This is a technical report published by the Department of Computing Science, Umeå University.

Frank Drewes, Joost Engelfriet
The Generative Power of Delegation Networks
UMINF reports 12.15
The Generative Power of Delegation Networks

Frank Drewes
Department of Computing Science, Umeå University
S–901 87 Umeå, Sweden
drewes@cs.umu.se

Joost Engelfriet
Leiden Institute of Advanced Computer Science, Leiden University
P.O. Box 9612, NL-2300 RA Leiden, The Netherlands
engelfri@liacs.nl

Abstract. A device that generates trees over a ranked alphabet Σ, together with an interpretation of the symbols in Σ as functions or relations on a domain A, generates subsets of A. This concept of tree-based generators is well known and essentially already present in the seminal paper by Mezei and Wright from 1967. A delegation network is a system consisting of a finite set of such generators that can “delegate” parts of the generation process to each other. It can be viewed as consisting of an (extended) IO context-free tree grammar and an interpretation. We investigate the language-theoretic properties of these systems and establish several characterizations of the generated languages. In particular, we obtain results in the style of Mezei and Wright. We also study the hierarchy of tree language classes obtained by iterating the concept of delegation, and show that this hierarchy is properly contained in the closure of the regular tree languages under nondeterministic macro tree transductions, but not in the IO-hierarchy.

1 Introduction

The theory of tree languages and tree transformations is an important and lively field of theoretical computer science [GS84, NP92, GS97, FV98, CDG02, FV09]. For the most part, this theory is concerned with formal devices that generate, recognize or transform trees over a finite ranked alphabet. In such an alphabet Σ, every symbol f ∈ Σ has a rank k that determines the number of (direct) subtrees that it requires: f[t₁,...,tₖ] denotes the tree with root label f and subtrees t₁,...,tₖ.

Let us use the term tree generator for any device γ that defines a tree language L(γ) ⊆ T₀, where T₀ denotes the set of all trees over Σ. A tree t ∈ T₀ can be seen as a formal expression composed of abstract operation symbols, where ranks are arities. The usefulness of devices dealing with such trees is to a large extent based on the fact that trees can be interpreted by choosing a domain
A and associating a (total) function $f: \mathcal{A}^k \to \mathcal{A}$ with each symbol $f$ of rank $k$. Thus, given such an interpretation $\sigma$, a tree $t$ evaluates recursively to an element $\sigma(t)$ of $\mathcal{A}$: $\sigma(f[t_1,\ldots,t_k]) = f(\sigma(t_1),\ldots,\sigma(t_k))$. Consequently, a device that generates trees provides the syntactic component of a tree-based generator – a pair consisting of the tree generator $\gamma$ with $L(\gamma) \subseteq T_\Sigma$ and an interpretation $\sigma$ of the symbols of $\Sigma$. By definition, it generates a subset of $\mathcal{A}$, viz. the set $\sigma(L(\gamma)) = \{\sigma(t) \mid t \in L(\gamma)\} \subseteq \mathcal{A}$. The application of this idea to the area of picture generation is studied in [Dre00, Dre01, Dre06].

In this paper we study so-called delegation networks, which were introduced in [Dre07a, Dre07b]. Such a delegation network consists of a finite number of generators that can “delegate” parts of the generation process to each other. The purpose of delegation is to make it possible to combine several generators, that may even be of different types, to generate complex objects. Intuitively, an individual generator in a delegation network can be compared to a nondeterministic procedure that can make use of other such procedures in order to accomplish its task. For example, one could envision the generation of 3D models of cities. In a delegation network for this purpose, one generator could generate the overall structure of the city, placing roads, parks and buildings in appropriate places while delegating the generation of these details to other generators. The “park generator” could in turn delegate the generation of plants to a generator that uses a suitable type of grammar to generate the branching structure of a plant, while the form of leaves may be generated by a different type of grammar and their texture by yet another one (see [Dre07a] for discussions of such devices). Delegation networks also make it possible to use non-grammatical generators by viewing them as so-called primitives – “black boxes” that turn input into output in a possibly nondeterministic way (see below). For example, in the hypothetical scenario above, textures may be generated by a primitive whose internal workings are based on random noise or cellular automata. Such a primitive could, for instance, take a 3D object as input and map a generated texture onto its surface. For concrete examples of delegation networks that combine different types of picture-generating mechanisms, see [Dre07b].

Let us discuss the formal notions underlying delegation networks in more detail. Just as tree-based generators, delegation networks use tree generators to generate subsets of a domain $\mathcal{A}$. However, in order to accomplish the ideas sketched above, they generalize tree-based generators in two ways.

The first generalization concerns the type of interpretations used. As in [ES77, ES78], where they are called nondeterministic $\Sigma$-algebras, we consider interpretations in which $f$, the interpretation of a symbol $f$ of rank $k$, may be an arbitrary relation $f \subseteq \mathcal{A}^k \times \mathcal{A}$ rather than being required to be a function from $\mathcal{A}^k$ to $\mathcal{A}$. Clearly, ordinary interpretations are a special case if we regard a function as a relation in the usual way. A tuple $((a_1,\ldots,a_k), a) \in f$ represents the fact that an application of $f$ to the arguments $a_1,\ldots,a_k$ may, nondeterministically, produce the value $a$. We write $f(a_1,\ldots,a_k)$ for the set $\{a \in \mathcal{A} \mid ((a_1,\ldots,a_k), a) \in f\}$.
of possible values of the application of $f$ to $a_1,\ldots,a_k$.

As a consequence of this generalization, the evaluation of a tree now determines a subset of $\mathbb{A}$, which is the set of its possible values:

$$\sigma(f[t_1,\ldots,t_k]) = \bigcup \{f(a_1,\ldots,a_k) \mid a_1 \in \sigma(t_1),\ldots,a_k \in \sigma(t_k)\},$$

and $\sigma(L(\gamma)) = \bigcup \{\sigma(t) \mid t \in L(\gamma)\} \subseteq \mathbb{A}$.

We note here that delegation networks were defined over many-sorted signatures in [Dre07a, Dre07b], using a finite set of domains instead of a single domain $\mathbb{A}$. Hence, in that setting, the interpretation of a symbol $f$ of rank $k$ is a relation $f \subseteq (\mathbb{A}_1 \times \cdots \times \mathbb{A}_k) \times \mathbb{A}_{k+1}$, where $\mathbb{A}_1,\ldots,\mathbb{A}_{k+1}$ may differ from each other. In view of the motivation of delegation networks sketched above, it seems natural to adopt this more general setting. However, many-sortedness does not affect any of the formal reasonings in this paper – its sole effect is to complicate notation. Therefore, we have chosen to restrict the exposition to one-sorted signatures (which are ranked alphabets) in the present paper.

The second generalization that leads from tree-based generators to delegation networks is obtained by evaluating trees that contain $n$ formal parameters. More precisely, we consider trees $t \in T_{\Sigma,n} = T_{\Sigma,\{x_1,\ldots,x_n\}}$, where the so-called parameter symbols $x_i$ are of rank $0$. Then $\sigma(t) \subseteq \mathbb{A}^n \times \mathbb{A}_i$ as follows. For $n$ arguments $a_1,\ldots,a_n$ (the actual parameters), $t$ is recursively evaluated as before, with the addition that each $x_i$ is interpreted as $a_i$. To be precise, $\sigma(x_i)(a_1,\ldots,a_n) = \{a_i\}$ and $\sigma(f[t_1,\ldots,t_k])(a_1,\ldots,a_n)$ is the union of all $f(b_1,\ldots,b_k)$ where $b_j \in \sigma(t_j)(a_1,\ldots,a_n)$.

Note that, if we now consider a tree generator $\gamma$ that generates trees with parameter symbols, i.e., $L(\gamma) \subseteq T_{\Sigma,n}$, then the straightforward definition of $\sigma(L(\gamma))$ yields a subset of $\mathbb{A}^n \times \mathbb{A}$: for $a_1,\ldots,a_n \in \mathbb{A}$, $\sigma(L(\gamma))(a_1,\ldots,a_n) = \bigcup \{\sigma(t)(a_1,\ldots,a_n) \mid t \in L(\gamma)\}$. Thus, a tree generator with $k$ parameter symbols can be used to define the interpretation $g \subseteq \mathbb{A}^k \times \mathbb{A}$ of a symbol $g \in \Sigma$ of rank $k$.

The last observation provides the formal basis for delegation, as it allows us to consider a finite set of tree generators that call each other recursively. More precisely, we divide $\Sigma$ into two ranked alphabets $\Pi$ and $\mathcal{G}$ of so-called primitives and generator symbols, respectively. The primitives come with an a priori interpretation $\pi$, whereas every generator symbol $g$ of rank $k$ is associated with a tree generator $\gamma_g$ that generates a tree language in $T_{\mathcal{G},\Pi,k}$. Intuitively, the purpose of $\gamma_g$ is to generate the interpretation of $g$ in the sense of the previous paragraph. Whenever $g$ appears in a tree generated by $\gamma_{g'}$ for some $g' \in \mathcal{G}$, this signals a delegation to $\gamma_g$ by $\gamma_{g'}$, i.e., $\gamma_{g'}$ calls $\gamma_g$. The semantics of delegation is straightforward in the case where $L(\gamma_g) \subseteq T_{\Pi,k}$. Then $g$ is interpreted as $\pi(L(\gamma_g))$.

The general case (in which $L(\gamma_g) \subseteq T_{\mathcal{G},\Pi,k}$) is slightly more complicated since delegation can be cyclic, which means that we will naturally adopt a least fixed point semantics.

Every delegation network $\mathcal{N}$ contains a designated main generator symbol $g_0$ of rank $0$. According to the semantics of delegation networks, $g_0$ generates a
subset of $A$. This set is considered to be the “language” $L(N)$ generated by $N$. As we will see, an appropriate operational view of delegation is that of parallel distributed processes executing (copies of) the tree generators $\gamma_g$ and evaluating the resulting trees, as follows. Initially, a process corresponding to $g_0$ is started. This process uses $\gamma_{g_0}$ to generate a tree. Then, for every individual occurrence of a symbol $g$ in that tree, a new process working in a similar manner (using $\gamma_g$ instead of $\gamma_{g_0}$) is started recursively. Evaluation is taken care of locally in these processes as well, as each process evaluates the tree it has generated. To be able to do this, its parent process must eventually provide it with each of its actual parameters (call-by-value). Evaluation of a tree works as follows: as soon as the subtrees of (an occurrence of) a symbol $f \in \Pi$ have been evaluated, $f$ is applied to them nondeterministically. When a subtree of a symbol $g \in G$ has been evaluated, the resulting value is passed to the child process that evaluates the occurrence of $g$, providing it with one of its actual parameters. When the child process has received all its actual parameters and has finished the evaluation of its tree (i.e., has arrived at the root), it reports the value it has computed to the parent process and terminates. The parent process can then continue the evaluation of its own tree. Eventually, when the main process has finished the evaluation of its tree, the resulting value is returned, being an element of the language generated by the delegation network.

A delegation network can be seen as an “extended” IO context-free tree grammar that is interpreted in a nondeterministic algebra $(A, \pi)$. In this sense, the present paper contributes to the theory of context-free tree grammars, which are of interest in, e.g., recursive program scheme theory, computational linguistics and data compression theory. The primitives and generator symbols of the network correspond to the terminals and nonterminals of the grammar, respectively, and the tree language $L(\gamma_g)$ is the set of right-hand sides of the rules of $g$ (which have the left-hand side $g[x_1, \ldots, x_k]$ when $g$ has rank $k$). Thus, as in the extended Backus-Naur Form, there may be infinitely many right-hand sides for a given left-hand side. Non-extended (“finitary”) IO context-free tree grammars and their interpretation in nondeterministic algebras were studied in [ES77, ES78]. In the present paper, we generalize some of the results of [ES77, ES78], showing that they carry over to the extended case. In particular, we prove several MW-like results. Here, ‘MW’ refers to the seminal paper [MW67], in which Mezei and Wright showed that the equational subsets of an algebra are exactly those that can be obtained by evaluating regular tree languages with respect to the algebra in question.

Let us discuss the main results of this paper. In Section 3 (Theorem 3.5), we prove an operational characterization of the language generated by a delegation network (defined as a least fixed point in Section 2). This characterization formalizes the operational view discussed above and shows that this view is adequate.

1Recent contributions in these fields are [Sch11], [KR11, ME12], and [LMSS12], respectively.
Theorem 5.6 in Section 5 is our main MW-like result. It solves a problem left open in [ES78] (for the finitary case), namely to find an MW-like characterization of the languages generated by delegation networks in the presence of nondeterministic primitives, i.e., primitives that are interpreted as non-functional relations. We accomplish this by generating and interpreting so-called jungles [HKP91] instead of trees.

Essentially, a jungle $J$ over the ranked alphabet $\Sigma$ is a DOAG (directed ordered acyclic graph) with a designated node, such that each node with $k$ outgoing edges (which are linearly ordered) is labeled with a symbol from $\Sigma$ of rank $k$, cf. [AG68]. Each node $v$ of $J$ determines a tree over $\Sigma$ obtained by unfolding the subgraph induced by all nodes that are reachable from $v$. Thus, a jungle $J$ can be viewed as a set of trees over $\Sigma$ of which the nodes can be shared, and it represents the tree determined by its designated node, denoted by tree($J$). The nodes that are not reachable from the designated node are said to be garbage, as they do not contribute to tree($J$). Given an interpretation $\sigma$, the (nondeterministic) evaluation of $J$ is defined inductively by using $\sigma$ to assign values to the nodes of $J$ in a bottom-up fashion.

In Section 5 we let each delegation network $\mathcal{N}$ generate a set of jungles, denoted $L_J(\mathcal{N})$, and we show that $L(\mathcal{N}) = \pi(L_J(\mathcal{N}))$, our main MW-like result. In fact, it is straightforward to turn an IO context-free tree grammar into a (context-free) graph grammar that generates jungles, by viewing each tree $t$ in $L(\gamma_g)$ as a jungle $J_k(t)$ that contains exactly one node with label $x_i$ for every parameter symbol $x_i$ (where $1 \leq i \leq k$ and $k$ is the rank of $g$). Thus, all nodes of $t$ with label $x_i$ are shared, and a garbage node with label $x_i$ is created when $x_i$ does not occur in $t$. Intuitively, this is in accordance with the call-by-value semantics of $\mathcal{N}$: whenever $x_i$ is used, the same (nondeterministic) value of $x_i$ must be used; moreover, every parameter $x_i$ must have at least one possible value.

Jungles are defined in Section 4, not as DOAGs with labeled nodes, but as acyclic hypergraphs with labeled hyperedges (as in [HKP91]; see also [HP91, CR93, Plu99]). This simplifies the definition of $L_J(\mathcal{N})$, which can then be generated by a so-called hyperedge-replacement graph grammar; see, e.g., [BC87, HK87, Hab92, Eng97, DHK97, CE12]. These (hyper)graph grammars are well known, well investigated and easy to understand.

In Section 6, we obtain further MW-like characterizations for the special case of deterministic primitives, i.e., primitives that are interpreted as functions.

- The delegation network $\mathcal{N}_{\text{free}}$, which is $\mathcal{N}$ with $\pi$ replaced by the free interpretation of symbols in $\Pi$, generates trees over $\Pi$. Our second MW-like result states that $L(\mathcal{N}) = \pi(L(\mathcal{N}_{\text{free}}))$ for delegation networks with deterministic primitives (Theorem 6.2).

- Alternatively, trees can be generated operationally by viewing $\mathcal{N}$ as an (extended) IO context-free tree grammar. We show in Theorem 6.4 that the language $L_T(\mathcal{N})$ generated in this way is equal to $L(\mathcal{N}_{\text{free}})$, which yields Theorem 6.6, our third MW-like characterization: $L(\mathcal{N}) = \pi(L_T(\mathcal{N}))$, still
under the assumption of deterministic primitives. This generalizes [ES78, Theorem 5.10 and Corollary 5.11].

• We also show, in Theorem 6.7, that these two MW-like characterizations hold in the presence of nondeterministic primitives, provided that \( N \) is linear and nondeleting, i.e., each parameter of \( g \) occurs exactly once in each tree of \( L(\gamma_g) \).

Since the tree language \( L_T(N) \) generated by a delegation network \( N \) only depends on the tree languages \( L(\gamma_g) (g \in G) \), delegation can be regarded as an operator \( \text{DEL} \) on classes of tree languages: for a class \( C \), \( \text{DEL}(C) \) is the class of tree languages \( L_T(N) \) such that \( L(\gamma_g) \in C \) for every \( g \in G \). In Sections 7 and 8 we study the hierarchy \( \text{DEL}^*(\text{FIN}) = \bigcup_{n \geq 0} \text{DEL}^n(\text{FIN}) \), which is obtained by starting with the finite tree languages and iterating the delegation operator, similar to the IO-hierarchy studied in [Mai74] and [ES78, Section 7]. Clearly, \( \text{DEL}^*(\text{FIN}) \) is the smallest nonempty class of tree languages that is closed under \( \text{DEL} \): if the tree generators of a delegation network \( N \) generate tree languages in that class, then so does \( N \).

In Section 7 we establish a result that can be used to prove that certain tree languages are not in \( \text{DEL}^*(\text{FIN}) \). As usual, we encode a path \( (v_1, \ldots, v_n) \) in a tree \( t \) over \( \Sigma \), from the root \( v_1 \) to some node \( v_n (n \geq 1) \), as the string \( \langle f_1, j_1 \rangle \cdots \langle f_{n-1}, j_{n-1} \rangle \langle f_n, 0 \rangle \) where \( f_i \) is the label of \( v_i \) in \( t \), and \( v_{i+1} \) is the \( j_i \)-th child of \( v_i \) in \( t \). The result states that the set of (encoded) paths in the trees of each tree language in \( \text{DEL}^*(\text{FIN}) \) is a context-free language (Theorem 7.23). The proof is long and complicated. It makes essential use of jungles, in particular of the fact that \( L_T(N) = \text{tree}(L_3(N)) \), which is proved in Corollary 6.5.

In Section 8 we prove that \( \text{DEL}^*(\text{FIN}) \) is included in \( \text{MTT}^*(\text{REG}) \), the closure of the class of regular tree languages under nondeterministic macro tree transductions (Theorem 8.15), but not in the smaller class \( \text{YIELD}^*(\text{REG}) \) of tree languages of the IO-hierarchy (Theorem 8.17). By the result of Section 7, \( \text{DEL}^*(\text{FIN}) \) is properly included in the first class and incomparable with the second. Since the inclusion of \( \text{DEL}^*(\text{FIN}) \) in \( \text{MTT}^*(\text{REG}) \) is effective, we immediately obtain the decidability of the membership, emptiness and finiteness problems for every language in \( \text{DEL}^*(\text{FIN}) \), as shown for \( \text{MTT}^*(\text{REG}) \) in [EV85, DE98]. Finally, we prove that, in fact, \( \text{MTT}^*(\text{REG}) \) is closed under \( \text{DEL} \) (Theorem 8.19).

2 Delegation Networks

Before turning to the definition of delegation networks, which will be given later in this section, let us summarize some basic terminology and notation.

We denote the set of all natural numbers (including zero) by \( \mathbb{N} \). For \( n \in \mathbb{N} \), we let \([n]\) denote \( \{1, \ldots, n\} \). By convention, this is the empty set if \( n = 0 \). The powerset of a set \( A \) is denoted by \( \wp(A) \). The set of all finite strings (or sequences) over \( A \) is denoted by \( A^* \); \( \lambda \) denotes the empty string, and \( A^+ = A^* \setminus \{\lambda\} \). For a string \( u = a_1 \cdots a_k \) over \( A \) (where \( a_1, \ldots, a_k \in A \)), we let \([u] = \{a_1, \ldots, a_k\} \). In
other words, \([u]\) is the smallest set \(A' \subseteq A\) such that \(u \in A'^*\). For sets \(A\) and strings \(u\), the cardinality of \(A\) and the length of \(u\) are denoted by \(|A|\) and \(|u|\), respectively. For a string \(u\) as above and \(i \in [k]\), \(a_i\) is also denoted by \(u(i)\). More generally, for a function \(f: S \rightarrow A^*\) and \(s \in S\), if \(f(s) = a_1 \cdots a_k\), then \(f(s,i)\) denotes \(a_i\), for all \(i \in [k]\).

We assume functions to be total, i.e., if \(f: A \rightarrow B\) is a function, then \(f(a)\) is defined for every \(a \in A\); functions from \(A\) to \(B\) that are not necessarily total, are called partial functions. For a function \(f: A \rightarrow B\), the canonical extensions of \(f\) to subsets of \(A\) and to strings over \(A\) are denoted by \(f\) as well, i.e., \(f(A') = \{f(a) \mid a \in A'\}\) for all \(A' \subseteq A\), and \(f(a_1 \cdots a_k) = f(a_1) \cdots f(a_k)\) for \(a_1, \ldots, a_k \in A\).

A binary relation \(r \subseteq A \times B\) may alternatively be viewed as a function \(r: A \rightarrow \wp(B)\) that maps elements of \(A\) to sets of elements of \(B\). As usual, we let \(r(a) = \{b \in B \mid (a,b) \in r\}\) for \(a \in A\), and \(r(A') = \bigcup_{a \in A'} r(a) = \{b \in B \mid \exists a \in A': (a,b) \in r\}\) for \(A' \subseteq A\). Note that this is consistent with the definition of \(f(A')\) above (where we identify a function \(f: A \rightarrow B\) with the relation \(\{(a,f(a)) \mid a \in A\}\), as usual). Observe that \(A\) can be a cartesian product \(A_1 \times \cdots \times A_k\), i.e., \(r\) may be of the form \(r \subseteq (A_1 \times \cdots \times A_k) \times B\). In that case, we let \(r(A'_1, \ldots, A'_k) = r(A'_1 \times \cdots \times A'_k)\) for \(A'_1 \subseteq A_1, \ldots, A'_k \subseteq A_k\). If \(k = 0\), we identify \(r\) with the set \(r() \subseteq B\). As usual, the composition of binary relations \(r \subseteq A \times B\) and \(s \subseteq B \times C\) is given by \(s \circ r = \{(a,c) \in A \times C \mid c \in s(r(a))\}\).

### 2.1 Signatures and Trees

A signature (or ranked alphabet) is a pair \((\Sigma, rk)\), where \(\Sigma\) is a set of symbols, and \(rk\) assigns to every \(\mathbf{f} \in \Sigma\) a rank \(rk(\mathbf{f}) \in \mathbb{N}\). In the following, we shall simply denote \((\Sigma, rk)\) by \(\Sigma\). If necessary, the rank \(k\) of a symbol \(\mathbf{f}\) is indicated by denoting \(\mathbf{f}\) as \(\mathbf{f}^{(k)}\). Symbols of rank 0 are called constant symbols.

The set of all trees over \(\Sigma\) is denoted by \(T_\Sigma\). By definition, it is the smallest set of strings satisfying the following condition:

For all \(\mathbf{f}^{(k)} \in \Sigma\) \((k \in \mathbb{N})\) and all \(t_1, \ldots, t_k \in T_\Sigma\), the string \(\mathbf{f}[t_1, \ldots, t_k]\) is in \(T_\Sigma\). (Here, the square brackets and the comma are assumed to be special symbols not in \(\Sigma\).)

A tree of the form \(\mathbf{f}[\lambda]\) (i.e., where \(\mathbf{f}\) is a constant symbol in \(\Sigma\)) is identified with the string \(\mathbf{f}\) of length 1. By this convention, all constant symbols are trees in \(T_\Sigma\).

A tree \(t \in T_\Sigma\) can be viewed as a graph whose nodes are labelled with symbols in \(\Sigma\). A node is a string in \((\mathbb{N} \setminus \{0\})^*\) which, intuitively, represents the path from the root to the node in question. Formally, we define the set \(V(t)\) of nodes of \(t\), the subtree \(t/v\) at a node \(v\), and the label \(\ell_t(v)\) of that node inductively, as follows. If \(t = \mathbf{f}[t_1, \ldots, t_k]\), then

\[
V(t) = \{\lambda\} \cup \{iv \mid i \in [k], v \in V(t_i)\}.
\]

Furthermore, \(t/\lambda = t\), \(\ell_t(\lambda) = \mathbf{f}\), and, for all \(i \in [k]\) and \(v \in V(t_i)\), \(t/iv = t_i/v\) and \(\ell_t(iv) = \ell_{t_i}(v)\). The node \(\lambda\) is the root of \(t\). As usual, a tree \(s\) is a subtree of
t if \( s = t/v \) for some \( v \in V(t) \). A node \( v \) of \( t \) is said to be an occurrence of the symbol \( \ell_t(v) \) and of the subtree \( t/v \). If \( t = f[t_1, \ldots, t_k] \), then the subtrees \( t_1, \ldots, t_k \) are also called the subtrees of (this particular occurrence of) the symbol \( f \). We will not always distinguish precisely between a symbol (or subtree) and one of its occurrences; this should be clear from the context.

For a signature \( \Sigma' \), we let \( V_{\Sigma'}(t) = \{ v \in V(t) \mid \ell_t(v) \in \Sigma' \} \). We say that \( t \) is linear in \( \Sigma' \) if each symbol in \( \Sigma' \) occurs at most once in \( t \) and nondeleting in \( \Sigma' \) if each symbol in \( \Sigma' \) occurs at least once in \( t \) (formally, \( |V_{\sigma}(t)| \leq 1 \), respectively \( |V_{\bar{\sigma}}(t)| \geq 1 \), for every \( \sigma \in \Sigma' \)).

For finite \( \Sigma \), a subset of \( T_\Sigma \) is a tree language. Any device \( \gamma \) that generates such a tree language is a tree generator; \( \Sigma \) is said to be its output signature, and the generated tree language is denoted by \( L(\gamma) \).

### 2.2 Interpretations

Let \( A \) be a domain, i.e., a set. We wish to be able to interpret every symbol of a signature \( \Sigma \), and to use this interpretation for evaluating trees over \( \Sigma \). Usually, the interpretation of a symbol \( f^{(k)} \) would be a \( k \)-ary function \( f : A^k \to A \). However, as mentioned in the Introduction, we generalize this in order to be able to model nondeterminism\(^2\). For this purpose, symbols are interpreted as relations \( f \subseteq A^k \times A \) rather than as functions. One may view such a relation \( f \) as a “nondeterministic function” \( f : A^k \to \wp(A) \) that, for an argument tuple in \( A^k \), yields the set of possible results. Of course, functions and partial functions are special cases.

A \( \Sigma \)-interpretation (into \( A \)) is a function \( \sigma \) that assigns to every symbol \( f^{(k)} \in \Sigma \) a relation \( \sigma(f) \subseteq A^k \times A \). If \( \sigma(f) \) is a function for all \( f \in \Sigma \), then we call \( \sigma \) a deterministic \( \Sigma \)-interpretation. Note that this implies totality. Throughout the rest of the paper, we will use the typographical convention that the interpretation \( \sigma(f) \) of a symbol \( f \) is denoted by \( f \). Thus, if \( f^{(k)} \in \Sigma \), then \( f \subseteq A^k \times A \) (and \( f : A^k \to A \) if \( \sigma \) is deterministic).

In the following, we want to represent complex operations by trees. For this purpose, let us define a parameter signature of rank \( n \in \mathbb{N} \) to be a signature \( Y = \{ y_1, \ldots, y_n \} \) consisting of pairwise distinct indexed symbols \( y_1, \ldots, y_n \), where \( y_i \) has rank 0 for every \( i \in [n] \). Note that the indices provide \( Y \) with a linear order. For a signature \( \Sigma \) disjoint with \( Y \), the set \( T_{\Sigma,Y} \) is denoted by \( T_{\Sigma,Y} \) to emphasize the special role of parameter symbols. For trees \( t, t_1, \ldots, t_n \in T_{\Sigma,Y} \), we denote by \( t(t_1, \ldots, t_n) \) the substitution of \( t_i \) for each occurrence of \( y_i \) in \( t \) (\( i \in [n] \)). (In this notation, the parameter signature in question will always be clear from the context.)

Given a \( \Sigma \)-interpretation \( \sigma \) and a parameter signature \( Y \) as above, we can evaluate trees \( t \) in \( T_{\Sigma,Y} \). The result of this evaluation, denoted by \( \sigma_Y(t) \), is the relation \( \varphi \subseteq A^n \times A \) such that \( \varphi(a_1, \ldots, a_n) \) is given as follows, for all \( a_1, \ldots, a_n \in A \):

---

\(^2\)See also [ES77, ES78].
Lemma 2.1 Let $\sigma$ be a $\Sigma$-interpretation, for a signature $\Sigma$, and let $Y$ be a parameter signature of rank $n$. For all trees $t \in T_{\Sigma,Y}$ and $t_1, \ldots, t_n \in T_{\Sigma}$, if $|\sigma(t_i)| = 1$ for all $i \in [n]$, then

$$\sigma(t(t_1, \ldots, t_n)) = \sigma(t)(\sigma(t_1), \ldots, \sigma(t_n)).$$

Proof This follows easily from the definition of $\sigma(t)$, using induction on $t$. Obviously, the equation holds if $t = y_i$ for some $i \in [n]$. Further, if $t = f[s_1, \ldots, s_k]$, and $a_i = \sigma(t_i)$ for every $i \in [n]$, we get

$$\sigma(t)(a_1, \ldots, a_n) = f(\sigma(s_1)(a_1, \ldots, a_n), \ldots, \sigma(s_k)(a_1, \ldots, a_n))$$
$$= f(\sigma(s_1(t_1, \ldots, t_n)), \ldots, \sigma(s_k(t_1, \ldots, t_n)))$$
$$= \sigma(f[s_1(t_1, \ldots, t_n), \ldots, s_k(t_1, \ldots, t_n)])$$
$$= \sigma(t(t_1, \ldots, t_n)),$$

as claimed. □

Note that the first equality of the display in the proof above does not hold without the requirement that $|\sigma(t_i)| = 1$: with $A_i = \sigma(t_i) \subseteq \Lambda$, we get

$$\sigma(t)(A_1, \ldots, A_n) = \bigcup_{a_i \in A_i} \sigma(t)(a_1, \ldots, a_n)$$
$$= \bigcup_{a_i \in A_i} f(\sigma(s_1)(a_1, \ldots, a_n), \ldots, \sigma(s_k)(a_1, \ldots, a_n)),$$

in contrast to the fact that

$$f(\sigma(s_1)(A_1, \ldots, A_n), \ldots, \sigma(s_k)(A_1, \ldots, A_n))$$
$$= f(\bigcup_{a_i \in A_i} \sigma(s_1)(a_1, \ldots, a_n), \ldots, \bigcup_{a_i \in A_i} \sigma(s_k)(a_1, \ldots, a_n))$$
$$= \bigcup_{a_i \in A_i} f(\sigma(s_1)(a_1^i, \ldots, a_n^i), \ldots, \sigma(s_k)(a_1^i, \ldots, a_n^i)).$$
This difference is what makes the usual MW-like result fail in the presence of nondeterministic interpretations.

2.3 Delegation Networks and Their Semantics

We are now ready to give the formal definition of delegation networks.

**Definition 2.2 (delegation network)** A delegation network is a system \( N = (G, \Pi, g_0, A, \pi) \), where

- \( G \) and \( \Pi \) are disjoint finite signatures of generator symbols and primitives, resp.,
- \( \Gamma = (\gamma_g, Y_g)_{g \in G} \) is a \( G \)-indexed family of pairs, each consisting of a tree generator \( \gamma_g \) and a parameter signature \( Y_g \) of the same rank as \( g \),
- \( g_0 \) is a constant symbol in \( G \),
- \( A \) is a domain, and
- \( \pi \) is a \( \Pi \)-interpretation into \( A \).

We require that each parameter signature \( Y_g, g \in G \), is disjoint with \( G \cup \Pi \), and that \( L(\gamma_g) \subseteq T_{G \cup \Pi, Y_g} \). The signature \( G \cup \Pi \) is denoted by \( \Sigma_N \).

By the definition above, \( \Sigma_N \) contains two kinds of symbols: the generator symbols \( g \in G \), each of which has an associated tree generator \( \gamma_g \), and the primitives \( f \in \Pi \). In the context of the particular delegation network \( N \), the latter are given a fixed interpretation by means of \( \pi \). Following our general convention, we shall therefore generally denote the interpretation \( \pi(f) \) of such a symbol by \( f \).

The semantics of \( N \) is obtained by constructing a \( \Sigma_N \)-interpretation \( \sigma_N \) that agrees with \( \pi \) on \( \Pi \). To take into account the fact that delegation can be cyclic, we choose a least fixed point semantics, using Tarski’s fixed point theorem.

**Proposition 2.3 ([Tar55, Theorem 1])** Let \( C \) be a complete lattice with partial order \( \leq \). Every monotonically increasing mapping \( \tau \) on \( C \) has a least fixed point, and this least fixed point is equal to the greatest lower bound of all \( c \in C \) such that \( \tau(c) \leq c \).

To continue, recall that the set of all relations \( r \subseteq A^k \) becomes a complete lattice if we choose \( \subseteq \) as its partial order. Its least element is the empty relation, and its largest is \( A^k \). We extend this ordering to \( \Sigma_N \)-interpretations \( \sigma, \sigma' \) in the obvious way: \( \sigma \leq \sigma' \) if and only if \( \sigma(f) \subseteq \sigma'(f) \) for all \( f \in \Sigma_N \).

We can now define the semantics of delegation networks. According to this semantics, a delegation network generates a subset of \( A \). Typically, the reader may think of generating strings, trees, graphs, pictures, etc. In contexts where the grammatical generation of sets of such objects is studied, they are sometimes called languages, generalizing the terminology commonly used in the string case. Here, we follow this tradition.
Definition 2.4 (semantics of delegation networks) Let \( \mathcal{N} = (\mathcal{G}, \Pi, \Gamma, g_0, A, \pi) \) be a delegation network, where \( \Gamma = (\gamma_g, Y_g)_{g \in \mathcal{G}} \).

1. The mapping \( \text{iterate}_{\mathcal{N}} \) on the complete lattice of all \( \Sigma_{\mathcal{N}} \)-interpretations into \( A \) is defined as follows: for every \( \Sigma_{\mathcal{N}} \)-interpretation \( \sigma \), and every symbol \( f \in \Sigma_{\mathcal{N}} \),
\[
\text{iterate}_{\mathcal{N}}(\sigma)(f) = \begin{cases} 
\sigma Y_g(L(\gamma_g)) & \text{if } f \in \mathcal{G} \\
\sigma & \text{if } f \in \Pi.
\end{cases}
\]

2. The least fixed point of \( \text{iterate}_{\mathcal{N}} \) is denoted by \( \sigma_{\mathcal{N}} \). (Note that by Proposition 2.3, \( \sigma_{\mathcal{N}} \) exists, as it is straightforward to verify that \( \text{iterate}_{\mathcal{N}} \) is monotonically increasing.)

3. The language generated by \( \mathcal{N} \) is \( L(\mathcal{N}) = \sigma_{\mathcal{N}}(g_0) \).

Note that the language generated by \( \mathcal{N} \) is a subset of \( A \), because the rank of \( g_0 \) is 0. Observe furthermore that, for \( f \in \Pi \), we have \( \sigma_{\mathcal{N}}(f) = f \). As a consequence, for all \( g \in \mathcal{G} \) with \( L(\gamma_g) \subseteq T_{\Pi} \), \( \sigma_{\mathcal{N}}(g) \) is just \( \pi(L(\gamma_g)) \). This shows that delegation networks can be seen as a generalization of the tree-based generators in [Dre06]: a tree-based generator is a delegation network \( \mathcal{N} \) with \( \mathcal{G} = \{g\} \), where \( \gamma_g \) has the output signature \( \Pi \) (i.e., does not delegate to itself), and \( \pi(f) \) is a function for all \( f \in \Pi \).

Another obvious fact that nevertheless may be worth pointing out is that the semantics of delegation networks does not depend on the particular devices \( \gamma_g \) that generate the tree languages \( L(\gamma_g) \). In fact, delegation networks are closely related to the systems of equations over nondeterministic \( \Pi \)-algebras studied in [ES78]. If we restrict delegation networks to the finitary case, i.e., require that all \( L(\gamma_g) \) (\( g \in \mathcal{G} \)) are finite, then the syntactic part \( (\mathcal{G}, \Pi, \Gamma) \) of a delegation network is what is called a system of context-free \( \Pi \)-equations in [ES78], Definition 5.1 and the least fixed point \( \sigma_{\mathcal{N}} \) is the solution of that system in the algebra of relations over \( \pi \) (see [ES78, Definition 5.3]). Furthermore, for every \( g \in \mathcal{G} \), \( \sigma_{\mathcal{N}}(g) \) is what is proposed to be called the call by value relation computed by \( (\mathcal{G}, \Pi, \Gamma, g) \) over \( \pi \) in the discussion after Corollary 5.7 in [ES78]. In the MW-like result [ES78, Theorem 5.10] it is shown that this relation is uniquely determined by the tree language generated by \( g \) in the IO context-free tree grammar \( (\mathcal{G}, \Pi, \Gamma) \). Note that, in the finitary case, there is no reason at all to distinguish between \( \gamma_g \) and \( L(\gamma_g) \), because finite tree languages can be specified by simply listing their elements. In this sense, a finite tree language is its own generator.

A simplification that can be made in large parts of the paper and, in particular, throughout Sections 3, 6 and 8 concerns the parameter signatures used by the tree generators \( \gamma_g \). Unless the contrary is clear from the context, we assume that \( Y_g \) is the standard parameter signature \( X_k = \{x_1, \ldots, x_k\} \), where \( k \) is the rank of \( g \). In this case, \( Y_g \) will not be specified explicitly, i.e., we let \( \Gamma = (\gamma_g)_{g \in \mathcal{G}} \).

\footnote{It is just another, more suggestive, name for a context-free tree grammar with terminal signature \( \Pi \).}
Moreover, we abbreviate $T_{\Sigma,X_k}$ as $T_{\Sigma,k}$. Thus, $L(\gamma_\mathcal{g}) \subseteq T_{\Sigma,k}$. More generally, we will assume that in an expression $t(t_1, \ldots, t_n)$ the parameter signature $X_n$ is used, unless mentioned otherwise.

We will also make use of special parameter symbols $\Box_1, \Box_2, \ldots$, where $\Box$ abbreviates $\Box_1$, mainly for the purpose of decomposing trees. To avoid confusion when using both the parameter signatures $X_n$ and $\{\Box_1, \ldots, \Box_n\}$, we denote an expression of the form $t(t_1, \ldots, t_n)$ by $t[t_1, \ldots, t_n]$ if it is to be evaluated with respect to the parameter signature $\{\Box_1, \ldots, \Box_n\}$. If, moreover, $t$ is linear and nondeleting in $\{\Box_1, \ldots, \Box_n\}$, then we denote by $t \cdot (t_1, \ldots, t_n)$ the *concatenation of $t$ and $t_1, \ldots, t_n$* given by $t \cdot (t_1, \ldots, t_n) = t[t_1, \ldots, t_n]$. Thus, denoting a tree $s$ in the form $t \cdot (t_1, \ldots, t_n)$ indicates that (a) $s = t(t_1, \ldots, t_n)$ where $\{\Box_1, \ldots, \Box_n\}$ is assumed to be the relevant parameter signature and (b) $t$ is assumed to be linear and nondeleting in $\{\Box_1, \ldots, \Box_n\}$. Hence, each $t_i$ corresponds to a unique occurrence of $t_i$ as a subtree of $s$. In particular, in the case where $n = 1$, $s = t \cdot t_1$ is a tree obtained from $t$ by replacing the unique occurrence of $\Box$ in $t$ with $t_1$.

We conclude this section with a series of small examples that illustrate some of the properties of delegation networks. Bigger examples that, moreover, discuss the many-sorted case of delegation networks, can be found in [Dre07b]. In the examples below, but also in Section 8, regular tree grammars will be used as generators.

**Definition 2.5 (regular tree grammar)** A *regular tree grammar* is a tuple $\gamma = (\Xi, \Sigma, R, \xi_0)$, where
- $\Xi$ is a finite signature of nonterminals of rank 0,
- $\Sigma$ is a finite signature of terminals disjoint with $\Xi$,
- $R$ is a finite set of rules $\xi \to r$, where $\xi \in \Xi$ and $r \in T_{\Sigma,\Sigma}$, and
- $\xi_0 \in \Xi$ is the initial nonterminal.

A derivation step $s \Rightarrow t$ (or simply $s \Rightarrow t$, if $\gamma$ is understood) consists of two trees $s, t \in T_{\Sigma,\Sigma}$ such that $t$ is obtained from $s$ by replacing a single occurrence of a nonterminal $\xi$ with $r$, such that the rule $\xi \to r$ belongs to $R$. The tree language generated by $\gamma$, called a *regular tree language*, is

$$L(\gamma) = \{ t \in T_\Sigma \mid \xi_0 \Rightarrow^* t \}.$$

The class of all regular tree languages is denoted by REG.

**Example 2.6** In this example and those that follow, we interpret primitives as picture operations in the sense of [Dre00], [Dre01], [Dre06]. Here, a *picture* is a nonempty subset of the Euclidean plane, typically being a union of objects such as squares and circles. In other words, $\mathcal{A}$ is the set $\mathbb{P}$ of all nonempty subsets of $\mathbb{R}^2$. A picture operation of arity $k$ is a function $f : \mathbb{P}^k \to \mathbb{P}$ which is given by a picture $P_0$ and affine transformations $\tau_1, \ldots, \tau_k$ of $\mathbb{R}^2$, where $f(P_1, \ldots, P_k) = P_0 \cup \bigcup_{i=1}^{k} \tau_i(P_i)$.

Let $\Pi = \{ f(3), x(1), d(0) \}$, and define a deterministic interpretation $\pi$ as follows.
- $\pi(d) = \bullet$ is the picture consisting of the unit disk centered at the origin.

12
\(\pi(r) = r\) scales its argument horizontally by a factor of 0.7, rotates it by 15 degrees and adds a unit circle.

\(\pi(f) = f\) keeps the first argument unchanged. The second argument is uniformly scaled by the factor 0.45, translated vertically by 1.45 units, and rotated by 75 degrees. The third argument is transformed in a similar way, except that the scaling factor is 0.75, the vertical translation is by 1.75 units, and the rotation is by \(-25\) degrees.

Two pictures that can be obtained from expressions over these operations are shown in Figure 1. In both pictures, the largest circle is the unit circle whose center is the origin of the coordinate system.

(a) Now, consider the delegation network \(N = (\{g(0)_0\}, \Pi, \Gamma, g_0, A, \pi)\), where \(\gamma_{g_0}\) and \(\gamma_h\) are the regular tree grammars with nonterminal signature \(\{\xi_0\}\), terminal signature \(\Pi\), and sets of rules \(R_{g_0}\) and \(R_h\), respectively, where

\[
R_{g_0} = \{\xi_0 \rightarrow f[h, \xi_0, \xi_0] \mid h\} \quad \text{and} \quad R_h = \{\xi_0 \rightarrow r[\xi_0] \mid d\}
\]

Thus, \(L(\gamma_h) = T_{\{r,d\}}\), i.e., it consists of all trees of the form \(r[\cdots r[d\cdots]]\), which means that \(\sigma_N(h) = \pi(L(\gamma_h))\). The fact that \(h\) occurs in the trees of \(L(\gamma_{g_0})\), means that the generator \(\gamma_{g_0}\) delegates parts of its task to the generator \(\gamma_h\).

As a consequence, the elements of \(L(N)\) are the pictures that are obtained by interpreting the trees in \(L(\gamma_{g_0})\) with respect to \(\pi'\), the extension of \(\pi\) to \(\Pi \cup \{h\}\) given by \(\pi'(h) = \pi(T_{\{r,d\}})\). One of the elements of \(L(N)\) is shown in Figure 2.

(b) Alternatively, \(L(N) = L(N')\), where \(N' = (\{g_0(0)\}, \Pi, \Gamma', g_0, A, \pi)\) is the tree-based generator given by the regular tree grammar \(\gamma_{g_0}' = (\{\xi_0, \xi\}, \Pi, R, \xi_0)\) with

\[
R = \{\xi_0 \rightarrow f[\xi, \xi_0, \xi_0] \mid \xi, \xi \rightarrow r[\xi] \mid d\}.
\]

---

4By convention, positive rotation direction means counterclockwise rotation.

5As usual, \(\xi \rightarrow t_1 \mid \cdots \mid t_n\) abbreviates \(n\) rules \(\xi \rightarrow t_1, \ldots, \xi \rightarrow t_n\).

6Cf. the remark following Definition 2.4
Instead of removing all the delegation from $N$, as in $N'$, we could also use the finitary delegation network that is identical to $N$, except that $L(\gamma g_0) = \{f[h, g_0, g_0], h\}$ and $L(\gamma h) = \{f[h], d\}$. In this way, we generate the same picture language $L(N)$ using self-delegation, i.e., the trees in $L(\gamma g_0)$ contain $g_0$, and similarly for $L(\gamma h)$.

**Example 2.7** We now modify the delegation network of Example 2.6 such that $\gamma g_0$ delegates additionally to $\gamma g$, where $g$ is of rank 1. Following our convention regarding parameter signatures, $Y_g = X_1 = \{x_1\}$. Throughout the example, $L(\gamma h)$ is as in Example 2.6.

(a) Let $L(\gamma g_0) = \{g[h]\}$, and let $L(\gamma g)$ be given by the regular tree grammar whose rules are $\xi_0 \rightarrow f[x_1, \xi_0, \xi_0] \mid x_1$. Thus, $L(\gamma g)$ is the set of all trees $t \in T_{\{t, x_1\}}$ such that the first child of every occurrence of $f$ is $x_1$. The interpretation of such a tree $t$ with respect to $\pi$ yields a (deterministic) unary operation on pictures that duplicates the argument as many times as there are occurrences of $x_1$ in $t$ and puts them into the corresponding places. Interpreting $x_1$ as the (for illustration purposes shaded and oriented) unit disk rather than as a parameter, we obtain a graphical representation of the operation $\sigma_N(t) = \pi(t)$ and its application to an element of $\sigma_N(h) = \pi(L(\gamma h))$, as shown in Figure 3. The interpretation $\sigma_N(g)$ of $g$ in $N$ is the nondeterministic unary operation $g$ such that, for a picture $P$, $g(P) = \{\pi(t)(P) \mid t \in L(\gamma g)\}$. Thus, $L(N) = \{g(P) \mid P \in \pi(L(\gamma h))\}$. 

![Figure 2: A picture generated by the delegation network in Example 2.6](image)
sample operation $\pi(t) = \sigma_N(t)$
sample argument in $\sigma_N(h)$

result in $\sigma_N(g[h]) = \sigma_N(g_0) = L(N)$

Figure 3: Graphical representation of applying $\pi(t)$ to an element of $\sigma_N(h)$ in Example 2.7(a)
It may be instructive to have a look at the relation between \( L(\mathcal{N}) \) and the tree language \( L(\mathcal{N}_{\text{free}}) \), where \( \mathcal{N}_{\text{free}} \) is identical to \( \mathcal{N} \), except that its interpretation \( \pi \) of primitives is the free interpretation \( \text{free}_{\Pi} \) with domain \( T_\Pi \). Thus, each tree in \( L(\mathcal{N}_{\text{free}}) \) is obtained from a tree in \( L(\gamma_g) \) by substituting a tree in \( L(\gamma_h) = T_{\{r,d\}} \) for the occurrences of the parameter \( x_1 \): \( L(\mathcal{N}_{\text{free}}) = \{ t(t_1) \mid t \in L(\gamma_g), t_1 \in T_{\{r,d\}} \} \). The fact that, in every picture in \( L(\mathcal{N}) \), the inner decorations of all circles are identical, corresponds to the fact that, in every tree in \( L(\mathcal{N}_{\text{free}}) \), the first subtrees of all occurrences of \( f \) are identical. By the MW-like Theorem 6.2 in Section 6, it holds that \( L(\mathcal{N}) = \pi(L(\mathcal{N}_{\text{free}})) \). However, this relies on the fact that the interpretations of the primitives are deterministic, i.e., they are functions. Suppose that we change \( \mathcal{N} \) into \( \mathcal{N}' \) by removing \( h \) from the signature of generator symbols and turning it into a nondeterministic primitive with \( \pi'(h) = \sigma_\mathcal{N}(h) \). From Definition 2.4, it is immediately clear that \( L(\mathcal{N}') = L(\mathcal{N}) \). In contrast, \( \pi'(L(\mathcal{N}_{\text{free}}')) \) is equal to the picture language generated in Example 2.6. This discrepancy is caused by the fact that the trees in \( L(\mathcal{N}_{\text{free}}') \) are of the form \( t(h) \) with \( t \in L(\gamma_g) \), in which \( h \) has several occurrences. When the interpretation \( \pi' \) is applied to \( t(h) \), these occurrences of \( h \) are evaluated individually. Thus, each occurrence of \( h \) gives nondeterministically rise to a potentially different element of \( \pi'(h) \). We will show in Section 5 how this can be avoided by generating and evaluating jungles instead of trees. In a jungle generated by \( \mathcal{N}' \), there will be only one occurrence of \( h \), which is shared. Hence, it is evaluated only once, which makes the evaluation of the entire jungle return the desired result.

Using self-delegation as in Example 2.6(c), \( \mathcal{N} \) can be turned into a finitary delegation network that generates the same picture language \( L(\mathcal{N}) \), by setting \( L(\gamma_g) = \{ f[x_1, g[x_1], g[x_1]], x_1 \} \). Then \( \mathcal{N}_{\text{free}} \) can be viewed as an IO context-free tree grammar that generates the language \( L(\mathcal{N}_{\text{free}}) \) using the rules \( g_0 \to g[h], g[x_1] \to f[x_1, g[x_1], g[x_1]], g[x_1] \to x_1, h \to r[h] \) and \( h \to d \). Similarly, a grammar for \( L(\mathcal{N}') \) is obtained by dropping the last two rules.

(b) A modification of \( L(\mathcal{N}) \) is obtained by taking as \( \gamma_g \) the regular tree grammar whose rules are \( \xi_0 \to f[x_1, g_0, \xi_0] \mid x_1 \). This is an example of cyclic delegation (similar to mutual recursion in programs), because \( \gamma_{g_0} \) delegates to \( \gamma_g \) and vice versa. The change implies that, while the inner decorations of the major spine of the picture (extending to the right) are still identical, the subpictures that correspond to the second subtrees of occurrences of \( f \) are recursively obtained (transformed versions of) arbitrary elements of \( \sigma_\mathcal{N}(g_0) = L(\mathcal{N}) \), thus differing with respect to their decorations. One of these pictures is shown in Figure 4.

(c) A variant of (b) with a slightly different domain is the following one. We use colored pictures\footnote{We omit a formal definition, since it is not needed to understand the example. See [Dre06, Chapter 8] for a discussion of color operations.} and change the interpretation of \( r \). It is now interpreted as a nondeterministic primitive that, when being applied to a picture, changes the color of each point in the picture by increasing the intensity of any of the
channels red, green, and blue by a certain fixed amount. The color of $\pi(d) = \bullet$ is black (i.e., the value of all channels is zero). Thus, $r^n(\bullet)$ yields unit disks in $(n + 1)^3$ different colors.

Apart from this change of $\pi$, the delegation network is as in (b). The operation $f$ is assumed to preserve color. Thus, similar to (b) the major spine of the picture is uniformly colored, whereas the subpictures that correspond to the second subtrees of occurrences of $f$ are, recursively, colored in the same fashion, but with their individual coloring. One of these pictures can be seen in Figure 5.

**Example 2.8** A phenomenon that must be taken into account when proving MW-like results is deletion. As an (artificial) example, let us modify the delegation network $N$ of Example 2.6(a) in the following way, yielding $N^{\bot}$. The generator symbol $h$ is turned into a symbol of rank 1. In the rules of $\gamma_{g_0}$, every occurrence of $h$ is replaced by $h[\bot]$, where $\bot$ is a new primitive of rank 0 that the extended interpretation $\pi^{\bot}$ interprets as undefined, i.e., $\pi^{\bot}(\bot) = \emptyset$. Keeping $\gamma_h$ unchanged, $\sigma_{N^{\bot}}(h)$ thus becomes the unary nondeterministic function $h$ given by $h(P) = \sigma_N(h)$, because $\gamma_h$ never makes use of the parameter $x_1$. Nevertheless, $L(N^{\bot}) = \emptyset$, because $\sigma_{N^{\bot}}(t)$ is undefined for all $t \in L(\gamma_{g_0})$, owing to the occurrences of $\bot$ in $t$.

In contrast to $\pi^{\bot}(\bot)$, the free interpretation of $\bot$ yields a result (namely $\bot$ itself), which implies that $\sigma_{N^{\bot}}(h[\bot]) = \sigma_N(h)$ and thus $L(N^{\bot}_\text{free}) = L(N_\text{free})$. In other words, if we use the free interpretation, the offending occurrences of $\bot$ are deleted, which causes $\pi^{\bot}(L(N^{\bot}_\text{free}))$ to be different from $L(N^\bot)$. We will show in Section 5 how this discrepancy can be removed by generating jungles instead of trees. In a generated jungle, the unused copies of $\bot$ will be retained (as garbage), so that the evaluation of the jungle yields the expected result.
Example 2.9 Since delegation networks with the free interpretation generate trees, they can be used as $\gamma_g$ in other delegation networks. This will be studied in detail in Sections 7 and 8. We consider two examples.

(a) Let $\mathcal{N}_0$ be the delegation network that uses the free interpretation, with the generator symbols $g_0^{(0)}$ and $g^{(1)}$, primitives $f^{(3)}$ and $h^{(0)}$, and $L(\gamma_{g_0}) = \{g[h]\}$ and $L(\gamma_g) = \{g[f[h, x_1, x_1]], x_1\}$. Then, $\mathcal{N}_0$ generates all trees in $T\{f, h\}$ such that the first child of each $f$ is $h$ and the binary tree obtained by considering only the second and third child of each occurrence of $f$ is fully balanced.

Now, let $\mathcal{N}$ use the interpretation in Example 2.6, with generators $g_0$ and $h$, where $\gamma_{g_0} = \mathcal{N}_0$. Again, $\gamma_h$ is as in the previous examples. Then all “arms” of each generated picture are equally refined (by the fact that $\mathcal{N}_0$ generates fully balanced trees), but the inner decorations of the circles vary, since each is generated by an individual copy of $\gamma_h$; see Figure 6 (in which the fully balanced tree has height 10). It is worth comparing this result with the one obtained by just combining the two delegation networks rather than using $\mathcal{N}_0$ as a tree generator in $\mathcal{N}$. More precisely, let $\mathcal{N}' = (\{g_0, g, h\}, \Pi, \Gamma, g_0, \pi)$, where $L(\gamma_{g_0}) = \{g[h]\}$, $L(\gamma_g) = \{g[f[h, x_1, x_1]], x_1\}$, and $\gamma_h$ and $\pi$ are again as in the preceding examples. Now, as shown in Figure 7, the two substructures attached to each circle are (transformed) copies of each other, including the inner decorations of the circles.

(b) Let us now assume, as in Example 2.7(c), that pictures are colored, and let us turn $d$ into a nondeterministic primitive such that $\pi(d)$ is the set of all uniformly colored unit disks. The primitives $r$ and $f$ are interpreted as in Example 2.6 (which implies that they preserve color). We wish to define a delegation network $\mathcal{N}''$ that, except for the coloring, generates the same pictures as $\mathcal{N}$.
Figure 6: A picture generated by the delegation network $\mathcal{N}$ in Example 2.9

Figure 7: A picture generated by the delegation network $\mathcal{N}'$ in Example 2.9
whereas the second yields a tree generation step using
of the following holds:
the tree language consisting of all trees of \( L(\gamma_0) \) in which every occurrence of \( h \) is replaced by \( h[x_1] \). The tree language \( L(\gamma_k) \) can be generated by an obvious vari-
above. In each picture, we want all (transformed) disks in the inner decorations
circles to have the same color. The network \( N'' \) has generator symbols \( g_0^{(0)}, \)
\( k^{(1)} \) and \( h^{(1)} \), with \( L(\gamma_g) = \{k[d]\}, L(\gamma_d) = \{r[h[x_1]], x_1\} \) and \( L(\gamma_k) \) being the
tree language consisting of all trees of \( L(\gamma_0) \) in which every occurrence of \( h \) is
\( \mathrm{N} \) symbol. Therefore, it uses the parameter signature \( Y = \{y_1\} \). Thus, in \( N'' \),
\( L(\gamma_g) = \{g[h[x_1]]\} \) and \( L(\gamma_k) = \{g[f[h[x_1]], y_1, y_1], y_1\} \). This explains why, in
general, we need other parameter signatures than \( X_k \).

3 An Operational Characterization

Throughout this section, let \( \mathcal{N} = (\mathcal{G}, \Pi, \Gamma, g_0, A, \pi) \) be a delegation network,
where \( \Gamma = (\gamma_g)_{g \in \mathcal{G}} \). The goal of the section is to develop an operational char-
acterization of the semantics of delegation networks. Such an operational char-
acterization allows us to avoid the explicit use of the fixed-point construction
of Definition 2.4 in proofs. To achieve this, we define a derivation relation \( \rightarrow_N \)
that, intuitively, works on trees over a larger (possibly infinite) signature, with
additional symbols \( \langle a \rangle \) of rank 0 that represent values \( a \in A \). Intuitively, \( \rightarrow_N \)
works bottom up and performs both evaluation and generation steps. Evaluation
steps apply the interpretation of a primitive to already computed values, i.e.,
they nondeterministically replace \( f(\langle a_1 \rangle, \ldots, \langle a_k \rangle) \) with \( f^{(a)} \in \Pi \) by any \( \langle a \rangle \) with
\( a \in f(a_1, \ldots, a_k) \). Generation steps nondeterministically replace \( g(\langle a_1 \rangle, \ldots, \langle a_k \rangle) \)
with \( g^{(a)} \in \mathcal{G} \) by any tree \( u(\langle a_1 \rangle, \ldots, \langle a_k \rangle) \), where \( u \in L(\gamma_g) \).

To start with, we have the following easy lemma.

Lemma 3.1 Let \( \Sigma \) and \( \sigma \) be a signature and a \( \Sigma \)-interpretation, resp. For every
tree \( s \cdot t \in T_\Sigma \), if \( t' \in T_\Sigma \) is such that \( \sigma(t) \subseteq \sigma(t') \), then \( \sigma(s \cdot t) \subseteq \sigma(s \cdot t') \).

Proof Straightforward structural induction on \( s \). □

We now define the derivation relation \( \rightarrow_N \) mentioned above, that will be the
basis for our operational characterization of the semantics of delegation networks.

Definition 3.2 Let \( \Sigma_N = \Sigma_N \cup \{\langle a \rangle^{(0)} \mid a \in A\} \). The derivation relation \( \rightarrow_N \)
on \( T_{\Sigma_N} \) is given as follows. For trees \( s \cdot t, s \cdot t' \in T_{\Sigma_N} \), we let \( s \cdot t \rightarrow_N s \cdot t' \) if one
of the following holds:

- \( t \) is of the form \( f(\langle a_1 \rangle, \ldots, \langle a_k \rangle) \), where \( f \in \Pi \), and \( t' = \langle a \rangle \) for some
  \( a \in f(a_1, \ldots, a_k) \), or

- \( t \) is of the form \( g(\langle a_1 \rangle, \ldots, \langle a_k \rangle) \), where \( g \in \mathcal{G} \), and \( t' = u(\langle a_1 \rangle, \ldots, \langle a_k \rangle) \) for
  a tree \( u \in L(\gamma_g) \).

Thus, the first item describes an evaluation step applying \( f \) to \( a_1, \ldots, a_k \),
whereas the second yields a tree generation step using \( \gamma_g \). We are going to show
\hfill \( \Sigma_N \).
that \( L(\mathcal{N}) = \{a \in \mathbb{A} \mid \mathcal{g}_0 \overset{*}{\rightarrow}_{\mathcal{N}} \langle a \rangle \} \). For the first inclusion of this equality, which is provided by the following lemma, it is technically convenient to extend \( \sigma_{\mathcal{N}} \) to \( \Sigma_{\mathcal{N}} \) in the obvious way: \( \sigma_{\mathcal{N}}(\langle a \rangle) = \{a\} \) for all \( a \in \mathbb{A} \).

**Lemma 3.3** For all trees \( t, t' \in T_{\Sigma_{\mathcal{N}}} \), \( t \overset{*}{\rightarrow}_{\mathcal{N}} t' \) implies \( \sigma_{\mathcal{N}}(t') \subseteq \sigma_{\mathcal{N}}(t) \).

*Proof* It suffices to consider the relation \( \rightarrow_{\mathcal{N}} \) instead of \( \overset{*}{\rightarrow}_{\mathcal{N}} \). According to the definition of \( \rightarrow_{\mathcal{N}} \), we have to consider two cases. Let \( t = s \cdot t_0 \) and \( t' = s \cdot t'_0 \).

**Case 1** We have \( t_0 = \mathbf{f}[\langle a_1 \rangle, \ldots, \langle a_k \rangle] \), where \( \mathbf{f} \in \Pi \), and \( t'_0 = \langle a \rangle \) for some \( a \in f(a_1, \ldots, a_k) \).

In this case, we immediately get

\[
\sigma_{\mathcal{N}}(t'_0) = \{a\} \subseteq f(a_1, \ldots, a_k) = \sigma_{\mathcal{N}}(t_0).
\]

Thus, Lemma 3.1 yields \( \sigma_{\mathcal{N}}(t') \subseteq \sigma_{\mathcal{N}}(t) \), as claimed.

**Case 2** We have \( t_0 = \mathbf{g}[\langle a_1 \rangle, \ldots, \langle a_k \rangle] \), where \( \mathbf{g} \in \mathcal{G} \), and \( t'_0 = u(\langle a_1 \rangle, \ldots, \langle a_k \rangle) \) for some \( u \in L(\gamma_\mathbf{g}) \).

This yields

\[
\sigma_{\mathcal{N}}(t'_0) = \sigma_{\mathcal{N}}(u)(a_1, \ldots, a_k) \quad \text{(Lemma 2.1)}
\]

\[
\subseteq \sigma_{\mathcal{N}}(L(\gamma_\mathbf{g}))(a_1, \ldots, a_k)
\]

\[
= \sigma_{\mathcal{N}}(\mathbf{g})(a_1, \ldots, a_k)
\]

\[
= \sigma_{\mathcal{N}}(t_0),
\]

where the equality in the third row holds because \( \sigma_{\mathcal{N}} \) is, by definition, a fixed point of iterate\(_{\mathcal{N}} \) (see Definition 2.4). As a consequence, again by Lemma 3.1, \( \sigma_{\mathcal{N}}(t') \subseteq \sigma_{\mathcal{N}}(t) \). □

By the previous lemma, \( \mathcal{g}_0 \overset{*}{\rightarrow}_{\mathcal{N}} \langle a \rangle \) only if \( a \in L(\mathcal{N}) \). The next lemma proves that the other direction is valid as well.

**Lemma 3.4** For all \( a \in L(\mathcal{N}) \), \( \mathcal{g}_0 \overset{*}{\rightarrow}_{\mathcal{N}} \langle a \rangle \).

*Proof* Let \( \sigma \) be the \( \Sigma_{\mathcal{N}} \)-interpretation such that \( \sigma(\mathbf{f}) = f \) for all \( \mathbf{f} \in \Pi \), and

\[
\sigma(\mathbf{g})(a_1, \ldots, a_k) = \{a' \in \mathbb{A} \mid \mathcal{g}[\langle a_1 \rangle, \ldots, \langle a_k \rangle] \overset{*}{\rightarrow}_{\mathcal{N}} \langle a' \rangle \}
\]

for all \( \mathbf{g}^{(k)} \in \mathcal{G} \) and \( a_1, \ldots, a_k \in \mathbb{A} \). We prove the inequality

\[
\text{iterate}_{\mathcal{N}}(\sigma) \leq \sigma.
\]

(1)

This finishes the proof of the lemma, because it yields \( \sigma_{\mathcal{N}} \leq \sigma \), by Proposition 2.3. In particular, \( \sigma_{\mathcal{N}}(\mathcal{g}_0) \subseteq \sigma(\mathcal{g}_0) \), which is the same as saying that \( a' \in L(\mathcal{N}) = \sigma_{\mathcal{N}}(\mathcal{g}_0) \) implies \( \mathcal{g}_0 \overset{*}{\rightarrow}_{\mathcal{N}} \langle a' \rangle \) (by the definition of \( \sigma(\mathcal{g}_0) \)).

By the definition of iterate\(_{\mathcal{N}} \), to prove (1), it suffices to show that \( \sigma(L(\gamma_\mathbf{g})) \subseteq \sigma(\mathbf{g}) \), for all \( \mathbf{g}^{(k)} \in \mathcal{G} \). In other words, we have to show that \( \mathcal{g}[\langle a_1 \rangle, \ldots, \langle a_k \rangle] \overset{*}{\rightarrow}_{\mathcal{N}} \langle a' \rangle \) for all \( a_1, \ldots, a_k \in \mathbb{A} \) and \( a' \in \sigma(L(\gamma_\mathbf{g}))(a_1, \ldots, a_k) \). For this, by the second item of Definition 3.2, it suffices to prove the following claim.
Claim Let \( k \in \mathbb{N} \). For all trees \( t \in T_{\Sigma, k} \) and all \( a' \in \sigma(t)(a_1, \ldots, a_k) \), we have
\[
 t(\langle a_1, \ldots, a_k \rangle) \rightarrow^* \mathcal{N} \langle a' \rangle.
\]

We proceed by structural induction on \( t \). If \( t = x_i \), then \( a' = a_i \), and there is nothing to show. Thus, let \( t = f[t_1, \ldots, t_l] \) and \( a' \in \sigma(f)(a'_1, \ldots, a'_l) \), for appropriate \( a'_i \in \sigma(t_i)(a_1, \ldots, a_k) \) \( (i \in [l]) \). Then the induction hypothesis yields
\[
 t_i(\langle a_1, \ldots, a_k \rangle) \rightarrow^* \mathcal{N} \langle a'_i \rangle \quad \text{for every} \ i \in [l],
\]
which gives
\[
 t(\langle a_1, \ldots, a_k \rangle) = f[t_1(\langle a_1, \ldots, a_k \rangle), \ldots, t_l(\langle a_1, \ldots, a_k \rangle)] \\

to \rightarrow^* \mathcal{N} \langle a'_1, \ldots, a'_l \rangle \\
\rightarrow^* \mathcal{N} \langle a' \rangle.
\]

Here, the last line holds by the first item of Definition \ref{def:func} if \( f \in \Pi \), and by the definition of \( \sigma \) if \( f \in \mathcal{G} \).

Combining the two previous lemmas, the desired operational characterization of the semantics of \( \mathcal{N} \) follows immediately.

**Theorem 3.5** \( L(\mathcal{N}) = \{ a \in A \mid g_0 \rightarrow^* \mathcal{N} \langle a \rangle \} \), for every delegation network \( \mathcal{N} = (\mathcal{G}, \Pi, \Gamma, g_0, A, \pi) \).

**Proof** By Lemmas \ref{lem:val} (where \( t = g_0 \) and \( t' = \langle a \rangle \)) and \ref{lem:gen}.

The previous theorem shows that the parallel distributed implementation of delegation networks discussed in the introduction is indeed correct. We repeat that discussion here in more detail, in order to relate it to the formal definitions. The main process takes care of the generation of an element of \( L(\mathcal{N}) \). For this, it first uses \( \gamma_{g_0} \) to generate a tree \( u_0 \) (if possible, i.e., if \( L(\gamma_{g_0}) \neq \emptyset \)). Then, for every individual occurrence of a symbol \( g \) in \( u_0 \), a similar process using \( \gamma_g \) instead of \( \gamma_{g_0} \) is started recursively. If desired, this process may run on a different machine. Each process also takes care of the nondeterministic evaluation of the tree \( u \) it has generated. For this to be possible, its parent process must eventually provide it with values for all the parameter symbols in \( u \), the actual parameters.

An evaluation step takes place as soon as the subtrees of (an occurrence of) a symbol \( f^{(k)} \in \Pi \) in \( u \) have been evaluated. In this situation, the process applies \( f \) to the values \( a_1, \ldots, a_k \) of the subtrees as in the first item of Definition \ref{def:func} obtains a value \( a' \) (provided that \( f(a_1, \ldots, a_k) \neq \emptyset \)), and replaces \( f \) by \( \langle a' \rangle \). To implement the second item of Definition \ref{def:func}, when a subtree of a symbol \( g \in \mathcal{G} \) has been evaluated, the resulting value is passed to the child process that evaluates the occurrence of \( g \), thus providing it with one of its actual parameters. Moreover, whenever a child process has received all its actual parameters and has finished the evaluation of its own tree, it reports the value \( a'' \) computed to its parent process and terminates.\(^9\) The parent process can then continue the evaluation of \( u \) at

\(^9\)Note that the parameter passing mechanism is call-by-value.

\(^{10}\)We wish to stress that, to implement the call-by-value semantics correctly, we must make sure that a child process does not terminate its computation before receiving all its actual
symbol $g$, changing that symbol into $\langle a'' \rangle$. In effect, all processes together carry out a derivation $g_0 \rightarrow^* \langle a \rangle$ according to Definition 3.2, where $a$ is the result of the evaluation performed by the main process. Of course, a real implementation must bound the number of processes created, and it must make sure that the depth of the recursion caused by cyclic delegation does not increase ad infinitum.

Whereas Theorem 3.5 shows that $L(N)$ can be generated by alternatingly using semantic evaluation steps and syntactic tree generation steps, in the next two sections we will prove MW-like results, to show that one can even perform all the syntactic steps first, and then the semantic ones. However, it turns out that such a result can be obtained for the general case only if the syntactic generation is defined in such a way that it yields trees with sharing and garbage, i.e., acyclic graphs, rather than ordinary trees. This case is treated in Section 5. In Section 6, we show that the use of trees is sufficient if all primitives are interpreted as functions.

4 Jungles and Hyperedge Replacement

In this section, we recall some notions regarding jungles [HP91, HKP91, Plu99], which will play a major role in Sections 5 and 7. Intuitively, a jungle is a directed ordered acyclic graph representing a tree. In such a jungle, subtrees can be shared and unreachable subtrees, so-called garbage, may occur. Technically, jungles are defined to be special hypergraphs, which makes it possible to generate them by means of hyperedge replacement, a context-free graph rewriting mechanism that has been studied extensively in the context of hyperedge-replacement graph grammars; see, e.g., [BC87, HK87, Hab92, Eng97, DHK97, CE12].

Our notion of hypergraph is made precise in the following definition.

**Definition 4.1 (Λ-hypergraph)** Let $\Lambda$ be a set of labels. A Λ-hypergraph is a tuple $H = (V, E, \text{att}, \text{lab}, \text{ext})$ consisting of

- a finite set $V$ of nodes,
- a finite set $E$ of hyperedges,
- an attachment function $\text{att}: E \rightarrow V^+$ assigning to every hyperedge $e$ a nonempty sequence $\text{att}(e)$ of attached nodes,
- a labelling function $\text{lab}: E \rightarrow \Lambda$, and
- a nonempty sequence $\text{ext} \in V^+$ of external nodes.

parameters, even those that do not occur in the tree it has generated. It does not necessarily have to receive them at the start of its computation.

11Recall that ‘MW-like’ refers to the seminal paper [MW67] by Mezei and Wright, where it was shown that a subset of a domain of an algebra is context-free if and only if it is the set of evaluations of a regular set of trees.

12Somewhat similar to the use of hyperedge replacement in this paper, [EH92] shows that the “term-generating” power of hyperedge-replacement graph grammars that generate jungles (which are then turned into trees) is equal to the power of attribute grammars. However, in the present paper, even the right-hand sides of rules (and, thus, all sentential forms) are jungles.

23
If $\Lambda$ is understood or of minor interest, we simply speak of hypergraphs rather than $\Lambda$-hypergraphs. The five components of a hypergraph $H$ may be denoted by $V_H, E_H, att_H, lab_H, ext_H$, respectively.

Before continuing, let us mention that we usually do not distinguish between isomorphic hypergraphs, i.e., hypergraphs which are identical up to a bijective renaming of nodes and hyperedges. In other words, operations and relations defined on hypergraphs are meant to work on abstract hypergraphs (isomorphism classes of hypergraphs), even though, for technical convenience, we use concrete representatives in definitions, proofs, and the like.

The notion of hyperedge replacement to be defined next relies on the disjoint union $S \uplus S'$ of sets $S, S'$ of nodes or hyperedges. To keep the technicalities as simple as possible, we shall generally assume that the hypergraphs in question are silently replaced by suitable isomorphic copies to make sure that $S \cap S' = \emptyset$. Using this convention, $S \uplus S'$ will be treated like the ordinary union of disjoint sets.

Given two hypergraphs $H$ and $H'$, the addition of $H'$ to $H$ yields the hypergraph $H \uplus H' = (V_H \uplus V_{H'}, E_H \uplus E_{H'}, att_H \cup att_{H'}, lab_H \uplus lab_{H'}, ext_H)$. Note that $H \uplus H'$ inherits its sequence of external nodes from $H$. Thus, $\uplus$ is not commutative.

The removal of a hyperedge $e$ from a hypergraph $H$ yields

$$H - e = (V_H, E_H \setminus \{e\}, att, lab, ext_H),$$

where $att$ and $lab$ are the restrictions of $att_H$ and $lab_H$, respectively, to $E_H \setminus \{e\}$.

Finally, given two sequences of nodes $v = v_1 \cdots v_k$ and $w = w_1 \cdots w_k$ of a hypergraph $H$, of the same length, $H_{v \leftarrow w}$ denotes the hypergraph obtained by identifying $v$ with $w$ in $H$. More precisely, let $\sim$ be the equivalence relation on $V_H$ generated by $\{(v_1, w_1), \ldots, (v_k, w_k)\}$, and let $img(v')$ denote the image of $v' \in V_H$ in the quotient set $V_H/\sim$. Then $H_{v \leftarrow w} = (V_H/\sim, E_H, img\circ att_H, lab_H, img(ext_H))$.

We are now ready to recall the definition of hyperedge replacement.

**Definition 4.2 (hyperedge replacement)** Let $H$ be a hypergraph. A hypergraph $H'$ fits a hyperedge $e \in E_H$ (in $H$) if $|att_H(e)| = |ext_H|$. In this case, $H[e \leftarrow H']$ is the hypergraph obtained from the disjoint union of $H - e$ and $H'$ by identifying $att_H(e)$ with $ext_H$, i.e.,

$$H[e \leftarrow H'] = ((H - e) \uplus H')_{att_H(e)=ext_H}.$$

Note that if $H'' = H[e \leftarrow H']$, then every node $v \in V_H \cup V_{H'}$ has an image $img(v)$ in $H''$, where $img$ is as in the paragraph preceding Definition 4.2. The function $img$ makes it possible to locate the nodes of $H$ and $H'$ in $H''$. In the following, it will therefore be called the locator function associated with the hyperedge replacement. By construction, $img$ maps the $i$th attached node of $e$ in $H$ to the same node as the $i$th external node of $H'$, but is injective on
As mentioned above, we are interested in a special kind of hypergraph, the so-called jungle. Let $\Sigma$ be a signature. A jungle is essentially a tree in $T_{\Sigma,n}$ (with garbage), in which subtrees may be shared and multiple occurrences of each parameter symbol $x_i$ must be shared. To make this precise, let $J$ be a $\Sigma$-hypergraph, with $ext_J = w_1 \cdots w_n w$. Assume that, for each hyperedge $e \in E_J$, if $lab_J(e) = f^{(k)}$, then $att_J(e)$ has length $k+1$, say $att_J(e) = v_1 \cdots v_k v$. Intuitively, the nodes $v_1, \ldots, v_k$ represent the parameters of $f$, and the nodes $w_1, \ldots, w_n$ represent the parameters of $J$. Therefore, we call $par_J(e) = v_1 \cdots v_k$ the sequence of parameter nodes of $e$, and $par(J) = w_1 \cdots w_n$ the sequence of parameter nodes of $J$. Similarly, $v$ and $w$ intuitively represent the results of $f$ and $J$, respectively, and so $res_J(e) = v$ and $res(J) = w$ are called the result nodes of $e$ and $J$, respectively. A parameter tentacle of $e$ is a pair $(e, j)$, where $j \in [k]$. We let $tent(J)$ denote the set of all parameter tentacles of hyperedges in $E_J$.

Now, let $v, w \in V_J$. For the definition of jungles below, and also for later use, we define a jungle path from $v$ to $w$ (in $J$) to be either the empty sequence $\lambda$, provided that $v = w$, or a nonempty sequence $(e_0, j_0) \cdots (e_m, j_m) \in tent(J)^*$ such that $v = res_J(e_0)$, $w = par_J(e_m, j_m)$, and $res_J(e_l) = par_J(e_{l-1}, j_{l-1})$ for all $l \in [m]$. The hypergraph $J$ is acyclic if, for every $v \in V_J$, $\lambda$ is the only jungle path from $v$ to $v$.

**Definition 4.3 (jungle)** Let $\Sigma$ be a signature. The set $J_{\Sigma,n}$ of jungles over $\Sigma$ with $n$ parameters ($n \in \mathbb{N}$) consists of all acyclic $\Sigma$-hypergraphs $J$ such that

- $par(J) = w_1 \cdots w_n$ for pairwise distinct nodes $w_1, \ldots, w_n$,
- for all $v \in V_J$, $v = res_J(e)$ for some $e \in E_J$ if and only if $v \notin \lbrace par(J) \rbrace$,
- for all hyperedges $e, e' \in E_J$, if $e \neq e'$ then $res_J(e) \neq res_J(e')$, and
- for every hyperedge $e \in E_J$, $|par_J(e)|$ is equal to the rank of $lab_J(e)$ in $\Sigma$.

We let $J_\Sigma = J_{\Sigma,0}$. For finite $\Sigma$, a jungle language is a subset of $J_{\Sigma,n}$, where $n \in \mathbb{N}$.

Intuitively, the $n$ pairwise distinct parameter nodes of a jungle $J$ correspond to the parameter symbols $x_i$ of a tree in $T_{\Sigma,n}$, and $res(J)$ corresponds to its root.

Every jungle $J \in J_{\Sigma,n}$ represents a unique tree $tree(J) \in T_{\Sigma,n}$, namely $tree(J) = tree(J, res(J))$, where $tree(J, v)$ is defined as follows, for all $v \in V_J$. If $par(J) = w_1 \cdots w_n$ and $v = w_i$, where $i \in [n]$, then $tree(J, v) = x_i$. Otherwise, let $e \in E_J$ be the unique hyperedge such that $res_J(e) = v$. Then $tree(J, v) = f[tree(J, v_1), \ldots, tree(J, v_k)]$, where $f^{(k)} = lab_J(e)$ and $par_J(e) = v_1 \cdots v_k$.

Conversely, a tree $t \in T_{\Sigma,n}$ can be turned into a jungle by creating a node that represents the root of $s$, for each (occurrence of a) subtree $s = f[s_1, \ldots, s_k]$. If $f \in \Sigma$, then there is a hyperedge labelled $f$ whose result node is this node, and whose parameter nodes are the nodes representing the roots of $s_1, \ldots, s_k$. If $f = \lambda$

\textsuperscript{13}Recall that, for a string $u$, $[u]$ denotes the set of all symbols occurring in $u$.

\textsuperscript{14}Recall from the end of Section 2 that $T_{\Sigma,n} = T_{\Sigma,X_n}$, where $X_n$ is the parameter signature $\{x_1, \ldots, x_n\}$.
distinct parameter nodes, such that 
\[ J \]

is closed under hyperedge replacement (see [EV94, Lemma 5.3]). Note that for every tree \( t \) and \( n \in \mathbb{N} \), \( \text{tree}(J_n(t)) = t \).

Fact 4.4 Let \( J \in J_{\Sigma,n} \) for a signature \( \Sigma \) and \( n \in \mathbb{N} \). For every jungle \( J' \in J_{\Sigma,k} \), if \( J' \) fits hyperedge \( e \in E_J \), then \( J[e \leftarrow J'] \in J_{\Sigma,n} \).

An important fact that needs to be kept in mind when working with jungles is that they may contain shared nodes as well as garbage. Here, a node \( v \) in a jungle \( J \) is shared if there are distinct parameter tentacles \( (e, j), (e', j') \in \text{tent}(J) \) such that \( \text{par}_J(e, j) = v = \text{par}_J(e', j') \). We say that \( v \) is garbage if there is no jungle path in \( J \) from \( \text{res}(J) \) to \( v \); in particular, \( v \) is a garbage root if \( v \neq \text{res}(J) \) and \( v \notin [\text{par}_J(e)] \) for all \( e \in E_J \). Shared nodes and garbage roots will play an important role in the proof of Theorem 5.6.
Garbage, except parameter nodes, can be removed from a jungle $J$ without affecting tree$(J)$. The following definitions related to this observation will only be needed in Section 7. Given a jungle $J$, we let $C_J$ denote the set of all nodes $v \in V_J$ that are not garbage. We say that $J$ is clean if $V_I = C_J \cup [par(J)]$, i.e., if all garbage nodes of $J$ are parameter nodes of $J$. The clean jungle clean$(J)$ is obtained from $J$ by restricting $J$ to $C_J \cup [par(J)]$. Thus, clean$(J) = (V,E,att,lab,ext)$, where $V = C_J \cup [par(J)]$, $E = \{ e \in E_J \mid res_J(e) \in V \}$, and $att$ and $lab$ are the restrictions of $att_J$ and $lab_J$ to $E$. As mentioned above, we have tree(clean$(J)$) = tree$(J)$. The reader should also note a few additional easy facts that will be used (in Section 7) without mention: (1) clean$(J)$ is clean (as the notation suggests), because $C_{clean(J)} = C_J$, (2) by construction, jungles of the form $J_n(t)$ are clean, and (3) if $J$ is clean, then clean$(J) = J$.

Another basic fact about jungles used in Section 7 is that, intuitively, a jungle $J$ maps to $J$, and that both jungles represent the same tree. Here, we say that a jungle $I$ maps to a jungle $J$ if there is a structure preserving mapping $\varphi : V_I \rightarrow V_J$, where structure preservation means that $\varphi(ext_I) = ext_J$ and, for every hyperedge $e \in E_I$, there exists a (unique) hyperedge $e' \in E_J$ with $att_J(e') = \varphi(att_I(e))$ and $lab_J(e') = lab_I(e)$. The formal statement below is similar to [Plu99] Lemmas 1.3.6 and 1.3.9.

**Lemma 4.5** For every jungle $J \in J_{\Sigma,n}$, the jungle $J_n(tree(J))$ maps to $J$. Furthermore, if $I$ is a jungle that maps to $J$, then tree$(I) = tree(J)$.

**Proof** As the second statement should be obvious\footnote{By induction on the structure of $I$, tree$(I,v) = tree(J, \varphi(v))$ for every $v \in V_I$ (cf. the proof of [Plu99] Lemma 1.3.6).}, we prove only the first. Let $v \in V_J$, and let $J_v$ be the jungle that is equal to $J$, except that res$_J(v) = v$. By induction on the number of hyperedges in clean$(J_v)$, we prove that $J_n(tree(J,v))$ maps to $J_v$. This is obvious if $v \in [par(J)]$, using the mapping $\varphi$ given by $\varphi(nod(x_j)) = par(J,j)$ for all $j \in [n]$ (where nod$(x_j)$ is as in the definition of $J_n(t)$). Now, consider the case where $v \notin [par(J)]$, and let $e \in E_J$ be the unique hyperedge such that res$_J(e) = v$, where $par_J(e) = v_1 \cdots v_k$. By the induction hypothesis, for all $i \in [k]$, $J_n(tree(J,v_i))$ maps to $J_{v_i}$. Let $\varphi_1, \ldots, \varphi_k$ be the corresponding structure-preserving mappings. Then the reader may easily check that $\varphi$ is a structure-preserving mapping from $J_n(tree(J,v))$ to $J_v$, where $\varphi(nod(\lambda)) = v$, $\varphi(nod(x_j)) = par(J,j)$ for $j \in [n]$, and $\varphi(nod(iw)) = \varphi_i(nod(w))$, for $i \in [k]$ and $w \in V_{\Sigma}(tree(J,v_i))$. \hfill \blacksquare

In examples, we will also use the jungle with the maximal amount of sharing that represents a tree $t \in T_{\Sigma,n}$, denoted $J_{n}^{\text{ms}}(t)$. It is defined as

$$J_{n}^{\text{ms}}(t) = (V,E,att,lab,ext)$$
Consider a delegation network $N$.

Example 5.1

As an example, if $\Sigma = \{\text{\textbullet}, \text{\textcircled{\textbullet}}\}$ of the hyperedge with label $1$ is not evaluated at all. The conclusion is that the result node $v_1$ of the hyperedge with label $a$ is shared (it is both the first and second parameter node of the hyperedge with label $f$), and the result node $v_2$ of the hyperedge with label $b$ is garbage (and, in fact, a garbage root). The evaluation of $J$ according to the definition below will evaluate the occurrences of $a$ and $b$ exactly once, thus yielding the correct result under both interpretations.

5 A Characterization Using Jungles

In this section, we show our first (and main) MW-like result stating that, for a delegation network $\mathcal{N}$, $L(\mathcal{N})$ can be obtained by first using $\mathcal{N}$ to generate a set of jungles and then evaluating the jungles obtained. As discussed in Examples 2.7(a) and 2.8, sharing and garbage in jungles are essential to achieve this if the interpretation of primitives is nondeterministic. Let us recall the problem by discussing another small example.

Example 5.1 Consider a delegation network $\mathcal{N}$ with primitives $f^{(2)}, a^{(0)}, b^{(0)}$ interpreted into strings, where $f$ is string concatenation, $a = \{\text{\textbullet}, \text{\textcircled{\textbullet}}\}$, and $b = \{\text{\textbullet}\}$. The generator symbols are $g_0^{(0)}$ and $g^{(2)}$ with $L(\gamma_{g_0}) = \{g[a, b]\}$ and $L(\gamma_{g}) = \{f[x_1, x_1]\}$. Then $L(\mathcal{N}) = \{\text{\textbullet}, \text{\textcircled{\textbullet}}\}$. In contrast, the tree language generated by $\mathcal{N}$ (using the free interpretation of $\Pi$) consists of the single tree $f[a, a]$, which evaluates to $\{\text{\textbullet}, \text{\textcircled{\textbullet}}, \text{\textcircled{\textbullet}}, \text{\textcircled{\textbullet}}\}$, because both occurrences of $a$ are evaluated independently. Moreover, if we turn $b$ into $\emptyset$, then $L(\mathcal{N}) = \emptyset$, whereas the evaluation of the tree language generated by $\mathcal{N}$ still yields the same result, because $b$ is not evaluated at all.

As might be expected, the jungle $J$ generated by $\mathcal{N}$ will look like this:

```
    f
   /|
  a b
```

The result node of $J$ is the result node $v_0$ of the hyperedge with label $f$. Note that the result node $v_1$ of the hyperedge with label $a$ is shared (it is both the first and second parameter node of the hyperedge with label $f$), and the result node $v_2$ of the hyperedge with label $b$ is garbage (and, in fact, a garbage root). The evaluation of $J$ according to the definition below will evaluate the occurrences of $a$ and $b$ exactly once, thus yielding the correct result under both interpretations.
We start by defining how jungles are evaluated.

**Definition 5.2 (jungle evaluation)** Consider a Σ-interpretation σ (into A) and a jungle \( J \in J_{\Sigma,n} \). Given \( a_1, \ldots, a_n \in A \), let \( \text{ASS}_{J,\sigma}(a_1, \ldots, a_n) \) be the set of all functions \( \alpha : V_J \rightarrow A \) such that the following hold:

- \( \alpha(\text{par}(J,i)) = a_i \), for all \( i \in [n] \);
- if \( \text{att}(J,e) = v_1 \cdots v_kv \), then \( \alpha(v) \in \sigma(\text{lab}(J,e))(\alpha(v_1), \ldots, \alpha(v_k)) \), for all \( e \in E_J \).

Now, \( \sigma(J) \) is the relation given by

\[
\sigma(J)(a_1, \ldots, a_n) = \{ \alpha(\text{res}(J)) \mid \alpha \in \text{ASS}_{J,\sigma}(a_1, \ldots, a_n) \},
\]

for all \( a_1, \ldots, a_n \in A \). For a set of jungles \( J \subseteq J_{\Sigma,n} \), \( \sigma(J) = \bigcup_{J \in J} \sigma(J) \).

For instance, if \( J \) is the jungle in Example 5.1 with the nodes \( v_0 \) (the root node), \( v_1 \) (the leftmost node), and \( v_2 \) (the rightmost node), then \( \text{ASS}_{J,\sigma}() = \{ \alpha_1, \alpha_2 \} \), where \( \alpha_1(v_1) = \diamond \) and \( \alpha_2(v_1) = \heartsuit \) and, for \( i \in [2], \alpha_i(v_2) = \diamond \) and \( \alpha_i(v_0) = \alpha_i(v_1)\alpha_i(v_1) \). Thus, \( \sigma(J) = L(N) \). Moreover, in the case where \( b = 0 \), we obtain \( \text{ASS}_{J,\sigma}() = \emptyset \).

The following lemma shows that the previous definition is consistent with the evaluation of trees: if we turn a tree \( t \) into the jungle \( J_n(t) \), then their evaluations coincide.

**Lemma 5.3** Let \( \sigma \) be a Σ-interpretation. For all trees \( t \in T_{\Sigma,n} \), \( \sigma(t) = \sigma(J_n(t)) \).

**Proof** Let \( J = J_n(t) \) and \( a_1, \ldots, a_n \in A \). We have to show that, for all \( a \in A \), \( a \in \sigma(t)(a_1, \ldots, a_n) \) if and only if there is some \( \alpha \in \text{ASS}_{J,\sigma}(a_1, \ldots, a_n) \) with \( \alpha(\text{res}(J)) = a \).

It follows directly from the definition of \( \sigma(t) \) (by using an obvious induction), that \( a \in \sigma(t)(a_1, \ldots, a_n) \) if and only if there is a function \( \alpha' : V(t) \rightarrow A \) such that \( \alpha'(\lambda) = a \), \( \alpha'(v) = a_i \) for all \( v \in V_{X_n}(t) \) with \( t_i(v) = x_i \), and \( \alpha'(v) \in f(\alpha'(v_{i1}), \ldots, \alpha'(v_{ik})) \) for all \( v \in V_{Z}(t) \) with \( t_i(v) = f(k) \). By the definition of \( \text{ASS}_{J,\sigma}(a_1, \ldots, a_n) \), this is the case if and only if there is some \( \alpha \in \text{ASS}_{J,\sigma}(a_1, \ldots, a_n) \) with \( \alpha(\text{res}(J)) = a \), which completes the proof. ■

It should be noticed that the validity of the previous proof depends on a subtlety whose importance might not be obvious at first sight, namely the fact that \( J_n(t) \) uses minimal sharing and has no garbage roots. In other words, it is not the case that \( \sigma(\text{tree}(J)) = \sigma(J) \) for every jungle \( J \). To see this, recall Example 5.1 where the sharing in \( J \) restricts nondeterminism, yielding \( \sigma(J) = \{ \diamond \diamond, \heartsuit \heartsuit \} \). In contrast, \( \sigma(\text{tree}(J)) = \sigma(f[a,a]) = \{ \diamond \diamond, \heartsuit \heartsuit, \heartsuit \diamond, \heartsuit \heartsuit \} \), because in this case the occurrences of \( a \) are evaluated independently. Moreover, as mentioned in Example 5.1, too, the existence of garbage in \( J \) may make \( \sigma(J) \) undefined.

We now show that interpretation of jungles is compatible with hyperedge replacement, similar to Lemma 3.1.
Lemma 5.4 Let \( \sigma \) be a \( \Sigma \)-interpretation, \( \pi^{(k)} \in \Sigma, J \in J_{\Sigma,n} \), and \( e \in E_J \) with \( \text{lab}_J(e) = \ell \) (where \( k, n \in \mathbb{N} \)). For all jungles \( J' \in J_{\Sigma,k} \), if \( \sigma(J') \subseteq \sigma(\ell) \), then \( \sigma(J[e \leftarrow J']) \subseteq \sigma(J) \).

Proof Let \( J'' = J[e \leftarrow J'] \), and let \( \alpha'' \in \text{ASS}_{J''}(a_1, \ldots, a_n) \) for \( a_1, \ldots, a_n \in A \), with \( \alpha''(\text{res}(J'')) = a \in A \). Let \( \text{img} : V_J \cup V_{J'} \rightarrow V_{J''} \) be the locator function associated with the hyperedge replacement (see the paragraph following Definition 4.2). With \( \alpha''(\text{img}(\text{ext}(J'))) = b_1 \cdots b_k b \), we get \( \alpha' \in \text{ASS}_{J',\sigma}(b_1, \ldots, b_k) \) by defining \( \alpha'(v) = \alpha''(\text{img}(v)) \), for all \( v \in V_{J'} \). It follows that \( b \in f(b_1, \ldots, b_k) \), which means that we get \( a \in A \cap J_{\Sigma,\sigma}(a_1, \ldots, a_n) \) by defining \( \alpha(v) = \alpha''(\text{img}(v)) \), for all \( v \in V_J \). Note that \( \alpha(\text{att}_J(e)) = b_1 \cdots b_k b \), because \( \text{img}(\text{att}_J(e)) = \text{img}(\text{ext}(J')) \). This proves that \( a \in \sigma(J)(a_1, \ldots, a_n) \) and completes the proof.

As mentioned in the introduction, it is straightforward to turn an I/O context-free tree grammar into a (context-free) graph grammar that generates jungles, by viewing each tree \( t \) in \( L(\gamma_g) \) as the jungle \( J_k(t) \) which contains exactly one node with label \( x_i \) for every parameter symbol \( x_i \) (where \( i \in [k] \) and \( k \) is the rank of \( g \)). With this in mind, the previous lemma motivates the following definition, which leads to Theorem 5.6, our main MW-like result.

Definition 5.5 For every delegation network \( \mathcal{N} = (G, \Pi, (\gamma_g)_{g \in G}, \mathbf{g}_0, A, \pi) \), the binary relation \( \Rightarrow_\mathcal{N} \) on \( J_{\Sigma,\gamma,n} \) (\( n \in \mathbb{N} \)) is given as follows. If \( J \in J_{\Sigma,\gamma,n} \) and \( e \in E_J \), then \( J \Rightarrow_\mathcal{N} J[e \leftarrow J_k(t)] \) for all trees \( t \in L(\gamma_g) \).

The jungle language generated by \( \mathcal{N} \) is given by

\[
L_\mathcal{N}(\mathcal{N}) = \{ J \in J_{\Pi} \mid J(\mathbf{g}_0) \Rightarrow_\mathcal{N} J \}\.
\]

Note that \( \Rightarrow_\mathcal{N} \) and, thus, \( L_\mathcal{N}(\mathcal{N}) \) do not depend on the last two components \( A \) and \( \pi \) of \( \mathcal{N} \). Thus, \( L_\mathcal{N}(\mathcal{N}) \) is generated in a purely syntactical way, cf. the last paragraph of Section 3. The reader should also note that the jungles generated by \( \mathcal{N} \), i.e., the elements of \( L_\mathcal{N}(\mathcal{N}) \), do not have parameter nodes, because \( \mathbf{g}_0 \) has rank 0.

Theorem 5.6 For every delegation network \( \mathcal{N} = (G, \Pi, (\gamma_g)_{g \in G}, \mathbf{g}_0, A, \pi) \),

\[
L(\mathcal{N}) = \pi(L_\mathcal{N}(\mathcal{N})).
\]

Proof The inclusion \( \pi(L_\mathcal{N}(\mathcal{N})) \subseteq L(\mathcal{N}) \) follows from Lemmas 5.3 and 5.4 as follows. By Lemma 5.3, \( \sigma_N(J(\mathbf{g}_0)) = \sigma_N(\mathbf{g}_0) \). Moreover, by the definition of iterate_\mathcal{N} (and the fact that \( \sigma_N \) is a fixed point of iterate_\mathcal{N}), we have \( \sigma_N(t) \subseteq \sigma_N(\mathbf{g}) \) for all \( \mathbf{g} \in G \) and \( t \in L(\gamma_g) \). Using this, together with Lemmas 5.3 and 5.4, induction on the length of derivations yields \( \sigma_N(J) \subseteq \sigma_N(\mathbf{g}_0) \), for all jungles \( J \in J_{\Sigma,\gamma} \) such that \( J(\mathbf{g}_0) \Rightarrow_\mathcal{N} J \). Since \( \sigma_N(J) = \pi(J) \) for all \( J \in J_{\Pi} \), this proves that \( \pi(L_\mathcal{N}(\mathcal{N})) \subseteq \sigma_N(\mathbf{g}_0) = L(\mathcal{N}) \).

It remains to prove that \( L(\mathcal{N}) \subseteq \pi(L_\mathcal{N}(\mathcal{N})) \). For this, we use the following definitions:
1. For a jungle $J \in J_{\Sigma^*}$, we let $ASS_J'$ denote the set of all partial functions $\alpha: V_J \rightarrow A$ such that the following hold:
   (a) $\alpha$ is defined on all shared nodes and garbage roots of $J$
   (b) for all $e \in E_J$ with $lab_J(e) = f^{(k)}$ and $att_J(e) = v_1 \cdots v_kv$, if $\alpha(v)$ is defined, then $f \in \Pi$, $\alpha(v_i)$ is defined for every $i \in [k]$, and $\alpha(v) \in f(\alpha(v_1), \ldots, \alpha(v_k))$.

   Note that if $\alpha(res(J))$ is defined, then $\alpha$ is total, and so $J \in J_\Pi$ and $\alpha(res(J)) \in \pi(J)$.

2. For $J \in J_{\Sigma^*}$, $\alpha \in ASS_J'$, and $v \in V_J$, define tree$_\alpha(J, v) \in T_{\Sigma^*}$ as follows: if $\alpha(v)$ is defined, then tree$_\alpha(J, v) = (\alpha(v))$. Otherwise, tree$_\alpha(J, v) = f[\text{tree}_\alpha(J, v_1), \ldots, \text{tree}_\alpha(J, v_k)]$, where $e \in E_J$ is the unique hyperedge such that $res_J(e) = v$, with $lab_J(e) = f^{(k)}$ and $par_J(e) = v_1 \cdots v_k$. We use the abbreviation $\text{tree}_\alpha(J)$ to denote $\text{tree}_\alpha(J, res(J))$.

   Note that if tree$_\alpha(J) = (a)$ for some $a \in A$, then $\alpha(res(J))$ is defined and equals $a$, which means that $J \in J_\Pi$ and $a \in \pi(J)$.

**Claim** For all trees $t \in T_{\Sigma^*}$, such that $g_0 \rightarrow^*_\Sigma t$, there are a jungle $J \in J_{\Sigma^*}$ and a partial function $\alpha \in ASS_J'$ such that $J(g_0) \rightarrow^*_{\Sigma^*} J$ and tree$_\alpha(J) = t$.

By Theorem 3.5, this proves the required inclusion since $a \in L(N)$ implies $g_0 \rightarrow^*_\Sigma (a)$, and hence there exists $J \in L_1(N)$ such that $a \in \pi(J)$.

We proceed by induction on the length of derivations. For $t = g_0$, the claim is true with $J = (g_0)$ and $\alpha$ being the totally undefined function. Now, let $g_0 \rightarrow^*_\Sigma t' \rightarrow^*_\Sigma t$, and assume that the claim holds for the initial part $g_0 \rightarrow^*_\Sigma t'$ of the derivation, where $J' \in J_{\Sigma^*}$ and $\alpha' \in ASS_{J'}$ satisfy the requirements given. Let $t' = s \cdot f[\langle a_1 \rangle, \ldots, \langle a_k \rangle]$, and let $v \in V_{J'}$ be the node yielding the root of $f[\langle a_1 \rangle, \ldots, \langle a_k \rangle]$ under tree$_{\alpha'}$, and $e \in E_{J'}$ the unique hyperedge such that $res_{J'}(e) = v$.\footnote{Recall the definition of shared nodes and garbage roots following Fact 4.4} Note that by the definition of tree$_{\alpha'}(J', v)$, $\alpha'(v)$ is undefined, and thus $v$ is not shared. Furthermore, $lab_{J'}(e) = f$, and if $par_{J'}(e) = v_1 \cdots v_k$, then $\alpha'(v_1) = a_i$ (also by the definition of tree$_{\alpha'}(J', v)$).

To complete the proof of the claim, we have to check two cases.

**Case 1** We have $f \in \Pi$ and $t = s \cdot \langle a \rangle$ for some $a \in f(a_1, \ldots, a_k)$.

Let $J = J'$, and let $\alpha$ be the extension of $\alpha'$ such that $\alpha(v) = a$. Clearly, this yields $\alpha \in ASS_J'$.

Moreover, the fact that $v$ is not shared implies that tree$_\alpha(J) = t$. (Note that if $v$ would be shared, setting $\alpha(v) = a$ could turn several subtrees of $t' = \text{tree}_{\alpha'}(J')$ into $\langle a \rangle$, thus yielding tree$_\alpha(J) \neq t$.)

**Case 2** We have $f \in G$ and $t = s \cdot s'[\langle a_1 \rangle, \ldots, \langle a_k \rangle]$ for a tree $t' \in L(\gamma_k)$.

Let $J = J'[e \leftarrow J_k(s')]$, and define $\alpha$ as follows. If $s' \notin X_k$, then $\alpha = \alpha' \circ \text{img}^{-1}$, where $\text{img}$ is the locator function associated with the hyperedge replacement, restricted to $V_{J'}$. (Note that $\text{img}$ is injective if $s' \notin X_k$, which makes $\alpha$ well defined.) It is straightforward to check that $\alpha \in ASS_J'$ and tree$_\alpha(J) = t$, again.

\footnote{More precisely, $v$ and $e$ are uniquely determined by the requirements that $v = res_{J'}(e)$ and, if $J' - e = (V, E, att, lab, w)$ and $J'' = (V, E, att, lab, vw)$, then tree$_{(J'')} = s$.}
using the fact that $v$ is not shared. Finally, if $s' = x_i$ for some $i \in [k]$, this means that $J$ is obtained from $J' - e$ by identifying $v$ with $v_i$, where $\text{par}_{J'}(e) = v_1 \cdots v_k$. Let $v'$ be the resulting node and define

$$
\alpha(w) = \begin{cases} 
\alpha'(v_i) & \text{if } w = v' \\
\alpha'(w) & \text{otherwise}
\end{cases}
$$

Again, it is straightforward to check that this fulfills the requirements. This completes the proof of the theorem. ■

6 Deterministic Primitives

In this section, we consider delegation networks with deterministic primitives. These are delegation networks whose interpretation $\pi$ of primitives is deterministic.

Looking at Example 5.1, the use of jungles in Theorem 5.6 seems to be essential only in the presence of nondeterministic primitives. In a delegation network with deterministic primitives, multiple evaluations of a tree always yield the same result, and never fail. Therefore, neither sharing nor garbage should be needed in the case of deterministic primitives. We confirm this intuition by proving another MW-like result, where the syntactic generation phase yields trees rather than jungles, but only delegation networks with deterministic primitives are considered.

We start by observing that, in the deterministic case, Lemma 5.3 holds even without the assumption of minimal sharing and absence of garbage roots; cf. the discussion following the proof of Lemma 5.3. (The next lemma can already be found in [AG68, Lemma 1.16].)

**Lemma 6.1** Let $\sigma$ be a deterministic $\Sigma$-interpretation. For all jungles $J \in J_{\Sigma,n}$, we have $\sigma(\text{tree}(J)) = \sigma(J)$.

**Proof** Consider $a_1, \ldots, a_n \in A$. Since $\sigma$ is deterministic, an obvious induction can be used to verify that $\text{ASS}_{J,\sigma}(a_1, \ldots, a_n) = \{\alpha\}$, where $\alpha$ is given by $\alpha(v) = \sigma(\text{tree}(J, v))(a_1, \ldots, a_n)$ for all $v \in V_J$. In particular, $\sigma(J)(a_1, \ldots, a_n) = \alpha(\text{res}(J)) = \sigma(\text{tree}(J))(a_1, \ldots, a_n)$, as claimed. ■

We can now prove our second MW-like result. For this, given a delegation network $N = (G, \Pi, \Gamma, g_0, A, \pi)$, let $N_{\text{free}} = (G, \Pi, \Gamma, g_0, T_{\Pi, \text{free} \Pi})$.

**Theorem 6.2** For every delegation network $N = (G, \Pi, \Gamma, g_0, A, \pi)$ with deterministic primitives, $L(N) = \pi(L(N_{\text{free}}))$.  

---

19 Note that $\alpha$ is defined on all shared nodes and garbage roots of $J$. This should be obvious for shared nodes. For garbage roots, it follows from the fact that all garbage roots of $J_k(s')$ are in $[\text{par}(J_k(s'))]$, which means that every garbage root of $J$ must be of the form $\text{img}(v)$, where $v$ is a garbage root of $J'$ or is in $[\text{src}_{J'}(e)]$. In both cases, $\alpha'(v)$ is defined.

20 Recall that $\pi$ is said to be deterministic if $\pi(f)$ is a function for every $f \in \Pi$.  

32
Proof. By Theorem 5.6 and Lemma 6.1, the equation
\[ L(N) = \pi(L_1(N)) = \pi(\text{tree}(L_1(N))) \]  
holds. Since \( N_{\text{free}} \) has deterministic primitives as well, \( \text{free}_\Pi \) is the identity on \( T_\Pi \), and \( L_1(N) = L_1(N_{\text{free}}) \), another instance of this equation is
\[ L(N_{\text{free}}) = \text{tree}(L_1(N)). \]
Thus, substituting the left-hand side of (3) for \( \text{tree}(L_1(N)) \) in (2) yields the claimed equality \( L(N) = \pi(L(N_{\text{free}})) \).

Motivated by the previous result, we now ask ourselves whether \( L(N_{\text{free}}) \) can be characterized in terms of derivations on trees. We show that such a characterization is obtained by using the well-known mechanism of IO context-free tree rewriting, using all IO context-free rules of the form \( g[x_1, \ldots, x_k] \rightarrow u \), where \( g^{(k)} \in G \) and \( u \in L(\gamma_g) \).

Definition 6.3 Let \( \mathcal{N} = (G, \Pi, (\gamma_g)_{g \in G}, g_0, A, \pi) \) be a delegation network, and \( t, t' \in T_{\Sigma^\prime} \). Then \( t \Rightarrow_{\mathcal{N}, \Pi} t' \) if \( t \) can be decomposed as \( t = t_0 \cdot g[s_1, \ldots, s_k] \) such that
- \( k \in \mathbb{N} \), \( g^{(k)} \in G \) and \( s_1, \ldots, s_k \in T_\Pi \), and
- \( t' = t_0 \cdot u(s_1, \ldots, s_k) \) for a tree \( u \in L(\gamma_g) \).
The tree language generated by \( \mathcal{N} \) is given by \( L_T(\mathcal{N}) = \{ s \in T_\Pi \mid g_0 \Rightarrow_{\mathcal{N}, \Pi}^* s \} \).

Note that as far as \( L_T(\mathcal{N}) \) is concerned, the last two components of a delegation network, which define the interpretation of primitives, are irrelevant. Thus, as mentioned after Definition 2.4, finitary delegation networks with the semantics \( L_T(\mathcal{N}) \) may be identified with IO context-free tree grammars. In fact, looking at the general case, we may consider the class \( F^\prime_N \) of all finitary delegation networks \( \mathcal{N}' = (G, \Pi, (\gamma'_g)_{g \in G}, g_0, A, \pi) \) such that, for all \( g \in G \), \( L(\gamma'_g) \subseteq L(\gamma_g) \). Then, obviously, \( L_T(\mathcal{N}) = \bigcup_{\mathcal{N}' \in F^\prime_N} L_T(\mathcal{N}') \).

Thus, still viewing \( \mathcal{N} \) as a tree-generating device in the sense of Definition 6.3, it may be seen as an IO context-free tree grammar which is extended in the sense that the set of right-hand sides for a given left-hand side \( g[x_1, \ldots, x_k] \) may be an infinite language taken from some class \( C \) of tree languages (i.e., \( L(\gamma_g) \subseteq C \)). This is similar to the notion of \( C \)-extended context-free string grammars and ETOI systems, which have been studied in the context of AFL theory; see, e.g., [Lee74, Asv78]. Recall also the extended Backus-Naur Form, where the adjective extended has a similar meaning.

These relationships explain why the results proved in this section (except Corollary 6.5) can be seen as “extended” versions of results in [ES77, ES78]. The next theorem shows that \( L_T(\mathcal{N}) \) provides the desired operational characterization of \( L(N_{\text{free}}) \). In the other direction, given the definition of \( L(N_{\text{free}}) \), this can be seen as a least fixed point characterization of \( L_T(\mathcal{N}) \), and thus as the extended version

\[ \text{proof continued...} \]

\[ \text{proof continued...} \]
of the least fixed point characterization of the IO context-free tree languages discussed after Definition 5.12 in [ES78].

**Theorem 6.4** For every delegation network \( \mathcal{N} \), \( L_T(\mathcal{N}) = L(\mathcal{N}_\text{free}) \).

**Proof** Let \( \mathcal{N} = (\mathcal{G}, \Pi, (\gamma_g)_{g \in \mathcal{G}}, g_0, T, \text{free}) \), assuming without loss of generality that \( \mathcal{N} = \mathcal{N}_\text{free} \). By Theorem 3.5 it suffices to show that

\[
L_T(\mathcal{N}) = \{ s \in T | g_0 \rightarrow^*_\mathcal{N} \langle s \rangle \}.
\]

Recall that \( \Sigma_\mathcal{N} \) consists of all symbols in \( \Sigma_\mathcal{N} \), and all constant symbols \( \langle s \rangle \) such that \( s \in T \). Given a tree \( t \in T_{\Sigma_\mathcal{N}} \), we make use of the following notation: \( \text{rem}(t) \) denotes the tree in \( T_{\Sigma,\{\circ\}} \) obtained from \( t \) by replacing every occurrence of all symbols of the form \( \langle s \rangle (s \in T) \) by the tree \( s \).

Observe that, for all trees \( t' \in T_{\Sigma_\mathcal{N}} \) with \( \text{rem}(t') \in T \), we have \( t' \rightarrow^*_\mathcal{N} \langle \text{rem}(t') \rangle \), by performing derivation steps according to the first case of Definition 3.2.

We show by induction on the length of derivations that, for every tree \( t \in T_{\Sigma_\mathcal{N}} \), there is a derivation \( g_0 \Rightarrow^*_\mathcal{N} t \) if and only if there is a tree \( z \in T_{\Sigma_\mathcal{N}} \) such that \( g_0 \rightarrow^*_\mathcal{N} z \) and \( \text{rem}(z) = t \). This completes the proof since, by the observation above, \( z \rightarrow^*_\mathcal{N} \langle s \rangle \) in the special case where \( \text{rem}(z) = s \in T \).

Clearly, the statement is trivially true in both directions for derivations of length 0, because \( \text{rem}(g_0) = g_0 \). It remains to be shown that single steps preserve the equivalence.

\( (\Rightarrow) \) Assume that \( t \Rightarrow^*_\mathcal{N} t' \), where the trees \( t = t_0 \cdot g[s_1, \ldots, s_k] \) and \( t' = t_0 \cdot u(s_1, \ldots, s_k) \) are as in Definition 6.3. In particular, \( s_1, \ldots, s_k \in T \). Let \( z \in T_{\Sigma_\mathcal{N}} \) be such that \( \text{rem}(z) = t \). Thus, \( z \) has the form \( z = z_0 \cdot g[z_1, \ldots, z_k] \), where \( \text{rem}(z_0) = t_0 \) and \( \text{rem}(z_i) = s_i \) for all \( i \in [k] \). By the observation above and the second case of Definition 3.2, this yields

\[
\rightarrow^*_\mathcal{N} z_0 \cdot g[\langle s_1 \rangle, \ldots, \langle s_k \rangle] \rightarrow^*_\mathcal{N} z_0 \cdot u(\langle s_1 \rangle, \ldots, \langle s_k \rangle),
\]

which ends this part of the proof, as \( \text{rem}(z_0 \cdot u(\langle s_1 \rangle, \ldots, \langle s_k \rangle)) = t_0 \cdot u(s_1, \ldots, s_k) \).

\( (\Leftarrow) \) Let \( z \rightarrow^*_\mathcal{N} z' \) with \( \text{rem}(z) = t \). If this step is performed according to the first case of Definition 3.2, then \( \text{rem}(z') = t \), so there is nothing to show, because \( t \Rightarrow^*_\mathcal{N} t \). Therefore, assume that \( z = z_0 \cdot g[\langle s_1 \rangle, \ldots, \langle s_k \rangle] \) and \( z' = z_0 \cdot u(\langle s_1 \rangle, \ldots, \langle s_k \rangle) \). Setting \( t_0 = \text{rem}(z_0) \), we get \( t = t_0 \cdot g[s_1, \ldots, s_k] \Rightarrow^*_\mathcal{N} t_0 \cdot u(s_1, \ldots, s_k) = \text{rem}(z') \), as required. \[ \blacksquare \]

Using (3), we get the following as an immediate consequence.

**Corollary 6.5** For every delegation network \( \mathcal{N} \), \( L_T(\mathcal{N}) = \text{tree}(L_J(\mathcal{N})) \).

\footnote{Intuitively, all that \( \text{rem} \) does is remove all angular brackets. Thus, as a recursive definition, \( \text{rem}(g[t_1, \ldots, t_k]) = g[\text{rem}(t_1), \ldots, \text{rem}(t_k)] \) for all \( g^{(k)} \in \Sigma_\mathcal{N} \) and \( t_1, \ldots, t_k \in T_{\Sigma_\mathcal{N},\{\circ\}} \), \( \text{rem}(\langle s \rangle) = s \) for all trees \( s \in T \), and \( \text{rem}(\langle \rangle) = \langle \rangle \).}
As another immediate consequence, we obtain an alternative formulation of our MW-like result in Theorem 6.2 similar to Theorem 5.6. For the finitary case, this was proved in [ES78, Theorem 5.10, Corollary 5.11]; see also the discussion after Definition 2.4.

**Theorem 6.6** For every delegation network $\mathcal{N} = (\mathcal{G}, \Pi, \Gamma, \mathcal{g}_0, A, \pi)$ with deterministic primitives, $L(\mathcal{N}) = \pi(L_T(\mathcal{N}))$.

The failure of (the finitary case of) Theorem 6.6 for delegation networks with nondeterministic primitives was pointed out in [ES78, p. 72]. Theorem 5.6, whose finitary counterpart is missing in [ES77, ES78], can be seen as a solution to this problem. Intuitively, it shows that one has to avoid copying (by sharing) and deletion (by garbage) to make Theorem 6.6 work in the presence of nondeterministic primitives. This intuition gives rise to an obvious conjecture, namely that the assumption of deterministic primitives in Theorem 6.6 can be dropped if neither copying nor deletion takes place. We conclude this section with the observation that this is indeed true. More precisely, let a linear nondeleting delegation network $\mathcal{N} = (\mathcal{G}, \Pi, \Gamma, \mathcal{g}_0, A, \pi)$ be a delegation network such that, for all $\mathcal{g}^{(k)} \in \mathcal{G}$, all trees $t \in L(\gamma_k)$ are linear and nondeleting in $X_k$. (Recall that this means that the parameter symbol $x_i$ occurs exactly once in $t$, for every $i \in [k]$.)

**Theorem 6.7** For every linear nondeleting delegation network $\mathcal{N} = (\mathcal{G}, \Pi, \Gamma, \mathcal{g}_0, A, \pi)$, we have $L(\mathcal{N}) = \pi(L_T(\mathcal{N}))$.

**Proof** For the proof, we make use of the “unrestricted” derivation relation $\Rightarrow_{\mathcal{N}, \text{unr}}$, which is defined exactly as $\Rightarrow_{\mathcal{N}, \text{IO}}$ in Definition 6.3, except that the trees $s_1, \ldots, s_k$ (the subtrees of the generator symbol that is being replaced) are allowed to be arbitrary trees in $T_{\Sigma, \{\Box\}}$. Thus, in contrast to $\Rightarrow_{\mathcal{N}, \text{IO}}$, arbitrary generator symbols rather than only the innermost ones can be replaced. To be able to use tree concatenation, we also consider rewriting of trees containing the parameter $\Box$. Thus, it makes sense to note that, for every tree $t = t_0 \cdot t_1$, if $t_0 \Rightarrow_{\mathcal{N}, \text{unr}} t'_0$ for a tree $t'_0$, then $t \Rightarrow_{\mathcal{N}, \text{unr}} t'_0 \cdot t_1$. In particular, $t'_0 \cdot t_1$ is defined, because $t'_0$ contains exactly one occurrence of $\Box$: thanks to the assumption that $\mathcal{N}$ is linear and nondeleting, the unique occurrence of $\Box$ in $t_0$ is neither deleted nor copied. We define the tree language generated by $\mathcal{N}$ in unrestricted derivation mode to be $L_{T, \text{unr}}(\mathcal{N}) = \{ s \in T_{\Pi} | \mathcal{g}_0 \Rightarrow_{\mathcal{N}, \text{unr}}^* s \}$.

For a derivation $J(\mathcal{g}_0) \Rightarrow_{\mathcal{N}}^* J$, using the assumption that $\mathcal{N}$ is linear and nondeleting, it follows by a straightforward induction on $n$ that $J$ is a jungle containing neither garbage roots nor shared nodes, and so $J = J_0(\text{tree}(J))$. In particular, every hyperedge $e \in E_J$ corresponds to a unique occurrence of $\text{lab}_J(e)$ in $\text{tree}(J)$. Therefore, every derivation $\mathcal{g}_0 \Rightarrow_{\mathcal{N}, \text{unr}}^* s$ can, in the obvious way, be turned into a derivation $J(\mathcal{g}_0) \Rightarrow_{\mathcal{N}}^* J_0(s)$; vice versa, every derivation $J(\mathcal{g}_0) \Rightarrow_{\mathcal{N}}^* J_0(s)$ can be turned into a derivation $\mathcal{g}_0 \Rightarrow_{\mathcal{N}, \text{unr}}^* s$. This shows that

$$L_J(\mathcal{N}) = J_0(L_{T, \text{unr}}(\mathcal{N})).$$

(4)
Next we show that $LT_{\text{unr}}(\mathcal{N})$, in fact, coincides with $LT(\mathcal{N})$:

$$LT(\mathcal{N}) = LT_{\text{unr}}(\mathcal{N}).$$  \hfill (5)

To prove (5), we show by induction on $n \in \mathbb{N}$ that the following is true for all trees $t = t_0 \odot g[s_1, \ldots, s_k] \in T_{\Sigma_N}^r$ with $g^{(k)} \in \mathcal{G}$ and $s_1, \ldots, s_k \in T_{\Pi}$: if $t \Rightarrow_{N_{\text{unr}}}^n s$ for a tree $s \in T_{\Pi}$, then there is a tree $u \in L(\gamma_g)$ such that $t \Rightarrow_{N_{\text{unr}}} t_0 \odot u(s_1, \ldots, s_k)$ holds. From this, it follows that $t \Rightarrow_{N_{\text{IO}}}^n s$ (by induction, since $t \Rightarrow_{N_{\text{IO}}} t_0 \odot u(s_1, \ldots, s_k)$), which proves (5). To prove the property claimed, if the derivation $t \Rightarrow_{N_{\text{unr}}} s$ is not already of the form required, it has the form $t \Rightarrow_{N_{\text{unr}}} t_0' \odot g[s_1, \ldots, s_k] \Rightarrow_{N_{\text{unr}}}^{n-1} s$, i.e., we have $t_0 \Rightarrow_{N_{\text{unr}}} t_0'$. By the induction hypothesis, $t_0' \odot g[s_1, \ldots, s_k] \Rightarrow_{N_{\text{unr}}} t_0' \odot u(s_1, \ldots, s_k) \Rightarrow_{N_{\text{unr}}}^{n-2} s$ for an appropriate tree $u \in L(\gamma_g)$. It follows that

$$t \Rightarrow_{N_{\text{unr}}} t_0 \odot u(s_1, \ldots, s_k) \quad \text{(by the definition of} \Rightarrow_{N_{\text{unr}}})$$
$$\Rightarrow_{N_{\text{unr}}} t_0' \odot u(s_1, \ldots, s_k) \quad \text{(since} t_0 \Rightarrow_{N_{\text{unr}}} t_0')$$
$$\Rightarrow_{N_{\text{unr}}}^{n-2} s,$$

as claimed.

Finally, using (4) and (5), we get

$$L(\mathcal{N}) = \pi(L(\mathcal{N})) \quad \text{(by Theorem 5.6)}$$
$$= \pi(J_o(L_{\text{unr}}(\mathcal{N}))) \quad \text{(by (4))}$$
$$= \pi(L_{\text{unr}}(\mathcal{N})) \quad \text{(by Lemma 5.3)}$$
$$= \pi(L(\mathcal{N})) \quad \text{(by (5))},$$

which completes the proof of the theorem. \hfill \blacksquare

We note here that the nondeleting property of $\mathcal{N}$ is essential for the correctness of (5). In [KM06, Corollary 5] it is wrongly stated that the corresponding equation holds for linear context-free tree languages, which is not the case unless the context-free tree grammar is assumed to be nondeleting. The erroneous corollary is, moreover, used in [SO07, Section 3].

7 Delegation as an Operator on Language Classes

In this section and the next, we view delegation networks as tree generators in the sense of Definition 6.3. Just as $L(\mathcal{N})$, the tree language $L_{T}(\mathcal{N})$ generated by a delegation network $\mathcal{N}$ depends only on the tree languages $L(\gamma_g)$; the specific devices $\gamma_g$ generating them are unimportant. Delegation, in the form realized by delegation networks, can therefore be considered as an operator on classes of tree languages (see also Example 2.9). The next definition formalizes this and introduces the resulting hierarchy of tree language classes, which we call the delegation hierarchy. The tree languages in this hierarchy are generated by delegation networks that themselves use delegation networks as tree generators.
**Definition 7.1** For every class \( C \) of tree languages, \( \text{DEL}(C) \) denotes the class of all tree languages of the form \( L_{\gamma}(\mathcal{N}) \), such that \( \mathcal{N} = (\mathcal{G}, \Pi, \Gamma, g_0, \mathcal{A}, \pi) \) is a delegation network with \( L(\gamma_g) \in C \), for all \( g \in \mathcal{G} \) (where \( \Gamma = (\gamma_g, Y_g)_{g \in \mathcal{G}} \)). Furthermore, \( \text{DEL}^0(C) = C \), \( \text{DEL}^{n+1}(C) = \text{DEL}(\text{DEL}^n(C)) \) for all \( n \in \mathbb{N} \), and \( \text{DEL}^*(C) = \bigcup_{n \in \mathbb{N}} \text{DEL}^n(C) \).

The *delegation hierarchy* is the hierarchy \( (\text{DEL}^n(\text{FIN}))_{n \in \mathbb{N}} \), where \( \text{FIN} \) denotes the class of all finite tree languages.

A delegation network \( \mathcal{N} = (\mathcal{G}, \Pi, (\gamma_g, Y_g)_{g \in \mathcal{G}}, g_0, \mathcal{A}, \pi) \) is *hierarchical* if it is \( n \)-hierarchical for some \( n \geq 1 \); it is 1-hierarchical if it is finitary (i.e., if \( L(\gamma_g) \) is finite for every \( g \in \mathcal{G} \)), and, for \( n \geq 2 \), it is \( n \)-hierarchical if \( \gamma_g \) is an \( (n-1) \)-hierarchical delegation network for every \( g \in \mathcal{G} \) (with \( L(\gamma_g) = L_{\gamma_g}(\mathcal{N}_g) \)). Thus, \( \text{DEL}^*(\text{FIN}) \) is the class of tree languages generated by hierarchical delegation networks, and, for \( n \geq 1 \), \( \text{DEL}^n(\text{FIN}) \) is the class generated by \( n \)-hierarchical ones.

Since \( L_T(\mathcal{N}) \) does not depend on the semantic components \( \mathcal{A} \) and \( \pi \) of \( \mathcal{N} \), as discussed after Definition 6.3, we will from now on denote a delegation network as a 4-tuple \( \mathcal{N} = (\mathcal{G}, \Pi, \Gamma, g_0) \).

We start our discussion of the delegation hierarchy with an example of a tree language in \( \text{DEL}^2(\text{FIN}) \). Note that \( \text{DEL}(\text{FIN}) \) is the class of tree languages generated by finitary delegation networks, i.e., by IO context-free tree grammars (cf. the last paragraph of Example 2.7(a)). Hence, \( \text{DEL}^2(\text{FIN}) \) is the class of tree languages generated by delegation networks of which each tree generator is an IO context-free tree grammar.

**Example 7.2** This example, of a tree language in \( \text{DEL}^2(\text{FIN}) \), is similar to Example 2.9(a). We first observe that the tree language consisting of all fully balanced binary trees of height \( n \geq 1 \) over the signature \( \{f^{(2)}, h^{(0)}\} \) is in \( \text{DEL}(\text{FIN}) \): it is \( L_T(\mathcal{N}_{\text{bin}}) \) for the delegation network \( \mathcal{N}_{\text{bin}} \) with generator symbols \( g^{(0)}_{0} \) and \( g^{(1)} \), and with \( L(\gamma_{g_0}) = \{g[h]\} \) and \( L(\gamma_g) = \{g[f[x_1, x_1]], f[x_1, x_1]\} \) where \( Y_g = \{x_1\} \).

Now consider the language \( \mathcal{L}_{\text{expsat}}(s, t_1, \ldots, t_2^n) \), \( n \geq 1 \), where \( s \) is a fully balanced binary tree of height \( n \) over the signature \( \{f, \sqcap_1, \ldots, \sqcap_2^n\} \) and \( t_1, \ldots, t_2^n \) are arbitrary trees over the signature \( \{k^{(2)}, a^{(0)}\} \). This language is equal to \( L_T(\mathcal{N}_{\text{expsat}}) \) for the 2-hierarchical delegation network \( \mathcal{N}_{\text{expsat}} \) with generator symbols \( h^{(0)}_0 \) and \( h^{(0)} \), and with \( L(\gamma_{h_0}) = L_T(\mathcal{N}_{\text{bin}}) \) and \( L(\gamma_h) = T_{\{k, a\}} \). Obviously, \( T_{\{k, a\}} \) is in \( \text{DEL}(\text{FIN}) \) (with \( L(\gamma_{g_0}) = \{k[g_0, g'_0, a]\} \)). Hence \( \mathcal{L}_{\text{expsat}} \) is in \( \text{DEL}^2(\text{FIN}) \).

In the preceding example, the tree generators \( \gamma_{h_0} = \mathcal{N}_{\text{bin}} \) and \( \gamma_h \) of the 2-hierarchical delegation network \( \mathcal{N}_{\text{expsat}} \) do not have parameters, whereas the tree generator \( \gamma_g \) of the 1-hierarchical delegation network \( \mathcal{N}_{\text{bin}} \) does have parameters. In general however, for an \( n \)-hierarchical delegation network \( \mathcal{N} = (\mathcal{G}, \Pi, (\gamma_g, Y_g)_{g \in \mathcal{G}}, g_0, \mathcal{A}, \pi) \), the trees generated by the \((n-1)\)-hierarchical delegation network \( \gamma_g \) contain the parameters in \( Y_g \). As a consequence, the tree
generators of the network $\gamma_g$ should use parameters that are not in $Y_g$ (see Example 2.9(b)). A careful treatment of this formal detail will be important in Lemma 7.22 and Theorem 7.23 (and their proofs); otherwise, we will assume as usual that $Y_g = X_k$ where $k$ is the rank of $g$.

In the remainder of the paper, we want to study the class $\text{DEL}^*(\text{FIN})$ of tree languages generated by hierarchical delegation networks, and compare it with known tree language classes. We start by looking at the so-called path languages of tree languages in $\text{DEL}^*(\text{FIN})$.

For a signature $\Sigma$, let $\langle \Sigma \rangle$ be the (finite unranked) alphabet

$$\langle \Sigma \rangle = \{ \langle \mathbf{f}, j \rangle \mid k \in \mathbb{N}, \mathbf{f}^{(k)} \in \Sigma, 0 \leq j \leq k \}.$$  

Given a tree $t \in T_{\Sigma,n}$ and $i \in [n]$, we let $\text{paths}_i(t) \subseteq \langle \Sigma \rangle^*$ be the string language of all downwards directed paths in $t$ from the root to an occurrence of $x_i$ (where $x_i$ is excluded). Formally,

$$\text{paths}_i(t) = \left\{ \begin{array}{ll} \{ \lambda \} & \text{if } t = x_i \\ \emptyset & \text{if } t = x_j, j \neq i \\ \{ \langle \mathbf{f}, j \rangle p \mid j \in [k], p \in \text{paths}_i(t_j) \} & \text{if } t = \mathbf{b}[t_1, \ldots, t_k], \text{ with } \mathbf{b} \in \Sigma. \end{array} \right.$$  

Thus, a symbol $\langle \mathbf{f}, j \rangle$ indicates that the path continues at the $j$th subtree of the symbol $\mathbf{f}$. The string language of paths from the root ending at a non-parameter symbol, a subset of $\langle \Sigma \rangle^+$, is denoted by $\text{paths}(t)$. Thus, $\text{paths}(x_i) = \emptyset$ for all $i \in [n]$, and, for $t = \mathbf{b}[t_1, \ldots, t_k]$ with $\mathbf{b}^{(k)} \in \Sigma$,

$$\text{paths}(t) = \{ \langle \mathbf{f}, 0 \rangle \} \cup \{ \langle \mathbf{f}, j \rangle p \mid j \in [k], p \in \text{paths}(t_j) \}.$$  

For technical convenience, we use ‘paths’ as a synonym for ‘paths’. For a tree language $L \subseteq T_{\Sigma,n}$ and $i \in \{0, \ldots, n\}$, we let $\text{paths}_i(L) = \bigcup_{t \in L} \text{paths}_i(t)$. For a jungle $J \in J_{\Sigma,n}$, let $\text{paths}_i(J) = \text{paths}_i(\text{tree}(J))$, and similarly for jungle languages. If $L$ is a tree language or a jungle language, then paths($L$) is called the path language of $L$. Note that, by Corollary 6.5, for every delegation network $\mathcal{N}$, $L_{\mathcal{N}}(\mathcal{N})$ and $L_{\mathcal{T}}(\mathcal{N})$ have the same path language. This will be the key observation that is used to prove the main result of this section.

Of course, paths$_i(J)$ can easily be obtained from the jungle paths in $J$ without taking the detour via tree($J$). For $i \in [n]$, it should be clear that paths$_i(J)$ is equal to the set of all $\langle \text{lab}_J(e_1), j_1 \rangle \cdots \langle \text{lab}_J(e_m), j_m \rangle$, such that $(e_1, j_1) \cdots (e_m, j_m)$ is a jungle path from res($J$) to par($J$, $i$). Similarly, paths($J$) is the set of all sequences of the form $\langle \text{lab}_J(e_1), j_1 \rangle \cdots \langle \text{lab}_J(e_m), j_m \rangle \langle \text{lab}_J(e), 0 \rangle$, where $e \in E_J$, and $(e_1, j_1) \cdots (e_m, j_m)$ is a jungle path from res($J$) to res($e$).

Our intention is to prove in this section that the path language of every tree language in $\text{DEL}^*(\text{FIN})$ is a context-free language. The proof would be easy if we would only consider proper delegation networks in Definition 7.1, where a delegation network $\mathcal{N} = (\mathcal{G}, \Pi, \Gamma, g_0)$ is proper if for every $t \in T_{\Sigma,\mathcal{N}}$ there exists $t' \in T_{\Pi}$ such that $t \Rightarrow_{\mathcal{N},IO} t'$.
To sketch the proof of this result (for the proper case), recall that a context-free grammar is a tuple $G = (\Xi, \Sigma, R, S)$ where $\Xi$ and $\Sigma$ are disjoint finite sets of nonterminals and terminals, $S \in \Xi$, and $R$ is a finite set of rules of the form $f \to \alpha$ with $f \in \Xi$ and $\alpha \in (\Xi \cup \Sigma)^*$. A derivation step is denoted $\beta_1 f \beta_2 \Rightarrow_G \beta_1 \alpha \beta_2$, and $L(G) = \{ w \in \Sigma^* \mid S \Rightarrow^*_G w \} = \{ w \in \Sigma^* \mid S \Rightarrow^*_{G, rm} w \}$ where $\Rightarrow^*_{G, rm}$ indicates rightmost derivations (meaning that $\beta_2 \in \Sigma^*$). A cf-extended context-free grammar is the same as a context-free grammar except that $R$ can be infinite, provided the language $\{ \alpha \mid (f \to \alpha) \in R \}$ is context-free, for every $f \in \Xi$. It is well known (and easy to see) that every cf-extended context-free grammar has an equivalent context-free grammar.

**Proposition 7.3** Let $N = (G, \Pi, \Gamma, g_0)$ be a proper delegation network such that $\text{paths}(\text{L}(\gamma_g))$ is context-free for every $g^{(k)} \in G$ and every $i \in \{0, \ldots, k\}$. Then $\text{paths}(L_T(N))$ is context-free.

**Proof sketch** The proof is similar to the one of [ES79, Corollary 4.2], which treats the case where $N$ is finitary. We construct the cf-extended context-free grammar $G = ((G), (\Pi), R, (g_0, 0))$ where $R$ is the set of all rules $(g, i) \to p$ with $p \in \text{paths}(\text{L}(\gamma_g))$. It can be shown (cf. Lemmas 4.2 and 4.3 of [ES79]) that for every $w \in (\Sigma_N)^*$, $\gamma_0, 0) \Rightarrow_{\gamma, \text{rm}}^{*} w$ if and only if there exists $t \in T_{\Sigma_N}$ with $w \in \text{paths}(t)$, such that $g_0 \Rightarrow_{\gamma_{\Sigma IO}} t$. Properness of $N$ is used in the proof of the only-if direction, and it is used to conclude from this statement that $L(G) = \text{paths}(L_T(N))$.

To illustrate the need for properness, consider the delegation network $N$ with $L(\gamma_g) = \{ g[a, g'] \}$, $L(\gamma_g') = \{ x_1 \}$ and $L(\gamma_g') = \emptyset$. Then $\text{paths}(L_T(N)) = \emptyset$, but $G$ has a derivation $(\gamma_0, 0) \Rightarrow (g, 1)(a, 0) \Rightarrow (a, 0)$. $lacksquare$

It would not be difficult to generalize this lemma in such a way that it can be used iteratively, showing that the path language of every tree language in DEL⁺(FIN) is context-free, provided only proper delegation networks are used. Now, it is easy to transform a delegation network $N$ into an equivalent proper one (assuming that $L_T(N) \neq \emptyset$). In fact, as for context-free grammars, it suffices to determine the set $\text{use}(N) \subseteq G$ of useful symbols: it is the smallest subset of $G$ such that for every $g^{(k)} \in G$ and $t \in L(\gamma_g)$, if $t \in T_{\text{use}(N) \cup \Pi \cup \Pi}$ then $g \in \text{use}(N)$. An equivalent proper network is then obtained from $N$ by removing all generator symbols that are not useful and by replacing $L(\gamma_g)$ with $L(\gamma_g) \cap T_{\text{use}(N) \cup \Pi \cup \Pi}$ for every $g^{(k)} \in \text{use}(N)$. The problem with this construction is that we do not know whether DEL⁺(FIN) is closed under intersection with regular tree languages, not even when they are of the form $T_{\Sigma}$ for some signature $\Sigma$. Thus, we also do not know whether the restriction to proper delegation networks leads to the same class DEL⁺(FIN) or to a smaller one, and for that reason we do not wish to impose that restriction (though, as we will show after Corollary 8.16, it is decidable whether a hierarchical delegation network is proper).
Thus, the straightforward approach to proving that the path language of every tree language in \( \text{DEL}^*(\text{FIN}) \) is context-free, is flawed. We now start our, rather technical, alternative proof. We first show that it suffices to consider a restriction of the operator \( \text{DEL} \), because this restriction will be needed later. Let us call a delegation network \( \mathcal{N} = (\mathcal{G}, \Pi, (\gamma_g, Y_g)_{g \in \mathcal{G}}, \mathbf{g}_0) \) propagating if \( L(\gamma_g) \cap Y_g \neq \emptyset \), for all \( g \in \mathcal{G} \). We let \( \text{pDEL} \) denote the operator which is defined in the same way as \( \text{DEL} \), except that only propagating delegation networks are considered. The definitions of \( \text{pDEL}^n(C) \) and \( \text{pDEL}^*(C) \) carry over from the general case in the obvious way.

Now, let \( *^{(1)} \notin \Pi \) be a special symbol reserved for its use in the following context. Define \( \text{erase}_s: T_{\Pi \cup \{*\}} \to T_{\Pi} \) to be the mapping given by \( \text{erase}_s(*[t]) = \text{erase}_s(t) \) and \( \text{erase}_s(f[t_1, \ldots, t_k]) = f[\text{erase}_s(t_1), \ldots, \text{erase}_s(t_k)] \) for all \( f^{(k)} \in \Pi \) and \( t_1, \ldots, t_k \in T_{\Pi \cup \{*\}} \). Hence, \( \text{erase}_s \) is a linear and nondeleting tree homomorphism.\(^{23}\)

We have the following lemma.

**Lemma 7.4** For every tree language \( L \in \text{DEL}^*(\text{FIN}) \), there is a tree language \( L' \in \text{pDEL}^*(\text{FIN}) \), such that \( L = \text{erase}_s(L') \).

**Proof** By induction on \( n \), where \( L \in \text{DEL}^n(\text{FIN}) \), we prove the more general statement that there is a tree language \( L' \in \text{pDEL}^*(\text{FIN}) \), such that \( L = \text{erase}_s(L') \) and the root label of every tree in \( L' \) is \(*\).

For \( n = 0 \), this is obvious, by defining \( L' = \{*[t] \mid t \in L\} \). For \( n > 0 \), let \( L = L_T(\mathcal{N}) \), where \( \mathcal{N} = (\mathcal{G}, \Pi, (\gamma_g, Y_g)_{g \in \mathcal{G}}, \mathbf{g}_0) \) with \( L(\gamma_g) \in \text{DEL}^{n-1}(\text{FIN}) \). Using the induction hypothesis, define \( \mathcal{N}' = (\mathcal{G}, \Pi \cup \{*\}, (\gamma'_g)_{g \in \mathcal{G}}, \mathbf{g}_0) \), where \( L(\gamma_g) = \text{erase}_s(L(\gamma'_g)) \) and each tree in \( L(\gamma'_g) \) has \(*\) as its root label, for all \( g \in \mathcal{G} \). Clearly, by looking at the very first step of a derivation in \( \mathcal{N}' \), this means that each tree in \( L_T(\mathcal{N}') \) has \(*\) as its root label.

It remains to argue that \( L = \text{erase}_s(L_T(\mathcal{N}')) \). This follows directly from the following claim, using induction on the length of derivations.

**Claim** Let \( s \in T_{\Sigma_N} \) and \( t \in T_{\Sigma_{N \cup \{*\}}} \) be such that \( \text{erase}_s(t) = s \). Then, for every derivation step \( s \Rightarrow_{N', \Pi} s' \), there is a derivation step \( t \Rightarrow_{N', \Pi} t' \) such that \( \text{erase}_s(t') = s' \). Vice versa, \( s \Rightarrow_{N', \Pi} \text{erase}_s(t') \) is a derivation step, for every derivation step \( t \Rightarrow_{N', \Pi} t' \).

For the first direction, following Definition 6.3, let \( s = s_0 \cdot (g[z_1, \ldots, z_k] \quad [s'] = s_0 \cdot u(z_1, \ldots, z_k), \quad \text{where} \quad s_0 \in T_{\Sigma_N, \{\square\}}, \quad g^{[k]} \in \mathcal{G}, \quad z_1, \ldots, z_k \in T_{\Pi}, \quad \text{and} \quad u \in L(\gamma_g) \). Since \( \text{erase}_s(t) = s \), it should be clear that we can write \( t \) as \( t_0 \cdot g(z'_1, \ldots, z'_k) \), where \( \text{erase}_s(t_0) = s_0 \) and \( \text{erase}_s(z'_i) = z_i \) for all \( i \in [k] \) (extending \( \text{erase}_s \) to trees in \( T_{\Sigma_N, \{\square\}} \) in the obvious way). In particular, since \( \text{erase}_s \) is linear and nondeleting, \( t_0 \) contains exactly one occurrence of \( \square \), because \( s_0 \) does. Hence, defining \( t' = t_0 \cdot u'(z'_1, \ldots, z'_k) \) for a tree \( u' \in L(\gamma'_g) \) such that \( \text{erase}_s(u') = u \), we get \( t \Rightarrow_{N', \Pi} t' \) and, by construction, \( \text{erase}_s(t') = s' \), as claimed.

\(^{23}\) Cf. the next section, after Definition 8.1.
We omit the proof of the other direction, because it is very similar to the preceding one.

Together with Lemma 7.4, the next lemma shows that it suffices to consider tree languages in $pDEL^{\ast}(FIN)$ to establish that the path languages of tree languages in $DEL^{\ast}(FIN)$ are context-free.

**Lemma 7.5** For every tree language $L$, if $\text{paths}(L)$ is a context-free language, then $\text{paths}(\text{erase}_{\ast}(L))$ is a context-free language.

**Proof** Let $L \subseteq T_{\Sigma,\{\ast\}}$. Obviously, $\text{paths}(\text{erase}_{\ast}(L)) = h(\text{paths}(L) \cap R)$, where $R$ is the regular language of all strings in $(\Sigma \cup \{\ast\})^*$ that do not end on $\langle \ast, 0 \rangle$, and $h$ is the string homomorphism that erases the symbol $\langle \ast, 1 \rangle$. The result follows from the fact that the class of context-free languages is closed under homomorphisms and intersection with regular languages.

Our proof of the context-freeness of $\text{paths}(L)$ for $L \in pDEL^{\ast}(FIN)$ makes use of the notion of context-free jungle languages. These are sets of jungles generated by the special case of hyperedge-replacement graph grammars in which all right-hand sides are jungles. Throughout the rest of this section, we assume that the reader is familiar with some of the most basic properties of hyperedge-replacement graph grammars.

**Definition 7.6 (context-free jungle grammar)** A context-free jungle grammar (abbreviated CFJG) is a tuple $G = (\Xi, \Sigma, R, f_0)$ such that

- $\Xi$ and $\Sigma$ are disjoint finite signatures of nonterminals and terminals, resp.,
- $R$ is a finite set of rules of the form $f \rightarrow K$, where $f^{(k)} \in \Xi$ and $K \in J_{\Xi,\Sigma,k}$ ($k \in \mathbb{N}$), and
- $f^{(n)}_0 \in \Xi$, where $n \in \mathbb{N}$, is the initial nonterminal.

For jungles $J, J' \in J_{\Xi,\Sigma,n}$, there is a derivation step $J \Rightarrow_G J'$ if there are a hyperedge $e \in E_J$ and a rule $\text{lab}_J(e) \rightarrow K$ in $R$, such that $J' = J[e \leftarrow K]$. The jungle language generated by $G$ is given by

$$L(G) = \{ J \in J_{\Sigma,n} \mid J(f_0) \Rightarrow^* G J \}. $$

A jungle language that can be generated by a CFJG is said to be context-free. An extended CFJG, its derivations, and the generated jungle language are defined in precisely the same way, with the exception that $R$ may be infinite. Clearly, for a delegation network $\mathcal{N} = (\mathcal{G}, \Pi, \Gamma, g_0)$, the jungle language $L_1(\mathcal{N})$, as defined in Definition 5.5, is generated by the extended CFJG $(\mathcal{G}, \Pi, R, g_0)$ where $R$ consists of all rules $g \rightarrow J_k(t)$ with $g^{(k)} \in \mathcal{G}$ and $t \in L(\gamma_k)$. A cf-extended CFJG is an extended CFJG such that the jungle language $L_\mathcal{G} = \{ J \mid (f \rightarrow J) \in R \}$ is context-free, for every $f \in \Xi$. In this case, we denote by $f \rightarrow L_\mathcal{G}$ the set of all rules in $R$ with left-hand side $f$.

For a CFJG (and similarly for an extended CFJG) $G = (\Xi, \Sigma, R, f_0)$, we say that a nonterminal $f^{(k)} \in \Xi$ is useful if there exists $J \in J_{\Sigma,k}$ such that
\[ J(f) \Rightarrow^*_G J. \] Obviously, all nonterminals (except \( f_0 \)) that are not useful can be discarded, together with the rules in which they occur, without changing \( L(G) \).

Therefore, except in the trivial case where \( L(G) = \emptyset \), we may always assume without loss of generality that all nonterminals of a CFJG considered are useful.

To prove that every tree language in \( \text{DEL}^*(\text{FIN}) \) has a context-free path language, we will show that for every tree language \( L \in \text{pDEL}^*(\text{FIN}) \) there is a context-free jungle language having the same path language as \( L \). More precisely, for every hierarchical delegation network \( N \) with \( L_T(N) = L \) we will define a CFJG \( G \) such that \( L(G) \) has the same path language as \( L(N) \). Then the following lemma yields the main result of this section (together with Lemmas 7.4 and 7.5).

**Lemma 7.7** For every context-free jungle language \( L \), \( \text{paths}(L) \) is context-free.

**Proof** Let \( L = L(G) \), where \( G = (\Xi, \Sigma, R, f_0^{(n)}) \) contains only useful nonterminals. Let \( G' \) be the context-free grammar given by \( G' = (\langle \Xi \rangle, \langle \Sigma \rangle, R', \langle f_0, 0 \rangle) \), where \( R' \) is the set of all rules \( \langle f, i \rangle \rightarrow p \) such that \( f^{(k)} \in \Xi, (f \rightarrow J) \in R, i \in \{0, \ldots, k\} \), and \( p \in \text{paths}(J) \). Then the following can be shown in a straightforward manner, by induction on the length of derivations: for all \( w \in \langle \Xi \cup \Sigma \rangle^+ \), \( \langle f_0, 0 \rangle \Rightarrow^*_G w \) if and only if there exists a jungle \( J \in J_{\Xi \cup \Sigma} \) with \( w \in \text{paths}(J) \), such that \( J(f_0) \Rightarrow^*_G J \). From this, together with the assumption that all nonterminals of \( G \) are useful, it follows that \( L(G') = \text{paths}(L(G)) \). □

The following two lemmas show that it suffices to construct a cf-extended CFJG if we want to prove that a certain jungle language is context-free, and that we can always remove garbage from the jungles generated.

**Lemma 7.8** For every cf-extended CFJG \( G \), the jungle language \( L(G) \) is context-free.

**Proof** This is similar to the proof of the fact that cf-extended context-free string grammars generate context-free languages. We only give the construction, whose correctness can be verified using the well-known fact that hyperedge replacement is confluent and associative [DHK97, Eng97].

Let \( G = (\Xi, \Sigma, R, \text{ini}) \) be a cf-extended CFJG, where \( R = \bigcup_{\xi \in \Xi} f \rightarrow L_\xi \). For every \( f \in \Xi \), let \( G_f = (\Xi_f, \Xi \cup \Sigma, R_f, \text{ini}_f) \) be a CFJG with \( L(G_f) = L_\xi \), and assume without loss of generality that the signatures \( \Xi_f, f \in \Xi \), are pairwise disjoint. Then \( L(G) = L(G') \), where \( G' = (\Xi', \Sigma, R', \text{ini}) \) is the CFJG given by \( \Xi' = \Xi \cup \bigcup_{\xi \in \Xi} \Xi_f \) and \( R = \{ f \rightarrow J(\text{ini}_f) \mid f \in \Xi \} \cup \bigcup_{\xi \in \Xi} R_\xi \). □

For the remainder of this section, recall the definitions regarding clean jungles after Fact 4.4 and, in particular, that \( C_J \) denotes the set of all non-garbage nodes.

---

\[ ^24 \text{Recall that we use } \Rightarrow^*_G \text{ to denote the derivation relation of } G', \text{ and } L(G') \text{ to denote its generated language.} \]
of a jungle $J$, and that clean($J$) denotes the clean jungle obtained from $J$ by restricting $J$ to $C_J \cup \{\text{par}(J)\}$.

**Lemma 7.9** For every CFJG $G$, there is a CFJG $G'$ with $L(G') = \text{clean}(L(G))$.

**Proof** Let $G = (\Xi, \Sigma, R, f_0^{(n)})$. Assuming that all nonterminals in $G$ are useful, we construct $G' = (\Xi', \Sigma, R', f_0^{(n)})$ based on an easy guess-and-verify strategy, in the following way. The set of nonterminals is given by

$$\Xi' = \{f_0\} \cup \{f_S^{(l)} | k \in \mathbb{N}, f^{(k)}_S \in \Xi, S \subseteq [k]\}.$$  

The intuition is that $f_S$ labels a hyperedge $e$ such that (a) res($e$) is not (and will not become) garbage, and (b) the parameter tentacles $(e, j)$ with $j \notin S$ have been removed as they have been guessed to be useless in the further course of the derivation, in the sense that, in the jungle $J$ derived from $e$, there will not be a jungle path from res($J$) to par($J$, $j$). Using finite sets INI and $K_S$, for $K \in J_{\Xi, \Sigma, k}$ and $S \subseteq [k]$, which implement the guess-and-verify strategy, the set of rules is given by

$$R' = \{f_0 \rightarrow I \mid I \in \text{INI}\} \cup \bigcup_{(t^{(k)} \rightarrow K) \in R} \{f_S \rightarrow K' \mid S \subseteq [k], K' \in K_S\}.$$  

The definition of INI and $K_S$ makes use of the following notation. Given a sequence $a = a_1 \cdots a_i$ and a set $S = \{i_1, \ldots, i_t\} \subseteq [k]$, where $i_1 < \cdots < i_t$, we let $a|_S = a_i \cdots a_i$.

Now, the set INI consists of all (clean) jungles $I$ with $V_I = \{w_1, \ldots, w_n, w\}$, $E_I = \{e\}$, lab$_I(e) = (f_0)_S$ and att$_I(e) = w_1 \cdots w_n w|_S$ for a set $S \subseteq [n]$, and ext$_I = w_1 \cdots w_n w$. Intuitively, in the first step of a derivation, we guess which parameter nodes of the generated jungle $J \in J_{\Sigma, n}$ are used, meaning that they belong to $C_J$.

To define $K_S$ for a jungle $J \in J_{\Xi, \Sigma, k}$ and a set $S \subseteq [k]$, let us say that a jungle $K'$ is garbage-free if $V_{K'} = C_{K'}$, i.e., if it is clean and all parameter nodes are used. Now, $K_S$ is defined to be the set of all garbage-free jungles $K' \in J_{\Xi, \Sigma, [S]}$ such that

- $V_{K'} \subseteq V_K$, $E_{K'} \subseteq E_K$, res($K'$) = res($K$), and res$_{K'}(e) = \text{res}_K(e)$ for all $e \in E_{K'}$,
- for all $e \in E_K$, with lab$_K(e) \in \Sigma$, lab$_{K'}(e) = \text{lab}_{K'}(e)$ and par$_{K'}(e) = \text{par}_{K}(e)$,
- for all $e \in E_{K'}$ with lab$_{K'}(e) = f^{(m)}_S \in \Xi$, there is $S' \subseteq [m]$ such that lab$_{K'}(e) = f_{S'}$ and par$_{K'}(e) = \text{par}_{K}(e)|_{S'}$, and
- par($K'$) = par($K$)$_{|S'}$.

Intuitively, the first three items above implement the guessing, whereas the last implements the verification in the guess-and-verify strategy mentioned above. By induction on the length of derivations, we can show for all $f^{(k)} \in \Xi, S \subseteq [k]$, and $J' \in J_{\Xi, \Sigma, [S]}$, that $J(f_S) \Rightarrow^*_G J'$ if and only if there is a jungle $J \in J_{\Xi, \Sigma, k}$ such that $J(f) \Rightarrow^*_G J$ and $J' \in J_S$. 

43
To complete the proof, note that, for $J \in J_{\Sigma,n}$, there is a unique $S \subseteq [n]$ such that $J_S \neq \emptyset$; moreover, for that $S$, $J_S$ is a singleton, say $J_S = \{J\}$, and $\text{clean}(J) = I[e \leftarrow J]$ with $I \in \text{INI}$ and $\text{lab}_I(e) = (\xi_0)_S$. Thus, taking into account the rules in $R'$ of the form $I_0 \rightarrow I$ with $I \in \text{INI}$, we get $L(G') = \text{clean}(L(G))$. (Note that the usefulness of the nonterminals of $G$ is required to obtain the inclusion $L(G') \subseteq \text{clean}(L(G))$.) This completes the proof. □

To turn a (propagating) hierarchical delegation network $N = (G, \Pi, (\gamma_g)_{g \in \mathcal{G}}, g_0)$ into a cf-extended CFJG $G$, and to be able to relate their generated jungle languages, we need a few rather technical preparations. Intuitively, we cannot define $G$ such that it generates exactly the jungles in $L_1(N)$, because of the following problem. Assume inductively that every $\gamma_g$ ($g^{(k)} \in \mathcal{G}$) is a delegation network for which we are given a CFJG $G_g$ such that $L(G_g) = L_1(\gamma_g) \subseteq J_{\Sigma,X_k}$. By a straightforward construction (see the proof of Lemma 7.22), $G_g$ can be transformed into a CFJG $G'_g$ such that $L(G'_g) = \{J' \mid J \in L(G_g)\} \subseteq J_{\Sigma,k}$ where the jungle $J'$ (with $k$ parameters) is obtained from the jungle $J$ (without parameters) as follows: for each $i \in [k]$, the $i$th parameter node $v_i$ of $J'$ is newly created, and $v_i$ is identified with all result nodes of hyperedges that are labeled with $x_i$; moreover, all hyperedges with labels in $X_k$ are removed. Now, to construct a CFJG $G$ such that $L(G) = L_1(N)$, it would be natural to use the (cf-extended) rules $g \rightarrow L(G'_g)$. However, by Definition 5.5, we should actually use the rules $g \rightarrow J_k(\text{tree}(L(G'_g)))$ instead. In other words, the inductively obtained jungles in $L(G_g)$ should be replaced by jungles with minimal sharing. Clearly this is not possible because there is, e.g., no CFJG that generates the set of all fully balanced binary trees (without sharing), cf. Example 7.2. Thus, intuitively, the problem is that a CFJG generates trees with too much sharing. To circumvent this problem, we will show (in Lemma 7.20) that we can define the grammar $G$ such that it generates a jungle language that differs from $L_1(N)$ but, nevertheless, has the same path language; more precisely, we will say that $L(G)$ is a “path variant” of $L_1(N)$, see Definition 7.17. For this, we invent a notion of redirecting parameter tentacles between two nodes $v$ and $w$ of a jungle $J$, an operation that can potentially increase the amount of sharing (but also can create garbage). Then $G$ can be constructed with the rules $g \rightarrow L(G'_g)$, where it is assumed that $L(G_g)$ is a path variant of $L_1(\gamma_g)$.

We now start the technical preparations. Consider a jungle $J$ and nodes $v, w \in V_J$. The set cand$_J(v, w)$ of all parameter tentacles which are candidates for redirection from $v$ to $w$ consists of all $(e, j) \in \text{tent}(J)$ with par$_J(e, j) = v$, such that there does not exist a jungle path in $J$ from $w$ to res$_J(e)$. Now, for every set $\Theta \subseteq \text{cand}_J(v, w)$, we let $J(\Theta, w)$ denote the jungle obtained from $J$ by redirecting par$_J(e, j)$ to $w$ for all $(e, j) \in \Theta$, and cleaning the result. More precisely, $J(\Theta, w) = \text{clean}(J_{\Theta ::= w})$, where $J_{\Theta ::= w}$ is equal to $J$, except that alt$_{J_{\Theta ::= w}}(e, j) = w$, for all $(e, j) \in \Theta$. By $J(v, w)$, we abbreviate the set $\{J(\Theta, w) \mid \Theta \subseteq \text{cand}_J(v, w)\}$. By the assumption that $\Theta \subseteq \text{cand}_J(v, w)$, redirect-
tion does not introduce cycles (i.e., nonempty jungle paths from a node to itself). Using this, one can easily show that \( J_{\Theta_0 := w} \) is a jungle, and hence \( J(\Theta, w) \) is a clean jungle. Note that we always have \( \text{clean}(J) \in J(v, w) \), by taking \( \Theta = \emptyset \). For technical convenience, \( J \) itself is not required to be clean. However, if \( v, w \in C_J \), then we have that \( C_J \supseteq C_{J_{\Theta_0 := w}} \), which means that \( J(\Theta, w) = \text{clean}(J)(\Theta', w) \), where \( \Theta' = \Theta \cap \text{tent}(\text{clean}(J)) = \{(e, j) \in \Theta \mid \text{res}_J(e) \in C_J\} \). (In particular, \( \text{clean}(J)(\Theta', w) \) is defined, because \( \Theta' \subseteq \text{cand}_{\text{clean}(J)}(v, w) \)). As a consequence, we then have \( J(v, w) = \text{clean}(J)(v, w) \). In the case that \( v \notin C_J \), we obviously have \( J(v, w) = \{\text{clean}(J)\} \).

Below, the interplay of hyperedge replacement and redirection (in the sense just defined) will be studied. For this, we now introduce a rewrite relation \( \Rightarrow \) on sets of clean jungles, which plays a central role in the reasoning on the following pages. The relation \( \Rightarrow \) is defined on sets of clean jungles in such a way that redirection from \( v_0 \) to \( v_1 \) is always paired with redirection from \( v_1 \) to \( v_0 \), thus replacing one jungle \( J \) in the set by two jungles \( J_0 \) and \( J_1 \). Intuitively, this makes sure that, in total, no paths of the original jungle are lost (by turning them into garbage).

**Definition 7.10** For sets \( \mathcal{J}, \mathcal{J}' \) of clean jungles, \( \mathcal{J} \Rightarrow \mathcal{J}' \) if there exist a jungle \( J \in \mathcal{J} \), nodes \( v_0, v_1 \in C_J \), and jungles \( J_i \in J(v_i, v_{1-i}) \) for \( i \in \{0, 1\} \), such that \( \mathcal{J}' = \mathcal{J} \setminus \{J\} \cup \{J_0, J_1\} \). If \( \mathcal{J} \) is a singleton \( \{J\} \), we may write \( J \Rightarrow \mathcal{J}' \) instead of \( \{J\} \Rightarrow \mathcal{J}' \).

Note that, in the preceding definition, any of \( J, J_0, \) and \( J_1 \) may be equal. Before giving an example, we introduce a notation that will also be used in the example, generalizing the operation \( \bullet \cdot \) of concatenation of trees to jungles. If \( J \) is a jungle in \( J_{\Sigma_1(\mathcal{G})} \) such that there is exactly one hyperedge \( e \in E_J \) with label \( \square \), and \( J' \) is a jungle in \( J_{\Sigma_2(\mathcal{G})} \), then \( J \cdot J' \) denotes the jungle \( J[e \leftarrow J'] \).

**Example 7.11** Let \( t = f_{[t_0, t_1]} \in T_\Sigma \) and let \( J = J_0(t) \). Consider the nodes \( v_0 = \text{nod}(1) \) and \( v_1 = \text{nod}(2) \) of \( J \), i.e., \( v_0 = \text{par}_J(e, 1) \) and \( v_1 = \text{par}_J(e, 2) \) where \( e \) is the unique hyperedge with \( \text{res}_J(e) = \text{res}(J) \) and \( \text{lab}_J(e) = \bullet \). Let \( \Theta_0 = \text{cand}_J(v_0, v_1) = \{(e, 1)\} \) and \( \Theta_1 = \text{cand}_J(v_1, v_0) = \{(e, 2)\} \). Then \( J_{\Theta_0 := v_1} = J(\Theta_0, v_1) \circ J_0(t_0) \), and cleaning this jungle (by removing \( J_0(t_0) \)) yields \( J(\Theta_0, v_1) = J_{\Theta_0 := v_1} = J_{\Theta_0 := v_0} = J(\Theta_0, v_0) \). Similarly, \( J_{\Theta_1 := v_0} = J(\Theta_1, v_0) \circ J_0(t_1) \) and \( J(\Theta_1, v_0) = J_{\Theta_1 := v_0} = J_{\Theta_1 := v_1} = J_{\Theta_1 := v_1} = J_{\Theta_1 := v_0} = J_{\Theta_1 := v_0} = J_{\Theta_1 := v_1} = J_{\Theta_1 := v_1} \). For an illustration see Figure \[\text{3}\]. This shows that \( J \Rightarrow \{J_0, J_1\} \) where \( J = J_0(f_{[t_0, t_1]}) \) and \( J_i = J_{0(i)}(f_{[t_0, t_1]} \cdot J_0(t_i)) \) for \( i \in \{0, 1\} \). ■

Note that, in the preceding example, \( \text{paths}(J) \subseteq \text{paths}(\{J_0, J_1\}) \). As we show next, this is not a coincidence.

**Lemma 7.12** For clean jungles \( J, J_0, \) and \( J_1 \), if \( J \Rightarrow \{J_0, J_1\} \), then \( \text{paths}(J) \subseteq \text{paths}(\{J_0, J_1\}) \).

\[\text{For } \circ \text{ see the beginning, and for } J_{0}^{\text{ms}} \text{ see the end of Section } 4\]
(a) Let Lemma 7.13 subderivations and (b) combine two derivations into one by taking unions.

Z
By the definition of
Proof first lemma makes proofs by induction on the length of derivations

Z

J
Let

Z

\( J \)
\( J_{\Theta_0} := v_1 \)
\( J_{\Theta_1} := v_0 \)

\( J \)
\( J_{\Theta_0} \)
\( J_{\Theta_1} \)

\( J_0(t_0) \)
\( J_0(t_1) \)

\( J_{\Theta_1} \)

\( J_{\Theta_0} \)

\( J_0(t_0) \)
\( J_0(t_1) \)

\( J_{\Theta_1} \)

\( J_{\Theta_0} \)

\( J_0(t_0) \)
\( J_0(t_1) \)

\( J_{\Theta_1} \)

\( J \)

\( J_{\Theta_0} \)
\( J_{\Theta_1} \)

Figure 8: Redirections in Example 7.11

Proof Let \( v_0, v_1 \in C_J \), and \( \Theta_i \subseteq \text{cand}_i(v_i, v_{i-1}) \), for \( i \in \{0, 1\} \). We have to show that every jungle path \( p = (e_1, j_1) \cdots (e_m, j_m) \) from \( v \) to \( w \) in \( J_{\Theta_0}, v_1 \) from \( v \) to \( w \) in \( J_{\Theta_1}, v_0 \). For this, note that \( p \) is a jungle path from \( v \) to \( w \) in \( J_{\Theta_1}, v_{i-1} \) unless one of its tentacles is in \( \Theta_i \). Hence, if \( p \) is not a jungle path from \( v \) to \( w \) in \( J_{\Theta_0}, v_1 \) nor in \( J_{\Theta_1}, v_0 \), we can find \( k, l \in [m] \) such that \( (e_k, j_k) \in \Theta_0 \) and \( (e_l, j_l) \in \Theta_1 \). Assuming without loss of generality that \( k > l \), it follows that there is a jungle path in \( J \) from \( \text{par}(e_l, j_l) = v_1 \) to \( \text{res}_J(e_k) \), which contradicts the assumption that \( (e_k, j_k) \in \text{cand}_J(v_0, v_1) \).

Let us prove two basic lemmas regarding clean jungles and redirection. The first lemma makes proofs by induction on the length of derivations \( J \Rightarrow^* J' \) possible. It allows us to (a) split a derivation \( J \Rightarrow^{n+1} J \) into a first step and two subderivations and (b) combine two derivations into one by taking unions.

Lemma 7.13
(a) Let \( J \) be a clean jungle. If \( J \) is a set of clean jungles such that \( J \Rightarrow^{n+1} J \), then there are \( J_i \) and \( J_i \) (\( i \in \{0, 1\} \)), such that \( J \Rightarrow \{J_0, J_1\} \), \( J_i \Rightarrow^* J_i \) for \( i \in \{0, 1\} \), and \( J = J_0 \cup J_1 \).

(b) Let \( J_0, J_1, J_2, J_3 \) be sets of clean jungles. If \( J_i \Rightarrow^* J'_i \) for \( i \in \{0, 1\} \), then \( J_0 \cup J_1 \Rightarrow^* J'_0 \cup J'_1 \).

Proof By the definition of \( \Rightarrow \), the derivation considered in (a) has the form \( J \Rightarrow \{J_0, J_1\} \Rightarrow^* J \). Thus, to prove (a), it suffices to show the following.

Let \( J_0 = \{J_1, J_2\} \) and \( J \) be sets of clean jungles, \( J_1 \) and \( J_2 \) not necessarily being distinct. For \( n \in \mathbb{N} \), if \( J_0 \Rightarrow^n J \), then there are \( J_1, J_2 \) such that (i) \( J = J_1 \cup J_2 \) and (ii) \( J_i \Rightarrow^n J_i \) for \( i \in \{1, 2\} \).

To prove this statement, we first observe that \( I \Rightarrow I \) for all nonempty sets \( I \) of clean jungles, because \( I \subseteq I(v, w) \) for every clean jungle \( I \) and \( v, w \in C_I \) (taking \( \Theta = \emptyset \)). We proceed by induction on \( n \). For \( n = 0 \), simply choose \( J_i = \{J_i\} \), \( i \in \{1, 2\} \). For \( n > 1 \), the derivation considered takes the form \( J_0 \Rightarrow^{n-1} J' \Rightarrow J \), where \( J = J' \setminus \{J\} \cup \{J', J'\} \), for jungles \( J, J' \), and \( J'' \).
Let \( J'_1, J'_2 \) be the subsets of \( J' \) obtained by applying the induction hypothesis to \( J_0 \mapsto^{n-1} J' \). For \( i \in \{1, 2\} \), let \( J_i = J'_i \setminus \{j\} \cup \{J', J''\} \) if \( J \in J'_i \), and \( J_i = J'_i \) if \( J \notin J'_i \). Then assertion (i) is obviously satisfied. Furthermore, \( J_i \Rightarrow J'_i \mapsto^{n-1} J'_1 \Rightarrow J_i \). The first part of this derivation exists due to the induction hypothesis. To see that also the last step exists, consider the two possible cases. If \( J \in J'_i \), then \( J'_i \Rightarrow J'_i \setminus \{j\} \cup \{J', J''\} = J_i \). If \( J \notin J'_i \), then \( J_i \Rightarrow J'_i \) by the observation mentioned above, since both sets are equal. This completes the proof of (a).

The proof of (b) consists mainly of the verification of two claims.

**Claim 1** For sets \( J \) and \( J' \) of clean jungles, if \( J \Rightarrow J' \), then \( J \Rightarrow J \cup J' \).

To see this, let \( J' = J \setminus \{j\} \cup \{J_0, J_1\} \), where \( J \in J \), \( v_0, v_1 \in C_J \), and \( J_i \in J(v_i, v_{1-i}) \) for \( i \in \{0, 1\} \). Exploiting again the fact that \( J \in J(v_i, v_{1-i}) \), we get
\[
J \Rightarrow J \setminus \{j\} \cup \{J, J_1\} = J \cup \{J_1\}
\Rightarrow J \cup \{J_1\} \setminus \{j\} \cup \{J_0, J\} = J \cup \{J_0, J_1\},
\]
as claimed.

**Claim 2** If \( J, J' \), and \( I \) are sets of clean jungles such that \( J \Rightarrow^* J' \), then \( J \cup I \Rightarrow^* J' \cup I \).

For derivations of length 0, Claim 2 is obviously valid. Thus, consider a derivation \( J \Rightarrow^a J'' \Rightarrow J' \). Let \( J \in J'' \) and \( v_0, v_1 \in C_J \) be such that \( J' = J'' \setminus \{j\} \cup \{J_0, J_1\} \) for jungles \( J_i \in J(v_i, v_{1-i}) \) (\( i \in \{0, 1\} \)). By the induction hypothesis, \( J \cup I \Rightarrow^* J'' \cup I \). Furthermore, \( J'' \cup I \Rightarrow J'' \cup I \setminus \{j\} \cup \{J_0, J_1\} = J' \cup (I \setminus \{j\}) \). For \( J \notin I \), the latter is obviously equal to \( J' \cup I \), whereas for \( J \in I \), Claim 1 yields \( J'' \cup I \Rightarrow J'' \cup I \cup J' \cup (I \setminus \{j\}) = J' \cup I \), as required.

Now, a twofold application of Claim 2 proves (b): \( J_0 \cup J_1 \Rightarrow^* J'_0 \cup J'_1 \Rightarrow^* J_0 \cup J_1 \).

It must be emphasized that Lemma 7.13 holds regardless of whether or not we consider \( \Rightarrow \) to be a relation on abstract or concrete jungles. In fact, let us denote the abstract jungle obtained from a concrete jungle \( J \) by \([J]\), and similarly for sets of jungles. Using (the two versions of) Lemma 7.13, one can easily verify the following fact, which allows us to switch freely between abstract and concrete jungles even in connection with \( \Rightarrow \):

Let \( J \) be a concrete jungle, and let \( J \) be a set of abstract jungles.

Then \([J] \Rightarrow J \) if and only if there is a set \( J' \) of concrete jungles such that \( J = [J'] \) and \( J \Rightarrow J' \).

Note that, even though this fact is easy to prove, it is not entirely trivial. In particular, the if direction needs Claim 1 (from the proof of Lemma 7.13) to “duplicate” an abstract jungle that is represented by two distinct but isomorphic concrete jungles.
The next lemma states a few facts regarding the interplay between redirection and hyperedge replacement. Owing to the application of ‘clean’, redirection may remove a hyperedge that would otherwise have been replaced. For example, we may consider a jungle $I = J[e \leftarrow K]$, and study the effect of turning from $J$ to $J' = J(\Theta, w)$. Since $e$ may or may not be in $J'$, we use the notation $J'[e \leftarrow K]$ to capture the two possible cases:

$$J'[e \leftarrow K] = \begin{cases} J'[e \leftarrow K] & \text{if } e \in E_{J'} \\ J' & \text{if } e \notin E_{J'} \end{cases}.$$

The lemma below considers a jungle $J[e \leftarrow K]$. Parts (b) and (c) of the lemma state that, if we perform redirections in $K$ or $J$ before replacing $e$, we only get results that are also obtained by applying redirection (to the same nodes) after having replaced $e$.

**Lemma 7.14** Let $I = J[e \leftarrow K]$ be a jungle, and let $img$ be the associated locator function.\(^{26}\)

(a) $\text{clean}(I) = \text{clean}(\text{clean}(J[e \leftarrow \text{clean}(K)]))$.

(b) For all $v, w \in V_K$,

$$I\langle img(v), img(w) \rangle \supseteq \{\text{clean}(J[e \leftarrow K']) \mid K' \in K\langle v, w \rangle\}.$$

(c) For all $v, w \in V_J$, if $\text{res}(K) \notin [\text{par}(K)]$, then

$$I\langle img(v), img(w) \rangle \supseteq \{\text{clean}(J'[e \leftarrow K]) \mid J' \in J\langle v, w \rangle\}.$$

**Remark** Before giving the proof of the lemma, let us note that the condition $\text{res}(K) \notin [\text{par}(K)]$ in (c) is indeed necessary. To see this, let $J = J_0(\mathfrak{g}[\mathfrak{h}[\mathfrak{a}]]), e$ and $e'$ are the hyperedges labelled with $\mathfrak{g}$ and $\mathfrak{h}$, resp., $\text{par}_J(e) = v$, and $\text{par}_J(e') = w$. Choosing $K = J_1(x_1)$ and $\Theta = \{(e, 1)\}$, we obtain $\text{clean}(I) = I = J_0(\mathfrak{h}[\mathfrak{a}]),$ whereas $J' = J(\Theta, w) = J_0(\mathfrak{g}[\mathfrak{a}])$ and thus $J'[e \leftarrow K] = J(a)$. Obviously, $J(a) \notin I\langle img(v), img(w) \rangle = \{1\}$.

**Proof** Property (a) is obvious from the fact that $\text{img}(C_J) \supseteq C_I \cap \text{img}(V_J)$ and $\text{img}(C_K \cup [\text{par}(K)]) \supseteq C_I \cap \text{img}(V_K)$.

For proving (b), we show that $\text{cand}_K(v, w) \subseteq \text{cand}_I(\text{img}(v), \text{img}(w))$ and

$$I\langle \Theta, \text{img}(w) \rangle = \text{clean}(J[e \leftarrow K(\Theta, w)]),$$

for all $\Theta \subseteq \text{cand}_K(v, w)$.

Consider $v, w \in V_K$ and $(e', j) \in \text{tent}(K)$ with $\text{par}_K(e', j) = v$, such that there is a jungle path $p$ in $I$ from $\text{img}(w)$ to $\text{res}_I(e') = \text{img}(\text{res}_K(e'))$. If $p$ is not a jungle path in $K$ (from $w$ to $\text{res}_K(e')$), then, since $\text{res}_K(e') \notin [\text{par}(K)]$, $p$ must contain a subsequence which is a jungle path in $I$ from some $\text{img}(\text{par}(K, i))$ to $\text{img}(\text{res}(K))$ and hence is a jungle path in $J$ from $\text{par}_J(e, i)$ to $\text{res}_J(e)$, which is impossible as

\(^{26}\)See the paragraph after Definition 4.2.
\( J \) is acyclic. We can thus conclude that \( \text{cand}_I(v, w) \subseteq \text{cand}_I(\text{img}(v), \text{img}(w)) \), which verifies the well-definedness of \( I(\Theta, \text{img}(w)) \).

Furthermore, it should be obvious that \( I_{\Theta := \text{img}(w)} = J[e \leftarrow K_{\Theta := w}] \). Using (a), this shows that \( I(\Theta, \text{img}(w)) = \text{clean}(J[e \leftarrow K(\Theta, w)]) \), as claimed.

Now, let us prove (c). We show that, for all \( \Theta \subseteq \text{cand}_J(v, w) \), there is \( \Theta' \subseteq \text{cand}_I(\text{img}(v), \text{img}(w)) \) such that

\[
I(\Theta', \text{img}(w)) = \text{clean}(J(\Theta, w)[e \not\xrightarrow{\sim} K]).
\]

Let \( \Theta(e) = \Theta \cap (\{e\} \times \text{N}) \) be the set of parameter tentacles of \( e \) in \( \Theta \). (Thus, \( \Theta(e) \) is empty, unless \( v \in [\text{par}_J(e)] \). Now, define

\[
\Theta' = (\Theta \setminus \Theta(e)) \cup \bigcup_{(e,j) \in \Theta(e)} \{(e', j') \in \text{tent}(K) \mid \text{par}_K(e', j') = \text{par}(K, j)\}.
\]

Note that the condition \( \text{par}_K(e', j') = \text{par}(K, j) \) is not equivalent to saying that \( \text{img}(\text{par}_K(e', j')) = \text{img}(v) \).

To see that \( \Theta' \subseteq \text{cand}_I(\text{img}(v), \text{img}(w)) \), consider the possible cases. For \( (e', j') \in \Theta \setminus \Theta(e) \), if there is a jungle path \( p \) in \( I \) from \( \text{img}(w) \) to \( \text{res}_I(e') \), then \( p \) is either a jungle path in \( J \) from \( w \) to \( \text{res}_J(e') \), or it contains a subsequence which is a jungle path in \( K \) from \( \text{res}(K) \) to \( \text{par}(K, j) \), for some \( j \in [\text{par}(K)] \). In the second case, \( p \) can be turned into a jungle path in \( J \) from \( w \) to \( \text{res}_J(e') \) by replacing this subsequence with \( (e, j) \). Thus, both cases contradict the assumption that \( \Theta \subseteq \text{cand}_J(v, w) \).

Now, consider a parameter tentacle \( (e', j') \in \text{tent}(K) \) such that \( \text{par}_K(e', j') = \text{par}(K, j) \) for a parameter tentacle \( (e, j) \in \Theta(e) \). By a similar reasoning as above, if there is a path in \( I \) from \( \text{img}(w) \) to \( \text{res}_I(e') \), then there is a path in \( J \) from \( w \) to \( \text{res}_J(e) \), contradicting the assumption that \( (e, j) \in \text{cand}_J(v, w) \). Thus, we can conclude that \( \Theta' \subseteq \text{cand}_I(\text{img}(v), \text{img}(w)) \).

As in the proof of (b), the correctness of the equation \( I(\Theta', \text{img}(w)) = \text{clean}(J(\Theta, w)[e \not\xrightarrow{\sim} K]) \) follows from (a), together with the obvious fact that \( I_{\Theta := \text{img}(w)} = J_{\Theta := w}[e \leftarrow K] \). Note, however, that this makes use of the assumption that \( \text{res}(K) \notin [\text{par}(K)] \).

We use Lemma 7.14 to show, in the next two lemmas, that \( \Rightarrow \) is compatible with hyperedge replacement.

**Lemma 7.15** Let \( I = J[e \leftarrow K] \) for jungles \( J \) and \( K \), where \( J \) is clean. If \( \text{clean}(K) \Rightarrow^* \mathcal{K} \), then \( \text{clean}(I) \Rightarrow^* \{\text{clean}(J[e \leftarrow K']) \mid K' \in \mathcal{K}\} \).

**Proof** We proceed by induction on the length of derivations \( \text{clean}(K) \Rightarrow^m \mathcal{K} \). By Lemma 7.14(c), the statement is true for \( m = 0 \), i.e., \( \mathcal{K} = \{\text{clean}(K)\} \). Therefore, let \( \text{clean}(K) \Rightarrow^{n+1} \mathcal{K} \) with \( \text{clean}(K) \Rightarrow \{K_0, K_1\} \) and \( K_i \Rightarrow^n K_i \) for \( i \in \{0, 1\} \), where \( \mathcal{K} = K_0 \cup K_1 \) (see Lemma 7.13(a)). Let \( v_0, v_1 \in C_K \) be such that \( K_i \in K(v_{i+1}) \cup C_K \).
Since $J$ is clean, $\text{img}(v_0)$ and $\text{img}(v_1)$ are in $C_I$. Hence, by Lemma 7.14(b), clean($[J[e \leftarrow K_i]] \in I(\text{img}(v_i), \text{img}(v_{i-1})) = \text{clean}(I(\text{img}(v_i), \text{img}(v_{i-1})))$. In other words, we have clean($I$) $\Rightarrow \{\text{clean}(I_0), \text{clean}(I_1)\}$, where $I_i = J[e \leftarrow K_i]$ for $i \in \{0, 1\}$. Applying the induction hypothesis, we get clean($I_i$) $\Rightarrow^*$ \{clean($J[e \leftarrow K'_i]$) $| K'_i \in K_i\}. Combining these derivations into one by means of Lemma 7.13(b) yields clean($I$) $\Rightarrow^*$ $\bigcup_{i \in \{0, 1\}} \{\text{clean}(J[e \leftarrow K'_i]) \mid K'_i \in K_i\} = \{\text{clean}(J[e \leftarrow K']) \mid K' \in K\}$, as claimed. 

**Lemma 7.16** Let $I = J[e \leftarrow K]$ for jungles $J$ and $K$, where $\text{res}(K) \notin \text{par}(K)$. If clean($J$) $\Rightarrow^* J'$, then there is a set $J' \subseteq \{\text{clean}(J'[e \leftarrow K']) \mid J' \in J\}$ such that clean($I$) $\Rightarrow^* J'$.

**Proof** Let clean($J$) $\Rightarrow^n J$. As in the previous proof, we proceed by induction on $n$, starting with the observation that the induction basis follows from Lemma 7.14(a). Again using Lemma 7.13(a), let clean($J$) $\Rightarrow^{n+1} J$ with clean($J$) $\Rightarrow \{J_0, J_1\}$ and $J_i$ $\Rightarrow^n J_i$ for $i \in \{0, 1\}$, where $J = J_0 \cup J_1$. Let $v_0, v_1 \in C_J$ be such that $J_i \in J\langle v_i, v_{i-1}\rangle$. If $e \in E_{J_i}$, then the induction hypothesis provides us with a set $J'_i$ such that $J'_i \subseteq \{\text{clean}(J'_i[e \leftarrow K]) \mid J'_i \in J_i\}$ and clean($J_i[e \leftarrow K]$) $\Rightarrow^* J'_i$. If $e \notin E_{J_i}$, then $J'_i = J_i$ has these properties.

We distinguish two cases.

**Case 1** For an $i \in \{0, 1\}$, $\text{img}(v_i) \notin C_I$ (where $\text{img}$ is the locator function).

In this case, $I\langle \text{img}(v_i), \text{img}(v_{i-1})\rangle = \{\text{clean}(I)\}$ and so Lemma 7.14(c) yields clean($I$) = clean($J_i[e \leftarrow K]$). Hence we can take $J' = J'_i$.

**Case 2** $\text{img}(v_0), \text{img}(v_1) \in C_I$.

By Lemma 7.14(c), clean($I$) $\Rightarrow \{\text{clean}(J_0[e \leftarrow K]), \text{clean}(J_1[e \leftarrow K])\}$. So, for $J' = J_0 \cup J'_1$, Lemma 7.13(b) yields clean($I$) $\Rightarrow \{\text{clean}(J_0[e \leftarrow K]), \text{clean}(J_1[e \leftarrow K])\} \Rightarrow^* J'$. Furthermore, we have $J' \subseteq \{\text{clean}(J'[e \leftarrow K]) \mid i \in \{0, 1\}, J' \in J_i\} = \{\text{clean}(J'[e \leftarrow K]) \mid J' \in J\}$, as claimed. 

We now define the previously mentioned notion of path variants that relates two jungle languages. As we shall prove directly after the definition (and an example), a path variant of a jungle language $L$ has the same path language as $L$. The main result of this section will be obtained by essentially proving that, for every tree language $L \in \text{pDEL}^\ast(\text{FIN})$, there exists a context-free path variant of $J_0(L)$.

**Definition 7.17 (path variant)** A jungle language $L' \subseteq J_{\Sigma, n}$ is a path variant of a jungle language $L \subseteq J_{\Sigma, n}$, if the following conditions are fulfilled.

(a) For every $J \in L$, there is a set $J \subseteq \text{clean}(L')$, such that clean($J$) $\Rightarrow^* J$. 

50
(b) tree(L') ⊆ tree(L).

Note that cleaning does not affect conditions (a) and (b). More precisely, let L' be a path variant of L. Then clean(L') is a path variant of L as well, and L' is a path variant of clean(L).

Example 7.18 Let L = J₀(L_{expseq}), where L_{expseq} is defined in Example 7.2, and let L' be the set of all jungle languages J₀_{ms}(s ⊢ (□, ..., □)) ∪ J₀(t) where s is a fully balanced binary tree of height n ≥ 1 over the signature \{f^{(2)}, □_1, ..., □_{2^n}\} and t is a tree over the signature \{k^{(2)}, a^{(0)}\}. Note that J₀_{ms}(s ⊢ (□, ..., □)) is a jungle with n + 1 nodes and with n f-labelled hyperedges and one □-labelled hyperedge. We claim that L' is a path variant of L. Clearly, tree(L) = L_{expseq} and tree(L') consists of all trees s ⊢ (t₁, ..., t₂^n) such that s is as above and t₁ = ... = t₂^n ∈ T_{[k,a]}. Thus, tree(L') ⊆ L_{expseq} = tree(L). We now prove, for every s as above and all trees t₁, ..., t₂^n ∈ T_{[k,a]}, that

J₀(s ⊢ (t₁, ..., t₂^n)) ⇒* \{J₀_{ms}(s ⊢ (□, ..., □)) ∪ J₀(t_i) | i ∈ [2^n]\}.

The proof is by induction on n. For n = 1, i.e., for s = f[□₁, □₂], it is immediate from Example 7.11. Now consider s ⊢ (t₁, ..., t₂^{n+1}) and note that it equals f[s'](t₁, ..., t₂^n), s'(t₂ⁿ⁺¹, ..., t₂ⁿ⁺²). Again from Example 7.11 we obtain that

J₀(s ⊢ (t₁, ..., t₂ⁿ⁺²)) ⇒ \{J₀_{ms}(f[□, □]) ∪ J₀(s ⊢ (t₁, ..., t₂ⁿ⁺²)), J₀_{ms}(f[□, □]) ∪ J₀(s' ⊢ (t₂ⁿ⁺２, ..., t₂ⁿ⁺³))\}.

By induction, J₀(s' ⊢ (t₁, ..., t₂ⁿ⁺²)) ⇒* \{J₀_{ms}(s' ⊢ (□, ..., □)) ∪ J₀(t_j) | 1 ≤ j ≤ 2ⁿ\}

and J₀(s' ⊢ (t₂ⁿ⁺２, ..., t₂ⁿ⁺³)) ⇒* \{J₀_{ms}(s' ⊢ (□, ..., □)) ∪ J₀(t_k) | 2ⁿ + 1 ≤ k ≤ 2ⁿ⁺²\}.

Hence, by Lemmas 7.15 and 7.13(b), J₀(s ⊢ (t₁, ..., t₂ⁿ⁺²)) ⇒* \{J₀_{ms}(f[□, □]) ∪ J₀_{ms}(s' ⊢ (□, ..., □)) ∪ J₀(t_i) | i ∈ [2ⁿ⁺²]\}.

Since J₀_{ms}(f[□, □]) ∪ J₀_{ms}(s' ⊢ (□, ..., □)) = J₀_{ms}(s ⊢ (□, ..., □)), this shows that condition (a) of Definition 7.17 is fulfilled, and thus that L' is a path variant of L.

Lemma 7.19 For all jungle languages L and L', if L' is a path variant of L, then paths(L) = paths(L').

Proof Condition (b) in the definition of path variant yields immediately that paths(L') ⊆ paths(L). For the converse direction, for sets J, J' of clean jungles with J ⇒ J', we have paths(J) ⊆ paths(J') by Lemma 7.12. Using condition (a) and the fact that paths(clean(J)) = paths(J), this yields paths(L) ⊆ paths(L'), as required.

The next lemma will be the major ingredient needed for the proof of the context-freeness of paths(L) for all L ∈ DEL*(FIN).

Lemma 7.20 Let N = (G, Π, (γ_δ)_{δ ∈ G}, g₀) be a propagating delegation network. If, for every g^{(k)} ∈ G, there exists a context-free path variant L_δ of J_k(L(γ_δ)), then there exists a context-free path variant of L_{j}(N).
Proof  By Lemma \[7.9\] and the fact that cleaning does not affect the property of being a path variant (see the remark following Definition \[7.17\]), we may assume that each \(L_g\) consists of clean jungles. To prove the lemma, instead of constructing a CFJG generating a context-free path variant of \(L_3(N)\), we construct a cf-extended CFJG \(G\) with this property. By Lemma \[7.8\] this suffices to prove the lemma. We let

\[
G = (\mathcal{G}, \Pi, \bigcup \{g \to L_g \mid g \in \mathcal{G}\}, g_0).
\]

We have to check that \(L(G)\) is a path variant of \(L_3(N)\), i.e., that conditions (a) and (b) of Definition \[7.17\] are satisfied.

For (a), similar to the notion of usefulness of nonterminals in \(G\), let us say that \(g \in \mathcal{G}\) is useful in \(N\) if \(J(g) \Rightarrow^* J\) for a jungle \(J\) over \(\Pi\). For \(Z \in \{G, N\}\), a derivation \(J(g_0) \Rightarrow^* J'\) is useful (in \(Z\)) if \(\text{lab}_{J'}(E_{J'}) \cap \mathcal{G}\) contains only symbols that are useful in \(Z\).

**Claim 1**  For every \(g \in \mathcal{G}\), if \(g\) is useful in \(N\), then \(g\) is useful in \(G\).

By the definition of \(\Rightarrow^\ast\), if \(K \Rightarrow^* K'\) and \(K' \in K\), then all hyperedge labels in \(K'\) occur also in \(K\), i.e., \(\text{lab}_{K'}(E_{K'}) \subseteq \text{lab}_K(E_K)\). Thus, by Definition \[7.17\](a), for all \(g^{(k)} \in \mathcal{G}\) and \(K \in J_k(L(\gamma_g))\), there is a rule \(g \to K'\) in \(G\), such that \(\text{lab}_{K'}(E_{K'}) \subseteq \text{lab}_K(E_K)\). By a straightforward induction on the length of derivations, this means that, for every derivation \(J(g) \Rightarrow^* J\), there is a derivation \(J(g) \Rightarrow^G J'\) such that \(\text{lab}_{J'}(E_{J'}) \subseteq \text{lab}_J(E_J)\). In particular, \(g\) is useful in \(G\) if it is useful in \(N\).

Now, we prove the following claim by induction on the length of derivations.

**Claim 2**  For every useful derivation \(J(g_0) \Rightarrow^* J\), there exists a set \(\mathcal{J}\) of clean jungles such that

(1) \(\text{clean}(J) \Rightarrow^* \mathcal{J}\) and
(2) for every \(J' \in \mathcal{J}\) there is a useful derivation \(J(g_0) \Rightarrow^G J''\) with \(\text{clean}(J'') = J'\).

Let us first argue that Claim 2 suffices to prove condition (a) of Definition \[7.17\]. For this, let \(J \in L_3(N)\), and let \(\mathcal{J}\) be such that \(J\) and \(\mathcal{J}\) satisfy (1) and (2). For \(J' \in \mathcal{J}\), by (2), there is a useful derivation \(J(g_0) \Rightarrow^*_G J''\) such that \(\text{clean}(J'') = J'\). As this derivation is useful, it can be extended to yield a jungle over \(\Pi\), say \(J'' \Rightarrow^* G J''' \in J\). Thus, \(J''' \in L(G)\). Since \(J \in J\), we have \(\mathcal{J} \subseteq J\), and thus \(\text{clean}(J'') \in J\), which means that \(\text{clean}(J'') = \text{clean}(J') = J' \in \mathcal{J}\). (Note that if \(I' = I[e \leftarrow K]\) for a hyperedge \(e \in E_I \setminus E_{\text{clean}(I)}\), then \(\text{clean}(I') = \text{clean}(I)\).) In other words, \(\mathcal{J} \subseteq \text{clean}(L(G))\).

Now, to prove Claim 2, first notice, using Claim 1, that it is trivially true for derivations of length 0 (with \(\mathcal{J} = \{J(g_0)\}\)). Thus, consider a useful derivation \(J(g_0) \Rightarrow^*_N J \Rightarrow^*_N I = J[e \leftarrow K]\), where \(\text{lab}_J(e) = g^{(k)}\) and \(K = J_k(t)\) for a tree \(t \in L(\gamma_g)\). Now, the derivation \(J(g_0) \Rightarrow^*_N J\) is then also useful. Assume that \(\mathcal{J}\) is a set of clean jungles satisfying (1) and (2) with respect to \(J\).

Since \(N\) is propagating, we have \(\text{res}(K) \notin \text{par}(K)\). By Lemma \[7.16\] this means that there is a set \(\mathcal{J'} \subseteq \{\text{clean}(J'[e \leftarrow K]) \mid J' \in \mathcal{J}\}\) such that \(\text{clean}(I) \Rightarrow^* G J'' \in J\).
Further, by the assumption that $L_g$ is a path variant of $J_k(L(\gamma_g))$, there is a set $K \subseteq L_g$, such that $K \Rightarrow^* K$. Let $I = \{\text{clean}(J'[e \xrightarrow{K'}]) \mid J' \in J, K' \in K\}$, we claim that $I$ satisfies (1) and (2) with respect to $I$. For checking (1), consider $I' = \{\text{clean}(J'[e \xrightarrow{K}]) \mid J' \in J, K' \in K\}$, where $J' \in J$. Using Lemma 7.15, we get $I' \Rightarrow^* \{\text{clean}(J'[e \xrightarrow{K'}]) \mid K' \in K\}$ (which is trivial if $e \notin E_{J'}$). Hence, repeated application of Lemma 7.13(b) gives

$$\text{clean}(I) \Rightarrow^* J' \Rightarrow^* \bigcup_{J' \in J} \{\text{clean}(J'[e \xrightarrow{K}]) \mid K' \in K\} = I,$$

as required. For checking (2), consider an element $\text{clean}(J'[e \xrightarrow{K'}])$ of $I$ (i.e., $J' \in J$ and $K' \in K$). Let $J(g_0) \Rightarrow^* G J''$ be a useful derivation such that $\text{clean}(J'') = J'$. By the construction of $G$, and since $K' \in L_g$, we have $J'' \Rightarrow^* G J''[e \xrightarrow{K'}]$ by either 0 or 1 derivation steps. By Lemma 7.14(a), $\text{clean}(J''[e \xrightarrow{K'}]) = \text{clean}(J'[e \xrightarrow{K'}])$, as required in (2). Furthermore, since all generator symbols in $J''$ and $K$ are useful in $G$ and $N$, respectively, and $\text{lab}_{K'}(E_{K'}) \subseteq \text{lab}_K(E_K)$, Claim 1 yields that the derivation $J(g_0) \Rightarrow^* G J''[e \xrightarrow{K'}]$ is useful.

It remains to prove that condition (b) of Definition 7.17 is satisfied. Instead of proving that $\text{tree}(L(G)) \subseteq \text{tree}(L_J(\mathcal{N}))$, we prove the more general statement that, for every jungle $J' \in L(G)$, there is a jungle $J \in L_J(\mathcal{N})$ that maps to $J'$. By Lemma 4.5, this proves that $\text{tree}(J) = \text{tree}(J')$.

We proceed by induction on the length of derivations to show that, for every derivation $J(g_0) \Rightarrow^* G J'$, there is a derivation $J(g_0) \Rightarrow^* N J$ such that $J$ maps to $J'$. The induction basis is trivial. Thus, consider a derivation

$$J(g_0) \Rightarrow^* G J' \Rightarrow^* I',$$

where $I' = J'[e' \xrightarrow{K'}], \text{lab}_{J'}(e') = g^{(k)}$, and $K' \in L_g$.

Let $K = J_k(\text{tree}(K'))$. By assumption, $K \in J_k(L(\gamma_g))$, and by Lemma 4.5, $K$ maps to $K'$. Furthermore, by the induction hypothesis, $J(g_0) \Rightarrow^* N J$, where $J$ maps to $J'$. Let $\varphi: V_J \rightarrow V_{J'}$ be the corresponding structure preserving mapping, and let $e_1, \ldots, e_m \in E_J$ be the pairwise distinct hyperedges $e$ such that $\varphi(\text{res}_J(e)) = \text{res}_{J'}(e')$. Then, using the fact that $J$ maps to $J'$ and $K$ maps to $K'$, it is straightforward to show that $I = J[e_1 \xrightarrow{K}] \cdots | e_m \xrightarrow{K}$ maps to $I'$. Since $J \Rightarrow^* N I$, this completes the inductive proof.

To finish the proof that $\text{tree}(L(G)) \subseteq \text{tree}(L_J(\mathcal{N}))$, consider $J' \in L(G)$. By the reasoning above, $J(g_0) \Rightarrow^* N J$ for a jungle $J$ that maps to $J'$. Furthermore, $J \in J_\Pi$ since $J' \in J_\Pi$, which means that $J \in L_J(\mathcal{N})$. \hfill \blacksquare

**Example 7.21** Let $\mathcal{N}$ be the propagating delegation network $\mathcal{N}_{\text{expseq}}$ of Example 7.2. Clearly, $L_J(\mathcal{N}) = J_0(L_T(\mathcal{N})) = J_0(L_{\text{expseq}})$. We first consider context-free path variants $L_{h_0}$ and $L_h$ of $J_0(L(\gamma_{h_0}))$ and $J_0(L(\gamma_{h}))$, respectively. Recall
that \( L(g_{h_0}) = L_T(N_{\text{bin}}) \) and \( L(g_h) = T_{\{k,a\}} \). Let \( L_{h_0} = L_J(N_{\text{bin}}) \). By Definitions 5.5 and 7.6, it is generated by the CFJG with rules \( g_0 \rightarrow J_0(g[h]) \), \( g \rightarrow J_1^{\text{ms}}(g[f[x_1,x_1]]) \), and \( g \rightarrow J_1^{\text{ma}}(f[x_1,x_1]) \). It is equal to \( J_0^{\text{ms}}(L_T(N_{\text{bin}})) \), which is a path variant of \( J_0(L_T(N_{\text{bin}})) \) as can be shown by an argument similar to the one in Example 7.18. Let \( L_h = J_0(L(g_h)) \). It is of course a path variant of itself, and it is generated by the CFJG with rules \( g'_0 \rightarrow J_0(k[g'_0,g'_0]) \) and \( g_0 \rightarrow J_0(a) \).

By (the proof of) Lemma 7.20, the jungle language generated by the cf-extended CFJG \( G \) with rules \( h_0 \rightarrow L_{h_0} \) and \( h \rightarrow L_h \) is a path variant of \( J_0(L_{\expseq}) \). By the construction in the proof of Lemma 7.8 (with some obvious simplifications), it is generated by the CFJG \( G' \) with rules \( g_0 \rightarrow J_0(g[h]) \), \( g \rightarrow J_1^{\text{ma}}(g[f[x_1,x_1]]) \), \( g \rightarrow J_1^{\text{ms}}(f[x_1,x_1]) \), \( h \rightarrow J_0(k[h,h]) \) and \( h \rightarrow J_0(a) \). It should be clear that \( L(G') = L' \), as defined in Example 7.18, where we already showed that \( L' \) is a path variant of \( J_0(L_{\expseq}) \).

Now, recall Corollary 6.5, which shows that \( \text{paths}(L_T(N)) = \text{paths}(L_J(N)) \). We would like to be able to apply Lemma 7.20 in an induction on the number of applications of pDEL to show that, for every propagating delegation network \( N \) generating a tree language \( L \in \text{pDEL}^{(\text{FIN})} \), there exists a context-free path variant of \( L_J(N) \). However, this yields an induction hypothesis which is too weak. What is needed is a context-free path variant of \( J_Y(L_T(N)) \), where \( Y \) is a parameter signature. Next, we prove a lemma that yields this link and completes the series of technical preparations required to prove the main result of this section.

To formulate the lemma, the following notation is used. For a jungle \( J \in J_{\Sigma,k} \) and a parameter signature \( Y = \{y_1, \ldots, y_n\} \), we let \( J_{[Y]} \in J_{\Sigma,Y,k+n} \) denote the jungle obtained from \( J \) by identifying, for each \( i \in [n] \), all result nodes of hyperedges \( e \in E_J \) with \( \text{lab}_J(e) = y_i \), which yields a new node \( v_i \), making \( v_i \) the \((k+i)\)th parameter node, and deleting all hyperedges with labels in \( Y \). If \( y_i \) does not occur in \( J \), then \( v_i \) becomes an isolated parameter node. For a jungle language \( L \), we let \( L_{[Y]} = \{ L_{[Y]} \mid J \in L \} \). Note that, for every tree \( t \in T_{\Sigma} \), \( J_Y(t) = J_0(t)_{[Y]} \).

**Lemma 7.22** Let \( L, L' \subseteq J_{\Sigma} \), and let \( Y \) be a parameter signature. If \( L' \) is a context-free path variant of \( L \), then \( L'_{[Y]} \) is a context-free path variant of \( J_Y(\text{tree}(L)) \).

**Proof** Let \( Y = \{y_1, \ldots, y_n\} \). We first prove that \( L'_{[Y]} \) is a path variant of \( J_Y(\text{tree}(L)) \) if \( L' \) is a path variant of \( L \).

Clearly, \( J_Y(\text{tree}(L)) \) and \( L'_{[Y]} \) satisfy condition (b) of Definition 7.17, because \( \text{tree}(J_Y(\text{tree}(L))) \) is obtained from \( \text{tree}(L) \) by turning each \( y_i \) into \( x_i \) for \( i \in [n] \), and \( \text{tree}(L'_{[Y]}) \) is obtained from \( \text{tree}(L') \) in the same way. Thus, it remains to be shown that condition (a) is satisfied. For this, we prove two claims.

**Claim 1** If \( J = J_0(\text{tree}(J')) \) for a clean jungle \( J' \in J_{\Sigma} \), then \( J \Rightarrow^* \{J'\} \).

54
We show that, for all clean jungles $J$ and $J'$, if $J$ maps to $J'$, then $J \equiv^* \{J'\}$.

Using Lemma 4.5, this proves Claim 1 by choosing $J = J_0(\text{tree}(J'))$.

Assume that $J$ maps to $J'$, and let $\varphi: V_J \rightarrow V_{J'}$ be the corresponding structure preserving mapping. Note that $\varphi$ is surjective, because $J'$ is clean. We proceed by induction on $m = |V_J| - |V_{J'}|$. If $m = 0$, then $\varphi$ is injective, and thus an isomorphism, which yields $J \equiv^* \{J'\}$. If $m > 0$, then there are distinct nodes $v_0, v_1 \in V_J$ such that $\varphi(v_0) = \varphi(v_1)$. Since $\varphi$ is structure preserving, we can choose these nodes in such a way that $\text{par}_J(e_0) = \text{par}_J(e_1)$ for the hyperedges $e_i$ with $\text{res}_J(e_i) = v_i$ ($i \in \{0,1\}$). Now, let $J_f$ be the jungle obtained from $J$ by identifying $v_0$ with $v_1$ and $e_0$ with $e_1$ (which is commonly called folding in the literature).

Obviously, $J_f$ is isomorphic to $J_i = J/\Theta_i$, for $i \in \{0,1\}$, where $\Theta_i = \{(e, j) \in \text{tent}(J) \mid \text{par}_J(e, j) = v_i\} = \text{cand}_J(v_i, v_i-1)$. In other words, $J \Rightarrow \{J_0, J_1\} = \{J_f\}$. Moreover, if $v$ is the node in $J_f$ resulting from the identification of $v_0$ and $v_1$, then $J_f$ maps to $J'$ by the mapping $\varphi'$ such that $\varphi'(v) = \varphi(v_0)$ and $\varphi'(v') = \varphi(v')$ for all $v \in V_J \setminus \{v_0, v_1\}$. By the induction hypothesis, this yields $J_f \equiv^* \{J'\}$, as claimed.

**Claim 2** Let $J \in J_\Sigma$ be a clean jungle, and let $\mathcal{J} \subseteq J_\Sigma$ be a set of clean jungles. If $J \equiv^* \mathcal{J}$, then $J[\mathcal{Y}] \equiv^* \mathcal{J}[\mathcal{Y}]$.

The correctness of the claim follows by an obvious inductive argument, using the fact that, for $v, w \in C_J$ and $\Theta \subseteq \text{cand}_J(v, w)$, we have $\Theta \subseteq \text{cand}_{J[\mathcal{Y}]}(\text{img}(v), \text{img}(w))$ and $J(\Theta, w)[\mathcal{Y}] = J[\mathcal{Y}](\Theta, \text{img}(w))$, where $\text{img}$ is the function mapping every node in $J$ to its image in $J[\mathcal{Y}]$. This is a direct consequence of the relevant definitions.

To conclude that $L[\mathcal{Y}]$ is a path variant of $J_Y(\text{tree}(L))$, notice first that, for every jungle $J \in J_\Sigma$, we have $\text{clean}(J[\mathcal{Y}]) = \text{clean}(J)[\mathcal{Y}]$. Now, consider a jungle $J \in L$. We have to show that there is a set $\mathcal{J} \subseteq \text{clean}(L'[\mathcal{Y}])$ such that $\text{clean}(J_Y(\text{tree}(J))) \Rightarrow^* \mathcal{J}$. By Claim 1 and the fact that $L'$ is a path variant of $L$, we have $J_0(\text{tree}(J)) \Rightarrow^* \{\text{clean}(J)\} \Rightarrow^* \mathcal{J}'$ for a set $\mathcal{J}' \subseteq \text{clean}(L')$. By Claim 2, this means that $J_Y(\text{tree}(J)) \Rightarrow^* \mathcal{J} \Rightarrow^* \{\text{clean}(L')\}$, for a set $\mathcal{J} \subseteq \text{clean}(L'[\mathcal{Y}])$. This set $\mathcal{J}$ is a subset of $\text{clean}(L'[\mathcal{Y}])$, which, by the remark above, is equal to $\text{clean}(L'[\mathcal{Y}])$. Thus, $L'[\mathcal{Y}]$ is indeed a path variant of $J_Y(\text{tree}(L))$.

To complete the proof of the lemma, it must be shown that $L'[\mathcal{Y}]$ is context-free if $L'$ is context-free. For this purpose, assume that $L' = L(G)$, for a CFJG $G = (\Xi, \Sigma, R, f_0)$. Let $\Xi'$ be obtained from $\Xi$ by turning the rank of every symbol $f^{(k)} \in \Xi$ into $k + n$; in particular, $f_0$ becomes a symbol of rank $n$. Let $\Sigma' = \Sigma \setminus Y$, and define, for every jungle $J \in J_{\Sigma \cup \Sigma'_k}$, $\widehat{J}[\mathcal{Y}]$ to be the jungle in $J_{\Sigma \cup \Sigma'_k} + n$ obtained from $J[\mathcal{Y}]$ by appending $\text{par}_{J[\mathcal{Y}]}(k + 1) \cdots \text{par}_{J[\mathcal{Y}]}(k + n)$ to $\text{par}_{J[\mathcal{Y}]}(e)$, for every hyperedge $e \in E_{J[\mathcal{Y}]}$, with $\text{lab}_{J[\mathcal{Y}]}(e) \in \Xi$.

As a direct consequence of its definition, $\widehat{J}[\mathcal{Y}] = J[\mathcal{Y}]$ for all $J \in J_\Sigma$. Now, for the CFJG $G' = (\Xi', \Sigma'_k, \{f \rightarrow \hat{K}[\mathcal{Y}] \mid (f \rightarrow K) \in R\}, f_0)$, it follows by a straightforward induction on the length of derivations, that $J(f_0) \Rightarrow^* J$ if and
only if \( J(\mathcal{L}_0) \Rightarrow^* J_{[\gamma]} \). In particular, this means that \( L(G') = \{ J_{[\gamma]} \mid J \in L(G) \} = L'_{[\gamma]} \). This completes the proof.

We are, finally, ready to prove the main result of this section.

**Theorem 7.23** For every tree language \( L \in \text{DEL}^*(\text{FIN}) \), \( \text{paths}(L) \) is context-free.

**Proof** By Lemmas 7.4 and 7.5, it suffices to prove the statement for \( L \in \text{pDEL}^*(\text{FIN}) \).

We prove first that, for every tree language \( L \in \text{pDEL}^*(\text{FIN}) \) and every parameter signature \( \gamma \), there is a context-free path variant of \( J_{\gamma}(L) \). The proof is by induction on \( n \). For \( n = 0 \), it suffices to note that finite jungle languages are context-free, and every jungle language is a path variant of itself. Now, let \( n > 0 \), and assume that the statement is true for \( n - 1 \). Let \( \mathcal{N} = (\mathcal{G}, \Pi, \Gamma, g_0) \) be a propagating delegation network such that \( L = L_T(\mathcal{N}) \), where \( \Gamma = (\gamma_g, \gamma_{\gamma_g})_{g \in \mathcal{G}} \) with \( L(\gamma_g) \in \text{pDEL}^{n-1}(\text{FIN}) \) for all \( g \in \mathcal{G} \). By the induction hypothesis, there is a context-free path variant \( L'_g \) of \( J_{\gamma_g}(L(g)) \), for every \( g \in \mathcal{G} \). Using Lemma 7.20, this shows that there is a context-free path variant \( L' \) of \( L_{[\gamma]} \). Hence, by Lemma 7.22 (for \( \Sigma = \Pi \)), \( L'_{[\gamma]} \) is a context-free path variant of \( J_{\gamma}(\text{tree}(L_{[\gamma]})) \).

In particular, we have shown that there is a context-free path variant \( L_0 \) of \( J_0(L) \). By Lemma 7.7, \( \text{paths}(L_0) \) is context-free, and by Lemma 7.19 it is equal to \( \text{paths}(J_0(L)) = \text{paths}(\text{tree}(J_0(L))) = \text{paths}(L) \).

Since the proof of Theorem 7.23 is effective, and since a tree language \( L \) is empty or finite if and only if the language \( \text{paths}(L) \) is, it follows that the emptiness and finiteness problems are decidable for \( L \in \text{DEL}^*(\text{FIN}) \). This result is, however, not very useful as such, because it is an open problem whether \( \text{DEL}^*(\text{FIN}) \) is closed under intersection with regular tree languages. (Usually, one is not interested in deciding whether \( L \) itself is empty (or finite), but rather whether \( L \cap R \) is, where \( R \) is a regular tree language that captures some type of correctness property, cf., e.g., the introduction of [DE98].) The result will be strengthened in the next section, where we obtain a more useful variant (Corollary 8.16).

**8 Delegation versus Macro Tree Transducers**

One of the main results of this section is that for every hierarchical delegation network \( \mathcal{N} \), the tree language \( L_T(\mathcal{N}) \) is the image of a regular tree language under a composition of several macro tree transductions. Macro tree transducers are a powerful type of tree transducer studied, e.g., in [EV85, EV94, FV98, DE98, EM02, CE12].

Several of the auxiliary results in this section concern classes of tree languages obtained by applying an operator such as \( \text{DEL} \) to a more or less arbitrary class \( C \) of tree languages. However, to avoid some very unnatural effects and technical
difficulties, we will restrict ourselves to genuine classes of tree languages. By definition, a class $C$ of tree languages is genuine if it contains all finite tree languages and, for every tree language $L \subseteq T_\Sigma$ and every bijective and rank-preserving renaming $r : \Sigma \rightarrow \Sigma'$ of symbols, if $L \in C$ then $r(L) \in C$ (where $r$ is extended to a function on trees in the canonical way). It is obvious that the operator $\text{DEL}$ preserves genuineness, i.e., if $C$ is genuine, then so is $\text{DEL}(C)$. It is left to the reader to check that this also holds for the operators $\text{TD}$, $\text{YIELD}$ and $r\text{DEL}$ introduced in the remainder of this section. Note that since $\text{FIN}$ is genuine, all the classes $\text{DEL}^n(\text{FIN})$ of the delegation hierarchy are genuine. Note also that the class $\text{REG}$ of regular tree languages (see Definition 2.5) is genuine.

A tree transduction is a binary relation $\tau \subseteq T_\Sigma \times T_\Delta$, where $\Sigma$ and $\Delta$ are finite signatures. Thus, viewed as a function $\tau : T_\Sigma \rightarrow \varphi(T_\Delta)$, $\tau$ maps trees in $T_\Sigma$ to sets of trees in $T_\Delta$. For a class $C$ of tree languages and a class $TR$ of tree transductions, we let $TR(C) = \{\tau(L) \mid \tau \in TR \text{ and } L \in C\}$, $TR^0(C) = C$ and $TR^{n+1}(C) = TR(TR^n(C))$, for $n \in \mathbb{N}$. Moreover, $TR^*(C) = \bigcup_{n \in \mathbb{N}} TR^n(C)$. As usual, $C$ is said to be closed under $TR$ if $TR(C) \subseteq C$. Similarly, $C$ is closed under $\text{DEL}$ if $\text{DEL}(C) \subseteq C$.

Next, we recall the definitions of two well-known classes of tree transductions, namely top-down tree transductions and $\text{YIELD}$ mappings. These are specific macro tree transductions and, vice versa, every macro tree transduction is a composition of these transductions. Thus, for the results in this section, we need not define macro tree transducers.

**Definition 8.1 (top-down tree transducer)** A top-down tree transducer (td transducer, for short) is a tuple $td = (\Sigma, \Delta, Q, R, q_0)$, where

- $\Sigma$ and $\Delta$ are finite signatures of input symbols and output symbols, resp.,
- $Q$ is a finite signature of states of rank 1, where $Q \cap (\Sigma \cup \Delta) = \emptyset$,
- $R$ is a finite set of rules, and
- $q_0 \in Q$ is the initial state.

Every rule in $R$ has the form $q[f[x_1, \ldots, x_k]] \rightarrow t \cdot (q_1[x_{i_1}], \ldots, q_l[x_{i_l}])$, where $k, l \in \mathbb{N}$, $q, q_1, \ldots, q_l \in Q$, $f^{(k)} \in \Sigma$, $i_1, \ldots, i_l \in [k]$, and $t \in T_{\Delta, (q_1, \ldots, q_l)}$.

Given a tree $s = s_0 \cdot q[f[s_1, \ldots, s_k]]$, if $R$ contains a rule as above, then there is a computation step $s \Rightarrow_{td} s_0 \cdot (t \cdot (q_1[s_{i_1}], \ldots, q_l[s_{i_l}])$. The subscript $td$ can be omitted if it is clear from the context. The top-down tree transduction $td : T_\Sigma \times T_\Delta$ computed by $td$ is given by

$$td(s) = \{t \in T_\Delta \mid q_0(s) \Rightarrow^* t\},$$

for all $s \in T_\Sigma$. The set of all top-down tree transductions (td transductions, for short) is denoted by $TD$.

An example of a td transducer that turns out to be helpful in our reasoning is the following. Consider a finite signature $\Sigma$, and let $\theta^{(0)}$ be special symbols not in $\Sigma$. Now, let $sel_\Sigma = (\Sigma \cup \{+, \theta\}, \Sigma, \{q\}, R, q)$, where $R$ consists of the rules $q[+[x_1, x_2]] \rightarrow q[x_i]$ ($i = 1, 2$) and, for all $f^{(k)} \in \Sigma$, $q[f[x_1, \ldots, x_k]] \rightarrow$
\[ f[q[x_1], \ldots, q[x_k]]. \] Obviously, for \( t \in T_{\Sigma \cup \{+\}} \), \( set_{\Sigma}(t) \) is the set of trees obtained from \( t \) by interpreting + as set union (of sets of trees) and \( \emptyset \) as the empty set.

In the following, we let \( \text{SET} \) denote the set of all td transductions \( set_{\Sigma} \), i.e., for all finite signatures \( \Sigma \). We usually drop the index \( \Sigma \), thus writing \( set \) instead of \( set_{\Sigma} \).

A td transducer is linear (nondeleting) if, for every rule \( q[f[x_1, \ldots, x_k]] \rightarrow t \), \( t \) is linear in \( X_k \) (nondeleting in \( X_k \), resp.). It is total (deterministic) if, for all \( q \in Q \) and \( f^{(k)} \in \Sigma \), there is at least one (at most one) rule with the left-hand side \( q[f[x_1, \ldots, x_k]] \). A td transduction is said to have one of these properties, or a combination thereof, if it can be computed by a td transducer that fulfills the requirements in question. A tree homomorphism is a tree transduction that can be computed by a total deterministic td transducer whose set of states is a singleton. For example, the td transducers \( set_{\Sigma} \) are linear, but neither nondeleting nor deterministic (because of the rules \( q[++x_1, x_2] \rightarrow q[x_i] \)) nor total (since there are no rules with the left-hand side \( q[\emptyset] \)).

\( \text{YIELD} \) mappings, which seem to have appeared for the first time in [Mai74], formalize the construction of trees using parameter substitution. Given a finite signature \( \Sigma \) and a number \( m \in \mathbb{N} \) such that \( \Sigma \cap X_m = \emptyset \) and \( k \leq m \) for all \( f^{(k)} \in \Sigma \), let the derived signature \( \Delta(\Sigma, m) \) consist of all symbols \( \text{subst}_{n+1} \), where \( 1 \leq n \leq m \), and all constant symbols \( f^{(k)} \), where \( f^{(k)} \in \Sigma \cup X_m \). In other words, in a derived signature, each of the original symbols \( f^{(k)} \in \Sigma \cup X_m \) is turned into a symbol \( f^{(k)} \) of rank 0. The only symbols of rank greater than 0 are the symbols \( \text{subst}_1, \ldots, \text{subst}_m \). For a signature \( \Omega \), we denote by \( \Omega^{(1)} \) the set of nullary symbols \( \{ f^{(k)} \mid f^{(k)} \in \Omega \} \); thus, if \( \Omega \) is a subset of \( \Sigma \), then \( \Delta(\Sigma, m) = \Delta(\Sigma \setminus \Omega, m) \cup \Omega^{(1)} \). For an arbitrary parameter signature \( \Sigma = \{ y_1, \ldots, y_m \} \), instead of \( X_m \), we similarly define the derived alphabet \( \Delta(\Sigma, Z) \).

**Definition 8.2 (YIELD mapping)**
For a derived signature \( \Delta(\Sigma, m) \), the YIELD mapping on \( T_{\Delta(\Sigma, m)} \) is the mapping \( Y_{\Sigma,m} : T_{\Delta(\Sigma, m)} \rightarrow T_{\Sigma,m} \) which is defined recursively, as follows. For every tree \( s \in T_{\Delta(\Sigma, m)} \),

\[
Y_{\Sigma,m}(s) = \begin{cases} 
Y_{\Sigma,m}(s_0)(Y_{\Sigma,m}(s_1), \ldots, Y_{\Sigma,m}(s_n)) & \text{if } s = \text{subst}_n[s_0, s_1, \ldots, s_n] \\
\text{f}[x_1, \ldots, x_k] & \text{if } s = f^{(k)}
\end{cases}
\]

where \( 1 \leq n \leq m \) and \( f^{(k)} \in \Sigma \cup X_m \).

For an arbitrary parameter signature \( Z = \{ y_1, \ldots, y_m \} \) (disjoint with \( \Sigma \)), we similarly define the YIELD mapping \( Y_{\Sigma,Z} : T_{\Delta(\Sigma,Z)} \rightarrow T_{\Sigma,Z} \). The set of all YIELD mappings is denoted by \( \text{YIELD} \).

\footnote{For \( t_0, t_1, \ldots, t_n \in T_{\Sigma,m} \), we use here, and in what follows, the notation \( t_0(t_1, \ldots, t_n) \) for \( t_0(t_1, x_{n+1}, \ldots, x_m) \) when \( n < m \). This is consistent with the fact that \( T_{\Sigma,m} = T_{\Sigma',n} \) where \( \Sigma' = \Sigma \cup \{ x_{n+1}, \ldots, x_m \} \).}
Although several slightly different definitions of \(YIELD\) are found in the literature \cite{ES77, ES78, EV85, DE98, FV98}, it is easy to see that the definition given above is equivalent to each of them, in all respects relevant for this paper.

In the following, we simply denote \(Y_{\Sigma,m}\) by \(Y\), assuming that \(\Sigma\) and \(m\) are appropriately chosen, so that all symbols in the trees \(Y_{\Sigma,m}\) are applied to are covered. Note that, in particular, \(Y(x_i^{(0)}) = x_i\) for all \(i \in [m]\). For technical convenience, we identify the symbol \(\square_i^{(0)}\) with \(\square_i\), for all \(i \geq 1\). As a consequence, \(Y(\square_i) = \square_i\).

We will need a technical lemma concerning a notion of \(G\)-preservation, which is defined as follows. For a delegation network \(N' = (G, \Pi, \Gamma, g_0)\) and a tree \(s \in T_{\Delta(\Sigma_N,m)}\), we say that \(s\) is \(G\)-preserving if, for every subtree of \(s\) of the form \(\text{subst}_n[s_0, s_1, \ldots, s_n]\) (where \(n \leq m\)), either \(s_1, \ldots, s_n \in T_{\Delta(\Pi,m)}\) or \(s_0 = f^{(n)}\) for some \(f \in \Sigma_N \cup X_m\) (or both). Note that \(\Delta(\Sigma_N, m) = \Delta(\Pi, m) \cup G^{(1)}\).

As the term \(G\)-preserving indicates, occurrences of symbols from \(G^{(1)}\) in \(s\) are preserved under \(Y\). More precisely, every occurrence of a symbol \(g^{(k)} \in G^{(1)}\) in \(s\) corresponds to a unique occurrence of \(g^{(k)}\) in \(Y(s)\). In fact, as the following technical lemma shows, even the property of \(g^{(k)}\) occurring rightmost in \(s\) is preserved (in both directions). To formulate this, for a tree \(s_0 \in T_{\Delta(\Sigma_N,m),\{\square\}}\) containing exactly one occurrence of \(\square\), let us say that \(s_0\) is rightmost if no symbol of \(G^{(1)}\) occurs to the right of \(\square\) in \(s_0\). Similarly, for a tree \(t_0 \in T_{\Sigma_N \cup X_m,\{\square\}}\) containing exactly one occurrence of \(\square\), we call \(t_0\) rightmost if no symbol of \(G\) occurs to the right of \(\square\) in it.

**Lemma 8.3** Let \(s \in T_{\Delta(\Sigma_N,m)}\) be \(G\)-preserving.

1. If \(s\) has the form \(s_0 \cdot g^{(k)}\) for a symbol \(g^{(k)} \in G\) and a rightmost tree \(s_0\), then \(Y(s)\) has the form \(Y(s_0) \cdot g[t_1, \ldots, t_k]\), where \(Y(s_0)\) is rightmost and \(t_1, \ldots, t_k \in T_{\Pi,m}\). Moreover, for \(u \in T_{\Delta(\Sigma_N,m)}\) with \(Y(u) \in T_{\Sigma_N,k}\), we have \(Y(s_0 \cdot u) = Y(s_0) \cdot (Y(u)(t_1, \ldots, t_k))\).
2. If \(Y(s) \in T_{\Pi,m}\), then \(s \in T_{\Delta(\Pi,m)}\).
3. If \(Y(s)\) has the form \(t_0 \cdot g[t_1, \ldots, t_k]\) for a symbol \(g \in G\), a rightmost tree \(t_0\), and trees \(t_1, \ldots, t_k \in T_{\Pi,m}\), then \(s\) can be written as \(s_0 \cdot g^{(k)}\), where \(s_0\) is rightmost and \(Y(s_0) = t_0\).

**Proof** (1) The proof is by induction on the structure of \(s_0\). The case that \(s_0 = \square\) is obvious, with \(t_i = x_i\) for \(i \in [k]\). Now let \(s_0 = \text{subst}_n[s'_0, s_1, \ldots, s_n]\).

**Case 1:** \(\square\) occurs in \(s_i\) for \(i \in [n]\). Then \(s = \text{subst}_n[s'_0, s_1, \ldots, s_i \cdot g^{(k)}, \ldots, s_n]\), and so \(s'_0 = f^{(n)}\) for some \(f \in \Sigma_N\) because \(s\) is \(G\)-preserving. Hence \(Y(s) = f[Y(s_1), \ldots, Y(s_i \cdot g^{(k)}), \ldots, Y(s_n)]\). By the induction hypothesis,

\[
Y(s_i \cdot g^{(k)}) = Y(s_i) \cdot g[t_1, \ldots, t_k].
\]
Consequently,
\[ Y(s) = f[Y(s_1), \ldots, Y(s_i) \cdot g[t_1, \ldots, t_k], \ldots, Y(s_n)] \]
\[ = f[Y(s_1), \ldots, Y(s_i), \ldots, Y(s_n)] \cdot g[t_1, \ldots, t_k] \]
\[ = Y(s_0) \cdot g[t_1, \ldots, t_k]. \]

The remaining requirements are easy to check.

Case 2: \( \sqcap \) occurs in \( s'_0 \). Then \( s = \text{subst}_n[s'_0 \cdot g^{(k)}, s_1, \ldots, s_n] \). Note that \( s_1, \ldots, s_n \) do not contain symbols of \( G^{(i)} \). By the induction hypothesis,
\[ Y(s'_0 \cdot g^{(k)}) = Y(s'_0) \cdot g[t_1, \ldots, t_k]. \]

Consequently,
\[ Y(s) = (Y(s'_0) \cdot g[t_1, \ldots, t_k])(Y(s_1), \ldots, Y(s_n)) \]
\[ = (Y(s'_0)(Y(s_1), \ldots, Y(s_n))) \cdot g[t'_1, \ldots, t'_k] \]
\[ = Y(s_0) \cdot g[t'_1, \ldots, t'_k] \]
where \( t'_i = t_i(Y(s_1), \ldots, Y(s_n)) \in T_{\Pi,m} \) for \( i \in [k] \). Again, the remaining requirements are easy to check.

(2) This is immediate from Statement 1: if \( s \) contains a symbol of \( G^{(i)} \), then one can consider the rightmost occurrence of such a symbol and write \( s \) as \( s_0 \cdot g^{(k)} \) where \( s_0 \) is rightmost.

(3) The proof is by induction on the structure of \( s \). The case that \( s \) is a constant symbol is easy: \( s \) must be equal to \( g^{(k)} \), \( Y(s) = g[x_1, \ldots, x_k] \), \( t_0 = \sqcap \), \( t_i = x_i \) for \( i \in [k] \), and \( s_0 = \sqcap \). Now let \( s = \text{subst}_n[u, s_1, \ldots, s_n] \). Then \( Y(s) = Y(u)(Y(s_1), \ldots, Y(s_n)) = t_0 \cdot g[t_1, \ldots, t_k] \). Let \( v \in V(t_0) \) be the node with label \( \sqcap \). We distinguish two cases.

Case 1: \( v \) is a node of \( Y(u) \) and its label (in \( Y(u) \)) is not in \( X_m \). Then \( Y(u) \) has the form \( t'_0 \cdot g[t'_1, \ldots, t'_k] \) and \( t_i = t'_i(Y(s_1), \ldots, Y(s_n)) \) for \( i = 0 \) and for all \( i \in [k] \). By the induction hypothesis, \( u \) can be written as \( u_0 \cdot g^{(k)} \) where \( Y(u_0) = t'_0 \). Let \( s_0 = \text{subst}_n[u_0, s_1, \ldots, s_n] \). Then \( s = \text{subst}_n[u_0 \cdot g^{(k)}, s_1, \ldots, s_n] = s_0 \cdot g^{(k)} \) and \( Y(s_0) = t'_0(Y(s_1), \ldots, Y(s_n)) = t_0 \).

It remains to be shown that no symbol of \( G^{(i)} \) occurs in \( s_1, \ldots, s_n \). Since \( s \) is \( G \)-preserving, it suffices to consider the case that \( u = g^{(k)} \) and \( k = n \). Then \( t_0 = t'_0 = \sqcap \) and \( Y(s) = g[Y(s_1), \ldots, Y(s_n)] = g[t_1, \ldots, t_n] \). Hence no symbol of \( G \) occurs in \( Y(s_1), \ldots, Y(s_n) \). Now Statement 2 gives the result.

Case 2: \( v \) is a node of one of the occurrences of \( Y(s_1), \ldots, Y(s_n) \) in \( Y(s) \). Then there exists \( i \in [n] \) such that \( Y(s_i) \) has the form \( t'_0 \cdot g[t_1, \ldots, t_k] \). Thus, \( s_i \) contains a symbol of \( G^{(i)} \). Since \( s \) is \( G \)-preserving, \( u = f^{(n)} \) for some \( f \in \Sigma_N \). Hence \( Y(s) = f[Y(s_1), \ldots, t'_0 \cdot g[t_1, \ldots, t_k], \ldots, Y(s_n)] \) and \( t_0 = f[Y(s_1), \ldots, t'_0, \ldots, Y(s_n)] \). By the induction hypothesis, \( s_i \) can be written as \( s_{i,0} \cdot g^{(k)} \) where \( Y(s_{i,0}) = t'_0 \). Take \( s_0 = \text{subst}_n[u, s_1, \ldots, s_{i,0}, \ldots, s_n] \). Then \( s = \text{subst}_n[u, s_1, \ldots, s_{i,0} \cdot g^{(k)}, \ldots, s_n] = \ldots \)
Given a derived signature \( \Delta(\Sigma, m) \), there is a useful linear and nondeleting tree homomorphism \( \text{comb}_{\Sigma, m} : T_{\Sigma, m} \rightarrow T_{\Delta(\Sigma, m)} \) (cf. \[ES77\] Definition 4.4). It is given by
\[
\text{comb}_{\Sigma, m}(f[t_1, \ldots, t_k]) = \text{sub}st_k[f^{(k)}, \text{comb}_{\Sigma, m}(t_1), \ldots, \text{comb}_{\Sigma, m}(t_k)],
\]
for all trees \( f[t_1, \ldots, t_k] \in T_{\Sigma, m} \) with \( k \geq 1 \), and \( \text{comb}_{\Sigma, m}(f) = f^{(0)} \) for \( f^{(0)} \in \Sigma \cup X_m \). In the following, we will often simply write ‘comb’ instead of ‘\( \text{comb}_{\Sigma, m} \)’. The comb mapping has the following obvious properties.

**Lemma 8.4** For every derived signature \( \Delta(\Sigma, m) \) and every tree \( t \in T_{\Sigma, m} \),
1. \( Y_{\Sigma, m}(\text{comb}_{\Sigma, m}(t)) = t \), and
2. \( \text{comb}_{\Sigma, m}(t) \) is \( \mathcal{G} \)-preserving if \( \Sigma = \Sigma_N \) for some \( N = (\mathcal{G}, \Pi, \Gamma, g_0) \).

The so-called TBY hierarchy \[DE98\] is obtained by starting with the regular tree languages, and then applying arbitrary td transducers and YIELD mappings to them.\[28\] Formally, let TBY be the set of all tree transductions of the form
\[
\tau_m \circ \cdots \circ \tau_1,
\]
where \( m \geq 1 \), \( \tau_1, \ldots, \tau_m \in \text{TD} \cup \text{YIELD} \) and \( \tau_i \in \text{YIELD} \) for at most one \( i \in [m] \). It is not known whether the TBY hierarchy, which is given by \( (\text{TBY}^n(\text{REG}))_{n \in \mathbb{N}} \), is proper at each level. It is closely related to the hierarchy obtained by applying \( n \) (nondeterministic) macro tree transducers to regular tree languages. In particular, \( \text{TBY}^*(\text{REG}) \) is equal to \( \text{MTT}^*(\text{REG}) \), the closure of \( \text{REG} \) under macro tree transductions. In this section, we show that \( \text{DEL}^*(\text{FIN}) \) is properly contained in \( \text{TBY}^*(\text{REG}) \), and that the latter is closed under \( \text{DEL} \).

**Example 8.5** Consider the monadic tree language \( L_{\text{monexp}} = \{f^{2^n}[a] \mid n \in \mathbb{N}\} \) consisting of all trees over \( \{f^{(1)}, a^{(0)}\} \) that contain \( 2^n \) occurrences of \( f \), for \( n \in \mathbb{N} \). It is well known that \( L_{\text{monexp}} \) is in \( \text{YIELD}(\text{TD}(\text{REG})) \subseteq \text{TBY}(\text{REG}) \). Use a regular tree grammar \( \gamma \) with the rules \( \xi_0 \rightarrow f[\xi_0] \) and \( \xi_0 \rightarrow a \) and a td transducer \( td \) with the rules
\[
q_0[f[x_1]] \rightarrow \text{sub}st_1[q[x_1], a^{(0)}] \\
q[f[x_1]] \rightarrow \text{sub}st_1[q[x_1], q[x_1]] \\
q[a] \rightarrow f^{(1)}.
\]
Thus, \( td(L(\gamma)) \) consists of all trees \( \text{sub}st_1[t, a^{(0)}] \) where \( t \) is a fully balanced binary tree over the signature \( \{\text{sub}st_1, f^{(1)}\} \). Obviously, \( Y(td(f^{n+1}[a])) = f^{2^n}[a] \) and, thus, \( Y(td(L(\gamma))) = L_{\text{monexp}} \). The tree language \( L_{\text{monexp}} \) is not in \( \text{DEL}^*(\text{FIN}) \),

---

\[28\] The three letters T, B, and Y refer to *top-down tree transducers*, *bottom-up tree transducers*, and *YIELD mappings*. In the definition used here, we omit bottom-up tree transducers, because it is known that each bottom-up tree transducer can be simulated by two top-down tree transducers.

\[29\] Note that, by Lemma 8.4(1) and the fact that \( \text{comb}_{\Sigma, m} \in \text{TD} \), the same class TBY is obtained if we require that \( \tau_i \in \text{YIELD} \) for exactly one \( i \in [m] \).
by Theorem 7.23, because \( \text{paths}(L_{\text{monexp}}) \) is not context-free (which is clear from the fact that \( \text{paths}(L_{\text{monexp}}) \cap \langle f, 1 \rangle^* \langle a, 0 \rangle = \{ \langle f, 1 \rangle^{2^n} \langle a, 0 \rangle \mid n \geq 0 \} \)).

Note that, of course, \( \text{DEL}^*(\text{FIN}) = \text{DEL}^*(\text{REG}) \), because \( \text{REG} \subseteq \text{DEL}(\text{FIN}) \). In fact, we can turn the subset sign in the latter formula into an equality sign if we restrict ourselves to delegation networks that work in a “regular” manner. More precisely, we say that a delegation network is regular if the rank of every generator symbol is 0. As a consequence, IO-derivations generate the same trees as unrestricted derivations. Intuitively, this means that a regular delegation network (viewed as a tree-generating device) is nothing else than a regular tree grammar with a possibly infinite set of rules, where the set of right-hand sides of rules for a given left-hand side is generated by the tree generator associated with it. Thus, regular delegation networks have the following property, similar to regular tree grammars.

**Lemma 8.6** Let \( \mathcal{N} = (\mathcal{G}, \Pi, (\gamma_g)_{g \in \mathcal{G}}, g_0) \) be a regular delegation network, and consider a generator symbol \( g \in \mathcal{G} \) and a tree \( t \in T_{\Sigma_\mathcal{N}} \). For \( n \in \mathbb{N} \), \( g \Rightarrow_n^{\mathcal{N}, \text{IO}} t \) if and only if there is a tree \( t_0 \cdot (g_1, \ldots, g_k) \in L(\gamma_g) \), where \( t_0 \in T_{\Pi, \{\mathcal{G}_1, \ldots, \mathcal{G}_k\}} \) and \( g_1, \ldots, g_k \in \mathcal{G} \), and there are derivations \( g_1 \Rightarrow_{\mathcal{N}_1, \text{IO}} t_1, \ldots, g_k \Rightarrow_{\mathcal{N}_k, \text{IO}} t_k \), such that \( t = t_0 \cdot (t_1, \ldots, t_k) \) and \( n = \sum_{i=1}^k n_i \).

For a class \( C \) of tree languages, let \( \text{rDEL}(C) \) be defined in the same way as \( \text{DEL}(C) \), except that the delegation networks considered are required to be regular. The operator \( \text{rDEL} \) will play an important role in the proofs of this section. Clearly, as illustrated by the three delegation networks of Example 2.6, \( \text{REG} = \text{rDEL}(\text{FIN}) = \text{rDEL}(\text{REG}) \). The second equality is a well-known result in disguise: extended regular tree grammars, with regular tree languages as right-hand sides, generate regular tree languages. (In turn, since trees are strings and regular tree grammars are a special case of context-free grammars, the latter is a special case of the fact that cf-extended context-free string grammars generate context-free languages.) More generally, using similar well-known arguments, we obtain the following result.

**Lemma 8.7** The operator \( \text{rDEL} \) is idempotent on genuine classes \( C \) of tree languages, i.e., \( \text{rDEL}(\text{rDEL}(C)) = \text{rDEL}(C) \).

**Proof** Since \( C' \subseteq \text{rDEL}(C') \) for every class \( C' \), one inclusion is obvious. For the other inclusion the construction is similar to the one of Lemma 7.8. Consider a regular delegation network \( \mathcal{N} = (\mathcal{G}, \Pi, (\gamma_g)_{g \in \mathcal{G}}, g_0) \) and, for every \( g \in \mathcal{G} \), a regular delegation network \( \mathcal{N}_g = (\mathcal{G}_g, \Sigma_\mathcal{N}_g, (\gamma_h)_{h \in \mathcal{G}_g}, h_0) \) such that \( L(T(\mathcal{N}_g)) = L(\gamma_g) \) and \( L(\gamma_h) \subseteq C \) for every \( g \in \mathcal{G} \) and \( h \in \mathcal{G}_g \). Since \( C \) is genuine, we may assume that the signatures \( \mathcal{G}_g, g \in \mathcal{G} \), are pairwise disjoint. Then \( L(\mathcal{N}) = L(\mathcal{N}') \), where \( \mathcal{N}' = (\mathcal{G}', \Pi, (\gamma_g')_{g \in \mathcal{G}}, g_0) \) is the regular delegation network such that \( \mathcal{G}' = \mathcal{G} \cup \bigcup_{g \in \mathcal{G}} \mathcal{G}_g \).
and for every \( g \in \mathcal{G} \) and \( h \in \mathcal{G}_g \), \( L(\gamma'_g) = \{ h_{g,0} \} \) and \( L(\gamma'_h) = L(\gamma_h) \). ■

We now prove a useful decomposition lemma. In the proof, we need the notion of rightmost derivation of a delegation network \( \mathcal{N} = (\mathcal{G}, \Pi, (\gamma_g)_{g \in \mathcal{G}}, g_0) \). A derivation step of \( \mathcal{N} \) as in Definition 6.3 is rightmost, denoted \( t_0 \cdot g[s_1, \ldots, s_k] \Rightarrow_{\mathcal{N}, \text{rm}} t_0 \cdot u(s_1, \ldots, s_k) \), if \( t_0 \) is rightmost. It is easy to see that \( g_0 \Rightarrow_{\mathcal{N}, \text{rm}} t \) for every \( t \in L_T(\mathcal{N}) \). In fact, if \( t_1 \Rightarrow_{\mathcal{N}, \text{rm}} t_2 \Rightarrow_{\mathcal{N}, \text{rm}} t_3 \) and the first derivation step is not rightmost, then there exists \( t'_3 \) such that \( t_1 \Rightarrow_{\mathcal{N}, \text{rm}} t'_3 \Rightarrow_{\mathcal{N}, \text{rm}} t_3 \). Thus, the rightmost derivation steps can be moved to the left until all have become rightmost (because the last derivation step is necessarily rightmost).

**Lemma 8.8** If \( C \) is a class of tree languages that is closed under linear nondeleting tree homomorphisms\(^{31}\), then \( \text{DEL}(C) \subseteq \text{YIELD}(r\text{DEL}(C)) \).

**Proof** Consider a delegation network \( \mathcal{N} = (\mathcal{G}, \Pi, (\gamma_g)_{g \in \mathcal{G}}, g_0) \), where \( L(\gamma_g) \in C \) for all \( g \in \mathcal{G} \). The construction of \( \mathcal{N}' \) is similar to the definition of \( G^D \) in [ES77, Definition 4.5]. Let \( \mathcal{N}' = (\mathcal{G}', \Delta(\Pi, m), (\gamma_g)_{g \in \mathcal{G}'}, g'_0) \), where

- \( m \) is the maximum rank of symbols in \( \Sigma_N \),
- \( \mathcal{G}' = \mathcal{G}^{(i)} \), hence \( \Sigma_{N'} = \Delta(\Sigma_N, m) \),
- \( L(\gamma_{g(k)}) = \text{comb}_{\Sigma_N, m}(L(\gamma_g)) \), for all \( k \in [m] \) and \( g^{(k)} \in \mathcal{G} \), and
- \( g'_0 = g_0^{(0)} \).

Note that \( L(\gamma_{g(k)}) \in C \) because \( \text{comb}_{\Sigma_N, m} \) is a linear nondeleting tree homomorphism. Note also that in the case where \( \Pi \) contains symbols \( x_i \), we use a derived alphabet \( \Delta(\Pi, Z) \) with \( Z = \{ y_1, \ldots, y_m \} \) disjoint with \( \Pi \), instead of \( \Delta(\Pi, m) \). We have to show that \( L_T(\mathcal{N}) = Y(L_T(\mathcal{N}')) \). To prove this, we proceed by induction on the length of rightmost derivations. We first observe that if \( g_0' \Rightarrow_{\mathcal{N}', \text{rm}} s \) then \( s \) is \( G \)-preserving. In fact, if \( s_0 \cdot g^{(k)} \Rightarrow_{\mathcal{N}', \text{rm}} s_0 \cdot \text{comb}(u) \) and \( s_0 \cdot g^{(k)} \) is \( G \)-preserving, then so is \( s_0 \cdot \text{comb}(u) \), as one easily proves by induction on the structure of \( s_0 \) using Lemma 8.4(2).

To show the inclusion \( L_T(\mathcal{N}) \subseteq Y(L_T(\mathcal{N}')) \), we prove that for every derivation \( g_0 \Rightarrow_{\mathcal{N}, \text{rm}} t \), there is a derivation \( g_0' \Rightarrow_{\mathcal{N}', \text{rm}} s \) such that \( Y(s) = t[32] \). For \( n = 0 \), this is trivial. Therefore, let

\[
g_0 \Rightarrow_{\mathcal{N}, \text{rm}} t_0 \cdot g[t_1, \ldots, t_k] \Rightarrow_{\mathcal{N}, \text{rm}} t_0 \cdot u(t_1, \ldots, t_k),
\]

where \( g \in \mathcal{G}, t_1, \ldots, t_k \in \Pi_T \), and \( u \in L(\gamma_g) \subseteq T_{\Sigma_N,k} \). By the induction hypothesis, \( g_0' \Rightarrow_{\mathcal{N}', \text{rm}} s \), where \( Y(s) = t_0 \cdot g[t_1, \ldots, t_k] \). By Lemma 8.3(3), this means that \( s = s_0 \cdot g^{(k)} \), for a tree \( s_0 \) such that \( Y(s_0) = t_0 \) and \( s_0 \) is rightmost.

---

\(^{30}\)To be precise, \( t_1 \) must be of the form \( t_0 \cdot (g[s_1, \ldots, s_k], g'[s'_1, \ldots, s'_l]) \) where \( \sqcup_2 \) occurs to the left of \( \sqcup_1 \) in \( t_0 \) and no symbol of \( G \) occurs to the right of \( \sqcup_2 \). Also, \( t_2 = t_0 \cdot u(s_1, \ldots, s_k), g'[s'_1, \ldots, s'_l] \) and \( t_3 = t_0 \cdot u(s_1, \ldots, s_k), u'(s'_1, \ldots, s'_l) \). Then \( t'_2 = t_0 \cdot (g[s_1, \ldots, s_k], u'(s'_1, \ldots, s'_l)) \).

\(^{31}\)Such a class \( C \) is genuine if and only if \( \text{FIN} \subseteq C \).

\(^{32}\)Note that this implies the inclusion: if \( t \in \Pi_T \), then \( s \in T_{\Delta n, \text{rm}} \) by Lemma 8.3(2).
By the definition of $\cN'$, we have $s_0 \cdot g^{(k)} \Rightarrow_{\cN', \text{IO}, \text{rm}} s'$ with $s' = s_0 \cdot \text{comb}(u)$. By Lemmas 8.3(1) and 8.4(1), $Y(s') = t_0 \cdot Y(\text{comb}(u))(t_1, \ldots, t_k) = t_0 \cdot u(t_1, \ldots, t_k)$.

The proof of the inclusion $Y(L_T(\cN')) \subseteq L_T(\cN)$ is similar. It only uses Lemmas 8.3(1) and 8.4.

Let us have a look at two examples that illustrate the preceding decomposition lemma.

**Example 8.9** The tree language $L_T(\cN_{\text{bin}})$ of fully balanced binary trees in Example 7.2 is in $\text{DEL}(\text{FIN})$ and hence in $\text{YIELD}(r\text{DEL}(\text{FIN}))$. It is equal to $Y(L_T(\cN_{\text{bin}}))$ where $L(\gamma_{g, h, 0}) = \{ \text{subst}_1[g^{(1)}, h^{(0)}] \}$ and $L(\gamma_{g, h, 1})$ consists of the trees $\text{subst}_1[g^{(1)}, \text{subst}_2[f^{(2)}, x_1^{(0)}, x_1^{(0)}]]$ and $\text{subst}_2[f^{(2)}, x_1^{(0)}, x_1^{(0)}]$.

**Example 8.10** The tree language $td(L(\gamma))$ from Example 8.5 is also in the class $\text{DEL}(\text{FIN})$, cf. Example 7.2. It is generated by the delegation network with generator symbols $g_0^{(0)}$ and $g^{(1)}$, and with $L(\gamma_{g_0}) = \{ \text{subst}_1[g^{(1)}, a^{(0)}] \}$ and $L(\gamma_g) = \{ g[\text{subst}_1[x, x_1]], x_1 \}$. Thus, by Example 8.5, the tree language $L_{\text{monexp}}$ is in $\text{YIELD}(\text{DEL}(\text{FIN}))$ and hence, trivially, in the class $\text{YIELD}(r\text{DEL}(C))$ where $C = \text{DEL}(\text{FIN})$. However, as observed in that example, $L_{\text{monexp}}$ is not in $\text{DEL}(\text{FIN})$.

The last example shows that the inclusion in Lemma 8.8 is not an equality for $C = \text{DEL}(\text{FIN})$. This is in contrast to the well-known fact that for $C = \text{FIN}$ equality holds: $\text{DEL}(\text{FIN}) = \text{YIELD}(\text{REG})$, see [ES77, Corollary 4.12]. The latter equality implies that $\text{DEL}(\text{REG}) = \text{DEL}(\text{FIN})$ (cf. the last paragraph of Example 2.7(a)); by Lemma 8.8, $\text{DEL}(\text{REG}) \subseteq \text{YIELD}(r\text{DEL}(\text{REG}))$ which is equal to $\text{YIELD}(\text{REG})$ by Lemma 8.7.

Next, we want to prove a lemma that allows us to reorder the operators $r\text{DEL}$ and $\text{YIELD}$. To be able to do that, we need an elementary property of $Y$.

**Lemma 8.11** For every derived signature $\Delta(\Sigma, m)$ and tree $s = s_0 \cdot (s_1, \ldots, s_i)$ in $T_\Delta(\Sigma, m)$, if $Y(s_1), Y(s_i) \in T_\Sigma, \text{ then } Y(s) = Y(s_0)[Y(s_1), \ldots, Y(s_i)]$.

**Proof** We prove the statement by structural induction on $s_0$, but for arbitrary $s_0 \in T_\Delta(\Sigma, m, \{ \Box_1, \ldots, \Box_i \})$, i.e., where $s = s_0[s_1, \ldots, s_i]$. For $i \in [\ell]$, let $t_i = Y(s_i)$.

If $s_0 = \Box_i$ for an $i \in [\ell]$, then $Y(s) = t_i = Y(s_0)[t_1, \ldots, t_i]$. If $s_0 = f^{(k)}$ for a symbol $f^{(k)} \in \Sigma \cup X_m$, then $s = s_0$ and, thus, $Y(s) = Y(s_0) = Y(s_0)[t_1, \ldots, t_i]$.

Finally, if $s_0 = \text{subst}_k[u_0, \ldots, u_k]$, using first the definition of $Y$, then the

---

Note that $Y(s_0)$ is in $T_{\Sigma \cup X_m \cup \{ \Box_1, \ldots, \Box_i \}}$ and recall furthermore from Section 2.3 that $Y(s_0)[Y(s_1), \ldots, Y(s_i)]$ is obtained from $Y(s_0)$ by replacing each occurrence of $\Box_i$ with $Y(s_i)$.
induction hypothesis, and then the assumption that \( t_1, \ldots, t_i \in T_\Sigma \), we get

\[
Y(s) = Y(u_0[s_1, \ldots, s_i])(Y(u_1[s_1, \ldots, s_i]), \ldots, Y(u_k[s_1, \ldots, s_i])) \\
= Y(u_0)[t_1, \ldots, t_i](Y(u_1)[t_1, \ldots, t_i], \ldots, Y(u_k)[t_1, \ldots, t_i]) \\
= Y(u_0)(Y(u_1), \ldots, Y(u_k))[t_1, \ldots, t_i] \\
= Y(s_0)[t_1, \ldots, t_i]
\]

as claimed.

We can now prove the mentioned lemma, that makes it possible to reorder rDEL and YIELD.

**Lemma 8.12** For every genuine class \( C \) of tree languages,

\[
\text{rDEL}(\text{YIELD}(C)) \subseteq \text{SET}(\text{YIELD}(\text{rDEL}(C))).
\]

**Proof** Consider a regular delegation network \( \mathcal{N} = (\mathcal{G}, \Pi, (\gamma_g)_{g \in \mathcal{G}}, g_0) \), where \( L(\gamma_g) = Y(L_g) \) with \( L_g \in C \). We first treat the special case where \( \Sigma_N \) does not contain parameters that are used by the YIELD mappings on the tree languages \( L_g \). In this case there exists a derived signature \( \Delta(\Sigma_N, m) \) such that \( L_g \subseteq T_{\Delta(\Sigma_N, m)} \) for all \( g \in \mathcal{G} \). We may assume that \( m \geq 2 \). Note that \( \Delta(\Sigma_N, m) = \Delta(\Pi, m) \cup \{ \langle g \rangle | g \in \mathcal{G} \} \).

Now, let \( \mathcal{G}' = \mathcal{G} \cup \{ \langle g \rangle | g \in \mathcal{G} \} \) and define \( \mathcal{N}' = (\mathcal{G}', \Pi', (\gamma'_g)_{g \in \mathcal{G}'}, g_0) \), where \( \Pi' = \Delta(\Pi) \cup \{ +, \theta \} \) and, for every \( g \in \mathcal{G} \), \( L(\gamma'_g) = L_g \) and \( L(\gamma'_{g_0}) = \{ \text{subst}_2[+,(2), g, \langle g \rangle], \theta^{(0)} \} \).

For \( g \in \mathcal{G} \) and \( n \in \mathbb{N} \), let \( \Lambda(g) = \bigcup_{n \in \mathbb{N}} \Lambda(g, n) \) and \( \Lambda'(g) = \bigcup_{n \in \mathbb{N}} \Lambda'(g, n) \), where

\[
\Lambda(g, n) = \{ t \in T_\Pi | g \Rightarrow^m_{\mathcal{N}, \text{IO}} t \text{ for some } m \leq n \} \text{ and } \\
\Lambda'(g, n) = \{ s \in T_\Pi | g \Rightarrow^m_{\mathcal{N}' \cup \theta, \text{IO}} s \text{ for some } m \leq n \},
\]

for all \( n \in \mathbb{N} \). In particular, \( \Lambda(g_0) = L_T(\mathcal{N}) \) and \( \Lambda'(g_0) = L_T(\mathcal{N}') \). We show, simultaneously for all \( g \in \mathcal{G} \), that \( \text{set}(Y(\Lambda'(g))) = \Lambda(g) \).

(‘\( \subseteq \)’) We divide the proof in two steps, in both cases proceeding by induction on the length of derivations, using Lemma 8.6.

First, consider a tree \( s \in T_\Pi \) such that \( g \Rightarrow^1_{\mathcal{N}', \text{IO}} s \) for some \( n \in \mathbb{N} \). By induction on \( n \), we show that \( \text{set}(Y(s)) \subseteq \text{set}(Y(\Lambda'(g, n))) \). If \( s = \theta^{(0)} \), then \( \text{set}(Y(s)) = \emptyset \). Otherwise, \( s = \text{subst}_2[+,(2), s_1, s_2] \), where \( g \Rightarrow^m_{\mathcal{N}, \text{IO}} s_1 \) and \( g \Rightarrow^m_{\mathcal{N}', \text{IO}} s_2 \) with \( n_1 + n_2 = n - 1 \). Thus,

\[
\text{set}(Y(s)) = \text{set}(Y(s_1)) \cup \text{set}(Y(s_2)) \subseteq \text{set}(Y(\Lambda'(g, n))),
\]

since the inclusion \( \text{set}(Y(s_1)) \subseteq \text{set}(Y(\Lambda'(g, n_1))) \) holds by the definition of \( \Lambda'(g, n_1) \) and \( \text{set}(Y(s_2)) \subseteq \text{set}(Y(\Lambda'(g, n_2))) \) is obtained from the induction hypothesis.

Now, to show by induction on \( n \) that \( \text{set}(Y(\Lambda'(g, n))) \subseteq \Lambda(g) \), let \( s \in \Lambda'(g, n) \). By Lemma 8.6 and the construction of \( \mathcal{N}' \), there is a tree \( s_0 \cdot \langle g \rangle \cdot \ldots \cdot \langle g \rangle \in \mathcal{G} \).

(‘\( \supseteq \)’) We prove that \( \text{set}(Y(\Lambda'(g, n))) = \Lambda(g) \).
Lemma 8.11 (for \(\Sigma = \Pi\) previous paragraph and the induction hypothesis, the latter yields \(\text{set}(Y(s_i)) \subseteq \text{set}(Y(\Lambda'(g_i, n_i))) \subseteq \Lambda(g_i)\) for all \(i \in \llbracket l \rrbracket\). In particular, \(\text{set}(Y(s_i)) \subseteq \Pi\), i.e., \(Y(s_i) \in \Pi_{\llbracket l \rrbracket}^{+}\) for all \(i \in \llbracket l \rrbracket\). Consequently \(Y(s) = Y(s_0) [ Y(s_1), \ldots, Y(s_i)]\) by Lemma 8.11 (for \(\Sigma = \Pi \cup \{+, \theta\}\)).

Since \(s_0 \in \Pi_{\llbracket l \rrbracket}^{\{c_1, \ldots, c_k\}}\), the tree \(Y(s_0)\) can be written in the form \(Y(s_0) = \gamma_0 \cdot (\Box, \ldots, \Box)\) with \(k \in \mathbb{N}\), \(t_0 \in \Pi_{\llbracket l \rrbracket}^{\{c_1, \ldots, c_k\}}\) and \(t_1, \ldots, t_k \in \llbracket l \rrbracket\). This means that

\[
Y(s_0) = Y(s_0)[g_1, \ldots, g_n] = (t_0 \cdot (g_1, \ldots, g_n))[g_1, \ldots, g_n] = t_0 \cdot (g_1, \ldots, g_n).
\]

Since \(Y(s_0) = \gamma_0 \cdot \Pi_{\llbracket l \rrbracket}^{\{c_1, \ldots, c_k\}}\), we obtain \(t_0 \cdot (g_1, \ldots, g_n) \in \Lambda(g)\) and \(t_0 \in \Pi_{\llbracket l \rrbracket}^{\{c_1, \ldots, c_k\}}\).

Thus, we get

\[
\text{set}(Y(s)) = \text{set}(Y(s_0) [ Y(s_1), \ldots, Y(s_i)]) \subseteq \Lambda(g).
\]

The last inclusion holds by Lemma 8.6 because \(t_0 \cdot (g_1, \ldots, g_n) \in \Lambda(g)\) and \(\text{set}(Y(s_i)) \subseteq \Lambda(g_i)\).

\(\Box\)

Note first that, for all trees \(s_1, \ldots, s_n \in \Lambda'(g)\), the tree

\[
s = \text{subst}_2[+^{(2)}, s_1, \text{subst}_2[+^{(2)}, s_2, \ldots, \text{subst}_2[+^{(2)}, s_n, \theta^{(0)}] \ldots]]
\]

satisfies \(g^{(0)} \Rightarrow_{N, IO}^n s\) and \(\text{set}(Y(s)) = \bigcup_{i \in [n]} \text{set}(Y(s_i))\). In particular, \(t = \theta^{(0)}\) if \(n = 0\).

We show that, for all trees \(t \in \Lambda(g)\), there is a tree \(s \in \Lambda'(g)\) such that \(Y(s) \in \Pi_{\llbracket l \rrbracket}^{\{c_1, \ldots, c_k\}}\) and \(t \in \text{set}(Y(s))\). Again, we proceed by induction on the length of derivations. Thus, suppose there is a derivation \(g \Rightarrow_{N, IO} t' \Rightarrow_{N, IO}^n t\) of \(t\), and let \(t' = Y(s_0) [ g^{(0)}, \ldots, g^{(0)}]\) with \(s_0 \in \Pi_{\llbracket l \rrbracket}^{\{c_1, \ldots, c_k\}}\). As in the previous step of the proof, \(Y(s_0)\) can be written as \(Y(s_0) = t_0 \cdot (g_1, \ldots, g_n)\) with \(t_0 \in \Pi_{\llbracket l \rrbracket}^{\{c_1, \ldots, c_k\}}\) and \(i_1, \ldots, i_k \in \llbracket l \rrbracket\), such that \(t' = t_0 \cdot (g_1, \ldots, g_n)\). It follows that \(t = t_0 \cdot (t_1, \ldots, t_k)\) for trees \(t_1, \ldots, t_k \in \Pi\) with \(g_i \Rightarrow_{N}^n t_j\) and \(\sum_{j \in [k]} n_j = n\).

Thus, the induction hypothesis yields derivations \(g_i \Rightarrow_{N'}^n s_j\) such that \(Y(s_j) \in \Pi_{\llbracket l \rrbracket}^{\{c_1, \ldots, c_k\}}\) and \(t_j \in \text{set}(Y(s_j))\) for all \(j \in \llbracket l \rrbracket\).

By the observation above, the latter gives rise to derivations \(g_i^{(0)} \Rightarrow_{N'}^n \hat{s}_i\) such that \(Y(\hat{s}_i) \in \Pi_{\llbracket l \rrbracket}^{\{c_1, \ldots, c_k\}}\) for all \(i \in \llbracket l \rrbracket\) and \(t_j \in \text{set}(Y(\hat{s}_j))\) for all \(j \in \llbracket l \rrbracket\). Using this and the construction of \(N'\), we get

\[
g \Rightarrow_{N'}^n s_0 \cdot (g_1^{(0)}, \ldots, g_l^{(0)}) \Rightarrow_{N'}^n s_0 \cdot (\hat{s}_1, \ldots, \hat{s}_l).
\]
and
\[
\begin{align*}
t & = t_0 \cdot (t_1, \ldots, t_k) \\
& \in \{ t_0 \cdot (t_1, \ldots, t_q) \mid t'_j \in \text{set}(Y(\widehat{s}_{i_j})) \text{ for } j \in [k] \} \\
& = \text{set}(t_0 \cdot (Y(\widehat{s}_{i_1}), \ldots, Y(\widehat{s}_{i_k}))) \\
& = \text{set}(Y(s_0)[Y(\widehat{s}_1), \ldots, Y(\widehat{s}_t)]) \\
& = \text{set}(Y(s_0(\widehat{s}_1, \ldots, \widehat{s}_t))) \quad \text{(by Lemma 8.11)}
\end{align*}
\]
as required. Moreover, \(Y(s_0(\widehat{s}_1, \ldots, \widehat{s}_t)) = t_0 \cdot (Y(\widehat{s}_{i_1}), \ldots, Y(\widehat{s}_{i_k}))\) is in \(\text{T}_{\text{NL}[+\theta]}\), because \(t_0 \in \text{T}_{\text{NL}[0,1,\ldots,\Delta_k]}\).

Finally, we consider the case where \(\Sigma_N\) does contain parameters that are used by the \(\text{YIELD}\) mappings on the tree languages \(L_g\). Suppose they use parameters \(y_1, \ldots, y_m\) (some of which are in \(\Sigma_N\)) and that the parameters \(x_1, \ldots, x_m\) are not in \(\Sigma_N\). For every \(g \in \mathcal{G}\), let \(L'_g\) be the tree language obtained from \(L_g\) by changing every \(y_i^{(0)}\) into \(x_i^{(0)}\). Since \(C\) is genuine, \(L'_g\) is in \(C\). Now define \(L''_g = \{ \text{subst}_m[t, y_i^{(0)}], \ldots, y_m^{(0)}] \mid t \in L'_g \}\). It should be clear that \(Y(L''_g) = Y(L_g) = L(\gamma_g)\) and that \(\Sigma_N\) does not contain parameters that are used by the \(\text{YIELD}\) mappings on the tree languages \(L'_g\). Hence we are in the previous case and we can define \(\mathcal{N}'' = (\mathcal{G}', \Pi', (\gamma'_h)_{h \in \mathcal{G}'}, g_0)\) as above, with \(L''_g\) instead of \(L_g\). Note, however, that \(L''_g\) need not be in \(C\). This can be corrected by defining the equivalent regular delegation network \(\mathcal{N}'' = (\mathcal{G}'', \Pi'', (\gamma''_h)_{h \in \mathcal{G}'}, g_0)\) with \(\mathcal{G}'' = \mathcal{G}' \cup \{ g \mid g \in \mathcal{G} \}\) and for every \(g \in \mathcal{G}, L(\gamma''_g) = \{ \text{subst}_m[g, y_1^{(0)}, \ldots, y_m^{(0)}] \}, L(\gamma''_g) = L'_g\) and \(L(\gamma''_g) = L(\gamma''_g) = \{ \text{subst}_2[+^{(2)}, g, g_0], \theta^{(0)} \}\). \(\blacksquare\)

Let us have a look at an example that illustrates the construction in the preceding proof.

**Example 8.13** The language \(L_{\text{expseq}}\) of Example 7.2 is in \(\text{rDEL}(\text{DEL}(\text{FIN}))\) and hence in \(\text{rDEL}(\text{YIELD}(C))\) with \(C = \text{rDEL}(\text{FIN})\), cf. Example 8.9. Hence, by Lemma 8.12, it is in \(\text{SET}(\text{YIELD}(\text{rDEL}(C))) = \text{SET}(\text{YIELD}(\text{REG}))\). Using the construction in the proof of Lemma 8.12 with some obvious simplifications, we obtain that \(L_{\text{expseq}} = \text{set}(Y(L(\gamma)))\) for the regular tree grammar \(\gamma\) with nonterminals \(h_0, g_0^{(0)}, g^{(1)}, h^{(0)}, h\), initial nonterminal \(h_0\), and the following rules:

\[
\begin{align*}
h_0 & \rightarrow g_0^{(0)} \\
g_0^{(0)} & \rightarrow \text{subst}_1[g^{(1)}, h^{(0)}] \\
g^{(1)} & \rightarrow \text{subst}_1[g^{(1)}, \text{subst}_2[f^{(2)}, x_1^{(0)}, x_1^{(0)}]] \\
g^{(1)} & \rightarrow \text{subst}_2[f^{(2)}, x_1^{(0)}, x_1^{(0)}] \\
h^{(0)} & \rightarrow \text{subst}_2[+^{(2)}, h, h^{(0)}] \\
h^{(0)} & \rightarrow \theta^{(0)} \\
h & \rightarrow \text{subst}_2[k^{(2)}, h, h] \\
h & \rightarrow a^{(0)}
\end{align*}
\]
Note that if the rules for \( h^{(0)} \) are replaced by the one rule \( h^{(0)} \rightarrow h \), then \( \text{set}(Y((L(\gamma)))) = Y((L(\gamma))) \) consists of all trees of the form \( s \cdot (t_1, \ldots, t_{2^n}) \) (as in Example 7.2) such that \( t_1 = \cdots = t_{2^n} \). ■

We can soon prove the main theorem of this section, which shows how the classes \( \text{DEL}^*(\text{FIN}) \) and \( \text{TBY}^*(\text{REG}) \) are related. To be able to apply Lemma 8.8 in the proof of this result, we must convince ourselves that the operator \( \text{DEL} \) preserves closedness under linear nondeleting tree homomorphisms, i.e., we need the following small result.

**Lemma 8.14** For every class \( C \) of tree languages, if \( C \) is closed under linear nondeleting tree homomorphisms, then so is \( \text{DEL}(C) \).\(^{34} \)

**Proof** Let \( N = (G, \Pi, (\gamma_g)_{g \in G}, g_0) \) be a delegation network, and let \( td: T_\Pi \rightarrow T_{\Pi'} \) be a linear nondeleting tree homomorphism. Let \( q \) be the unique state of the td transducer \( td \). For \( m \in \mathbb{N} \), we extend \( td \) to a linear nondeleting tree homomorphism from \( T_{\Sigma_{X,m}} \) to \( T_{G \cdot \Pi',m} \) by adding the rules \( q[g[x_1, \ldots, x_k]] \rightarrow g[q[x_1], \ldots, q[x_k]] \) and \( q[x_i] \rightarrow x_i \) for every \( g^{(k)} \in G \) and \( i \in [m] \). Note that in the last rule, \( x_i \) is an input symbol. Now define \( N' = (G, \Pi', (\gamma'_g)_{g \in G}, g_0) \) such that \( L(\gamma'(g)) = td(L(\gamma(g))) \) for \( g \in G \). It is straightforward to prove, by induction on the length of the derivations, that \( g_0 \Rightarrow_{X,10}^* t' \) if and only if there exists \( t \) such that \( g_0 \Rightarrow_{X,10}^* t \) and \( td(t) = t' \). The proof uses that \( td(s \cdot s') = td(s) \cdot td(s') \) (with the additional rule \( q[\boxtimes] \rightarrow \boxtimes \)) and that \( td(u(t_1, \ldots, t_n)) = td(u(td(t_1), \ldots, td(t_n))) \), which can be shown by induction on the structure of \( s \) and \( u \). Hence \( td(L_T(N)) = L_T(N') \). ■

As mentioned, we now have all ingredients needed for the proof of the main theorem of this section.

**Theorem 8.15** The class \( \text{DEL}^*(\text{FIN}) \) is a proper subset of \( \text{TBY}^*(\text{REG}) \).

**Proof** To show that \( \text{DEL}^*(\text{FIN}) \) is a subset of \( \text{TBY}^*(\text{REG}) \), we prove that, in fact, \( \text{DEL}^*(\text{FIN}) \subseteq (\text{YIELD} \circ \text{SET})^n(\text{YIELD}(\text{REG})) \) for all \( n \in \mathbb{N} \). Notice first that, for every genuine class \( C \) of tree languages closed under linear nondeleting tree homomorphisms, we have

\[
\text{rDEL}(\text{DEL}(C)) \subseteq \text{rDEL}(YIELD(\text{rDEL}(C))) \quad \text{(by Lemma 8.8)}
\]
\[
\subseteq \text{SET}(YIELD(\text{rDEL}^2(C))) \quad \text{(by Lemma 8.12)}
\]
\[
= \text{SET}(YIELD(\text{rDEL}(C))) \quad \text{(by Lemma 8.7)}.
\]

For \( n \in \mathbb{N} \), \( \text{DEL}^*(\text{FIN}) \) is closed under linear nondeleting tree homomorphisms by Lemma 8.14. Hence, \( n \)-fold application of the inclusion above shows that

\[
\text{rDEL}(\text{DEL}^*(\text{FIN})) \subseteq (\text{SET} \circ \text{YIELD})^n(\text{rDEL}(\text{FIN}))
\]
\[
= (\text{SET} \circ \text{YIELD})^n(\text{REG}).
\]

\(^{34}\) We note here, without proof, that a much stronger result holds: \( \text{DEL}(C) \) is closed under arbitrary tree homomorphisms, for every genuine class \( C \).
and, therefore,
\[
\text{DEL}^{n+1}(\text{FIN}) \subseteq \text{YIELD}(\text{rDEL}(\text{DEL}^n(\text{FIN}))) \quad \text{(by Lemma 8.8)}
\]
\[
\subseteq \text{YIELD}((\text{SET} \circ \text{YIELD})^n(\text{REG}))
\]
\[
= (\text{YIELD} \circ \text{SET})^n(\text{YIELD}(\text{REG})).
\]

The properness of the inclusion \(\text{DEL}^*(\text{FIN}) \subseteq \text{TBY}^*(\text{REG})\) is immediate from Example 8.5. Since the tree languages in the class \(\text{TBY}^*(\text{REG})\) are decidable, so are those in \(\text{DEL}^*(\text{FIN})\). In fact, the proof of Theorem 8.15 is effective, in the sense that for every hierarchical delegation network \(N\) one can construct a regular tree grammar \(\gamma\) and a finite sequence \(T_1, T_2, \ldots, T_n\) where each \(T_i\) is either a top-down tree transducer \(td\) or a derived signature \(\Delta(\Sigma, m)\), such that \(L_T(N) = T_n(\cdots T_2(T_1(L(\gamma)))\cdots)\) where we also use \(T_i\) to denote either the top-down tree transduction \(td\) or the YIELD mapping \(Y_{\Sigma, m}\). From this we obtain that the uniform membership problem is decidable for hierarchical delegation networks (see [EV85, Theorem 7.5]). We also obtain from [EV85, Theorem 7.4] and [DE98, Theorem 4.5] the following decidability result, where it should be noted (again) that we do not know whether \(\text{DEL}^*(\text{FIN})\) is (effectively) closed under intersection with regular tree languages, whereas \(\text{TBY}^*(\text{REG})\) is.

**Corollary 8.16** It is decidable, for \(L \in \text{DEL}^*(\text{FIN})\) and \(R \in \text{REG}\), whether \(L \cap R = \emptyset\). It is also decidable whether \(L \cap R\) is finite, and if the answer is yes, the elements of \(L \cap R\) can be computed.

In fact, the much stronger corollary with \(\tau(L)\) instead of \(L \cap R\) holds as well, where \(\tau\) is a composition of macro tree transductions (i.e., a composition of top-down tree transductions and YIELD mappings).

One consequence of Corollary 8.16 is that it is decidable whether a delegation network \(N = (G, \Pi, \Gamma, g_0)\) is proper (see Proposition 7.3), if \(\gamma_g\) is a hierarchical delegation network for every \(g \in G\) (and hence so is \(N\)). We have to check that all generator symbols of \(N\) are useful, i.e., that use\((N) = G\). Clearly, use\((N)\) can be computed (as for context-free grammars), because, for \(g^{(k)} \in G\) and \(U \subseteq G\), it is decidable whether \(L(\gamma_g) \cap T_{td,\Pi,k} \neq \emptyset\). Note also that every delegation network \(N\) such that \(L(\gamma_g) \in \text{TBY}^*(\text{REG})\) for every \(g \in G\), can effectively be transformed into an equivalent proper delegation network \(N'\) with the same property, where ‘equivalent’ means that \(L_T(N) = L_T(N')\).

Let \(\text{dTD}\) denote the class of td transductions computed by deterministic td transducers. The class \((\text{dTD} \cup \text{YIELD})^*(\text{REG})\) equals the class \(\text{dMTT}^*(\text{REG})\), the closure of \(\text{REG}\) under deterministic macro tree transductions. Recall that \(\text{TBY}^*(\text{REG}) = (\text{TD} \cup \text{YIELD})^*(\text{REG}) = \text{MTT}^*(\text{REG})\). The class \(\text{YIELD}^*(\text{REG})\) is the so-called IO-hierarchy, see [Mai74] and [ES78, Section 7].

---

35Here, \(L\) is given by a hierarchical delegation network and \(R\) by a regular tree grammar.
The next theorem strengthens the properness of the inclusion of $\text{DEL}^*(\text{FIN})$ in $\text{TBY}^*(\text{REG})$.

**Theorem 8.17** The class $\text{DEL}^*(\text{FIN})$ is incomparable with $\text{YIELD}^*(\text{REG})$ and with $\text{TD}^* \cup \text{YIELD}^*(\text{REG})$.

**Proof** By Example 8.10, the tree language $L_{\text{monexp}}$ of Example 8.5 is in the class $\text{YIELD}(\text{DEL}(\text{FIN}))$. Thus, by Lemma 8.8, it is in $\text{YIELD}(\text{YIELD}(\text{REG}))$, as is well known. Consequently, $L_{\text{monexp}}$ is in $\text{YIELD}^*(\text{REG})$, but not in $\text{DEL}^*(\text{FIN})$.

It remains to present a tree language in $\text{DEL}^*(\text{FIN})$ that is not in $\text{TD}^* \cup \text{YIELD}^*(\text{REG})$. For a tree $t$, let $\text{yield}(t)$ denote the string of constant symbols in $t$, read from left to right. Now, let $L'_{\text{expseq}} \in \text{DEL}^2(\text{FIN})$ be defined in the same way as $L_{\text{expseq}}$ in Example 7.2, except that $L'_{\text{expseq}}(\gamma_h)$ is the regular tree language consisting of all trees $t$ over the signature \{f(2), a(0), b(0), \ldots, 1(0)\} such that $\text{yield}(t) = wbw'$ for some $w, w' \in \{a, 0, 1\}^*$. Using arguments from [EM02], we shall show that $L'_{\text{expseq}}$ is not an element of $\text{TD}^* \cup \text{YIELD}^*(\text{REG})$. The proof of this fact is similar to (the second half of) the proof of [EM02, Theorem 25]. It reads as follows.

For a set $A$ of symbols and a string language $L$, let $\text{rub}_A(L)$ denote the set of all strings of the form $w_0a_1w_1 \cdots a_nw_n$ with $w_0, \ldots, w_n \in A^*$ and $a_1 \cdots a_n \in L$. Intuitively, $\text{rub}_A$ inserts “rubbish”, arbitrary strings in $A^*$, between the symbols of strings in $L$. Clearly, $\text{yield}(L'_{\text{expseq}}) = \text{rub}_{\{0,1\}}(L_{2^n}) = \text{rub}_{\{0,1\}}(\text{rub}_{\{a\}}(L_{2^n}))$, where $L_{2^n} = \{b^{2^n} \mid n \in \mathbb{N}\}$. Now, the “bridge theorem” [EM02, Theorem 20] tells us that $\text{yield}(L'_{\text{expseq}}) \in \text{yield}((\text{TD} \cup \text{YIELD})^*(\text{REG}))$ implies that $\text{rub}_{\{a\}}(L_{2^n}) \in \text{yield}(\text{TD}(\text{REG}))$. Furthermore, by [ERS80, Theorem 3.2.14], the latter implies $L_{2^n} \in \text{yield}(\text{TD}_\text{fc}(\text{REG}))$, where $\text{TD}_\text{fc}$ denotes the set of all finite-copying top-down tree transductions. However, by [ERS80, Corollary 3.2.7], this could only be the case if the Parikh image of $L_{2^n}$ was semilinear, which it is not. Hence, $\text{yield}(L'_{\text{expseq}}) \notin \text{yield}((\text{TD} \cup \text{YIELD})^*(\text{REG}))$ and thus, as claimed, $L'_{\text{expseq}} \notin (\text{TD} \cup \text{YIELD})^*(\text{REG})$.

In the proof above we have defined a language $L'_{\text{expseq}}$ that is in $\text{DEL}^2(\text{FIN})$, but not in $\text{YIELD}(\text{REG})$ and hence not in $\text{DEL}(\text{FIN})$ (see the paragraph after Example 8.10). It is an open problem whether the delegation hierarchy $(\text{DEL}^n(\text{FIN}))_{n \in \mathbb{N}}$ is proper at each level, i.e., whether $\text{DEL}^n(\text{FIN})$ is properly included in $\text{DEL}^{n+1}(\text{FIN})$ for every $n \in \mathbb{N}$. Properness of the IO-hierarchy of (classes of) tree languages was shown in [Dam82]. The properness of the corresponding string IO-hierarchy (obtained by taking yields) was established in [EM02, Section 7] by means of a so-called bridge theorem.

As a last result, we will prove that the class $\text{TBY}^*(\text{REG})$ is closed under $\text{DEL}$. For that we need, in addition to Lemmas 8.8 and 8.12, a lemma that allows us to reorder the operators rDEL and TD.

70
Lemma 8.18 For all genuine classes $C$ of tree languages, $\text{rDEL}(\text{TD}(C)) \subseteq \text{TD}(\text{rDEL}(C))$.

Proof Consider a regular delegation network $\mathcal{N} = (G, \Pi, (g_g)_{g \in \mathcal{G}}, g_0)$ such that $L(g_g) = td_g(L_g)$ for all $g \in \mathcal{G}$, where $td_g$ is a td transducer and $L_g \in C$. Since $C$ is genuine, we may assume that $L_g \subseteq T_{\Sigma_g}$ for pairwise disjoint signatures $\Sigma_g$. Consequently, we may also assume that there is a single td transducer $td \subseteq T_{\Sigma} \times T_{\Sigma}$, where $\Sigma = \bigcup_{\Sigma_g} \Sigma_g$, such that $L(g_g) = td(L_g)$ for all $g \in \mathcal{G}$. Finally, it is easy to modify $td$ in such a way that the right-hand sides of rules $q[f[x_1, \ldots, x_k]] \rightarrow t$ with $k \geq 1$ do not contain generator symbols (i.e., symbols in $\mathcal{G}$). In other words, $td$ outputs generator symbols only when it reaches a leaf of its input tree.

We have to construct a regular delegation network $\mathcal{N}' = (G', \Pi', (g'_g)_{g \in \mathcal{G}'}, g'_0)$ and a td transducer $td'$, such that $td'(L_T(\mathcal{N}')) = L_T(\mathcal{N})$ and $L(g'_g) \in C$ for all $g \in \mathcal{G}'$. To see how $td'$ and $\mathcal{N}'$ can be obtained, consider a derivation $g_0 \Rightarrow_{\mathcal{N}, IO} t_0 \cdot (g_1, \ldots, g_m) \Rightarrow_{\mathcal{N}, IO} t_0 \cdot (t_1, \ldots, t_l)$, where $t_0 \cdot (g_1, \ldots, g_m) \in td(s)$ for a tree $s \in L_{g_0}$. The td transducer $td'$ will, intuitively, start by working in the same manner as $td$, on an input tree (generated by $\mathcal{N}'$) whose topmost part looks like $s$. However, when $td$ would generate one of the generator symbols $g_i$ from a leaf $a$ of $s$, $td'$ must be able to continue the computation on a subtree of which (recursively) the topmost part looks like a tree from $L_{g_i}$. Clearly, when $\mathcal{N}'$ generates this input to $td'$, it “knows” only $a$, but not $g_i$. Therefore, we simply let $\mathcal{N}'$ generate one subtree for each generator symbol in $G$, i.e., $a$ is turned into a symbol $#_a$ of rank $|G|$, and we let $td'$ select the subtree at the position corresponding to $g_i$ when it processes $#_a$. Another detail to be taken into account is that $td$ may be nonlinear, which means that a single occurrence of $a$ in the input tree $s$ may give rise to any number of generator symbols $g_{i_1}, \ldots, g_{i_m}$ in the output tree $td(s)$, with $i_j = i$ for all $j \in [m]$. To achieve the right effect in $td'$, we have to make sure that it may work on different input trees from $L_{g_i}$ in different places, even though the input subtrees it works on are necessarily copies of each other. We solve this problem by generating not only one tree from $L_{g_i}$, (modified as described above), but an arbitrary sequence of such trees, from which $td'$ nondeterministically chooses one. This is similar to the implementation of nondeterminism in the proof of Lemma 8.12.

To give the formal construction, let $\mathcal{G} = \{h_1, \ldots, h_p\}$. We define the relevant components of $\mathcal{N}'$ as follows.

- $G'$ contains all symbols $a^{(0)} \in \Sigma$ and all $h_i, h_i^*$ for $i \in [p]$ (which are assumed to be pairwise distinct).
- $\Pi'$ consists of all symbols $f^{(k)} \in \Sigma$ with $k \geq 1$, the symbols $+, (2)$ and $\theta(0)$, and all symbols $#_a^{(p)}$ such that $a^{(0)} \in \Sigma$. Of course, even these symbols are assumed to be pairwise distinct.
- For every $g \in \mathcal{G}$, $L(g'_g) = L_g$ and $L(g'_g) = \{+[g, g^*], \theta\}$, and for every $a^{(0)} \in \Sigma$, $L(g'_a) = \{#_a[h_1^*, \ldots, h_p^*]\}$.
- Finally, $g'_0 = g_0$.

71
Next, let us construct $t^d'$, assuming that $t^d = (\Sigma, \Sigma^*, Q, R, q_0)$. Clearly, the input and output signatures of $t^d'$ are $\Pi^*$ and $\Pi$, respectively. Its set of states is $Q' = Q \cup \{ q_{sel} \}$, where $q_{sel} \notin Q$, and $q_0$ is the initial state also in $t^d'$. The set $R'$ contains all rules $q[[x_1, \ldots, x_k]] \rightarrow t$ in $R$ such that $k \geq 1$. For every rule $q[a] \rightarrow t_0 \cdot (h_1, \ldots, h_l)$ in $R$ (where $t_0$ does not contain symbols belonging to $G$), $R'$ contains the rule $q[\#a[x_1, \ldots, x_p]] \rightarrow t_0 \cdot (q_{sel}[x_i], \ldots, q_{sel}[x_i])$. Thus, these rules account for the choice of the “right” input subtrees to continue the computation with, i.e., using $L_{h_{ij}}$ for $h_{ij}$. Finally, $R'$ also contains the rules $q_{sel}[[x_1, x_2]] \rightarrow q_0[x_1]$ and $q_{sel}[[x_1, x_2]] \rightarrow q_{sel}[x_2]$. These account for the nondeterministic choice of a tree in $L_{h_{ij}}$.

For the correctness of the construction, we prove that for all $g \in G$ and $t \in T_\Pi$, $g \Rightarrow_{N,10}^* t$ if and only if there exists $s \in T_\Pi$ such that $g \Rightarrow_{N,10}^* t$ and $t \in t^d(s)$.  

As in the proof of Lemma 8.12, we proceed by induction on the length of the derivations, using Lemma 8.8.

(*\Rightarrow\*) Consider a derivation $g \Rightarrow_{N,10}^* t$. Then $s = s_0 \cdot (s_1, \ldots, s_k)$ for some $s_0 \cdot (a_1, \ldots, a_k) \in L_g$ where $a_0(0), \ldots, a_k(0) \in \Sigma$ and $s_0$ does not contain any symbol $a(0) \in \Sigma$, and $a_i \Rightarrow_{N,10}^* s_i$ with $n_i \leq n$ for all $i \in [k]$. Moreover, $s_i = \#a_i[s_1, \ldots, s_i,p]$ with $h_i^* \Rightarrow_{N,10}^* s_i,j$ and $m_{i,j} \leq n$ for all $j \in [p]$. Now consider $t \in t^d(s)$. It follows from the construction of $t^d$ that there is a tree of the form $t_0 \cdot (h_1, \ldots, h_l)$ in $td(s_0 \cdot (a_1, \ldots, a_k))$ such that $t_0$ does not contain symbols from $G$ and $t = t_0 \cdot (t_1, \ldots, t_l)$ for some $t_1, \ldots, t_l$; moreover, there exist $i_1, \ldots, i_l \in [k]$ such that $q_{sel}[s] \Rightarrow_{t^d}^* t_0 \cdot (q_{sel}[s_{i_1,j}], \ldots, q_{sel}[s_{i_l,j}])$ and $q_{sel}[s_{i_j,j}] \Rightarrow_{t^d}^* t_r$ for all $r \in [l]$. Hence, by the rules for $q_{sel}$ and the definition of $L(\gamma_h^*)$, there exists $\bar{s}_r$ such that $t_r \in t^d(\bar{s}_r)$ and $h_{r,j} \Rightarrow_{N,10}^* \bar{s}_r$ with $\bar{s}_r \leq m_{i_j,j}$. This, by induction, $h_{r,j} \Rightarrow_{N,10}^* t_r$, and so $g \Rightarrow_{N,10}^* t$ because $t_0 \cdot (h_1, \ldots, h_l)$ is in $td(L_g) = L(\gamma_g)$.

(*\Rightarrow\*) Now consider a derivation $g \Rightarrow_{N,10}^{n+1} t$. Then $t = t_0 \cdot (t_1, \ldots, t_l)$ for some $t_0 \cdot (h_1, \ldots, h_l) \in td(L_g)$ where $t_0$ does not contain symbols from $G$ and $h_{r,j} \Rightarrow_{N,10}^* t_r$ with $n_r \leq n$, for all $r \in [l]$. By induction, there exist $s_1, \ldots, s_l$ such that $h_{r,j} \Rightarrow_{N,10}^* s_r$ and $t_r \in t^d(s_r)$. This gives derivations $h_{r,j} \Rightarrow_{N,10}^* \hat{s}_j$ for $j \in [p]$ such that $q_{sel}[\hat{s}_j] \Rightarrow_{t^d}^* t_r$ for every $r \in [l]$. Now let $s_0 \cdot (a_1, \ldots, a_k) \in L_g$ be as above such that $t_0 \cdot (h_1, \ldots, h_l) \in td(s_0 \cdot (a_1, \ldots, a_k))$, and let $s = s_0 \cdot (\#a_1[s_1, \ldots, \hat{s}_p], \ldots, \#a_k[s_1, \ldots, \hat{s}_p])$. Then $g \Rightarrow_{N,10}^* s_0 \cdot (a_1, \ldots, a_k) \Rightarrow_{N,10}^* s_0 \cdot (\#a_1[h_1^*, \ldots, h_l^*], \ldots, \#a_k[h_1^*, \ldots, h_l^*]) \Rightarrow_{N,10}^* t$ and so $g \Rightarrow_{N,10}^* s$. Moreover, $q_{sel}[s] \Rightarrow_{t^d}^* t_0 \cdot (q_{sel}[\hat{s}_1], \ldots, q_{sel}[\hat{s}_l]) \Rightarrow_{t^d}^* t_0 \cdot (t_1, \ldots, t_l) = t$ and so $t \in t^d(s)$.

This proves that $L_T(N) = td'(L_T(N'))$.

Theorem 8.19 The class TBY*(REG) is closed under DEL.

Proof By Lemma 8.8 and the fact that TBY*(REG) is, by definition, closed under tree homomorphisms and YIELD, it suffices to show that it is closed under DEL. We do this by proving that, in fact, even TBY"(REG) is closed under
rDEL, for $n \in \mathbb{N}$. For $n = 0$, this is just the fact that REG is closed under rDEL by Lemma 8.7 (for $C = \text{FIN}$). It implies, by Lemma 8.18, that TD$^*$ (REG) is closed under rDEL. For the inductive step, let $C = \text{TD}^* (\text{TBY}^n (\text{REG}))$ (which equals TBY$^n (\text{REG})$ for $n > 0$) be closed under rDEL. Then we get
\[
\begin{align*}
\text{rDEL}(\text{TBY}^{n+1}(\text{REG})) &= \text{rDEL}(\text{TD}^*(\text{YIELD}(C))) \\
&\subseteq \text{TD}^*(\text{rDEL}(\text{YIELD}(C))) \quad \text{(by Lemma 8.18)} \\
&\subseteq \text{TD}^*(\text{SET}(\text{YIELD}(\text{rDEL}(C)))) \quad \text{(by Lemma 8.12)} \\
&= \text{TBY}^{n+1}(\text{REG}),
\end{align*}
\]
as claimed.

By the above proof, TBY$^n (\text{REG})$ is closed under rDEL, for every $n \in \mathbb{N}$. Also, TBY$^n (\text{REG})$ is closed under linear tree homomorphisms – by definition for $n > 0$, and by [Tha73, Theorem 10] for $n = 0$. Hence Lemma 8.8 implies the following more precise result: DEL(TBY$^n (\text{REG})) \subseteq \text{TBY}^{n+1}(\text{REG})$ for every $n \in \mathbb{N}$.

9 Conclusion

Several directions of research remain.

In view of the MW-like result of Theorem 5.6 it would be natural to generalize delegation networks in such a way that the tree generators $\gamma_g$ are turned into jungle generators, i.e., $L(\gamma_g) \subseteq J_{\Sigma, k}$ if $g$ has rank $k$. In Definition 2.4, the first case of the definition of iterate$_{\Sigma}(\sigma)(f)$ would thus evaluate jungles rather than trees. By Lemma 5.3, this does indeed yield a generalization of delegation networks. We guess that, with the definition of $L_{\Sigma}(\mathcal{N})$ adapted in the obvious way, Theorem 5.6 still holds. We also guess that this generalization does not really extend the generating power of delegation networks. In particular, we think that for every CFJG $G$ of which the initial nonterminal has rank 0, and for every II-interpretation $(A, \pi)$, there exists a finitary delegation network $\mathcal{N} = (G, \Pi, \Gamma, g_0, A, \pi)$ such that $L(\mathcal{N}) = \pi(L(G))$.

A question left open by Theorem 5.6 is whether $L_{\Sigma}(\mathcal{N}) = L(\mathcal{N})$ for a delegation network $\mathcal{N}$ obtained from $\mathcal{N}$ by changing its interpretation $(A, \pi)$ into some $(A', \pi')$, similar to the fact that $L_T(\mathcal{N}) = L(\mathcal{N}_{\text{free}})$ (see Section 6) and similar to the result of [MW67]. We guess that such an interpretation $(A', \pi')$ does not exist. Maybe a more general notion of interpretation can be used to obtain a similar result.

Let us repeat some open questions mentioned before. Is DEL$^n (\text{FIN})$ a proper subset of DEL$^{n+1}(\text{FIN})$ for every $n \in \mathbb{N}$? Are DEL$^n (\text{FIN})$ and DEL$^* (\text{FIN})$ closed under intersection with regular tree languages? More generally, what are the closure properties of DEL$^* (\text{FIN})$?
In addition to studying these open theoretical questions, an implementation of (many-sorted) delegation networks should be made. Such an implementation may combine the ideas of the system Treebag [DK00, DK01] (see also http://www.cs.umu.se/~drewes/treebag), which implements tree-based generation, with the evaluation strategy sketched in the Introduction and after Theorem 3.5. In this way, a very general core system would be obtained, to which implementations of any desired classes of tree generators and interpretations may be added. To increase efficiency, such a system should be able to work in a parallel and perhaps distributed manner. While the implementation of the theoretical concept as such is rather straightforward, there are two major practical challenges. The first is to handle nondeterminism in an appropriate way, including cyclic delegation which may easily lead to infinite recursion. The second challenge is to design and implement a suitable user interface that makes it possible to interact with the delegation network and its individual generators in a convenient way. Such an interface should be based on an abstract interaction model which can easily be instantiated for various combinations of tree generators and interpretations. Again, the ideas realized in Treebag can be used as a starting point, but a much more flexible model must be developed.

References


