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Design of energy efficient walking gaits for a three-link planar biped walker with two unactuated degrees of freedom.

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Abstract—We consider the example of a three-link planar biped walker with two passive links. The main objective is to design symmetric periodic gaits in flat ground, that can be exponentially stabilized by feedback control. To this end, we apply recent advances in nonlinear control, to propose a systematic procedure to the problems of gait synthesis and control design. The core of the method lays on a nontrivial coordinate transformation, in order to approach the problem in a state-dependent form. For gait synthesis, such procedure allows a reduction of the search space, with the feasibility of considering energetic performance for optimization. For control design, this allows to apply concepts of transverse linearization, to design a nonlinear feedback control law, which performance is studied by numerical simulations.

Index Terms—Walking robots, underactuated mechanical systems, limit cycles, virtual holonomic constraints, transverse linearization.

I. INTRODUCTION

Passive walkers have served as standard platforms to introduce the concept of dynamic walking [1]. This refers to a behavior that is dynamically stable, and presents the characteristics of being *a)* periodic, *b)* energetically efficient, and *c)* dexterous, with close resemblance to human walking. From the theoretical point of view, such behaviors are analyzed as limit cycles of nonlinear dynamical systems, which can exist either as a natural response of the system, or by the use of feedback control. Within this topic, recent advances in nonlinear control have opened the possibility to comprehensively approach systems with underactuated dynamics, which are inspired on the design of anthropomorphic robots. Approaches that fall into this category are: the zero-moment point [2], passivity based control [3], hybrid zero dynamics [4], [5], and hybrid transverse linearization [6], [7].

In order to validate theoretical arguments, some of these methods have also presented successful experimental studies [8]. Nevertheless, most of these existing results usually consider the special case of legged robots with passive ankle, i.e. underactuation degree one. For systems with a higher degree of under-actuation, only the hybrid zero dynamics [9], and the hybrid transverse linearization approach have proposed some alternatives [10], [11].

The aim of this article is to present complementary arguments, which result from a nontrivial combination in the use of these approaches. The intention is to define a clarified systematic procedure, to be used for gait synthesis

and control design. To this end, we revisit the example¹ introduced in [4], [7], which is modified in [11] to exhibit a challenging problem of underactuation degree two, as described below.

II. MODEL OF THE BIPED WALKER

We consider the model shown in Fig. 1, which represents a planar biped walker with two symmetric legs and upper torso.

Fig. 1. Schematics of the biped in the sagittal plane and level ground. The walking motion is described by the evolution of the stance leg angle q_1 , the swing leg angle q_2 , and the torso's angle q_3 . The length of each leg is denoted by r and the torsos by l . The masses of the legs, denoted by m , are lumped at $r/2$. The hip mass is denoted by M_h , the torso's by M_t , and it is lumped at a distance l from the hip. To keep the torso around upright, we consider the use of torsional springs with similar stiffness coefficient K , and located in between each leg and the torso.

A. Swing Phase Model

Assuming non-slippery and single support contact with the ground, dynamics during the stance phase can be approximated by the standard second order Euler-Lagrange model [12]:

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + G(q) = \begin{bmatrix} u \\ -u \\ 0 \end{bmatrix}, \quad (1)$$

where $M(q)$ denotes the inertia matrix, $C(q, \dot{q})$ the matrix of Coriolis and centrifugal forces, $G(q)$ the vector of gravity, and u the torque acting in between the legs. The physical parameters considered in this case study are listed in Table I².

¹This example considers a planar compass biped with upper torso, and passive ankle

²The stiffness coefficient K is a design parameter used to shape different walking behaviors, and therefore not directly given.

TABLE I
PHYSICAL PARAMETERS OF THE THREE LINK ROBOT

Parameters	Mass [kg]	CoM [m]	Gravity [m/s ²]
Legs	$m = 5$	$r/2 = 0.5$	$g=9.81$
Hip	$M_h = 15$	$r = 1$	
Torso	$M_t = 15$	$l = 1$	

B. Impact Model

The impact of the swing leg with the ground, at the end of each step, can be modeled as an impulse effect provoked to the model (1), and attributed to the convergence of the state space to the switching surface

$$\Gamma^+ = \Gamma^- = \{q_l \in \mathbb{R}^2 : \cos(q_1 + \varphi) - \cos(q_2 + \varphi) = 0\},$$

$$q_l = \{q_1, q_2\} \subset q, \quad (2)$$

representing the ground, and for which φ denotes angular declination. Applying the notion of conservation of angular momentum [4], [13], [14], [15], the instantaneous jump in the values of the states (post-impact) can be computed as follows³:

$$\begin{aligned} q^+ &= \Delta \cdot q^-, \\ \dot{q}^+ &= \Delta(q^-) \cdot \dot{q}^-, \end{aligned} \Rightarrow q^- \in \Gamma^-, q^+ \in \Gamma^+ \quad (3)$$

where the subscripts $(\cdot)^-$ and $(\cdot)^+$ denote the values prior and immediately after the impact event respectively. The coordinate relabeling Δ is used for switching the roles of the legs, so that the same dynamical model (1) can be applied for all single support phases.

III. PROBLEM FORMULATION

A walking gait is the result of an interplay between the discrete and continuous dynamic behaviors of the models (1) - (3). Under certain conditions, this interacting phenomena can be made to be periodic, whenever the resulting solution presents the following characteristics.

Definition 1: Given a set of initial conditions $[q_0, \dot{q}_0] \in \mathbb{R}^{6 \times 1}$, the input $u(t)$, and the period $T_e > 0$, then:

- (i) The solution $q = [q_*(t), \dot{q}_*(t)]$ of the differential equation in (1) uniquely defines the continuous in time phase of the motion.
- (ii) Impact occurs whenever the trajectory $q_*(t)$ holds (2), i.e.,
- (iii) The trajectory is periodic, i.e. $q = q_*(t) = q_*(t + T_e)$, if and only if the vector of initial conditions $[q_0, \dot{q}_0]$ can be computed from (3) at the moment of impact, i.e.

$$\begin{aligned} \Delta \cdot q(T_e) &= q_0, \\ \Delta(q(T_e)) \cdot \dot{q}(T_e) &= \dot{q}_0. \end{aligned} \quad (5)$$

- (iv) The gait $q_*(t)$ is referred to as symmetric, if it satisfies that

$$q_{2*}(T_e) = -q_{1*}(T_e) \in q_*(t), \quad (6)$$

³These matrices are defined in [4], [11]

provided that $\varphi = 0$ (i.e. flat ground). As part of the gait analysis, the values of different dynamic criteria are often used to determine the qualitative and quantitative performance of the gait [5], [13]. An example is the energetic efficiency, which is defined as the energy spent per unit of distance traveled, i.e.

$$J_1(u) = \frac{1}{L_{step}} J_0(u), \quad (7)$$

where $L_{step} = 2r \sin(q_1^+)$ is the step length, and $J_0(u)$ is an integral of the absolute power needed to generate this trajectory, i.e.,

$$J_0(u) = \int_0^{T_e} |(\dot{q}_{1*}(t) - \dot{q}_{2*}(t)) u| dt \quad (8)$$

IV. GAIT SYNTHESIS

A. Re-parametrization of the trajectories

The main objective of our gait synthesis approach consists of finding a set of time independent geometric functions, which uniquely describe the instantaneous postures of the robot along the gait. This implies that rather than searching for the time evolution of the generalized coordinates

$$q = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} q_{1*}(t) \\ q_{2*}(t) \\ q_{3*}(t) \end{bmatrix}, \quad (9)$$

we are interested on an alternative state-dependent parametric representation of the motion. One example is

$$\begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} \theta_*(t) \\ \phi_2(\theta_*(t)) \\ \phi_3(\theta_*(t)) \end{bmatrix}, \quad (10)$$

where the angle of the stance leg is used as a new parameterizing variable, with $\phi_2(\cdot)$ and $\phi_3(\cdot)$ being the functions mentioned above⁴, and defined for the interval $\theta_* \in [\Theta_b, \Theta_e]$ ⁵. With this scheme, the explicit dependence of time disappears, and θ becomes the new trajectory generator, which dynamics can be found by substituting the time derivatives

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix} = \begin{bmatrix} \dot{\theta} \\ \phi_2'(\theta) \dot{\theta} \\ \phi_3'(\theta) \dot{\theta} \end{bmatrix}, \quad (11)$$

and

$$\begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \\ \ddot{q}_3 \end{bmatrix} = \begin{bmatrix} \ddot{\theta} \\ \phi_2''(\theta) \dot{\theta}^2 + \phi_2'(\theta) \ddot{\theta} \\ \phi_3''(\theta) \dot{\theta}^2 + \phi_3'(\theta) \ddot{\theta} \end{bmatrix}, \quad (12)$$

into the dynamics model (1). This yields a set of differential equations of the form:

$$\begin{aligned} \alpha_1(\theta) \ddot{\theta} + \beta_1(\theta) \dot{\theta}^2 + \gamma_1(\theta) &= u, \\ \alpha_2(\theta) \ddot{\theta} + \beta_2(\theta) \dot{\theta}^2 + \gamma_2(\theta) &= -u, \\ \alpha_3(\theta) \ddot{\theta} + \beta_3(\theta) \dot{\theta}^2 + \gamma_3(\theta) &= 0, \end{aligned} \quad (13)$$

⁴In the context of control theory, these functions are known as virtual holonomic constraints, see e.g. [6].

⁵The evolution of the stance leg q_1 varies monotonically from its initial to its final configuration.

where the coefficients $\alpha_j(\cdot)$, $\beta_j(\cdot)$, and $\gamma_j(\cdot)$ are smooth scalar functions defined by the variable θ , the functions ϕ_2 , ϕ_3 , and their partial derivatives, i.e.,

$$\begin{aligned}\alpha_j(\theta) &= f_j(\theta, \phi_2(\theta), \phi_3(\theta), \phi_2'(\theta), \phi_3'(\theta)), \\ \beta_j(\theta) &= h_j(\theta, \phi_2(\theta), \phi_3(\theta), \phi_2'(\theta), \phi_3'(\theta), \phi_2''(\theta), \phi_3''(\theta)), \\ \gamma_j(\theta) &= l_j(\theta, \phi_2(\theta), \phi_3(\theta)), \text{ where } j = \{1, 2, 3\}.\end{aligned}\quad (14)$$

For the case of underactuation degree one, we can recall from [7], that given the initial conditions $[\theta_0, \dot{\theta}_0]$, and the constraint functions ϕ_2 , ϕ_3 , the motion can be explicitly found as solution of the unactuated dynamics, i.e. a single differential equation of the form (13) with no control input u .

As the degree of underactuation increases, so does the number of differential equations⁶, for which it is difficult to a) prove existence, and b) find a common solution $\theta_*(t)$. Considering (13), this number sums to two nonlinearly dependent relations, for which a non-trivial choice of ϕ_2 , ϕ_3 , $[\theta_0, \dot{\theta}_0]$, would allow to find the continuous in time phase of the gait $\theta_*(t)$, assuming that it exists.

In the remaining part of this section, we introduce a transformation of the system (13), such that ϕ_3 or ϕ_2 can be postulated in differential form⁷. This allows to ease the task in the search for the solution $\theta_*(t)$ in (13), by avoiding arbitrary choices of ϕ_2 and ϕ_3 .

B. Equations for constraint functions

Notice that the set of differential equations (13) are linear in $\ddot{\theta}$, $\dot{\theta}^2$ and u . Based on this fact, we can rearrange (13) as a linear system of equations of the form $Ax = b$, and given by

$$\begin{bmatrix} \alpha_1(\theta) & \beta_1(\theta) & -1 \\ \alpha_2(\theta) & \beta_2(\theta) & 1 \\ \alpha_3(\theta) & \beta_3(\theta) & 0 \end{bmatrix} \begin{bmatrix} \ddot{\theta} \\ \dot{\theta}^2 \\ u \end{bmatrix} = \begin{bmatrix} -\gamma_1(\theta) \\ -\gamma_2(\theta) \\ -\gamma_3(\theta) \end{bmatrix}, \quad (15)$$

which gives the solution for $\ddot{\theta}$, $\dot{\theta}^2$ and u , as

$$\begin{bmatrix} \ddot{\theta} \\ \dot{\theta}^2 \\ u \end{bmatrix} = \begin{bmatrix} \alpha_1(\theta) & \beta_1(\theta) & -1 \\ \alpha_2(\theta) & \beta_2(\theta) & 1 \\ \alpha_3(\theta) & \beta_3(\theta) & 0 \end{bmatrix}^{-1} \begin{bmatrix} -\gamma_1(\theta) \\ -\gamma_2(\theta) \\ -\gamma_3(\theta) \end{bmatrix}, \quad (16)$$

provided that the inversion of such a matrix exists. A short form to represent (16) is

$$\begin{bmatrix} \ddot{\theta} \\ \dot{\theta}^2 \\ u \end{bmatrix} = \begin{bmatrix} D_1 \\ D_2 \\ D_3 \end{bmatrix}, \quad (17)$$

which given the relation between the derivatives [6]

$$\ddot{\theta}_*(t) = \frac{1}{2} \frac{d}{d\theta} \left(\dot{\theta}_*^2(t) \right), \quad (18)$$

⁶We refer to the number of differential equations in the form (13) that do not contain u .

⁷These results are inspired upon results presented in [11], [16]

allows to define the following equality

$$D_1 = \frac{1}{2} \frac{d}{d\theta} (D_2). \quad (19)$$

Taking the derivative of the function D_2 , we find that

$$\begin{aligned}2D_1 &= \frac{\partial D_2}{\partial \theta} + \frac{\partial D_2}{\partial \phi_2} \phi_2' + \frac{\partial D_2}{\partial \phi_3} \phi_3' + \frac{\partial D_2}{\partial \phi_2'} \phi_2'' + \\ &\quad \frac{\partial D_2}{\partial \phi_3'} \phi_3'' + \frac{\partial D_2}{\partial \phi_2''} \phi_2''' + \frac{\partial D_2}{\partial \phi_3''} \phi_3''',\end{aligned}\quad (20)$$

which yields the differential equation

$$\phi_3''' = f(\theta, \phi_2, \phi_3, \phi_2', \phi_3', \phi_2'', \phi_3'', \phi_2'''), \quad (21)$$

describing the torso's trajectory ϕ_3 as a function of the legs θ and ϕ_2 . Therefore, given the function ϕ_2 , and the initial condition vector,

$$\zeta = [\theta^+, \phi_3^+, \phi_3'^+, \phi_3''^+] \in \mathbb{R}^{4 \times 1}, \quad (22)$$

solution of (21) can be found for the interval of motion θ^+ and θ^- , representing the initial and final stance-leg angles. Additionally, the initial condition $\phi_3''^+$ in (22), which is explicitly defined by dynamics of the robot, can be found as solution of (13), similarly as done in (16), i.e.,

$$\begin{bmatrix} \ddot{\theta} \\ \phi_3'' \\ u \end{bmatrix} = \begin{bmatrix} H_1(\theta, \dot{\theta}^2, \phi_2, \phi_3, \phi_2', \phi_3', \phi_2'') \\ H_2(\theta, \dot{\theta}^2, \phi_2, \phi_3, \phi_2', \phi_3', \phi_2'') \\ H_3(\theta, \dot{\theta}^2, \phi_2, \phi_3, \phi_2', \phi_3', \phi_2'') \end{bmatrix}. \quad (23)$$

In this form, the original problem of finding the two functions ϕ_i for solving (13), gets reduced to an appropriate choice of ϕ_2 to solve (21).

C. Defining a function for ϕ_2

Following the arguments of [5], we present an example with a fifth order Bézier polynomial⁸, i.e.,

$$\begin{aligned}\phi_2(\theta) &= B_N P_i, \\ B_N &= \sum_{i=0}^{N=5} \binom{N}{i} (1 - \hat{\theta})^{N-i} (\hat{\theta})^i, \\ P_i &= \{P_0, P_1, P_2, P_3, P_4, P_5\},\end{aligned}\quad (24)$$

where $\hat{\theta}$ denotes the normalization of the variable θ within the convex hull $[\theta^+, \theta^-]$, i.e.,

$$\hat{\theta} = \frac{\theta - \theta^+}{\theta^- - \theta^+}, \quad (25)$$

and for which the higher order derivatives can be computed as

$$\begin{aligned}\phi_2^k(\theta) &= B_{N-k} \prod_{m=0}^{k-1} (N-m) C_k(\hat{\theta}')^k, \\ C_k &= \sum_{j=0}^k \binom{k}{j} P_{i+j} (-1)^{k+j},\end{aligned}\quad (26)$$

⁸Notice that any type of parametric function can satisfy this procedure.

with k being the derivative's order, and

$$\hat{\theta}' = \frac{\partial \hat{\theta}}{\partial \theta} = \frac{1}{\theta^- - \theta^+}. \quad (27)$$

The values of the parameters P_i are used to shape different walking behaviors. However, in order to introduce the gait characteristics described in section III, some of these parameters cannot be arbitrarily chosen. In fact, as presented in [5], it follows that evaluating (24) at the boundaries of the trajectory, i.e., when $\theta = \theta^+$ and $\theta = \theta^-$, allows to define that

$$\begin{aligned} P_0 &= \theta^- = \phi_2^+, \\ P_5 &= \theta^+, \end{aligned} \quad (28)$$

provided that the ground declination $\varphi = 0$. In addition, evaluating (26) for $k = 1$ at the beginning of the gait, i.e. $\phi_2'(\theta^+)$, yields

$$P_1 = \frac{\phi_2'^+}{5\hat{\theta}'} + P_0. \quad (29)$$

Finally, the vector composed by

$$\eta_\star = [K, \theta^+, P_2, P_3, P_4, \phi_3^+, \dot{q}_1^+, \dot{q}_2^+, \dot{q}_3^+] \in \mathbb{R}^{9 \times 1}, \quad (30)$$

can be *a)* used for solving (21), and *b)* used for determining the periodicity of the gait.

D. Optimization procedure to find periodic gaits

The proposed steps in the iterative search are summarized below.

- 1) Choose a vector $\eta_\star$ (30), such that the initial conditions

$$\begin{aligned} \chi_0 &= [\theta^+, \phi_2^+, \phi_3^+, \dot{q}_1^+, \dot{q}_2^+, \dot{q}_3^+] \in \Gamma^+, \\ \chi_0 &\subset \eta_\star. \end{aligned} \quad (31)$$

- 2) Calculate $\phi_3''^+$ from (23) using (31).
- 3) The symmetrical configuration of the legs allows to define $\theta^- = -\theta^+$, to solve the differential equation (21) for the interval $[\theta^+, \theta^-]$. If the solution is not well-defined for the whole interval go back to the first step.
- 4) Once the solution $\phi_3(\theta)$ has been found, compute $\dot{\theta}$ from (17), i.e.

$$\dot{\theta}^- = \sqrt{D_2(\theta^-)} = \dot{q}_1^-. \quad (32)$$

- 5) Use the end-values of the solution from the previous steps to define the vector

$$\begin{aligned} \chi_\star^- &= [\theta^-, \phi_2^-, \phi_3^-, \dot{q}_1^-, \dot{q}_2^-, \dot{q}_3^-] \in \Gamma^-, \\ \chi_\star^- &\subset \eta_\star. \end{aligned} \quad (33)$$

- 6) Apply the impact equation (3), i.e.

$$\chi_\star^+ = \Delta \cdot \chi_\star^-. \quad (34)$$

- 7) Verify that (31) with (34) yields

$$\|\chi_0 - \chi_\star^+\| = 0. \quad (35)$$

If such an equality does not comply, adjust the vector (31) and repeat all the process.

- 8) This procedure can be formulated to additionally attain to minimize a cost function

$$J = J_1(u), \quad (36)$$

as explained for (7), so that energy can be considered as part of the performance criteria of the search.

E. Example of a Gait

This procedure has been implemented in the programming environment of MATLAB, resulting in several solutions, one of which has the vector of initial conditions given by

$$\chi_\star \approx [-0.2503, 0.2503, 0.3131, 1.2512, 0.4956, -0.2695], \quad (37)$$

with spring stiffness coefficient $K = 23.9770 \text{ N/m}$. The evolution of θ is within the step interval $[-0.2503, 0.2503]$, and the period is $T_e = 0.4878 \text{ sec}$. The numerical value of (8) calculated for this gait is $21,16 \frac{\text{Nm}}{\text{sec}}$. The constraint functions $\phi_2(\theta)$, $\phi_3(\theta)$, and the control signal $u(\theta)$ are shown in Fig. 2.

Fig. 2. Solutions of (13) as functions of θ with initial conditions (37). Top Left: Support leg. Top right: Swing Leg. Bottom left: Torso. Bottom right: Nominal input torque. The motion goes from left to right.

V. CONTROL DESIGN

The technique proposed in [7], [17], [18] is applied. The aim is to compute a special time variant impulsive linear system, that represents a linearization of the dynamics transverse to the motion. Finding a linear stabilizing feedback controller for this system leads to a straightforward design of an orbitally exponentially stabilizing nonlinear feedback controller for the hybrid nonlinear model (1) - (3). The steps to follow for computing this controller is given below.

A. Dynamics along the target orbit and transverse coordinates

Given a motion of the mechanical system (1), in the form (10), allows to introduce a vector of new generalized coordinates in the vicinity of the orbit, i.e.

$$\begin{bmatrix} \theta \\ y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} q_1 \\ q_2 - \phi_2(\theta) \\ q_3 - \phi_3(\theta) \end{bmatrix}, \quad (38)$$

with its time derivative being

$$\begin{bmatrix} \dot{\theta} \\ \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 - \phi_2'(\theta)\dot{\theta} \\ \dot{q}_3 - \phi_3'(\theta)\dot{\theta} \end{bmatrix}. \quad (39)$$

The purpose of the control action is to stabilize the steady state

$$\theta = \theta_*(t), \dot{\theta} = \dot{\theta}_*(t), y = 0, \dot{y} = 0, \quad (40)$$

from any given initial condition, and for $t > 0$. Dynamics (1) in the new coordinates can be found by introducing (38), (39), and

$$\begin{bmatrix} \ddot{\theta} \\ \ddot{y}_1 \\ \ddot{y}_2 \end{bmatrix} = \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 - (\phi_2''(\theta)\dot{\theta}^2 + \phi_2'(\theta)\ddot{\theta}) \\ \ddot{q}_3 - (\phi_3''(\theta)\dot{\theta}^2 + \phi_3'(\theta)\ddot{\theta}) \end{bmatrix}, \quad (41)$$

into dynamics of the robot (1). This substitution yields the dynamics model:

$$\alpha_i(\theta)\ddot{\theta} + \beta_i(\theta)\dot{\theta}^2 + \gamma_i(\theta) = g_i(\theta, \dot{\theta}, y, \dot{y}, \ddot{y}), \quad (42)$$

$$R_1(\theta, y, \dot{\theta}, \dot{y}) + N_1(\theta, y)u = \ddot{y}_1, \quad (43)$$

$$R_2(\theta, y, \dot{\theta}, \dot{y}) + N_2(\theta, y)u = \ddot{y}_2. \quad (44)$$

where $g_i(\cdot)$ is a smooth function that is equal to zero on the desired orbit, and α_i, β_i and γ_i is obtained from any of the equations in (13) with the form⁹:

$$\alpha(\theta)\ddot{\theta} + \beta(\theta)\dot{\theta}^2 + \gamma(\theta) = 0, \quad (45)$$

B. Partial feedback linearization

Introducing the feedback transformation

$$u = \frac{1}{N_1}u_\perp - \frac{R_1}{N_1}, \quad (46)$$

with $N_1(0, \theta_*(t)) \neq 0$ for all $t \in [0, T_e]$, brings the y_1 -dynamics (43) into the linear form $\ddot{y}_1 = u_\perp$. With this transformation the nominal control input $u_*(t)$ is given by¹⁰

$$u_*(t) = U(\theta_*(t), \dot{\theta}_*(t)) = -\frac{R_1(0, \theta_*(t), 0, \dot{\theta}_*(t))}{N_1(0, \theta_*(t))}, \quad (47)$$

which substitution into (44) yields the y -dynamics as:

$$\begin{bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ R_2 - \frac{N_2}{N_1}R_1 \end{bmatrix} + \begin{bmatrix} 1 \\ \frac{N_2}{N_1} \end{bmatrix} u_\perp. \quad (48)$$

C. Integral of the reduced dynamics

The solution of a scalar second order differential equation of the form (45) is given by the integral function [6]

$$I^{(i)}(\theta, \dot{\theta}, \theta_0, \dot{\theta}_0) = \dot{\theta}^2 - \Psi(\theta_0, \theta) \left[\dot{\theta}_0^2 + \int_{\theta_0}^{\theta} \Psi(s, \theta) \frac{2\gamma(s)}{\alpha(s)} ds \right], \quad (49)$$

with

$$\Psi(\theta_1, \theta_2) = \exp \left\{ -2 \int_{\theta_1}^{\theta_2} \frac{\beta(\tau)}{\alpha(\tau)} d\tau \right\}, \quad (50)$$

which strictly preserves its zero value along a solution $\theta_*(t)$, initiated at $(\theta_0, \dot{\theta}_0) = (\theta_*(0), \dot{\theta}_*(0))$.

⁹The form (45) can be found on (13) by summing up the first two rows, the three of them, or just by considering the third equation.

¹⁰Notice that u can also be calculated from (17) prior any controller design, and its form can be visualized in Fig. 2.

D. Coordinates measuring the distance

The dynamical system (42), (48) possesses a natural choice of $(2n - 1)$ so-called transverse coordinates

$$x_\perp = [I^{(i)}(\theta, \dot{\theta}, \theta_*(0), \dot{\theta}_*(0)), y_1, y_2, \dot{y}_1, \dot{y}_2]^T, \quad (51)$$

with $I^{(i)}(\cdot)$ being the scalar function (49)-(50) associated with any of the equations (45). The integral function qualifies as transverse coordinate since a) it preserves zero value along solutions of (45), and b) quantifies the distance orthogonal to the vector field along the orbit from any point in its vicinity. Thus, the nonlinear dynamical system (42)-(48) presents a trivial solution defined for $t \in [0, T_e]$, and given by

$$y_1 \equiv 0, y_2 \equiv 0, \theta \equiv \theta_*(t), u_\perp \equiv 0. \quad (52)$$

E. Hybrid Transverse Linearization

The linearization of dynamics of the transverse coordinates (51) in the equilibrium (52) is defined by the linear time variant system

$$\begin{aligned} \frac{d}{d\tau}[\delta x_\perp] &= A(\tau)\delta x_\perp + B(\tau)\delta v, \quad \tau \in [0, T]. \\ \delta x_\perp(\tau^+) &= L\delta x_\perp(\tau^-), \quad \tau = T_e \cdot k, k = 1, 2, \dots \end{aligned} \quad (53)$$

The coefficients $A(\cdot)$ and $B(\cdot)$ are computed by deriving the first-order terms in the Taylor multi-series expansion for $g_i(\cdot)$ in (42), and the right hand side of (48), see e.g. [18]. This procedure leads to the matrices of the form

$$\begin{aligned} A(\tau) &= \begin{bmatrix} a_{11}(\tau) & a_{12}(\tau) & a_{13}(\tau) & a_{14}(\tau) & a_{15}(\tau) \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ A_{21}(\tau) & A_{22}(\tau) & A_{23}(\tau) & A_{24}(\tau) & A_{25}(\tau) \end{bmatrix}, \\ B(\tau) &= \begin{bmatrix} b_1(\tau) & 0 & 0 & 1 & B_2(\tau) \end{bmatrix}^T. \end{aligned} \quad (54)$$

The computation of the matrix L in (53) is done exactly as in [7], [17], and it is related to a linearization of the impact map (3).

F. Feedback controller design

Considering a C^1 -smooth vector of gains $\Lambda(\tau)$, a controller that stabilizes the equilibrium of (53) can be designed as:

$$\delta v = \Lambda(\tau) \cdot \delta x_\perp(\tau). \quad (55)$$

In this form, a controller for the nonlinear system (42)-(48) can be constructed from (55) as

$$u_\perp(t) = \Lambda(s) \cdot x_\perp(t), \quad (56)$$

where $s(\theta_*(t)) = t$ for $t \in [0, T_e]$. The controller $\Lambda(\tau)$ can be chosen via a few steps of a numerical minimization procedure for the maximum absolute value of the eigenvalues of the state transition matrix $\Phi(T_{e+})$, and computed as follows

$$\begin{aligned} \frac{d}{d\tau}\Phi(\tau) &= [A(\tau) + B(\tau)\Lambda(\tau)]\Phi(\tau), \quad \Phi(0^+) = I, \\ \Phi(T_e^+) &= L\Phi(T_e^-) \end{aligned} \quad (57)$$

G. Control Design Simulations

After running the numerical search from above, we found the controller gains

$$\Lambda \approx [-0.8406, -15, -0.50, -40.011, -5.0988]. \quad (58)$$

To evaluate the performance of this controller, we simulated the model (1) - (3) with all our designing parameters, but varying the initial conditions. As visualized in Fig. (3), the controller is capable of converging to the desired cycle, and sustain a final periodic behavior. To verify this statement, we can refer to Fig. (4), which shows the convergence of the transverse coordinates (51) to the equilibrium (52).

Fig. 3. Walking gait in the phase space q_t vs $\dot{q}_t$, i.e. stance and swing legs positions and velocities. The red line represents the nominal gait, and ∇ points out the initialization of the simulation.

Fig. 4. Approach of the transverse coordinates to the zero equilibrium, and recorded from the nonlinear model as stated in (38), (39), and (49).

VI. CONCLUSIONS

We presented an example of a three-link planar biped walker with two symmetric legs and upper torso. In this case study, the use of torsional springs was considered to keep the torso upright, while one actuator was thought to act in between the legs. These design considerations lead to a system of underactuation degree two, for which we designed walking behaviors in flat ground. To this end,

we proposed certain arguments useful for designing hybrid limit cycles. This lets us define a systematic procedure to approach the problem of gait synthesis, by a nontrivial search for state-dependent functions. Such results allowed us to derive feedback control laws, recalling concepts of hybrid transverse linearization.

The main results drawn from this presentation are: a) a step-by-step procedure for planning quasi-optimal gaits, and b) an exemplified approach for control design of mechanical systems with underactuation degree higher than one.

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