. dual optimality) is achieved, all the while maintaining the primal optimality of intermediate subproblems (i.e. dual feasibility). The dual algorithm is of the active set type. This means that a subset of the \( m \) constraints in (1.1b) that are satisfied as equalities by the current estimate \( x \) of the solution to the QP problem. Letter \( K \) is used to denote the set \( \{1, 2, ..., m\} \) of indices of the constraints (1.1b) and \( A \subseteq K \) to denote the indices of the active set. For further explanation see Goldfarb & Idnani (1983).

Appendix D2 – Solving Mean-Conditional Value at Risk optimization problem
If it is not possible to determine the joint density function \( p(y) \) of the random events, one might instead apply a number of scenarios which may represent some historical values of the returns. The function in (2.29) can then be approximated by the following function:

\[
F_\alpha(x, \zeta) = \zeta + (1 - \alpha)^{-1} \sum_{s=1}^{S} (f(x, r_s) - \zeta)^+,
\]

where \( r_s \) denotes the scenarios (i.e. the historical values of the returns). The optimization problem in (2.28) can then be approximated by:

\[
\min_{x, \zeta} \zeta + (1 - \alpha)^{-1} \sum_{s=1}^{S} (f(x, r_s) - \zeta)^+.
\]

In order to solve the optimization problem above one then need to introduce artificial variables \( z_s \) that will replace \( (f(x, r_s) - \zeta)^+ \). The constraints imposed on the artificial variables are \( z_s \geq f(x, r_s) - \zeta \) and \( z_s \geq 0 \). The optimization problem then becomes:

\[
\min \zeta + (1 - \alpha)^{-1} \sum_{s=1}^{S} z_s
\]
subject to

\[ z_s \geq f(x, r_s) - \zeta \]
\[ z_s \geq 0. \]

If \( f(x, r_s) \) is linear in \( x \), all the expressions \( z_s \geq f(x, r_s) - \zeta \) represent linear constraints. The optimization problem then becomes a linear programming problem that can be solved by an appropriate linear programming algorithm (Rockafellar & Uryasev, 2000 and Rockafellar & Uryasev, 2002). A linear programming algorithm can be found in Ogryczak & Sliwinski (2011).