Wolff's inequality in multi-parameter Morrey spaces

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Abstract

We prove Wolff inequalities for multi-parameter Riesz potentials and Wolff potentials in Lebesgue spaces $L^p(R^d)$ and multi-parameter Morrey spaces $L^p_\lambda(R^d)$, where $R^d = R^{n_1} \times R^{n_2} \times \cdots \times R^{n_k}$, $\lambda = (\lambda_1, \dots, \lambda_k)$ and $0 < \lambda_i \leq n_i$, $1 \leq i \leq k$, in the dyadic case as well as in the non-dyadic (continuous) case.

Key words and phrases: Riesz potential, Wolff potential, multi-parameter potential, dyadic potential, Wolff's inequality, Morrey spaces

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1 Introduction.

The nonlinear potential theory for Riesz potentials and more general potentials of functions in the Lebesgue class $L^p(R^d)$ is well known, see [1] and [10] and the references contained there. A major step in this theory was the so called Wolff's inequality [4] and its applications to various parts of the theory. The corresponding theory for functions in the Morrey spaces $L^p_\lambda(R^d)$, with its roots in [3], [5], [7] and [9], is more recent and a part of it was developed in [2]. In this paper we consider the extension of two basic inequalities in this theory to a multi-parameter setting, see [8]. Our main result (Theorem 2) is a generalization of Wolff's inequality, referred to above,

to the multi-parameter Morrey spaces $L_\lambda^p(R^d)$.

We consider multi-parameter Riesz potentials and Wolff potentials in an Euclidean space $R^d = R^{n_1} \times R^{n_2} \cdots \times R^{n_k}$, where $d = n_1 + n_2 + \cdots + n_k$ and each R^{n_i} is a standard Euclidean space. The multi-parameter Riesz potential $R_\rho \mu$ of a nonnegative measure μ in R^d is defined by

$$R_\rho \mu(\mathbf{x}) = \int |x_1 - y_1|^{\rho_1 - n_1} \cdots |x_k - y_k|^{\rho_k - n_k} d\mu(y_1, \dots, y_k),$$

where $\mathbf{x} = (x_1, \dots, x_k)$. Here $\rho = (\rho_1, \dots, \rho_k)$ and $0 < \rho_i < n_i$, $1 \leq i \leq k$. The case $k = 1$ gives the classical Riesz potential in R^d , see [1]. We are mainly going to work with the dyadic counterpart $R_\rho^{dy} \mu$ of $R_\rho \mu$, defined in (1) below and comment on the non-dyadic (continuous) case at the end of Section 3.

We also define the corresponding dyadic multi-parameter Wolff potential $W_{\rho,p}^{dy} \mu$ in (2) and prove Wolff's inequality in this setting (Theorem 1). The case $k = 1$ was proved by Th. Wolff in [4], while the present author settled the case $k = 2$ in [8]. D. R. Adams and Jie Xiao [2] studied the potential theory for the Riesz kernel in the Morrey space $L_\lambda^p(R^d)$ and proved a Wolff inequality. We define the dyadic multi-parameter Morrey-Wolff potentials $W_{\rho,\lambda,p}^{dy} \mu$ and prove the corresponding Wolff inequality (Theorem 2) in the multi-parameter Morrey spaces $L_\lambda^p(R^d)$.

The plan of this paper is as follows. Section 2 contains our notation and definitions, our results are stated in Section 3 and proved in Section 4.

2 Preliminaries.

We work in Euclidean spaces $R^d = R^{n_1} \times R^{n_2} \times \cdots \times R^{n_k}$, where $d = n_1 + n_2 + \cdots + n_k$ and each R^{n_i} is a standard Euclidean space. Points in R^d are written $\mathbf{x} = (x_1, x_2, \dots, x_k)$, where $x_i \in R^{n_i}$, $1 \leq i \leq k$. Our basic geometric object is a dyadic rectangle $I = I_1 \times I_2 \times \cdots \times I_k$, where each I_i is a standard dyadic cube in R^{n_i} , $1 \leq i \leq k$. The side length of a cube I in R^m is denoted by $l(I)$ and $|I| = l(I)^m$ is the m -dimensional Lebesgue measure of I . If $\mathbf{a} = (a_1, a_2, \dots, a_k)$, where $a_i > 0$ for $1 \leq i \leq k$, we define a multi-parameter ball $B(\mathbf{x}; \mathbf{a})$ as the Cartesian product of standard balls $B(x_i, a_i)$ in R^{n_i} , $1 \leq i \leq k$. We let $M_+(R^d)$ denote the class of nonnegative Borel measures in R^d , finite on compact sets. The Lebesgue measure of a set $E \subset R^d$ is denoted by $|E|$ and Lebesgue integration is written $\int dx$.

The standard Lebesgue space of p -integrable functions is denoted by $L^p(R^d)$

We use the following vector notation to simplify our presentation. A k -vector is a vector $\alpha = (\alpha_1, \dots, \alpha_k)$, with real numbers α_i , $1 \leq i \leq k$. We say that $\alpha > 0$, if all $\alpha_i > 0$. If β is another k -vector we say that $\alpha > \beta$ if $\alpha - \beta > 0$. When $\alpha > 0$ and β are k -vectors we define $\alpha^\beta = \alpha_1^{\beta_1} \cdot \alpha_2^{\beta_2} \cdots \alpha_k^{\beta_k}$. Let $I = I_1 \times \cdots \times I_k$ be a dyadic rectangle in $R^d = R^{n_1} \times \cdots \times R^{n_k}$ and define k -vectors $l(I) = (l(I_1), \dots, l(I_k))$ and $n = (n_1, \dots, n_k)$. Then $l(I)^\alpha = l(I_1)^{\alpha_1} \cdot l(I_2)^{\alpha_2} \cdots l(I_k)^{\alpha_k}$, for any k -vector α .

We can now define the dyadic multi-parameter Riesz potential $R_\rho^{dy} \mu$ of the measure $\mu \in M_+(R^d)$ using this notation by

$$R_\rho^{dy} \mu(\mathbf{x}) = \sum_I l(I)^{\rho-n} \cdot \mu(I) \cdot \chi_I(\mathbf{x}) \quad (1)$$

and the dyadic multi-parameter Wolff potential by

$$W_{\rho,p}^{dy} \mu(\mathbf{x}) = \sum_I l(I)^{p(\rho-n)+n} \cdot \mu(I)^{p-1} \chi_I(\mathbf{x}) \quad (2)$$

where in both cases the sum is over all dyadic rectangles $I = I_1 \times I_2 \times \cdots \times I_k$ in R^d .

We study the dyadic multi-parameter Riesz potentials and Wolff potentials in the Lebesgue spaces $L^p(R^d)$ and more generally in the multi-parameter Morrey spaces $L_\lambda^p(R^d)$ defined as follows.

Definition 1 Let $1 < p < \infty$, $\lambda = (\lambda_1, \dots, \lambda_k)$ and $0 < \lambda \leq n$. A function f belongs to $L_\lambda^p(R^d)$ if

$$\|f\|_{p,\lambda}^p = \sup_I l(I)^{\lambda-n} \int_I |f|^p dx$$

is finite, where supremum is over all dyadic rectangles I in R^d .

The spaces $L_\lambda^p(R^d)$ for $k = 1$ go back to C. B. Morrey in the 1930's and were further studied by Stampacchia [9] and S Campanato [3]. See [5] and [7] and the references contained there. For applications to partial equations, see [6]. Note that when $\lambda_i = n_i$, for all $1 \leq i \leq k$, we get $L_\lambda^p(R^d) = L^p(R^d)$.

We define the corresponding multi-parameter Wolff potentials as follows.

Let $1 < p < \infty$ and let $0 < \rho < n$, $0 < \lambda \leq n$ be as above, then the dyadic Morrey–Wolff potential $W_{\rho,\lambda,p}^{dy}\mu$ of a measure $\mu \in M_+(R^d)$ is defined by

$$W_{\rho,\lambda,p}^{dy}\mu(\mathbf{x}) = \sum_I l(I)^{p(\rho-n)+\lambda} \cdot \mu(I)^{p-1} \chi_I(\mathbf{x})$$

where the sum is over all dyadic rectangles I in R^d .

3 Main results.

The following multi-parameter Wolff Inequality in $L^p(R^d)$ is our first result.

Theorem 1 *Let $1 < p < \infty$, $1/p + 1/q = 1$, $0 < \rho < n/q$ and let $\mu \in M_+(R^d)$. Then*

$$\int W_{\rho,p}^{dy}\mu(\mathbf{x}) d\mu(\mathbf{x}) \leq \int [R_\rho^{dy}\mu(\mathbf{x})]^p d\mathbf{x} \leq C \cdot \int W_{\rho,p}^{dy}\mu(\mathbf{x}) d\mu(\mathbf{x}),$$

where the constant C only depends on d, p and ρ .

Theorem 1 is a special case of the next theorem, which is our multi-parameter Wolff Inequality in the Morrey space $L_\lambda^p(R^d)$ and is our main result.

Theorem 2 *Let $1 < p < \infty$, $1/p + 1/q = 1$, ρ, d and λ be as above, $0 < \rho < n/q$, $0 < \lambda \leq n$ and let $\mu \in M_+(R^d)$. Then*

$$\|R_\rho^{dy}\mu\|_{p,\lambda}^p \leq C \cdot \int W_{\rho,\lambda,p}^{dy}\mu(\mathbf{x}) d\mu(\mathbf{x}),$$

where C only depends on d, p, ρ and λ .

In the non-dyadic (continuous) case we define a multi-parameter Wolff potential by

$$W_{\rho,p}\mu(\mathbf{x}) = \int_0^\infty \dots \int_0^\infty \mathbf{a}^{p(\rho-n)} \cdot \mu B(\mathbf{x}; \mathbf{a})^{p-1} \frac{d\mathbf{a}}{\mathbf{a}},$$

where $\mathbf{a} = (a_1, a_2, \dots, a_k)$ and $\frac{d\mathbf{a}}{\mathbf{a}} = \frac{da_1}{a_1} \frac{da_2}{a_2} \dots \frac{da_k}{a_k}$. Then Theorem 1 holds, modulo a multiplicative constant depending on d, p and ρ , with $R_\rho^{dy}\mu$ and $W_{\rho,p}^{dy}\mu$ replaced by $R_\rho\mu$ and $W_{\rho,p}\mu$ respectively. The main point is that

$$R_\rho\mu(\mathbf{x}) \leq \tilde{R}_\rho^{dy}\mu(\mathbf{x}) = C \cdot \sum_I l(I)^{\rho-d} \cdot \mu(\tilde{I}) \cdot \chi_I(\mathbf{x}),$$

where $\tilde{I}$ denotes the rectangle concentric to I with $l(\tilde{I}) = 3 \cdot l(I)$. Then (3) remains true by inequality $(\star)$ in [4], p. 170 and the result follows from standard estimates. An analogous argument proves Theorem 2 in the non-dyadic case with $R_\rho^{dy} \mu$ and $W_{\rho, \lambda, p}^{dy}$ replaced by $R_\rho \mu$ and $W_{\rho, \lambda, p}$ respectively, where

$$W_{\rho, \lambda, p} \mu(\mathbf{x}) = \int_0^\infty \dots \int_0^\infty \mathbf{a}^{p(\rho-n)+\lambda} \cdot \mu B(\mathbf{x}; \mathbf{a})^{p-1} \frac{d\mathbf{a}}{\mathbf{a}},$$

denotes the corresponding non-dyadic Morrey-Wolff potential.

4 Proofs.

For the readers convenience and for the clearence of the exposition we give a proof of the special case in Theorem 1 before we prove the more complicated Theorem 2.

Proof of Theorem 1. The first inequality follows from the elementary inequality $(\sum_i a_i)^p \geq \sum_i a_i^p$, for any $a_i \geq 0$ and $p \geq 1$. To prove the second inequality we first assume that the support of μ is contained in a fixed dyadic rectangle $\bar{I} = \bar{I}_1 \times \dots \times \bar{I}_k$ in R^d . We claim that then

$$\int_{\bar{I}} \left(\sum_I l(I)^{\rho-n} \cdot \mu(I) \cdot \chi_I(\mathbf{x}) \right)^p d\mathbf{x} \leq C \cdot \sum_{I \subset \bar{I}} l(I)^{p(\rho-n)} \cdot l(I)^n \cdot \mu(I)^p, \quad (3)$$

where both sums are over all dyadic rectangles $I = I_1 \times \dots \times I_k$ and C only depends on d, p and ρ . It is clearly no loss of generality to assume that $I \subset \bar{I}$ in the sum on the left hand side of (3). In the case $k = 1$, (3) was proved in [4], p. 170, for $l(\bar{I}) = 1$. The case of a general $\bar{I}$ follows by homogeneity. The case $k = 2$ was proved by the author in [8], Theorem 4.2. The case of a general k is now proved by induction over the number k of factors.

Assume that (3) holds for k factors, where $k \geq 1$. We keep the space R^d above and consider $R^d \times R^m$, for some Euclidean space R^m , where points are denoted by $(\mathbf{x}, \mathbf{s})$, $\mathbf{x} \in R^d$ and $\mathbf{s} \in R^m$. Let $\bar{J}$ be a cube in R^m , $0 < \rho_m < m$ and let $\mu \in M_+(R^d \times R^m)$ be a measure with support in $\bar{I} \times \bar{J}$. For a fixed point s in $\bar{J}$ we define a measure ν_s in R^d , supported in $\bar{I}$, by

$$\nu_s(E) = \sum_{J \subset \bar{J}} l(J)^{\rho_m-m} \cdot \mu(E \times J) \cdot \chi_J(\mathbf{s}).$$

Then

$$\int_{\bar{I}} \left(\sum_I l(I)^{\rho-n} \cdot \nu_{\mathbf{s}}(I) \cdot \chi_I(\mathbf{x}) \right)^p d\mathbf{x} \leq C \cdot \sum_{I \subset \bar{I}} l(I)^{p(\rho-n)} \cdot l(I)^n \cdot \nu_{\mathbf{s}}(I)^p \quad (4)$$

by our induction hypothesis applied to the measure $\nu_{\mathbf{s}}$ and

$$\int_{\bar{J}} \nu_{\mathbf{s}}(I)^p d\mathbf{s} \leq C \cdot \sum_{J \subset \bar{J}} l(J)^{p(\rho_m-m)} \cdot l(J)^m \cdot \mu(I \times J)^p \quad (5)$$

by the case $k = 1$ for the measure μ . Combining (4) and (5) proves (3) for $(k+1)$ factors and settles our claim.

To complete the proof of Theorem 1 we take $\mu \in M_+(R^d)$ and let μ_M be the restriction of μ to the dyadic cube $I_M = [0, 2^M]^d$, for a positive integer M . Then

$$\int_{I_M} [R_{\rho}^{dy} \mu_M(\mathbf{x})]^p d\mu(\mathbf{x}) \leq C \cdot \int W_{\rho, \lambda, p}^{dy} \mu(\mathbf{x}) d\mu_M(\mathbf{x}) \leq C \cdot \int W_{\rho, \lambda, p}^{dy} \mu(\mathbf{x}) d\mu(\mathbf{x})$$

by (3) and letting $M \rightarrow \infty$ gives

$$\int_D [R_{\rho}^{dy} \mu(\mathbf{x})]^p d\mu(\mathbf{x}) \leq C \cdot \int W_{\rho, \lambda, p}^{dy} \mu(\mathbf{x}) d\mu(\mathbf{x})$$

by Monotone Convergence, where D is the quadrant $[0, \infty)^d$. Repeating this argument for the other quadrants proves Theorem 1. $\triangle$

Our proof of Theorem 2 uses the result in Theorem 1 in an essential way and also contains a much simplified proof of the case $k = 1$ in [2], p. 1656.

Proof of Theorem 2. Let $\bar{I} = \bar{I}_1 \times \cdots \times \bar{I}_k$ be a fixed dyadic rectangle in R^d . We must prove that

$$\int_{\bar{I}} [R_{\rho}^{dy} \mu(\mathbf{x})]^p d\mu(\mathbf{x}) \leq C \cdot l(\bar{I})^{n-\lambda} \cdot \int W_{\rho, \lambda, p}^{dy} \mu(\mathbf{x}) d\mu(\mathbf{x}), \quad (6)$$

for some constant C that only depends on d, ρ, λ and p . We split μ in three terms as follows. Let $\mu = \mu_1 + \mu_2 + \mu_3$, where μ_1 and μ_2 are the restrictions of μ to $\bar{I}$ and $\bar{I}_1^c \times \cdots \times \bar{I}_k^c$ respectively. We start with the principal term μ_1 . Then

$$\int_{\bar{I}} [R_{\rho}^{dy} \mu_1(\mathbf{x})]^p d\mu(\mathbf{x}) \leq C \int_{\bar{I}} \left(\sum_{I \subset \bar{I}} l(I)^{\rho-n} \cdot \mu_1(I) \cdot \chi_I(\mathbf{x}) \right)^p d\mathbf{x} \leq$$

$$\begin{aligned}
&\leq C \cdot \sum_{I \subset \bar{I}} l(I)^{p(\rho-n)} \cdot l(I)^n \cdot \mu(I)^p = C \cdot l(\bar{I})^{n-\lambda} \cdot \sum_{I \subset \bar{I}} \left(l(I)^{\rho-n+\lambda/p} \cdot \mu(I) \right)^p \leq \\
&\leq C \cdot l(\bar{I})^{n-\lambda} \cdot \int W_{\rho,\lambda,p}^{dy} \mu(\mathbf{x}) d\mu(\mathbf{x}),
\end{aligned}$$

by (3) and the definition of $W_{\rho,\lambda,p}^{dy} \mu(\mathbf{x})$. This proves (6) for the principal term μ_1 .

Next we consider the second and easiest term μ_2 . We show that $R_\rho^{dy} \mu_2(\mathbf{x})$ is bounded on $\bar{I}$. For each i , $1 \leq i \leq k$, we let $\bar{I}_i^{j_i}$ denote the unique dyadic cube containing $\bar{I}_i$ with $l(\bar{I}_i^{j_i}) = 2^{j_i} \cdot l(\bar{I}_i)$, $j_i \geq 0$. Define a k -vector $j = (j_1, j_2, \dots, j_k)$ and put $\bar{I}^j = \bar{I}_1^{j_1} \times \bar{I}_2^{j_2} \times \dots \times \bar{I}_k^{j_k}$. Then for any $x \in \bar{I}$

$$R_\rho^{dy} \mu_2(\mathbf{x}) = \sum' l(I)^{\rho-n} \cdot \mu(I) \cdot \chi_I(x) + \sum_{j \geq 0} l(\bar{I}^j)^{\rho-n} \cdot \mu_2(\bar{I}^j),$$

where the first sum $\sum'$ is over all dyadic rectangles $I = I_1 \times I_2 \times \dots \times I_k$ with $I_j \subset \bar{I}_j$ for some $1 \leq j \leq k$ and consequently equals zero. Hence we get the upper bound

$$\begin{aligned}
R_\rho^{dy} \mu_2(\mathbf{x}) &\leq \sum_{j \geq 0} l(\bar{I}^j)^{\rho-n} \cdot \mu_2(\bar{I}^j) = \sum_{j \geq 0} l(\bar{I}^j)^{-\lambda/p} \cdot l(\bar{I}^j)^{\rho-n+\lambda/p} \mu_2(\bar{I}^j) \leq \\
&\leq \left(\sum_{j \geq 0} l(\bar{I}^j)^{-\lambda q/p} \right)^{1/q} \cdot \left(\sum_{j \geq 0} l(\bar{I}^j)^{(\rho-n)p+\lambda} \cdot \mu_2(\bar{I}^j)^p \right)^{1/p} \leq \\
&\leq C \cdot l(\bar{I})^{-\lambda/p} \cdot \left(\sum_{j \geq 0} l(\bar{I}^j)^{(\rho-n)p+\lambda} \cdot \mu_2(\bar{I}^j)^p \right)^{1/p},
\end{aligned}$$

by Hölder's inequality, with $1/p + 1/q = 1$. Integrating to the p -th power over $\bar{I}$ gives that

$$\int_{\bar{I}} [R_\rho \mu_2(\mathbf{x})]^p d\mathbf{x} \leq C \cdot l(\bar{I})^{n-\lambda} \cdot \int_{\bar{I}} W_{\rho,\lambda,p}^{dy} \mu(\mathbf{x}) d\mu(\mathbf{x})$$

by the definition of $W_{\rho,\lambda,p}^{dy} \mu(\mathbf{x})$, which proves (6) for μ_2 .

Before we proceed to the last term we need some more notation. If $A = \{a_1, a_2, \dots, a_l\}$, $a_1 < a_2 < \dots < a_l$, is a nonempty subset of $\{1, 2, \dots, k\}$ and $v = (v_1, v_2, \dots, v_k)$ is a k -vector, we define the l -vector $v_A = (v_{a_1}, v_{a_2}, \dots, v_{a_l})$.

If $v > 0$ and α is any other k -vector we put $v_A^\alpha = (v_A)^{\alpha_A}$, as defined above. In particular, if $\mathbf{x} = (x_1, x_2, \dots, x_k)$ is in R^d then $\mathbf{x}_A$ is a point in $R^A = R^{n_{a_1}} \times R^{n_{a_2}} \times \dots \times R^{n_{a_l}}$ and if I is a dyadic rectangle we put $I_A = I_{a_1} \times I_{a_2} \times \dots \times I_{a_l}$.

Now we consider the last term μ_3 , which is a mixture of μ_1 and μ_2 in a sense to be described below. We split $\{1, 2, \dots, k\}$ into two nonempty and disjoint sets $A = \{a_1, a_2, \dots, a_l\}$ and $B = \{1, 2, \dots, k\} \setminus A$ and consider the class of dyadic rectangles $I = I_1 \times I_2 \times \dots \times I_k$ such that $I_i \subset \bar{I}_i$, for $i \in A$, and $2\bar{I}_i \subset I_i$, for $i \in B$. Let $\mathbf{x} \in \bar{I}$ then $R_\rho^{dy} \mu_3(\mathbf{x})$ only depends on $\mathbf{x}_A$ and we have

$$\begin{aligned} R_\rho^{dy} \mu_3(\mathbf{x}) &= R_\rho^{dy} \mu_3(\mathbf{x}_A) \leq \\ &\leq C \cdot \sum_{I_i \subset \bar{I}_i, i \in A} (l(I)^{\rho-n})_A \cdot \left(\sum_{\bar{I}_i \subset I_i, i \in B} (l(I)^{\rho-n})_B \cdot \mu(I) \right) \cdot \chi_{I_A}(\mathbf{x}_A), \end{aligned}$$

where

$$\nu(I_A) = \sum_{\bar{I}_i \subset I_i, i \in B} l(I)_B^{\rho-n} \cdot \mu(I)$$

defines ν on dyadic rectangles I_A and can be extended to a nonnegative measure in R^A in the standard way. We now get

$$\begin{aligned} \int_{\bar{I}} [R_\rho^{dy} \mu_3(\mathbf{x})]^p d\mathbf{x} &= l(\bar{I})_B^n \cdot \int_{\bar{I}_A} [R_\rho^{dy} \mu_3(\mathbf{x}_A)]^p d\mathbf{x}_A \leq \\ &\leq C \cdot l(\bar{I})_B^n \cdot l(\bar{I})_A^{n-\lambda} \cdot \sum_{I_i \subset \bar{I}_i, i \in A} \left(l(I_A)_A^{(\rho-n)+\lambda/p} \cdot \nu(I_A) \right)^p \end{aligned}$$

by the first case, with μ replaced by ν , and

$$\nu(I_A)^p = \left(\sum_{\bar{I}_i \subset I_i, i \in B} l(I)_B^{\rho-n} \cdot \mu(I) \right)^p \leq C \cdot l(\bar{I})_B^{-\lambda} \cdot \sum_{\bar{I}_i \subset I_i, i \in B} l(I_B)^{p(\rho-n)+\lambda} \cdot \mu(I)^p$$

by the second case applied to the measure μ . Combining the last two inequalities with the definition of $W_{\rho, \lambda, p}^{dy} \mu$ and summing over all choices of A and B proves (6) also in this last case. $\triangle$

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