Umeå University

This is an accepted version of a paper published in *Physics essays*. This paper has been peer-reviewed but does not include the final publisher proof-corrections or journal pagination.

Citation for the published paper:

Wennerström, H., Westlund, P. (2013)

"On Stern-Gerlach coincidence measurements and their application to Bell's theorem"

Physics essays, 26(2)

Access to the published version may require subscription.

Permanent link to this version:

<http://urn.kb.se/resolve?urn=urn:nbn:se:umu:diva-64735>

<http://umu.diva-portal.org>

On Stern-Gerlach coincidence measurements and their application to Bell's theorem

By

Håkan Wennerström, Division of Physical Chemistry

Chemical Center, P.O.Box 124,

University of Lund, SE 22100 Lund, Sweden

and

Per-Olof Westlund, Department of Chemistry:

Biological and Computational Chemistry,

Umeå University, 901 87 Umeå, Sweden

Corresponding authors: P.-O. Westlund, Email: perolof.westlund@chem.umu.se ;

H. Wennerstöm Email: Hakan.wennerstrom@fkem1.lu.se

Keywords:

Stern-Gerlach device, coincident measurement, entangled state, spin relaxation, Bell's theorem

Abstract

We analyze a coincidence Stern-Gerlach measurement often discussed in connection with the derivation and illustration of Bell's theorem. The treatment is based on our recent analysis of the original Stern-Gerlach experiment (PCCP, 14, 1677-1684 (2012)), where it is concluded that it is necessary to include a spin relaxation process to account for the experimental observations. We consider two limiting cases of a coincidence measurement using both an analytical and a numerical description. In one limit relaxation effects are neglected. In this case the correlation between the two spins present in the initial state is conserved during the passage through the magnets. However, at exit the z coordinate along the magnetic field gradient is randomly distributed between the two extreme values. In the other limit T_2 relaxation is assumed to be fast relative to the time of flight through the magnet. In this case the z coordinate takes one of two possible values as observed in the original Stern-Gerlach experiment. Due to the presence of a relaxation process involving transfer of angular momentum between particle and magnet the initially entangled spin state changes character leading to a loss of correlation between the two spins. In the original derivations of Bell's theorem based on a coincidence Stern-Gerlach setup one assumes both a perfect correlation between the spins and only two possible values for the z -coordinate on exit. According to the present calculations one can satisfy either of these conditions but not both simultaneously.

Resumé

Le théorème de Bell est le plus souvent considéré pour l'interprétation de l'expérience de Stern et Gerlach. Notre analyse s'inspire d'une de nos récentes études (PCCP, 14, 1677-1684 (2012)) démontrant l'importance des processus de relaxation de spin dans le phénomène observé par Stern et Gerlach. Nous examinons la mesure en coïncidence dans deux cas limites en nous appuyant sur une description analytique et numérique. Dans le premier cas, où les effets de relaxation sont négligés, la corrélation entre les deux spins présents à l'état initial est préservée lors du passage au travers du champ magnétique. Néanmoins, les particules se retrouvent à leur sortie du champ, distribuées de façon aléatoire suivant la direction z définie par l'orientation du champ magnétique. Dans le second cas, assumant une relaxation T_2 rapide comparativement au temps de vol au travers de l'aimant, les particules se distribuent suivant seulement deux valeurs de z similairement à l'expérience de Stern et Gerlach. L'état intriqué original du spin change cependant en raison d'un processus de relaxation impliquant un transfert de moment angulaire entre aimant et particule ayant pour conséquence une diminution de la corrélation entre les deux spins. En comparaison avec l'approche du théorème de Bell, qui assume une corrélation parfaite entre les spins ainsi que deux valeurs possibles pour z à la sortie du champ, nos calculs, quant à eux, démontrent que ces deux conditions peuvent être satisfaites séparément, mais pas simultanément.

1 Introduction

Since the original publication 1922 [1] the Stern-Gerlach experiment has been used to provide a concrete illustration of the concept of spins [2]. More than 30 years later Bohm [3] introduced the coincidence Stern-Gerlach gedanken experiment triggered by his interest in hidden variable theories. The arguments were further developed by Bohm and Arahonov [4]. The basic idea was that if a system of two spin particles initially in a state with zero combined spin angular momentum disintegrated into two separated particles their spin states would be correlated. Thus a measurement on the spin state on one particle provides information on the spin state of the other. In the early 1960-ties the coincident determination of spin states using Stern-Gerlach method has been extensively discussed by Bell at conferences presentations and meetings. These discussions lead to the formulation of Bell's theorem, although Bell apparently never published a formal derivation of the theorem [5]. A lucid account of the formal argument applied for Stern-Gerlach case can be found in refs. [6]-[8]. In more recent years Bell's theorem has been extensively applied and discussed in relation to coincidence measurements of double photon decays[9], [10]. However, the present paper is entirely focused on coincidence measurements on spin $1/2$ -particles using the Stern-Gerlach device and how Bell's theorem can be applied to this particular case.

Two spin particles emanating from a common initial $S=0$ state pass separate Stern-Gerlach magnets. Due to the entangled initial state there is a correlation between the observations in the two magnets. Based on Bohms original conclusions about the correlations an application of Bell's theorem leads to the remarkable conclusion that the outcome of the measurement in one of the magnets depends on the orientation of the other magnet. The existence of such an interdependence has far-reaching philosophical implications [11].

In the Stern-Gerlach experiment particles with mass m , and spin S , with magnetogyric ratio γ_S , passes through a magnet with a field gradient, β , so that the coupling between the field and the spin angular momentum produces a force on the particle in a direction perpendicular to the main direction of propagation. The observation is that on exiting the magnet, after a time of flight t_f , the particle deviates from the original straight trajectory in the direction along the magnetic field. For a particle with spin quantum number $S=1/2$ the observed deviation amounts to

$$\Delta z_{\pm} = \pm \frac{\hbar \gamma_S \beta t_f^2}{2m}, \quad (1)$$

and this corresponds to spin states $S_z = \frac{1}{2}$ and $S_z = -\frac{1}{2}$, respectively for the particles as they leave the magnet. In a coincidence measurement two particles from a common known source pass through two separate Stern-Gerlach magnets labeled 1 and 2. For the case with $S=1/2$ there are four different possible results of a combined experiment; $S_z^{(1)} = \pm \frac{1}{2}$; $S_z^{(2)} = \pm \frac{1}{2}$. To characterize a series of coincidence measurements it is convenient to define a correlation coefficient

$$\chi = 4 \langle S_z^{(1)} S_z^{(2)} \rangle. \quad (2)$$

When $\chi = -1$ there is a perfect (anti)correlation between the two states as the particle leave the magnet, while for $\chi = 0$ there is no correlation. Bohm conclude that for similar oriented magnets the correlation should be -1. This would follow directly if $S_z(tot) = S_z^{(1)} + S_z^{(2)}$ was a constant of the motion, as it would be if one only considers the z-component of the field. However, Bohm realized that there are also other components to the interaction between the magnet and the spin. He analyzed one of these contributions and concluded that it was of negligible consequence [3], [12]. Later researches, like Bell, have trusted Bohm's call on these issues and based their arguments on $S_z(tot)$ being a

constant of the motion in practice, if not in strict theory. Thus in the existing literature discussions of the Stern-Gerlach coincidence experiments one uses the value $\chi = -1$.

Recently we presented a detailed model of the original Stern-Gerlach measurement [13]. Thus we were able to, for the first time, account for the experimental result in quantitative detail using explicit equations of motion for both the translational and spin degrees of freedom. Our main point was that in the standard analysis of the experiment one has overlooked the significance of spin relaxation processes during the time the particle with spin travels through the magnet. The conclusion was that the observations of Stern and Gerlach could only be accounted for if the transverse T_2 relaxation time was shorter than the time of flight in the magnet. A T_2 relaxation process implies a transfer of angular momentum between the particle and the magnet. To illustrate the effect of T_2 -relaxation we preformed simulations of particle trajectories using the equation of motion for the spin containing terms for the deterministic Zeeman interaction with the local average field and terms for the stochastic relaxation. In the present paper we extend the analysis to the case of coincidence measurements.

Fig.1 The experimental setup for coincidence measurement using two Stern-Gerlach devices at different distances L_1 and L_2 from the source. In this configuration the angle between the two SG-magnets is set to zero, $\gamma = 0$.

2 Model system

In line with previous discussions of Stern-Gerlach coincidence measurements we consider an initial state of spherical (S) symmetry with respect to both position and spin degrees of freedom. At time $t = 0$ the system is at rest but it decomposes into two identical spin one half non-charged particles. In practice it is difficult to realize such an initial state and this is presumably the reason why, to our knowledge, such Stern-Gerlach coincidence measurements have so far not been reported in the literature. The setup, illustrated in Fig. 1, also involves two aligned Stern-Gerlach magnets on opposite sides of the sample emitting the particle. In the general case, the main, z-direction, of the two magnets form an angle γ , but we will mainly discuss the case $\gamma = 0$. Between the sample and the magnet there are slits ensuring that only particles propagation along the main axis, to given accuracy, enters the magnets.

We describe the dynamics of the system with explicit equations of motion for both the translational and spin degrees of freedom. As in our previous analysis of the original Stern-Gerlach experiment [13] the translational motion is described by Newtonian mechanics, while the spin dynamics is formulated in terms of the propagation of the spin density operator within a Liouville formalism.

At time $t = 0$ the system, initially at rest disintegrates with an excess energy E_0 so that the separating particles acquire a momentum $p_0 = \sqrt{mE_0}$. Using a constraint particle pairs are selected to travel along

the x -axis. The particles will continue on a straight trajectory until they at $t = t_e$ enter the S-G magnet. For the effect of the spin on the translation we consider only the z -component of the magnetic field and

$$B \approx (0,0,B_0 + z\beta) \quad (3)$$

where we have neglected stray fields present as the particle enters the magnet as well as the gradient along the y -direction. Due to the presence of the field gradient there is a force

$$F_z^i(t) = \hbar\gamma_S\beta \langle S_z^{(i)}(t) \rangle \quad (4)$$

acting in the z -direction on particle i . From classical mechanics it follows that the position of the particles travelling along the positive x -axis is

$$\vec{r}_+(t) = (\frac{p_0 t}{m}, 0, \int_0^t v_z(\tau) d\tau) \quad (5)$$

The velocity v_z in the z -direction is determined by the force through

$$\frac{dv_z^i}{dt} = \frac{F_z^i(t)}{m} \quad (6)$$

The magnitude of this force depends on the spin state as follows from eq.(4). It is a central theme of the present paper to determine the dynamics of the spin system in order to calculate the force $F_z^i(t)$. This will determine the positions of the particles as they leave the S-G magnet, which constitute the primary observable in the experiment.

3 The spin dynamics

Seen microscopically the spins of the particles interact with the (permanent) magnet through a magnetic dipolar coupling with its spins. In a more macroscopic view this dipolar coupling can be seen as consisting of a Zeeman coupling to a magnet field B and fluctuating component caused by the thermal excitations of the spin system of the magnet. The equation of motion of the spin density operator $\hat{\rho}(t)$ can then be written [13]:

$$\frac{d}{dt}\hat{\rho}(t) = -i(L_0 + \Delta L(t))\hat{\rho}(t) \approx -\frac{i}{\hbar}[H_Z, \hat{\rho}(t)] + R\hat{\rho}(t) \quad (7)$$

Where the Hamiltonian H_Z is

$$H_Z = -\hbar\gamma_S \vec{B}(\vec{r}) \left(S_z^{(1)} + S_z^{(2)} \right) \quad (8)$$

Lacking detailed information on the states of all spins in the magnet the relaxation process is best described as a stochastic process. For the case when $\hat{\rho}(t)$ represents the state of a single pair of particles, rather than a statistical ensemble of such particles, the relaxation superoperator R in eq. (7) is stochastic in nature[13]. Below we will first describe the spin dynamics analytically for the two limiting cases of negligible relaxation (3.1) and very rapid T_2 –relaxation (3.2). In the following section we give an account of explicit simulations of the combined spin and translational equations of motion.

3.1 Spin dynamics due to Zeeman interaction only

In the absence of relaxation the spin dynamics is completely determined by the first term on the right hand side of eq.(7). By assumption the initial $t=0$ state is a, spherically symmetric, singlet state represented by the spin state vector

$$|\varphi_B\rangle = \frac{1}{\sqrt{2}}\{|\alpha\rangle_1 |\beta\rangle_2 - |\beta\rangle_1 |\alpha\rangle_2\} \quad (9)$$

Due to the spherical symmetry the spin states can be defined with respect to any reference direction. The corresponding density operator can be represented by the 4×4 density matrix

$$\rho(0) = \frac{1}{2} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (10)$$

It follows from the Hamiltonian eq.(8) that both operators $S_z^{(1)}$ and $S_z^{(2)}$ are constants of the motion.

For the initial state one has

$$\langle S_z(0) \rangle \equiv \langle S_z^{(1)}(0) \rangle + \langle S_z^{(2)}(0) \rangle = 0 \quad (11)$$

and consequently

$$\langle S_z^{(1)}(t) \rangle = -\langle S_z^{(2)}(t') \rangle \quad (12)$$

for all $t, t' \geq 0$. When a particle at $t = t_e$ enters the magnetic field the Zeeman interaction will affect the x and y components of the spin as in a Larmor precession. The density matrix then takes the form

$$\rho(t > t_e) = \frac{1}{2} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -e^{i(\delta\tau)} & 0 \\ 0 & -e^{-i(\delta\tau)} & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (13)$$

in the basis where $|\alpha\rangle$ and $|\beta\rangle$ are defined with respect to the z-direction. The angle $\delta\tau \equiv (\omega_0^{(1)} - \omega_0^{(2)})\tau$ rotates at a rate determined by the Larmor frequencies. Such a density matrix represents a mixed singlet ($S = 0, S_z = 0$) and triplet ($S = 1, S_z = 0$) state. That the triplet state is mixed into the original $S=0$ state is due to the fact that the particle permutation symmetry is broken when the particles enter non-equivalent magnets [14]. For both the initial, eq.(10) and the final, eq. (13) density matrices the expectation value of the correlation function in eq.(2) is

$$\chi = 4 \langle S_z^{(1)} S_z^{(2)} \rangle = -1 \quad (14)$$

since the off diagonal elements have no influence on this particular expectation value for the given basis.

By combining eqs.(3) and (13) it follows that for $t \geq t_e$ there is a constant force F_z acting on a particle.

By simple integration the z-coordinate on exiting the magnet at $t = t_e + t_f$ is

$$z(exit) = \hbar\gamma_S\beta t_f^2 \frac{\langle S_z^{(i)}(t) \rangle}{m} \quad (15)$$

However, the value of $\langle S_z^{(i)}(0) \rangle$ is not specified by the density matrix of eq.(10), only the condition of eq.(11). The initial state is spherically symmetrical and there is thus no orientational preference in this state. It follows that the expectation value can take any value allowed by an isotropic distribution.

Thus the expectation value can be expressed as

$$\langle S_z^i \rangle = \text{tr} \left\{ S_z^i R^{-1}(\theta, \varphi) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right\} R(\theta, \varphi) \quad (16)$$

Here $R(\theta, \varphi)$ is the rotation operator for a spin 1/2 particle and the angles (θ, φ) are taken from an isotropic distribution. From eq.(16) we find

$$\langle S_z^{(1)} \rangle = \left(\frac{1}{2} \right) \cos(\theta). \quad (17)$$

where all $\cos(\theta)$ occur with equal probability. When eq.(17) is valid for one of the particles then according to eq.(12)

$$\langle S_z^{(2)} \rangle = -\left(\frac{1}{2} \right) \cos(\theta). \quad (18)$$

for the other particle (2) in the same pair.

For an isotropic distribution of the expectation value $\langle S_z^{(1)} \rangle$ the calculated position for particle 1 on exit is evenly distributed between the extreme values given by eq.(1). For particle 2 the exit z-coordinate has the same value but with opposite sign according to eq.(18). This is not consistent with the observations in the original S-G experiment [1],[13] .

The original analysis of the S-G coincidence measurement by Bohm, Bell and others was based on the assumption that only the two exit coordinates given in eq.(1) are possible. Thus even if the exit z-coordinate is the primary observable there is a one to one correspondence with spin states $S_z = \mp 1/2$. However, for the case when the exit z-coordinate can take a range of values as found above it is useful to introduce the correlation function

$$\chi' = \frac{\langle z_+(exit) z_-(exit) \rangle}{\Delta z_+^2}. \quad (19)$$

For a system where one has the S-G behavior the two correlation functions χ and χ' are equal, while this is not the case when the $z(exit)$ is continuously distributed between the two extreme values. For the case where the spin dynamics is solely determined by the Zeeman interaction it follows from eqs (15), (17) and (18) that

$$\chi' = -\frac{1}{2} \int_{-1}^1 \cos(\theta)^2 d\cos(\theta) = -\frac{1}{3} \quad (20)$$

In contrast to the value $\chi=-1$ in eq.(14).

3.2 Spin dynamics with relaxation

Thermal excitations of the spin system in the magnet can cause transfer of angular momentum between the particles in the gap of the magnet and spins of the magnet itself. The most efficient process is the T_2 -relaxation. For the limiting case when T_2 is much shorter than the time of flight, t_f , in the magnet there will be a change in the density operator due to a relaxation process as the particle encounters the magnet at $t \approx t_e$.

To illustrate the time evolution of the density operator consider the case when the particle that propagates along the positive x-axis meets the magnet first so that $t_e^{(1)} < t_e^{(2)}$. A T_2 relaxation for particle 1 implies that $\langle S_x^{(1)} \rangle = 0$. With the further condition that the density operator has to describe a pure state the density matrix for $t_e^{(2)} > t > t_e^{(1)}$ takes one of two possible forms

$$\rho(t_e^{(2)} > t > t_e^{(1)}) = \begin{bmatrix} \rho_{\alpha\alpha}^{(2)} & \rho_{\alpha\beta}^{(2)} & 0 & 0 \\ \rho_{\beta\alpha}^{(2)} & 1 - \rho_{\alpha\alpha}^{(2)} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (21)$$

or

$$\rho(t_e^{(2)} > t > t_e^{(1)}) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \rho_{\alpha\alpha}^{(2)} & \rho_{\alpha\beta}^{(2)} \\ 0 & 0 & \rho_{\beta\alpha}^{(2)} & 1 - \rho_{\alpha\alpha}^{(2)} \end{bmatrix} \quad (22)$$

In this case $S_z^{(1)}$ is not a constant of motion due to the stochastic relaxation process. The result of the relaxation process depends on the initial value $\langle S_z^{(1)}(0) \rangle$. The probabilities [13]

$$P(\alpha^{(1)}) = \frac{1}{2} + \langle S_z^{(1)}(0) \rangle; \quad P(\beta^{(1)}) = \frac{1}{2} - \langle S_z^{(1)}(0) \rangle \quad (23)$$

determine which of the alternatives eq.(21) or eq.(22) is realized. When the second particle reaches its magnet at $t = t_e^{(2)}$ there is, in the limit considered, a second similar stochastic T_2 relaxation process.

This eliminates all off diagonal elements of the density matrix. The combined result of the two relaxation processes is that the density matrix can take one of four different forms at $t > t_e^{(2)}$:

$$\rho_{\alpha\alpha}^{(1)} = 1, \rho_{\alpha\alpha}^{(2)} = 1; \quad \rho = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (24)$$

$$\rho_{\alpha\alpha}^{(1)} = 1, \rho_{\beta\beta}^{(2)} = 1; \rho = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (25)$$

$$\rho_{\beta\beta}^{(1)} = 1, \rho_{\alpha\alpha}^{(2)} = 1; \rho = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (26)$$

$$\rho_{\alpha\alpha}^{(1)} = 1, \rho_{\beta\beta}^{(2)} = 1; \rho = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (27)$$

For a given value of $\langle S_z^{(1)}(0) \rangle = -\langle S_z^{(2)}(0) \rangle$ it follows from eq.(23) that these four alternatives occur for a given pair of particles with probabilities

$$P(\alpha, \alpha) = \frac{1}{4} - \langle S_z^{(1)}(0) \rangle^2, \quad P(\alpha, \beta) = \left(\frac{1}{2} - \langle S_z^{(1)}(0) \rangle\right)^2, \\ P(\beta, \alpha) = \left(\frac{1}{2} + \langle S_z^{(1)}(0) \rangle\right)^2, \quad P(\beta, \beta) = \frac{1}{4} - \langle S_z^{(1)}(0) \rangle^2, \quad (28)$$

Since the spins align along the magnetic field directly on entry there is a constant force acting on the particles and when leaving the magnet they have acquired a position as given in eq.(1) and observed in a conventional S-G experiment. As in the absence of relaxation the average value of the correlation function χ is obtained by considering all values of $\cos(\theta)$ as equally probable and by averaging over all directions

$$\chi = 4 \left\{ \langle S_z^{(1)} \rangle \langle S_z^{(2)} \rangle \right\} = -1/3 \quad . \quad (29)$$

In this case, the two correlations functions χ and χ' are the same. The final result is thus the correlation function χ' is the same both in presence and absence of relaxation, while there is a difference in χ .

4 Trajectory simulation

In section 3 we analyzed two limiting cases where it is possible to obtain analytical results for the correlation function. To further substantiate the conclusions we have performed simulations of particle trajectories using the same methodology as in our previous paper [13].

The central features are that the translational motion of the particles is described explicitly by calculating particle trajectories using Newton's equations (cf. Eqs. (4)-(6)). The coupling between the translation and the spin degree of freedom comes from a force in the z-direction determined by the spin state and the gradient of the magnetic field as in eq.(4). There are two contributions to the spin dynamics. First there is a Zeeman term due to the coupling between the spin of the particle and the average local magnetic field. In addition we allow for relaxation of the spin in the field caused by the difference between the local average field and the local dipolar couplings to the dynamic spin system of the permanent magnet. The relaxation is, within such a description, a stochastic process making the complete equation of motion (eq.(7)) a stochastic differential equation[13].The stochastic spin relaxation was modeled to yield on average a proper T_2 relaxation of an ensemble of spins. The numerical integration of the equation of motion were made as in ref.[13].

The difference with respect to the previous simulations is found in the initial conditions. At the initial decays of the composite system the particles acquire a momentum of a given magnitude, assuming that

a fixed amount of energy is liberated. However, the direction of the momentum is undetermined and due to the presence of the slits only particles pairs with the momentum as close as possible along the x-axis are selected for measurements. The width of the slit will determine the spread of the momentum the y and z directions. With perfectly aligned magnets and a point source a deviation $\Delta y, \Delta p_y, \Delta z, \Delta p_z$ in the magnet for $x > 0$ is by conservation of momentum matched by a particle reaching the slit of the other magnet. However, to illustrate our main point it is, in the present context, sufficient to consider the ideal case where the particles enter the magnet purely along the x-axis. The slit does not impose a selection of the spin states. On entering the magnet the entangled pairs can be represented by a pure state but with value of $\langle S_z^{(1)} \rangle$ corresponding to a spherical distribution. Since the spins emerge from a composite rotationally symmetric initial S-state the forces on the two particles of a given pair are initially of the same magnitude but of opposite sign.

At this stage there is a perfect (anti)correlation between the spin states of the two particles. The magnetic field is along the z-direction and the magnetic moments of the spins precess around this direction. The Larmor precession frequency is large relative to the residence time. In the absence of T_2 -relaxation the forces remain constant throughout the passage, while this is in general not the case if a relaxation process occurs.

It is a practical difficulty to interpret the outcome of the trajectory calculations in terms of correlations of final spin states. We have in the calculations the final value of $\cos(\theta)$ as the particle leaves the magnet, but the experimental observable is the position on a detector and in particular the z_f -coordinate. In the idealized experiment the z-coordinate is $\Delta z_{\mp}$ corresponding to spin states $S_z = -1/2$

and $S_z = 1/2$. However, in the simulation there are deviations so that $\Delta z_- \leq z_f \leq \Delta z_+$. In analogy with eq.(19) we have chosen to measure the correlation by defining:

$$\chi'(i,j) = \frac{\langle z_f(1)z_f(2) \rangle}{\Delta z_+^2}. \quad (30)$$

for each pair of correlated trajectories i and j . In a first step a correlation coefficient is determined by averaging over all trajectories (i,j) of a given value of $\cos(\theta)$,

$$\chi'(\cos(\theta)) = \langle \chi'(i,j) \rangle \quad (31)$$

Then the average is taken over all of $\cos(\theta)$, to obtain a measure of the correlation observed when the spins enter the magnet in a arbitrary pure state so that

$$\chi' = -\frac{1}{2} \int_{-1}^1 \chi'(\cos(\theta)) d\cos(\theta) \quad (32)$$

For the ideal case were $z_f = \pm \Delta z_+$ the correlation χ' of Eq.(32) corresponds to χ of Eq.(2).

Table 1 The parameters used for the simulation of a pair of spin bearing particles ($s=1/2$) initially in spin state $S=0$, moving through two SG apparatus.

B_0 (T)	β (T/cm)	V_x (m/s)	t_f (μ s)	T_2 (μ s)
0.30	10	600	60	1

The right and left trajectories for particles are displayed in Fig.1, travelling in the positive and negative x -direction, respectively. These trajectories were calculated for a set of initial conditions, $(\cos(\theta), \varphi)$, of the angles in Eq.(16). Using the parameters of Table 1 simulations were performed for each choice of

initial conditions in the absence, $T_2 \gg t_f$, and then in the presence, $T_2 \ll t_f$, of relaxation effects. For each value of $\cos(\theta)$, 200 φ -angles were generated from 1.8 to 360 in steps of 1.8, and it was verified that this initial condition had no effect on the end result.

The values of $\cos(\theta)$ was increased in steps of 0.1 from -1.0 to 1.0. For each $\cos(\theta)$, with the initial conditions corresponding to the $S=0$ state $N_t=200$ left trajectories ($x<0$) and $N_t=200$ right ($x>0$) trajectories were simulated. In a coincidence experiment the values of $\cos(\theta)$, φ , at $x>0$ corresponds to values $\cos(\theta)$, $360-\varphi$ at the magnet at $x<0$. The result of a coincidence measurement obtained by combining the deviations $z_f(1)$ of the left and the deviation $z_f(2)$ of right trajectories according to Eq.(30) for each initial condition $(\cos(\theta), \varphi)$, and $(-\cos(\theta), 360-\varphi)$ and then averaging over φ we obtain $\chi'(\cos(\theta))$ as in eq.(31). By averaging over $\cos(\theta)$ according to eq.(32) χ is then obtained.

Consider first the case with no relaxation so that the spin dynamics is totally governed by the Zeeman coupling to the averaged local field. For the central trajectories with $y=0$ the z -component of the spin is a constant of motion and there is a constant force, proportional to $\cos(\theta)$. Thus we expect the final z -value to be proportional $\cos(\theta)$, as is indeed found in the simulations. The calculated correlation factor $\chi'(\cos(\theta))$, according to eq.(30) is shown in figure 2. The values varies from 0 to -1 and the average as in eq.(32) gives $\chi' \approx -0.36$.

In a discussion of the Stern-Gerlach experiment Bell notes that the pattern of z_f -values we obtain in the simulation considering only the Zeeman term corresponds to "a naive classical expectation" [15, 16].

He continues by stating that the values $z \approx \pm z_{max}$ follows from a proper quantum description, but in a

way that is hard to understand conceptually. It is our conclusion that the introduction of a T_2 -relaxation process provides a basis for such a conceptual understanding of the Stern-Gerlach-experiment. When the T_2 -relaxation is rapid we find in the simulation that the final z_f -value on exit is in each case close to one of $\Delta z_{\pm}$. Thus for each individual case i,j , the factor $\chi'(i,j)$ is close to either -1 or 1. However, the value averaged over both N_t independent trajectories with different values of the angle φ depend on $\cos(\theta)$ in a similar way as in the absence of relaxation as illustrated in fig. 2B. Through numerical integration we find

$$\chi' \approx -0.36$$

Fig. 2 A, B The correlation $\chi' \cos(\theta)$ is displayed for the case with no relaxation (A) and for the case with T_2 relaxation (B) with $T_2 < t_f$ (time of flight).

The variation of $\chi'(\cos(\theta))$ with $\cos(\theta)$ can in this case be understood as a consequence of that the transition probabilities given in eq.(23). Thus if the initial $\langle S_z^{(1)} \rangle$ expectation value is close to $\pm \frac{1}{2}$ the relaxation has only a minor effect and the original correlation between the spins of the two particles is predominantly preserved and $\chi' = -1$ in the trajectory. In the other limit, $\langle S_z^{(1)} \rangle \approx 0$, the relaxation gives a transfer of angular momentum to a state of either $\langle S_z \rangle = 1/2$ or $\langle S_z^{(1)} \rangle = -1/2$ with nearly equal probability. The relaxation processes, being stochastic in nature, are not correlated between two magnets and for the factor $\chi'(i, j)$ the values +1 and -1 are equally probable.

5 Discussion

The coincidence Stern-Gerlach thought experiment was originally used as a basis for the derivation of Bell's theorem [11]. The remarkable conclusion of these arguments was that changes in the orientation of one magnet should influence the measurement in the other magnet even if this change takes place after the separation of the two particles so that there is no direct physical interaction between the two subsystems. The derivation is based on two essential elements of the description of the S-G coincidence measurement. One is that in a single S-G experiment with a spin $S = \frac{1}{2}$ particle only two outcomes are possible. As the particle leaves the magnet the z-coordinate takes, within experimental accuracy, either of the two values $\pm \Delta z$ given in eq.(1). The second element is that the correlation function χ takes the value -1. The first of these assertions is based on the actual observations in S-G experiments, while the

second follows if state of the system at exit can be described by a density matrix of eq.(10) or, more generally as in eq. (13).

Rather than letting the theoretical description rely on a measurement rule, we have above analyzed the coincidence measurement using explicit equations of motion for the spin dynamics. Our approach is in this respect in line with the coherence program [17] for measurements in quantum systems. For the case, when the spin dynamics is taken to be solely determined by the (time independent) Zeeman interaction between spin and magnet, the solution does give that the correlation function $\chi = -1$. However, the z-coordinate on exit can take any value between the two extremes Δz , which is not compatible with the conventional derivation. On the other hand, we find, for the case where there is a rapid T_2 -relaxation of the spins in the magnet, that we reproduce the result $z(exit) = \Delta z_{\pm}$, in agreement with the experimental observations for S-G measurements on single particles. However the changes in the density matrix due to the stochastic relaxation process result in a change in the correlation function so that $\chi = -1/3$ in this case. It is thus our conclusion that the original derivation of Bell's theorem is based on two elements that are mutually incompatible for the case of a coincidence Stern-Gerlach experiment. Thus our calculations falsify the original derivation and the application of the theorem to this particular case. Above we have considered two limiting cases of negligible or rapid T_2 relaxation. In practice one can envisage the possibility to gradually cool the magnets reducing thermal fluctuations of their spin systems. This could lead to a transition from rapid relaxation at high to slow relaxation at low temperatures and thus a transition between the two extreme cases envisaged above. The numerical trajectory calculations allow for describing such a situation, but we see no reason why such an intermediate case should give results that are qualitatively different from those of the two limiting cases.

Acknowledgements

This work was supported by Swedish Research Council.

References

- [1] W. Gerlach and O. Stern, *Zeit. Phys.* **9**, 349 (1922).
- [2] R. P. Feynman, R.B. Leighton and M. Sand, *Feynman lecture on Physics*, Vol III, Ch 5 and 6 (Addison-Wesley, Sixteenth printing 1983).
- [3] D. Bohm, *Quantum Theory*, p 594, (Dover Publication, 1989).
- [4] D. Bohm and Y. Aharonov, *Phys. Rev.*, 1957, **108**, 1070(1957).
- [5] J.S. Bell, *Speakable and unspeakable in quantum mechanics* (Cambridge University Press, Cambridge 1987).
- [6] L. E. Ballentine, *Rev. Mod. Phys.* **42**, 358(1970); *ibid*, *Quantum Mechanics. A Modern development*, Ch. 20, (World Scientific, Singapore 1998).
- [7] H.P. Stapp. *Phys.Rev. D*.3. 1303 (1971).
- [8] M.O. Scully and M. Suhail Zubarry, *Quantum Optics Ch 8*, (Cam. Univ. Press. 1997).
- [9] A. Aspect, J. Dalibard and G. Roger, *Phys. Rev. Lett.* **47**, 460(1981); *ibid*: *Phys. Rev. Lett.* **49**, 1804(1982).
- [10] M. A. Rowe, D. Kielpinski, V. Meyer, C. A. Sackett, W. M. Itano, C. Monroe and D. J. Wineland, *Nature*, **409**,791(2001).

- [11] J.T. Custing and E McMullin Eds. *Philosophical Consequences of Quantum Theory: Reflections on Bell's Theorem* (University of Notre Dame Press, Notre Dame Indiana 46556 , 1989).
- [12] M. le Bellac, *Quantum Physics*, Ch. 3.2.2 (Cam. Univ. Press. 2006).
- [13] H. Wennerström and P-O Westlund, *Physical Chemistry Chemical Physics*, **14**, 1677 (2012).
- [14] M. Carravetta and M. H. Lewitt, *J. Phys. Chem.***122**, 214505 (2005).
- [15] J.S. Bell , *Journal de Physique*, Colloque C2, Supplr Au nr 3, Tome 42(1981).
- [16] J.S. Bell ,*Physics*, **1**, 195(1964).
- [17] M. Schlosshauer, *Rev. Mod. Phys.* **76**, 1267(2004).