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A FEM-Based Method Using Harmonic Overtones to Determine the Effective Elastic, Dielectric, and Piezoelectric Parameters of Freely Vibrating Thick Piezoelectric Disks

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Abstract

To gain an understanding of the electro-elastic properties of tactile piezoelectric sensors used in the characterization of soft tissue, the frequency-dependent electric impedance response of thick piezoelectric disks has been calculated using finite element modeling. To fit the calculated to the measured response, a new method was developed using harmonic overtones for tuning of the calculated effective elastic, piezoelectric, and dielectric parameters. To validate the results, the impedance responses of 10 piezoelectric disks with diameter to thickness ratios of 20, 6, and 2 have been measured from 10 kHz to 5 MHz. A two-dimensional, general purpose finite element partial differential equation solver with adaptive meshing capability run in the frequency-stepped mode, was used. The equations and boundary conditions used by the solver are presented. Calculated and measured impedance responses are presented, and resonance frequencies have been compared in detail. The comparison shows excellent agreement, with average relative differences in frequency of 0.27%, 0.19%, and 0.54% for the samples with diameter to thickness ratios of 20, 6, and 2, respectively. The method of tuning the effective elastic, piezoelectric and dielectric parameters is an important step towards a finite element model that describes the properties of tactile sensors in detail.

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I. INTRODUCTION

Piezoelectric transducers are used today in many different areas: from non-destructive testing of the structural properties of hard materials, to the use of ultrasonic waves to penetrate human soft tissue to diagnose medical conditions. Recently, a series of studies has been reported utilizing piezoelectric tactile sensors for the characterization of soft tissue. It has been shown that these sensors can distinguish between normal and cancerous prostate tissue, and could therefore constitute a new diagnostic instrument [1]. Moreover, such sensors can be used to measure the intra-ocular pressure, providing information for the evaluation of edema [2], [3].

The basic principle involves a piezoelectric element vibrating at resonance. When the sensor element is placed in contact with soft tissue, the resonance frequency is changed slightly. This frequency shift is used to obtain information about the structural properties of the tissue. The interpretation of the sensor output signal requires further investigation in these applications. The creation of an electro-elastic model of the sensor-tissue system and fitting the model to measurements of the vibrational spectra will be an important step towards a deeper understanding of the interaction between the vibrating sensor and the soft tissue with which it is in contact.

Our general understanding of piezoelectric vibrations is based on the solution of the constitutive piezoelectric equations, supported by three-dimensional Maxwell equations and the equations of linear elasticity with suitable boundary conditions. The solutions to the equations are complex, and a closed-form solution could in general not be obtained without to resort to computer-based numerical methods.

Different techniques have been used to measure the vibrational surface pattern, or mode shape, of a piezoelectric ceramic. The surface motion of thick piezoelectric BaTiO_3 disks with diameter to thickness ratios between 1 and 7 has been measured by an optical method in which radial, thickness extensional, and edge mode vibrations were identified [4]. The resonant frequency response of thin piezoelectric ceramic PbTiO_3 disks has been measured using a transmission circuit, showing three groups of thickness modes and two groups of radial modes [5]. The vertical vibration velocity distributions and the frequency spectra of beveled and unbeveled $\text{Pb}(\text{Zr}\cdot\text{Ti})\text{O}_3$ disks have also been measured [6].

Two principal methods have been used to take into account vibrations in piezoelectric ceramics by numerical methods: the variational approximation method [7] and the finite

element method, FEM [8]. The strength of the latter lies in its flexibility when modeling different types of geometries, layered structures, material properties, and the coupling between the piezoelectric sensor and the medium with which it is in contact. Vibrational modes in piezoelectric PZT-5H disks have been analyzed using FEM [9]. Radial, edge, length expander, thickness shear, and thickness extensional modes were identified. FEM-calculated and measured electric impedance (henceforth called impedance) frequency spectra of PZT5A samples with radial (R), edge (E), thickness shear (TS), thickness extensional (TE), and high-frequency radial vibrational modes have been reported [10].

A problem when comparing experimental results to results predicted by models lies in the reliability of the values of the electro-elastic (elastic, piezoelectric, and dielectric permittivity) physical constants that characterize the piezoelectric ceramic. The data provided by manufacturers are often incomplete, and the disagreement is often as high as $\pm 20\%$ [11]. The large spread in the parameter input data makes it difficult to compare the results of FEM modeling to real measured results.

The development of computer technology has made it possible to perform FEM modeling on ordinary desktop computers. Moreover, FEM software has been developed to use dynamic mesh adaptive refinement and multi-threading technology, which speeds up the computation. The idea of using a FEM model of a piezoelectric ceramic as part of an iterative procedure to adjust the values of the electro-elastic constants, until the calculated and measured impedance spectra match, is now realizable. The asymptotic waveform evaluation method was adopted for a fast frequency sweep of the FEM-analysis and a design sensitivity method was used for a material inversion scheme of piezoelectric transducers [12]. Using a three-dimensional finite-element simulation of the constitutive equations, combined with an inexact Newton or Landweber iterative inversion scheme, the piezoelectric material parameters were reconstructed from electric impedance and mechanical displacement measurements [13]. Using a four step method: experimental measurements, identification of vibration modes and their sensitivity to material constants, a preliminary identification algorithm, and final refinement of material constants using an optimization algorithm, the piezoelectric constants were determined [14].

The aim of this study was: 1) to develop a FEM model for piezoelectric disks that calculates the electric impedance response, 2) to develop a new method that minimizes the differences between the FEM-calculated and measured responses employing effective parameter grouping and harmonic overtones for tuning of the electro-elastic parameters, 3) to validate the tuning

method by a comparison of the calculated and measured responses of thick piezoelectric disks with diameter to thickness ratios of 20, 6, and 2.

II. METHOD

A small, compact, low-cost finite element partial differential equations solver, FlexPDE [15] was used. The solver has a dynamic mesh adaptive refinement capability and can be run in a multi-threaded environment on an ordinary desktop computer. The solver and the formulated partial differential equations with boundary conditions will hereafter be denoted PSM (Partial differential equation Solver-based Model).

A. The PSM solver, equations, and boundary conditions

1) *The finite element partial differential equation solver of the PSM:* The solver of the PSM transforms a system of partial differential equations and boundary conditions into a finite element model. The finite elements are created by a triangular mesh generator that allocates an unstructured mesh to the problem area. The solution process incorporates steps according to the weighted residual method [15] where, at the vertices of the triangles, the continuous problem is transformed into a discrete minimization problem.

This method of solving the partial differential equations does not guarantee that the solution is correct at every point in the problem domain, but the solution is “correct” in an integral sense at each vertex of all the triangles that constitute the domain. During the problem-solving cycle, the mesh is modified by an adaptive procedure that measures the adequacy of the mesh and increases the mesh density by subdividing triangles in areas where the error is large. The system iterates the mesh refinement and solution until a user-defined error tolerance is achieved. An example of adaptive mesh generation can be seen in Fig. 1. In this work, either adaptive or static (fixed) mesh generation was chosen to optimize the trade-off between accuracy and computational time.

2) *Derivation of the PSM equations in axisymmetric cylindrical format:* In the steps used to derive the equations of the PSM model, the definitions and nomenclature given in the IEEE standards on Piezoelectricity [16] were adopted. In a rectangular coordinate system (x_1, x_2, x_3) , the mechanical system variables of the disk are the strain, S_{ij} , and the stress, T_{ij} . The mechanical strain is given by: $S_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$, where $u_{i,j}^1 = \partial u_i / \partial x_j$ and

¹A comma followed by an index denotes partial differentiation with respect to a space coordinate [16].

u_i is the mechanical displacement vector. The electric flux density, D_i , and the quasistatic electric field, E_k , are the electric system variables. The stiffness, piezoelectric, and dielectric permittivity constants are denoted c_{ijkl}^E , e_{kij} , and ϵ_{ik}^S , respectively. Superscripts E and S indicate that the constants are measured under constant electric field and constant strain conditions, respectively.

The mechanical and electrical systems interact through the linear constitutive piezoelectric stress equations: $T_{ij} = c_{ijkl}^D S_{kl} - h_{kij} D_k$ and $E_i = -h_{ikl} S_{kl} + \beta_{ik}^S D_k$; with supporting equations:

$$(a) D_{i,i} = 0, \quad (b) E_k = -\varphi_{,k}, \quad (c) T_{ij,i} = -\omega^2 \rho u_i, \quad (1)$$

where ρ is the density of the ceramic and φ is the electric potential [16], [17]. Equation (1c) represents the equation of motion, where we assume harmonic oscillations of type $e^{i\omega t}$ with the angular frequency $\omega = 2\pi f$, f being the frequency. The tensors, c_{ijkl}^D , h_{ikl} , T_{ij} , and S_{kl} are converted into compressed matrix versions, i.e. $c_{ijkl}^D \rightarrow c_{pq}^D$, $h_{ikl} \rightarrow h_{iq}$, $T_{ij} \rightarrow T_p$, and $S_{kl} \rightarrow S_q$ where i, j, k, l take the values 1,2,3 and p, q takes the values 1,2,3,4,5,6 [16].

The PSM was configured for an axisymmetric piezoelectric disk in a two-dimensional (r, z) cylindrical coordinate system, as shown in Fig. 2. The disk has a diameter to thickness ratio denoted a/h , where a is the radius and h is half the thickness. The disk was assumed to vibrate freely, with no traction forces on its surfaces, poled in the thickness direction and of $6mm$ crystal hexagonal symmetry [18]. Compressed $6mm$ versions [17] of the elastic (A.1), piezoelectric (A.2), and dielectric (A.3) constant matrices were used. The flat surfaces were completely covered by thin electrodes. The system was assumed to be linear with small deviations from its equilibrium state. A vertical cross-section of the disk, shown in Fig. 3, was identified as the problem area for solving the partial differential equations that model the piezoelectric vibrations. The boundaries Γ_2 and Γ_0 are defined in the planes of the electrodes in the figure. Boundaries Γ_1 and Γ_3 were set parallel to the z -axis at $r = a$ and $r = 0$ respectively. The electrodes were connected to an alternating voltage source $V e^{i\omega t}$, generating a vertical electric field.

The piezoelectric constitutive equations with S_q and D_k as independent variables can be expressed [16], [18]:

$$a) T_p = c_{pq}^D S_q - \tilde{h}_{kp} D_k, \quad b) E_i = -h_{iq} S_q + \beta_{ik}^S D_k, \quad (2)$$

where $\sim$ means the transpose of the matrix and the definitions of the constants c_{pq}^D , h_{kp} , and β_{ik}^S can be found in the Appendix. Due to the axis-symmetry of the model, the derivatives of the displacement vector components with respect to θ can be ignored. Equation (2a) yields the components of stress in cylindrical format:

$$\begin{aligned} a) \quad T_{rr} &= u_{r,r} c_{11}^D + \frac{u}{r} c_{12}^D + c_{13}^D u_{z,z} - \frac{e_{31}}{\varepsilon_{33}^S} D_z, \\ b) \quad T_{\theta\theta} &= \frac{u}{r} c_{11}^D + u_{r,r} c_{12}^D + c_{13}^D u_{z,z} - \frac{e_{31}}{\varepsilon_{33}^S} D_z, \\ c) \quad T_{zz} &= \left(u_{r,r} + \frac{u}{r} \right) c_{13}^D + u_{z,z} c_{33}^D - \frac{e_{33}}{\varepsilon_{33}^S} D_z, \\ d) \quad T_{rz} &= (u_{z,r} + u_{r,z}) c_{44}^D - \frac{e_{15}}{\varepsilon_{11}^S} D_r, \end{aligned} \quad (3)$$

where $T_{r\theta} = T_{\theta z} = 0$. Equation (1c) relates the divergence of the stress to the motion of the vibrating body. Application of the axisymmetric cylindrical divergence operator [17] to the stress yields the equations that describe the periodic elastic vibrations of the piezoelectric disk:

$$\begin{aligned} T_{rr,r} + T_{rz,z} + \frac{1}{r} (T_{rr} - T_{\theta\theta}) + \omega^2 \rho u_r &= 0, \\ T_{zz,z} + \frac{T_{rz}}{r} + T_{rz,r} + \omega^2 \rho u_z &= 0. \end{aligned} \quad (4)$$

The second constitutional equation (2b), gives the components of the electric field:

$$\begin{aligned} E_r &= \frac{1}{\varepsilon_{11}^S} (D_r - e_{15} (u_{z,r} + u_{r,z})), \\ E_z &= \frac{1}{\varepsilon_{33}^S} \left(D_z - e_{33} u_{z,z} - e_{31} \left(u_{r,r} + \frac{u}{r} \right) \right). \end{aligned} \quad (5)$$

Substitution of (1b) into (5) gives the electric flux density components:

$$\begin{aligned} D_r &= e_{15} (u_{z,r} + u_{r,z}) - \varepsilon_{11}^S \varphi_{,r}, \\ D_z &= e_{31} \left(u_{r,r} + \frac{u}{r} \right) + e_{33} u_{z,z} - \varepsilon_{33}^S \varphi_{,z}. \end{aligned} \quad (6)$$

Application of (1a), the divergence of the electric flux density, yields:

$$D_{r,r} + \frac{D_r}{r} + D_{z,z} = 0. \quad (7)$$

Equations (7), (4), and (3) make up the system of partial differential equations solved by the partial differential equation solver of the PSM.

3) *Calculation of the impedance response function:* The solution of the PSM equations will give values of the elastic displacements (u_r, u_z) and the electric potential, φ , for each frequency, ω , at every node of the mesh. Using (6), the total charge, Q , on an electrode with area A can be calculated:

$$Q = \int_A \mathbf{D} \cdot d\mathbf{A} = 2\pi \int_0^a D_z r dr. \quad (8)$$

The electric current can be expressed as $I = dQ/dt = i\omega Q$. The frequency-dependent impedance of the piezoelectric element can be expressed as $\varphi/(i\omega Q)$, where φ is the potential at one electrode when the other electrode is grounded.

4) *The PSM boundary conditions:* The boundary conditions of the four boundaries of the problem area $\Gamma_0, \Gamma_1, \Gamma_2$, and Γ_3 (Fig. 3) are listed in Table I. The keyword “value”, specifies the value that a dependent variable must take at a boundary of the domain. The keyword “natural” boundary condition specifies the value that the normal component of the gradient of the dependent variable should have at the boundary of the domain. If a natural boundary condition is set to zero, the dependent variable is continuous across that boundary. Equation (1c) combined with the boundary conditions $\text{natural}(u_r) = \text{natural}(u_z) = 0$ allows the boundary to move freely. The divergence of the electric flux density, Eq. (1a), relates the boundary condition $\text{natural}(\varphi) = 0$ to the continuity of the E-field across a boundary. At the origin (0,0), a single point value(u_r, u_z) was set equal to zero to allow only axisymmetric extensional mode vibrations [10]. The electric potential φ was set to zero at Γ_0 and to the value of the exciting voltage at boundary Γ_2 .

B. A method to tune the effective elastic, piezoelectric and dielectric parameters

To minimize the differences between the calculated (PSM) and the measured impedance responses, a procedure for tuning the effective electro-elastic model parameters was devised. For simplification, the superscripts, E and S will be dropped in the effective parameter symbols, and the term “parameter” takes the same meaning as the term “effective parameter”.

Ten electro-elastic parameters are required by the PSM model: the dielectric permittivity parameters $\hat{\epsilon}_{11}$, $\hat{\epsilon}_{33}$, the elastic parameters $\hat{c}_{11}$, $\hat{c}_{12}$, $\hat{c}_{13}$, $\hat{c}_{33}$, $\hat{c}_{44}$, and the piezoelectric parameters $\hat{e}_{33}$, $\hat{e}_{31}$, $\hat{e}_{15}$. The circumflex denotes that the parameters are effective PSM model parameters and not the physical constants. The density of the disk was not considered in the

tuning procedure, and was set to the value given by the manufacturer or set to a measured value.

The basic principle of the method was to group together electro-elastic parameters which, when varied, demonstrated a significant correlation to the variation in a set of selected resonance frequencies and impedance, here denoted target mode frequencies and target impedance. Moreover, the identification procedure started from the high end of the frequency spectrum, where the parameters associated with the thickness extensional resonance modes are dominant, proceeding to the low end where radial and edge modes dominate. The procedure was completed by identification of two parameters that most affect resonance modes superimposed on the fundamental thickness extensional mode.

Three groups of parameters were formed with associated frequency intervals, and their determination was carried out in three sequential tuning stages.

- 1) The high-frequency tuning stage: the dielectric permittivity and thickness extensional mode group with the parameter set $\hat{\epsilon}_{33}$, $\hat{c}_{33}$, $\hat{e}_{33}$, and frequency interval characterized by the first thickness extensional resonance mode and its harmonic overtones. Target mode frequencies: the first and second thickness extensional modes TE_1 and TE_2 . Target impedance, Z^h : impedance in decibel averaged over a frequency interval located between TE_1 and TE_2 .
- 2) The low-frequency tuning stage: the radial and edge mode group with the parameter set $\hat{c}_{11}$, $\hat{c}_{12}$, $\hat{c}_{13}$, $\hat{e}_{31}$, and frequency interval characterized by the first and second radial, and the edge mode resonance. Target mode frequencies: the first radial mode resonance and anti-resonance R_1 , R_1^a , the second radial mode R_2 , and the first edge mode E .
- 3) The superimposed mode tuning stage: the thickness extensional superimposed mode group with the parameter set $\hat{c}_{44}$, $\hat{e}_{15}$, and the frequency interval characterized by two modes superimposed on the first thickness extensional resonance mode. Target mode frequencies: the two resonance modes that showed the greatest variation while varying parameters $\hat{c}_{44}$ and $\hat{e}_{15}$ were selected.

1) Parameter sensitivity ratios: To identify the target mode frequencies and target impedance, an investigation of the parameter sensitivity ratios [19] was carried out using a PSM model for a prototype piezoelectric disk, PD20, with $a/h = 20$. As a representative for the PZT5 family, the PZT5A electro-elastic material constants were chosen as PSM model parameters for the prototype disk as the material is well known and the manufacturer has supplied a

complete set of parameters [20]. In the sensitivity investigation, the sensitivity ratios were determined by varying one parameter at a time by $\pm 2.5\%$ while keeping the other parameters constant. The parameters have almost linear behavior for a $\pm 20\%$ variation, except in the region near the TE_1 resonance frequency [14].

The sensitivity ratios were calculated as relative ratios, which in turn formed a sensitivity matrix, ξ_{nm} , where n, m takes the values 1,2,3 for the high-frequency tuning stage, the values 1,2,3,4 for the low-frequency tuning stage and the values 1,2 for the superimposed mode tuning stage. As an example, the relative sensitivity ratio, ξ_{23} , for the high-frequency tuning stage, can be expressed as the ratio of the relative variation in target mode frequency f_{TE1} to the relative variation in parameter $\hat{e}_{33}$: $\xi_{23} = \frac{f_{TE1}^+ - f_{TE1}^-}{f_{TE1}} / \frac{\hat{e}_{33}^+ - \hat{e}_{33}^-}{\hat{e}_{33}}$. The superscript $+$ indicates the result of a positive increase in the value of $\hat{e}_{33}$. In this way, a set of target mode frequencies and a target impedance were determined for the PD20 disk. The complete version of ξ_{nm} for the PD20 disk can be seen in section IV.

It was decided not to incorporate the $\hat{e}_{11}$ parameter in the tuned parameter set. The main reason being that the disk was polarized in the thickness direction of the disk. To investigate the influence of $\hat{e}_{11}$, sensitivity ratios for $\hat{e}_{11}$ were determined for the high-frequency tuning stage where the grouped parameters for the dominant TE_1 resonance are determined (Section IV).

The method of subdividing the electro-elastic parameter set into three groups is based on the assumption that the variation in parameters in a group has a negligible effect on the target mode frequencies belonging to the foregoing groups in the sequence. For example, after determining the parameter set in the first high-frequency tuning stage, the variation of the parameters in the following two stages should not significantly affect the target mode frequencies and impedance in the first stage. To check for group cross-sensitivity, sensitivity ratios for the PD20 sample were calculated.

2) *Feedback algorithm for the determination of grouped electro-elastic parameters:* In the tuning of the electro-elastic parameters, a feedback algorithm was used to iteratively determine the grouped parameter values. The structure of the algorithm is outlined in Fig. 4 as a flowchart diagram forming the feedback loop. The basic mechanism of the loop is to modify the the grouped parameter values, input to the PSM, until the termination criteria is met. Below follows a description of the execution of the feedback loop for the first high-frequency tuning stage using the PD20 as template. The procedure is identical for the

low-frequency tuning stage and the superimposed mode tuning stage:

The loop is initialized by setting $\psi_n^{cl}|_w = \psi_n^{init}$ where the values of the elements of ψ_n^{init} are set to the values of the PD20 high-frequency tuning stage parameters ($\hat{e}_{33}$, $\hat{c}_{33}$, $\hat{e}_{33}$) and the loop counter w is set to zero.

Start of loop: the closed loop group parameter vector, $\psi_n^{cl}|_w$, is entered into the PSM which calculates the resulting group parameter response vector, $\gamma_n^{PSM}|_w$, where the elements of $\gamma_n^{PSM}|_w$ are the PSM calculated values of the target mode impedance and frequencies Z^h, TE_1, TE_2 .

The target relative difference vector, Δ_n^γ , is calculated by element-wise calculation of the relative difference between $\gamma_n^{PSM}|_w$ and γ_n^{meas} , where the elements of γ_n^{meas} is the measured target impedance and frequencies Z^h, TE_1, TE_2 . The loop error factor $\delta^{err} = \sqrt{(\Delta_1^\gamma)^2 + (\Delta_2^\gamma)^2 + \dots + (\Delta_n^\gamma)^2}$ is calculated. If δ^{err} is less than 10^{-3} , the loop terminates and the tuned group parameter values are the values of $\psi_n^{cl}|_w$, else, the loop continues with the calculation of the loop correction vector $\eta_n|_w$.

For each tuning stage, the PSM is regarded as a nearly linear system with the group parameter vector, ψ_n , as input. This assumption holds if the step in parameter variation is not too large. From the sensitivity analysis, the selected grouped parameter set for the stage are the parameters that show the greatest variation. Thus, a relative difference of target impedance and frequencies, Δ_n^γ , can be expressed as $\Delta_n^\gamma \approx \xi_{nm} \cdot \Delta_m^\psi$ where ξ_{nm} is the tuning stage sensitivity matrix and Δ_m^ψ is a one column vector of relative variations of the grouped parameters. From this, the parameter correction vector η_n can be expressed as $\eta_n = [1]_n + \kappa[\xi_{nm}]^{-1}\Delta_m^\gamma$ where $[1]_n$ is a column vector containing ones and κ is a feedback factor to ensure a smooth convergence of the parameter set ($\kappa = 0.6$). The $[\]^{-1}$ operation symbolizes matrix inversion. The purpose of η_n is to modify the elements of the loop grouped parameter vector so that after the next PSM calculation, the loop error factor δ^{err} decreases.

The next $(w + 1)$ th closed loop group parameter vector is formed by correction of the previous state group parameter vector: $\psi_n^{cl}|_{w+1} = \eta_n \times \psi_n^p|_w$, where ψ_n^p is the previous loop state group parameter vector and $\times$ indicate element by element multiplication. The loop counter is incremented by one ($w = 1$) and the next $\psi_n^p|_w$ is set to the corrected group parameter vector $\psi_n^{cl}|_w$. The loop is repeated by entering the start of loop point above.

For the second low-frequency tuning stage, the group parameter vector ψ_n^{init} is populated with elements containing values of parameters ($\hat{c}_{13}$, $\hat{c}_{12}$, $\hat{c}_{11}$, $\hat{c}_{31}$) and the PSM calculates the target mode frequencies ($E(R_{12})$, R_2 , R_1 , R_1^a). The low-frequency tuning stage sensitivity

matrix given in Section IV is used to calculate the parameter correction vector η_n .

The final tuning of the complete PSM parameter set is ended with the superimposed mode tuning stage. The group parameter vector ψ_n^{init} is populated with elements containing values of parameters ($\hat{c}_{44}$, $\hat{e}_{15}$) and the PSM calculates the target mode frequencies ($TE_1(R_6)$, $TE_1(E)$). The superimposed mode tuning stage sensitivity matrix given in Section IV is used to calculate the parameter correction vector η_n .

III. EXPERIMENTAL PROCEDURE

To verify the tuning method, three sets of 6mm piezoelectric disks, poled in the thickness direction, were acquired from a manufacturer (Morgan Electro Ceramics, Wrexham, UK). Each set contained ten disks with the same a/h ratio. The sets, of type PZT5A1, will hereafter be referred to as samples SA20, SA06, and SA02, with average a/h ratios of 19.64 ± 0.04 , 5.98 ± 0.02 , and 1.98 ± 0.01 , respectively. The average thickness of samples SA20, SA06, and SA02 was 2.03 ± 0.01 mm, 5.01 ± 0.03 mm, and 5.04 ± 0.01 mm, respectively. The average density of each sample was measured to: 7675 ± 42 kg/m³ for sample SA20, 7860 ± 63 kg/m³ for sample SA06, and 7743 ± 56 kg/m³ for sample SA02.

The electrodes of the disks, deposited by the manufacturer, were made of silver and had a thickness of less than 10 μ m [11]. The 0.1 mm diameter electric leads that connected the electrodes to the voltage source were soldered to the electrodes. The soldered connections were of semi-spherical shape with a radius less than 0.5 mm. Each connection was positioned at the edge of the electrode as shown in Fig. 2.

Cross-sections of the sample holder are shown in Fig. 5. The freely vibrating piezoelectric disk was supported by three 0.5 mm spring-loaded needles. The sample holder was oriented so that the plane of the needles coincided with the $z = 0$ plane. The measurements were carried out at room temperature with air as surrounding medium. A network analyzer (Agilent E5100A/B, Agilent, Santa Clara, CA) was configured to measure the real and imaginary parts of the impedance as a function of frequency.

The impedance of each piezoelectric disk was measured in two frequency sweeps, from 20 kHz to 1.4 MHz with a frequency resolution of 1.7 kHz, followed by a sweep from 1.4 MHz to 5.0 MHz with 5.0 kHz step resolution. The power input to the piezoelectric samples was set to -60 dBmW to ensure that the system was linear.

The measured data were corrected for instrumental scaling factors. The real part of the correction term was found by measuring the response of a reference resistor. The quadra-

ture correction term was measured with a parallel-plate reference capacitor. The reference components were mounted in the sample holder when measured. The values of the reference resistor and the reference capacitance were determined by a measuring bridge (Wayne Kerr B605, Wilmot Breeden Electronics, Sussex, UK). It was discovered that the sample holder and the connected electrical cables contributed a parasitic inductance of $L_S = 0.13 \mu\text{H}$. The parasitic reactance was subtracted from the imaginary part of the measured impedance. After completion of the measuring sequences, the data were transferred to a desktop computer for post-processing.

IV. RESULTS

Following the procedure for the PD20 disk, outlined in Section II-B1, two prototype disks, PD06 and PD02, with a/h ratios of 6 and 2 were constructed. The sensitivity ratios are presented in matrix form, ξ_{nm} , for the prototype disks PD20, PD06, and PD02 with the corresponding target mode frequencies and target impedance for the high-frequency tuning stage in Table II, for the low-frequency tuning stage in Table III, and for the superimposed mode tuning stage in Table IV. The value of the target impedance, Z^h , was averaged for all calculated frequencies in the frequency interval Δf^h .

It was decided not to incorporate the $\hat{e}_{11}$ parameter in the set of tuned parameters. To investigate the influence of the $\hat{e}_{11}$ parameter on the tuned high-frequency grouped parameters, sensitivity ratios were calculated (Table II). For the PD20 disk, the influence of the $\hat{e}_{11}$ parameter on the determination of the $\hat{e}_{33}$, $\hat{c}_{33}$ and $\hat{e}_{33}$ parameters were less than 0.2%, 0.2% and 0.4% respectively.

The group cross sensitivity ratios for the PD20 and PD02 prototype disks are presented in Table V and Table VI, respectively. The sensitivity ratios show that subdividing the electro-elastic parameter set into three almost independent groups is justified in the cases of PD20 and PD02 (PD06 ratios are not given here). For the the PD02 sample, the cross sensitivity ratios indicate that there is a non-negligible cross coupling between grouped parameters. Tuning of the PD02 sample is discussed in section V.

PSM models of the samples SA20, SA06 and SA02, were created. The electro-elastic parameters of the corresponding prototype disk, with the same a/h ratio, was selected as initial parameters. The tuning of the parameter set, for each sample, followed the stages described in Section II-B using the iterative feedback loop described in Section II-B2. The

number of loops needed for the termination criteria to be met varied between 4 to 7.

The PSM calculations were carried out in the discrete frequency step mode. The impedance was calculated in the frequency range 10 kHz to 1.4 MHz with 0.5 kHz frequency steps and 1.4 MHz to 5.0 MHz with 2 kHz frequency steps. The electrode surfaces were regarded as being infinitely thin with perfect electrical conductivity. The piezoelectric disks were assumed to be lossless. The boundary conditions were set according to Section II-A4. In the frequency range 10 kHz to 1.4 MHz the adaptive mesh mode was selected with the error limit set to 10^{-3} . In the range 1.4 MHz to 5.0 MHz both the adaptive mesh mode and the fixed grid were used to optimize the computational speed. The error limit was varied between 10^{-3} and 10^{-2} in the higher frequency range to achieve reasonable computational times. To identify resonance mode types from the mode shape plots, a fixed grid covering the problem area was used with the number of elements: 8×82 for sample SA20; 16×50 for sample SA06, and 16×30 for sample SA02. To identify target mode frequencies, a simplified resonance mode naming scheme was used. A resonance mode superimposed on another mode was named with the main characteristic mode and an identification mode within parentheses. For example, the mode denoted $TE_1(R_6)$ has a low-frequency mode shape of the first thickness extensional mode TE_1 with a superimposed mode that has the same characteristic surface wavelength as mode R_6 [5].

The results of the PSM calculations and measurements are presented in Fig. 6 and Fig. 7, where the measured impedance is plotted versus the frequency, together with the response calculated for the three samples using the PSM after tuning. The impedance is presented in decibel with one ohm as reference impedance. For increased readability, the graphs show only frequency intervals where resonances are present allowing direct comparison between the calculated and measured results. The extreme peak values in the calculated results are a consequence of the PSM being assumed lossless. The relative differences between the sample resonance frequencies calculated using the PSM and the measured frequencies are summarized in Table VIII.

The target mode frequencies are indicated in the frequency plots for sample SA20 in Fig. 6 and for sample SA06 and SA02 in Fig. 7. To compare the effect of tuning the electro-elastic parameters, smoothed impedance responses of the prototype disks are plotted in Fig. 6 for disk PD20 and in Fig. 7 for disks PD06 and PD02.

The initial prototype parameters differed from the final tuned parameters depending on the sample and type of parameter. The maximum deviation of the elastic, piezoelectric and

dielectric parameters was +6.3% , +11.7%, and -16.7% for sample SA20; -12.3%, -14.7%, and -42.9% for sample SA06; and -2.3%, +11.2%, and -5.7% for sample SA02.

The tuned parameters for the samples SA20, SA06, and SA02 are listed in Table VII together with the prototype parameters that served as the initial parameter set. To control the uncertainty in the parameter values while executing the algorithm, the PSM frequency step was adjusted. To relate the calculated parameter values to the uncertainty, an estimated uncertainty, Δ_u , based on the used frequency step, is presented with each parameter value in Table VII.

V. DISCUSSION

The FEM-based method presented in this work uses both fixed and adaptive meshing to optimize the trade-off between accuracy and computational time when performing the sensitivity calculations and the electro-elastic parameter iterations. Moreover, in comparison with earlier work, the frequency range of interest is wider, including the harmonic overtone resonances of the thickness extensional modes. This simplifies the determination of the parameters that govern the thickness mode resonances. The parameter tuning method presented in this paper therefore provides a valuable contribution to the concept.

When using the tuned electro-elastic parameters, the PSM impedance calculations for samples SA20 and SA06 are in excellent agreement with the measured impedances. However, the cross-sensitivity ratios for sample SA02 indicate that the method used to group parameters was not optimal for a disk with a small a/h ratio. The geometry of the disk allows only one radial and one edge resonance mode to exist. Hence, modes of increasing frequency are strongly affected by the first thickness extensional mode. The $TE_1(R_3)$ target mode frequency (Fig. 7), belonging to the high-frequency tuning stage, was affected by the low-frequency tuning stage ($\hat{c}_{13}$, $\hat{c}_{11}$) and the superimposed mode tuning stage ($\hat{c}_{44}$) parameter determinations, as can be seen in Table VI. The effect of this resonance mode cross-coupling between stages can be seen in Table VIII. The relative deviation for the mode $TE_1(R_3)$ in the SA02 sample is 2.03% while the other mode resonances deviate by at most 0.8%. Moreover, in the SA02 impedance response, shown in Fig. 7, one resonance mode was measured at 387 kHz and several modes in the 550-750 kHz frequency range that were not reproduced by the PSM model. The absence of these modes in the PSM response could be the result of considering only axisymmetric extensional modes and neglecting flexural modes.

The spread in the impedance calculated using the PSM, shown in Fig. 6 and Fig. 7, can be

understood in terms of a combination of the fact that the model is assumed lossless and the solver operating mode. As the solver iterates a unique solution for each discrete frequency, and if the frequency belongs to a region where the system exhibits instability (near a resonance), the defined error limit and that the model is lossless, may interact causing the solver to stop iterating and deliver a single offset value.

Although the tuning method does not determine the real physical electro-elastic constants, the method offers a simple and fast way of constructing accurate FEM models of thick piezoelectric disks.

VI. CONCLUSIONS

The work presented in this paper is part of an ongoing project to construct an electro-elastic model of piezoelectric tactile sensors in contact with soft tissue. The first step was to set up a frequency-stepped FEM model, the PSM, utilizing an adaptive meshing technique to calculate the frequency-dependent electric impedance of thick piezoelectric disks. To tune the effective electro-elastic PSM model parameters, in order to accurately reproduce the impedance response of real piezoelectric thick disks, a new method using harmonic overtones was developed. A comparison of the impedance calculations with the corresponding measured impedance showed excellent agreement. The average relative differences between the calculated and measured resonance frequencies were 0.27% for sample SA20, 0.19% for sample SA06, and 0.54% for sample SA02, while the maximum relative differences were 0.71% in mode R_9 for sample SA20, 0.84% in mode $E(R_4)$ for sample SA06, and 2.03% in mode $TE_1(R_3)$ for sample SA02.

APPENDIX

The elastic stiffness constant, c_{pq}^D , at constant electric flux density, D , can be expressed:

$$c_{pq}^D = \begin{bmatrix} c_{11}^D & c_{12}^D & c_{13}^D & 0 & 0 & 0 \\ c_{12}^D & c_{11}^D & c_{13}^D & 0 & 0 & 0 \\ c_{13}^D & c_{13}^D & c_{33}^D & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44}^D & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{44}^D & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{66}^D \end{bmatrix}, c_{66}^D = \frac{(c_{11}^D - c_{12}^D)}{2} \quad (A.1)$$

where:

$$\begin{aligned} c_{11}^D &= c_{11}^E + \frac{e_{31}^2}{\varepsilon_{33}^S}, & c_{12}^D &= c_{12}^E + \frac{e_{31}^2}{\varepsilon_{33}^S}, \\ c_{13}^D &= c_{13}^E + \frac{e_{31}e_{33}}{\varepsilon_{33}^S}, & c_{33}^D &= c_{33}^E + \frac{e_{33}^2}{\varepsilon_{33}^S}, \\ c_{44}^D &= c_{44}^E + \frac{e_{15}^2}{\varepsilon_{11}^S}. \end{aligned}$$

The piezoelectric constant, h_{kp} , is given by:

$$h_{kp} = \begin{bmatrix} 0 & 0 & 0 & 0 & h_{15} & 0 \\ 0 & 0 & 0 & h_{15} & 0 & 0 \\ h_{31} & h_{31} & h_{33} & 0 & 0 & 0 \end{bmatrix}, \quad (\text{A.2})$$

where:

$$h_{15} = \frac{e_{15}}{\varepsilon_{11}^S}, \quad h_{31} = \frac{e_{31}}{\varepsilon_{33}^S}, \quad h_{33} = \frac{e_{33}}{\varepsilon_{33}^S}.$$

The inverted dielectric permittivity constant, β_{ik}^S , at constant strain, S, is:

$$\beta_{ik}^S = \begin{bmatrix} \beta_{11}^S & 0 & 0 \\ 0 & \beta_{11}^S & 0 \\ 0 & 0 & \beta_{33}^S \end{bmatrix}, \quad (\text{A.3})$$

where:

$$\beta_{11}^S = \frac{1}{\varepsilon_{11}^S}, \quad \beta_{33}^S = \frac{1}{\varepsilon_{33}^S}.$$

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TABLE I
BOUNDARY VALUE CONDITIONS FOR THE PSM
PROBLEM AREA

Boundary	Type of condition		
Γ_0	natural(u_r)	natural(z)	value(φ)
Γ_1	natural(u_r)	natural(u_z)	natural(φ)
Γ_2	natural(u_r)	natural(u_z)	value(φ)
Γ_3	value(u_r)	natural(u_z)	natural(φ)

The four boundaries Γ_0 , Γ_1 , Γ_2 , and Γ_3 are located as in Fig. 3. All boundary value conditions natural(), value() are set to zero except for boundary Γ_2 , where value(φ) is set to the excitation voltage. At the origin (0,0) a single point value(u_r, u_z) was set to zero to allow only axisymmetric extensional mode vibrations.

TABLE II

HIGH-FREQUENCY TUNING STAGE PARAMETER SENSITIVITY MATRIX ξ_{nm} FOR PROTOTYPE DISKS PD20, PD06 AND PD02; TARGET MODE FREQUENCIES AND TARGET IMPEDANCE FOR SAMPLES SA20, SA06 AND SA02

	Target Imp./Mode	Freq. [kHz]	Imp. [dB]	Δf^h [kHz]	$\hat{\epsilon}_{33}$	$\hat{c}_{33}$	$\hat{e}_{33}$	$\hat{\epsilon}_{11}$
PD20/SA20	Z^h		27.24 ^a	1690-1720	$\xi_{11} = -0.353$	$\xi_{12} = 0.022$	$\xi_{13} = 0.036$	-0.017
	TE ₁	987.1			$\xi_{21} = -0.020$	$\xi_{22} = 0.405$	$\xi_{23} = 0.061$	-0.020
	TE ₂	3280.0			$\xi_{31} = -0.116$	$\xi_{32} = 0.366$	$\xi_{33} = 0.210$	0.006
PD06/SA06	Z^h		38.30 ^a	1650-1740	$\xi_{11} = -0.225$	$\xi_{12} = 0.019$	$\xi_{13} = 0.007$	-0.006
	TE ₁	411.4			$\xi_{21} = -0.024$	$\xi_{22} = 0.387$	$\xi_{23} = 0.044$	-0.015
	TE ₂	1316.6			$\xi_{31} = -0.091$	$\xi_{32} = 0.420$	$\xi_{33} = 0.167$	0.008
PD02/SA02	Z^h		54.53 ^a	2740-2780	$\xi_{11} = -0.169$	$\xi_{12} = 0.009$	$\xi_{13} = -0.029$	0.008
	TE ₁ (R ₃) ^b	341.0			$\xi_{21} = -0.061$	$\xi_{22} = 0.375$	$\xi_{23} = 0.064$	-0.038
	TE ₃	2171.8			$\xi_{31} = -0.099$	$\xi_{32} = 0.385$	$\xi_{33} = 0.156$	-0.018

The values are the ratios of the relative variations of target impedance Z^h and target mode frequencies to the relative variations in the high-frequency tuning stage parameters ($\hat{\epsilon}_{33}, \hat{c}_{33}, \hat{e}_{33}, \hat{\epsilon}_{11}$). ^a The average impedance of Z^h for all calculated frequencies in the frequency interval Δf^h . ^b Nomenclature explained in Section IV.

TABLE III

THE LOW-FREQUENCY TUNING STAGE PARAMETER SENSITIVITY MATRIX ξ_{nm} FOR PROTOTYPE DISKS PD20, PD06, AND PD02; TARGET MODE FREQUENCIES FOR SAMPLES SA20, SA06 AND SA02

	Target mode	Freq. [kHz]	$\hat{c}_{13}$	$\hat{c}_{12}$	$\hat{c}_{11}$	$\hat{e}_{31}$
PD20/SA20	E(R ₁₂) ^a	644.6	$\xi_{11} = -0.682$	$\xi_{12} = <0.004$	$\xi_{13} = 0.495$	$\xi_{14} = 0.020$
	R ₂	128.6	$\xi_{21} = -0.737$	$\xi_{22} = 0.040$	$\xi_{23} = 0.858$	$\xi_{24} = <0.002$
	R ₁	49.99	$\xi_{31} = -1.046$	$\xi_{32} = 0.325$	$\xi_{33} = 0.858$	$\xi_{34} = <0.004$
	R ₁ ^a	59.55	$\xi_{41} = -0.553$	$\xi_{42} = 0.0168$	$\xi_{43} = 0.813$	$\xi_{44} = 0.086$
PD06/SA06	E(R ₄) ^a	269.8	$\xi_{11} = -0.545$	$\xi_{12} = 0.035$	$\xi_{13} = 0.360$	$\xi_{14} = 0.014$
	R ₂	168.1	$\xi_{21} = -0.736$	$\xi_{22} = 0.035$	$\xi_{23} = 0.763$	$\xi_{24} = 0.001$
	R ₁	69.08	$\xi_{31} = -0.835$	$\xi_{32} = 0.281$	$\xi_{33} = 0.623$	$\xi_{34} = 0.003$
	R ₁ ^a	79.50	$\xi_{41} = -0.498$	$\xi_{42} = 0.174$	$\xi_{43} = 0.528$	$\xi_{44} = 0.081$
PD02/SA02	E(R ₂) ^a	269.8	$\xi_{11} = -0.668$	$\xi_{12} = 0.072$	$\xi_{13} = 0.409$	$\xi_{14} = 0.034$
	TE ₁ (R ₄) ^a	419.2	$\xi_{21} = -0.260$	$\xi_{22} = 0.002$	$\xi_{23} = 0.274$	$\xi_{24} = 0.022$
	R ₁	175.9	$\xi_{31} = -1.108$	$\xi_{32} = 0.295$	$\xi_{33} = 0.637$	$\xi_{34} = 0.003$
	R ₁ ^a	208.9	$\xi_{41} = -0.686$	$\xi_{42} = 0.143$	$\xi_{43} = 0.545$	$\xi_{44} = 0.095$

The values are the ratios of the relative variations in the target mode frequencies to the relative variations in the low-frequency tuning stage parameters ($\hat{c}_{11}, \hat{c}_{12}, \hat{c}_{13}, \hat{e}_{31}$).

R₁^a is the anti-resonance mode of R₁. ^a Nomenclature explained in Section IV.

TABLE IV
 THE SUPERIMPOSED MODE TUNING STAGE PARAMETER
 SENSITIVITY MATRIX ξ_{nm} FOR PROTOTYPE DISKS PD20, PD06,
 AND PD02; TARGET MODE FREQUENCIES FOR SAMPLES SA20,
 SA06 AND SA02

	Target mode	Freq. [kHz]	$\hat{c}_{44}$	$\hat{e}_{15}$
PD20/SA20	$TE_1(R_6)^a$	1153.1	$\xi_{11} = 0.125$	$\xi_{12} = 0.043$
	$TE_1(E)^a$	1044.5	$\xi_{21} = 0.106$	$\xi_{22} = 0.058$
PD06/SA06	$TS_1(R_7)^a$	340.1	$\xi_{11} = 0.282$	$\xi_{12} = -0.023$
	$TE_1(R_9)^a$	484.3	$\xi_{21} = 0.161$	$\xi_{22} = 0.045$
PD02/SA02	$TE_2(R_7)^a$	1255.8	$\xi_{11} = 0.236$	$\xi_{12} = 0.064$
	$TE_2(R_5)^a$	1198.5	$\xi_{21} = 0.108$	$\xi_{22} = 0.125$

The values are the ratios of the relative variations in target mode frequencies to the relative variations in the superimposed mode tuning stage parameters ($\hat{c}_{44}, \hat{e}_{15}$). ^a Nomenclature explained in Section IV.

TABLE V
PROTOTYPE DISK PD20 GROUP CROSS-SENSITIVITY RATIOS CSR1 AND CSR2

	Target mode	Target imp.	$\hat{c}_{13}$	$\hat{c}_{12}$	$\hat{c}_{11}$	$\hat{e}_{31}$	$\hat{c}_{44}$	$\hat{e}_{15}$
CSR1 ^a		Z^h	-0.006	-0.003	0.007	0.002	-0.002	-0.001
	TE ₁		0.017	<0.002	-0.006	0.010	0.034	0.018
	TE ₂		0.006	<0.006	0.012	-0.012	0.012	<0.006
CSR2 ^b	E						<0.002	< 0.002
	R ₂						<0.002	< 0.002
	R ₁						<0.004	< 0.004
	R ₁ ^a						<0.003	< 0.003

^a The values are the relative variations in target impedance, Z^h , and target mode frequencies (TE₁, TE₂), belonging to the high-frequency tuning stage, to the relative variations of parameters ($\hat{c}_{13}, \hat{c}_{12}, \hat{c}_{11}, \hat{e}_{31}$) and parameters ($\hat{c}_{44}, \hat{e}_{15}$) belonging to the low-frequency tuning stage and the superimposed mode tuning stage, respectively.

^b The values are the relative variations in target mode frequencies (E, R₂, R₁, R₁^a) to the relative variations in the superimposed mode tuning stage parameters ($\hat{c}_{44}, \hat{e}_{15}$). R₁^a is the anti-resonance mode of R₁.

TABLE VI
PROTOTYPE DISK PD02 GROUP CROSS-SENSITIVITY RATIOS CSR1 AND CSR2

	Target mode	Target imp.	$\hat{c}_{13}$	$\hat{c}_{12}$	$\hat{c}_{11}$	$\hat{e}_{31}$	$\hat{c}_{44}$	$\hat{e}_{15}$
CSR1 ^a		Z^h	-0.011	-0.001	0.004	0.007	0.014	-0.028
	$\text{TE}_1(\text{R}_3)^c$		-0.459	0.039	0.367	0.042	0.102	0.043
	TE_3		-0.016	0.002	0.023	-0.009	0.016	-0.014
CSR2 ^b	$\text{E}(\text{R}_2)^c$						0.037	-0.022
	$\text{TE}_1(\text{R}_4)^c$						0.060	0.048
	R_1						<0.001	<0.001
	R_1^a						<0.001	<0.001

^a The values are the relative variations in target impedance, Z^h , and target mode frequencies $\text{TE}_1(\text{R}_3)$ and TE_3 , belonging to the high-frequency tuning stage, to the relative variations of parameters ($\hat{c}_{13}, \hat{c}_{12}, \hat{c}_{11}, \hat{e}_{31}$) and parameters ($\hat{c}_{44}, \hat{e}_{15}$) belonging to the low-frequency tuning stage and the superimposed mode tuning stage, respectively.

^b The values are the relative variations in target mode frequencies $\text{E}(\text{R}_2)$, $\text{TE}_1(\text{R}_4)$, R_1 , and R_1^a to the relative variations in the superimposed mode tuning stage parameters ($\hat{c}_{44}, \hat{e}_{15}$). R_1^a is the anti-resonance mode of R_1 .

^c Nomenclature explained in Section IV.

TABLE VII
 PROTOTYPE DISK MATERIAL CONSTANTS (PZT5A) AND TUNED ELECTRO-ELASTIC
 PARAMETERS OF SAMPLES SA20, SA06, AND SA02 WITH ESTIMATED UNCERTAINTY Δ_u

	PZT5A	SA20	Δ_u	SA06	Δ_u	SA02	Δ_u
$\hat{c}_{11}$ [$10^{10} N/m^2$]	11.5	12.18	± 0.148	12.81	± 0.032	13.09	± 0.053
$\hat{c}_{12}$	7.54	7.776	± 0.066	8.075	± 0.065	6.284	± 0.024
$\hat{c}_{13}$	7.52	7.853	± 0.061	8.013	± 0.028	7.677	± 0.026
$\hat{c}_{33}$	11.1	11.80	± 0.006	12.40	± 0.025	11.41	± 0.028
$\hat{c}_{44}$	2.11	2.098	± 0.014	2.240	± 0.049	2.159	± 0.020
$\hat{e}_{31}$ [C/m^2]	-5.4	-5.759	± 0.104	-5.670	± 0.093	-3.086	± 0.052
$\hat{e}_{15}$	12.3	10.78	± 0.222	10.49	± 0.140	8.854	± 0.195
$\hat{e}_{33}$	15.8	16.39	± 0.021	14.97	± 0.068	15.25	± 0.052
$\epsilon_{11}^b/\epsilon_0$	916	916	-	916	-	916	-
$\hat{\epsilon}_{33}/\epsilon_0$	830	811.1	± 0.972	923.3	± 1.04	782.8	± 0.289
ρ^b [kg/m^3]	7750	7750	-	7750	-	7750	-

^b Parameter not included in the tuning process. $\epsilon_0 = 8.854 \times 10^{-12}$ F/m.

TABLE VIII

THE RELATIVE DEVIATION Δ_d BETWEEN THE TARGET MODE RESONANCE FREQUENCIES AND THE MEASURED RESONANCE FREQUENCIES OF SAMPLES SA20, SA06, AND SA02

SA20			SA06			SA02		
Freq. [kHz]	Mode	Δ_d [%]	Freq. [kHz]	Mode	Δ_d [%]	Freq. [kHz]	Mode	Δ_d [%]
49.99	R_1	0.03	69.08	R_1	0.11	175.06	R_1	0.04
59.55	R_1^a	0.08	79.50	R_1^a	0.01	268.89	$E(R_2)$	0.78
128.22	R_2	0.22	168.11	R_2	0.07	341.00	$TE_1(R_3)^{a,b}$	2.03
548.09	$R_9^{a,b}$	0.71	269.76	$E(R_4)^{a,b}$	0.84	418.31	$TE_1(R_4)^b$	0.31
644.58	$E(R_{12})^b$	0.32	340.10	$TS_1(R_7)^b$	0.03	1200.19	$TE_2(R_5)^b$	0.10
987.52	TE_1	0.05	411.36	TE_1	0.09	1254.05	$TE_2(R_7)^b$	0.12
1044.50	$TE_1(E)^b$	0.19	484.34	$TE_1(R_9)^b$	0.03	1300.96	TE_2	0.69
1153.10	$TE_1(R_6)^b$	0.18	1316.6	TE_2	0.12	2174.00	TE_3	0.28
3280.00	TE_2	0.03						
$\overline{\Delta_d}^c$		0.27			0.19			0.54

R_1^a is the anti-resonance mode of R_1 .

^a The mode with maximum Δ_d for each sample.

^b Nomenclature explained in Section IV.

^c $\overline{\Delta_d}$ denotes the average value of the deviation.

Fig. 1. Example of the mode shape of a piezoelectric circular disk, the area of which is equal to half the cross-sectional area of the disk. The unstructured mesh was generated using the adaptive mesh refinement technique. The displacements in the radial and axial directions are amplified to illustrate the variations in the mesh density.

Fig. 2. A piezoelectric disk of thickness $2h$ and diameter $2a$ oriented in the two coordinate systems (x_1, x_2, x_3) and (r, θ, z) . The flat surfaces are covered by electrode material. The disk is electrically driven by a voltage source V connected to the electrodes.

Fig. 3. The PSM problem area defined as half the vertical cross-sectional area of the cylindrical disk. The boundaries Γ_2 and Γ_0 are in the planes of the two electrodes.

Fig. 4. Flowchart version of the parameter identification feedback loop for tuning of grouped parameter values. GPV is short for group parameter vector. ψ_n^{cl} - closed loop GPV, ψ_n^{init} - initial GPV (start value), ψ_n^p - previous state GPV, γ_n^{meas} - measured target mode frequency and impedance response vector, γ_n^{PSM} - PSM-calculated target mode frequency and impedance response vector, Δ_n^γ - relative difference vector, ξ_{nm} - tuning stage sensitivity matrix, κ - feedback factor, η_n - correction vector, $\times$ - element by element multiplication, δ^{err} - loop error factor. n, m take the values 1,2,3 for the high-frequency tuning stage, the values 1,2,3,4 for the low-frequency tuning stage and the values 1,2 for the superimposed mode tuning stage.

Fig. 5. Cross-sections of the sample holder: (a) piezoelectric cylindrical disk; (b) electrodes with soldered wire connections; (c) PVC sample holder; (d) 0.5 mm spring-loaded needles.

Fig. 6. Measured and calculated impedance response of the SA20 sample and the PD20 prototype disk. Filled circles - Tuned PSM-calculated results; open circles - measured data; solid line - smoothed prototype disk, PD20, result. (a) The low-frequency tuning stage target modes R_1 , R_2 , R_1^a and $E(R_{12})$. (b) The superimposed mode tuning stage target modes $TE_1(E)$ (enlarged in the inset), $TE_1(R_6)$ and the high-frequency tuning stage target mode TE_1 . (c) The high-frequency tuning stage target mode TE_2 . The nomenclature is explained in Section IV and the numerical values are given in Section IV.

Fig. 7. Measured and calculated impedance responses of the SA06/PD06 and SA02/PD02 samples/prototype disks. Filled circles - Tuned PSM-calculated results; open circles - measured data; solid line - smoothed result for prototype disk PD6 in (a), (b) and PD2 in (c), (d). (a) Sample SA06: the low-frequency tuning stage target modes R_1 , R_1^a , R_2 , $E(R_4)$; high-frequency tuning stage target mode TE_1 ; the superimposed mode tuning stage target modes $TE_1(R_9)$ and $TS_1(R_7)$ (enlarged in the inset). (b) Sample SA06: the high-frequency tuning stage target mode TE_1 . (c) Sample SA02: the low-frequency tuning stage target modes R_1 , R_1^a , $E(R_2)$ and $TE_1(R_4)$; the high-frequency tuning stage target mode $TE_1(R_3)$; the superimposed mode tuning stage target modes $TE_2(R_5)$ and $TE_2(R_7)$. (d) Sample SA02: the high-frequency tuning stage target mode TE_3 . The nomenclature is explained in Section IV and the numerical values are given in Section IV.