PIECEWISE RATIONAL INTERPOLATION

by

Jan Gelfgren

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UNIVERSITY OF UMEÅ
DEPARTMENT OF
MATHEMATICS
S-901 87 UMEÅ
SWEDEN
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1. Introduction.

When \( f \) is a formal power series at \( z = 0 \), the \([n,m]\) Padé approximant can be defined to be the unique rational function \( \frac{P_n(z)}{Q_m(z)} \), which has contact of highest order at the origin, where \( P_n \) and \( Q_m \) are polynomials of degree \( n \) and \( m \) respectively (see [1], [2]).

If \( f \) is analytic in a small region containing the origin, we can use Hermite's interpolation formula there to get an estimate of the error \( e(z) = |(f - \frac{P_n}{Q_m})(z)| \).

If, however, the function \( f \) is to be approximated on a real interval, the picture changes. If we use just one Padé approximant, we cannot in general use Hermite's interpolation formula to get a good error estimate in the whole region. To avoid this we partition the interval and determine one rational approximating function on each subinterval, and then we try to tie them together. The approximation result is then compared with the outcome of piece wise polynomial approximation.

The same technique can be used for error estimation in the complex plane. In this case, however, we cannot tie the approximating rational functions together.

Another approach is to use rational splines (see [11], [15], [12], [1]). With this kind of approximation a key problem is to solve the nonlinear equations that arise.

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1. Definition of the Padé approximant.

If \( f \) is a formal power series

\[
(1.1) \quad f(z) = \sum_{k=0}^{\infty} a_k z^k, 
\]

then (see [13]) the \([n,m]\) Padé approximant to \( f \) is a rational function \( P_n/Q_m \), where

\[
(1.2) \quad P_n(z) = \sum_{k=0}^{n} a_k z^k, \quad Q_m(z) = \sum_{k=0}^{m} \beta_k z^k, \quad Q_m(z) \neq 0, 
\]

and the coefficients \( a_k \) and \( \beta_k \) are determined so that

\[
(1.3) \quad f(z) \cdot Q_m(z) - P_n(z) = A \cdot z^{m+n+1} + \text{higher degree terms}, 
\]

where \( A \) is a constant. From (1.1), (1.2) and (1.3) it follows that the numbers \( a_k \) and \( \beta_k \) must satisfy the system of equations

\[
(1.4) \quad \sum_{k=0}^{m} a_{j-k} \beta_k = \begin{cases} 
\alpha_j & 0 \leq j \leq n \\
0 & n+1 \leq j \leq n+m, \quad a_j = 0 \text{ if } j < 0. 
\end{cases} 
\]

Since the last \( m \) equations have \( m+1 \) unknowns \( \beta_k \), it is possible to choose the \( \beta_k \)'s to satisfy these equations. After that the numbers \( a_k \) are determined from the \( n+1 \) first equations.

Although the numbers \( a_k \) and \( \beta_k \) are not uniquely determined by (1.4), the \((n,m)\) Padé approximant \( P_n/Q_m \) is, which is proved in the following way [5, p. 2].

Let \( P_n^{*}/Q_m^{*} \) be another \((n,m)\) Padé approximant to \( f \) and consider the following expression

\[
(1.5) \quad (f Q_m - P_n^{*})Q_m^{*} - (f Q_m - P_n)Q_m = P_n^{*}Q_m^{*} - P_n Q_m. 
\]

On the left side all the coefficients of terms of degree at most \( n+m \) are zero by the relation (1.3). The right side is a polynomial of degree at most \( n+m \) and hence identically zero, which means that \( P_n^{*}/Q_m^{*} = P_n/Q_m \).
2. The approximation procedure.

If a function $f$ is analytic at the points $z_0$ and $z_1$, then we can determine the following special Newton series at these points (see [14, p. 52-54]).

$$f(z) \sim \sum_{k=0}^{\infty} a_k k(z), \text{ where}$$

$$w_k(z) = (z-z_0)^k - [k/2](z-z_1)^{[k/2]}.$$ 

**Def. 2.1.** The symmetric $[n,m]$ Padé approximant to $f$ at $z_0$ and $z_1$ is a rational function $P_n/Q_m$, where

$$P_n(z) = \sum_{k=0}^{n} a_k w_k(z), \quad Q_m(z) = \sum_{k=0}^{m} b_k z^k, \quad Q_m(z) \neq 0,$$

$n+m+1 = 2p$, and the coefficients $a_k$ and $b_k$ are determined so that

$$f(z)Q_m(z) - P_n(z) = A(z-z_0)^p(z-z_1)^p + \text{higher degree terms},$$

where $A$ is a constant.

In [8, p. 6 def. 1.1] Karlsson has defined the best interpolating function $P_n/Q_m$ of type $[n,m]$ to $f$ at $\beta_1, \beta_2, \ldots, \beta_{n+m+1}$, to be the one that is determined by the relation $(fQ_m - P_n)(z) = 0$ for $z = \beta_1, \beta_2, \ldots, \beta_{n+m+1}$, and we see that our Padé approximant satisfies this definition if we let $z_0$ and $z_1$ be multiple zeros of order $p$.

By using a lemma of Karlsson (see [8, p. 10]) to express $f(z)Q_m(z)$ as a Newton series we can solve (2.4), and in the same way as before we can prove the uniqueness of $P_n/Q_m$.

Let $I = [a,b]$ be a real interval, and let $r$ be a real number that is greater than zero. Let $\Omega = \{z|d(z,I) < r\}$, where $d$ is the usual distance function. Let $I$ have the partition $\tau: a = x_0 < x_1 < \ldots < x_N = b$, where every subinterval $\Delta_j = [x_{j-1}, x_j]$ has equal length $\Delta = |\Delta_j| = x_j - x_{j-1}$ for $j = 1, 2, \ldots, N$ and $\Delta < r$. 


When approximating \( f \) on \( I \) we determine \( P_{n,j}/Q_{m,j} \), that is the symmetric \([n,m]\) Padé approximant to \( f \) at the endpoints \( x_{j-1} \) and \( x_j \) of each subinterval \( \Delta_j \). In this way we get a piecewise rational function \( S_{n,m}(I)(z) \). This function, however, does not necessarily have \( p-1 \) continuous derivatives at the nodes, but from (2.4) we see that if \( P_{n,j} \) and \( Q_{m,j} \) have no common factors, it does.

**Remark.** When \( m = 0 \), then by \( S_{n,0}(\tau)(z) \) we denote the piecewise polynomial function which on \( \Delta_j \) is defined to be

\[
(2.5) \quad P_{2p-1}(z) = \sum_{k=0}^{2p-1} a_k w_k(z), \quad \text{where} \quad w_k(z) = (z-x_{j-1})^{k-[k/2]}(z-x_j)^{[k/2]}.
\]

We see, that it is the partial sum of the Newton series (2.1) of \( f \) at \( x_{j-1} \) and \( x_j \).

### 3. Error estimation.

The capacity (transfinite diameter) of the compact set \( F \subset \mathbb{C} \) can be defined by

\[
(3.1) \quad \text{cap. } F = \lim_{k \to \infty} \inf_{h \in p_k} \max_{z \in F} |h(z)|^{1/k},
\]

where \( p_k \) is the class of all polynomials \( h(z) = z^k + \ldots \) of degree \( k \) (see [6, ch. 7], [7, ch. 16]). It satisfies

\[
(3.2) \quad \text{meas } F \leq \Pi(\text{cap } F)^2 \quad (\text{see } [10]).
\]

**Lemma 3.1** Let \( r > 0 \) be given and let \( g(z) \) be a polynomial of degree \( \leq m \) that satisfies

\[
\max_{z \in \gamma} |g(z)| > 1, \quad \text{where} \quad \gamma = \{z|z = g(t), \ 0 \leq t \leq 1, \ g(0) = g(1) \text{ and } g_1 \text{ cont}\}
\]

and has diameter \( 2r \).

Let \( 0 < \varepsilon < 1/(2\sqrt{5}+1) \) and let \( B = \{z|z \in \text{the interior of } \gamma, \ |g(z)| \leq \varepsilon^m\} \). Then \( \text{cap } B \leq (2\sqrt{5}+1)\varepsilon r \).
This lemma is a modification of a lemma by Pommerenke [10] and is proved in the same way. Pommerenke considers $\gamma = \{z||z| = r\}$ and instead of $2\sqrt{3}+1$ his constant is 3.

Now let $I$, $\tau$, $\Omega$ and $S_{n,m}(\tau)(z)$ be as in section 2. Let $G(\Delta) = \{z = x+iy| x \in I, \ |y| < (3/4)\Delta\}$. Then we have

Theorem 3.2 Let $f$ be analytic in $\Omega$, and let $I$ have the partition $\tau$. Let $\eta > 0$ be arbitrarily chosen. Then it is possible to find a real number $k$ so that, when $n > k\cdot m$ and $z \in G(\Delta)$, we have

$$|f - S_{n,m}(\tau)(z)| \leq K(\Delta) \max_{\Omega} |f| \left(\sqrt{\frac{15}{16}} \frac{\Delta}{r}\right)^{n-km+1},$$

except on a set $F$ of capacity $< \eta$.

Remark 1. $K(\Delta)$ is a constant which is less than $4+4/\pi$, and $k$ is the smallest real number that satisfies

$$\left(\sqrt{\frac{15}{16}} \frac{\Delta}{r}\right)^k \left(\frac{(b-a)^2(9/4)\Delta^2}{\eta}\right)^N \frac{5(r + \Delta/2)}{\sqrt{(b-a)^2 + (9/4)\Delta^2}} < 1.$$

Remark 2. When $m = 0$ we get the polynomial case, where we have no exceptional set.

Remark 3. If we approximate just on $I$, that is if $z \in I$ in (3.3), then the constant $\sqrt{\frac{15}{16}}$ can be replaced by 1/2. We see that the polynomial case is to prefer mainly for two reasons. In the first place the Newton series is easier to determine than the Padé approximant, and in the second place we do not get any exceptional set in the polynomial case.

If $f$ is real valued on $[a,b]$, we get the following stronger result for piece wise polynomial approximation. We use the same notation as before.

Theorem 3.3. Let $f$ be analytic in $\Omega$ and real valued on $[a,b]$. Let $[a,b]$ have the partition $\tau$. Then, for $x \in [a,b]$, we have
(3.5) \[ |(f^{(j)} - S_{2p-1,0}(\tau))(x)| \leq \frac{\Omega}{r^{2p}} \frac{(2p)!}{j!(2p-j)!} \Delta^{2p-j}, \]

where \( j = 0, 1, 2, \ldots, p-1 \) and \( (2p)! = \frac{(2p)!}{(2p-j)!} \).

Remark 4. If we put \( j = 0 \) we get

\[ (3.6) \quad |(f - S_{2p-1,0}(\tau))(x)| \leq \frac{\max |f|}{\Omega} \left( \frac{\Delta}{2\pi} \right)^{2p}, \]

From Remark 3 and (3.3) we see, that now the constant \( K(\Delta) \) has been reduced to unity. All the time we have had \( n+m+1 = 2p \).

4. Proofs of Theorem 3.2 and Theorem 3.3.

Proof of Theorem 3.2. If \( z \) belongs to the region

\( G(\Delta,j) = \{z = x+iy| x \in \Delta_j, |y| \leq (3/4)\Delta\} \), then Hermite's interpolation formula (see [14, p. 50]) gives

\[ (4.1) \quad \frac{(f Q_{m,j} - P_{n,j})(z)}{(z-x_{j-1})^p(z-x_j)^p} = \frac{1}{2\pi i} \int \frac{(f Q_{m,j})(t)}{(t-x_{j-1})^p(t-x_j)^p(t-z)} dt, \]

where \( Q_{m,j} \) is normed so that \( \max_{t \in \gamma_j} |Q_{m,j}(t)| = 1 \) and

\( \gamma_j = \{z|z = x_{j-1} + r \cdot e^{is}, s \in [\pi/2, 3\pi/2]\} \cup \{z|z = x_j + r \cdot e^{is}, s \in [-\pi/2, \pi/2]\} \cup \{z = x+iy| x \in \Delta_j, |y| = r\}, \)

\( P_{n,j}/Q_{m,j} \) is the symmetric Padé approximant to \( f \) at the endpoints of \( \Delta_j \).

From (4.1) we get

\[ (4.2) \quad |(f Q_{m,j} - P_{n,j})(z)| \leq \frac{1}{2\pi} \max_{\gamma_j} \frac{2\pi r + 2\Delta}{r - (3/4)\Delta} \cdot \frac{\max_{\gamma_j} (z-x_{j-1})^p(z-x_j)^p}{\min_{\gamma_j} (t-x_{j-1})^p(t-x_j)^p}. \]
Since $\max_{z \in \Omega(\Delta, j)} |z - x_{j-1}| |z - x_j| = (15/16)\Delta^2$, and
\[
\min_{t \in \gamma_j} |t - x_{j-1}| |t - x_j| = r^2(1 + (\Delta/2r)^2),
\]
we get
\[
(4.3) \quad |(f Q_{m,j} - P_{n,j})(z)| \leq K(\Delta) \max_{\gamma_j} |f|(\sqrt{\frac{15}{16}} \Delta)^{m+n+1}.
\]
Now
\[
(4.4) \quad |(f - P_{n,j})(z)| \leq K(\Delta) \max_{\gamma_j} |f| \cdot (\sqrt{\frac{15}{16}} \Delta)^{m+n+1} \cdot (\frac{1}{\epsilon_j})^m,
\]
except on a set $F$ of capacity $\leq 5(r + \Delta/2)\epsilon_j$ (see Lemma 4.1).

We put $\epsilon_j = \epsilon > 0$ for $j = 1, 2, \ldots, N$, and then use the definition of $S_{n,m}(\tau)(z)$. For $z \in G(\Delta)$ this gives
\[
(4.5) \quad |(f - S_{n,m}(\tau)(z)| \leq K(\Delta) \max_{\Omega} |f| \cdot (\sqrt{\frac{15}{16}} \Delta)^{m+n+1} \cdot (\frac{1}{\epsilon})^m,
\]
except on a set $F = \bigcup F_j$.

The "subadditivity" of the capacity (see [9, p. 259]) gives
\[
(4.6) \quad \text{cap } F_j \leq 5(r + \Delta/2)\epsilon \quad \text{for all } j \Rightarrow \text{cap } F \leq (5(r + \Delta/2)\epsilon)^{1/N} (\text{diam } F)^{1-1/N}
\]
\[
(4.7) \quad \text{Let } \epsilon = \left(\frac{\eta}{\sqrt{(b-a)^2 + (9/4)\Delta^2}}\right)^N \cdot \frac{\sqrt{(b-a)^2 + (9/4)\Delta^2}}{5(r + \Delta/2)}
\]
then $\text{cap } F \leq \eta$.

If $k$ is the smallest real number that satisfies (3.4), then (4.5) gives (3.3) except on a set $F$ of capacity $\leq \eta$. From (4.2) we see, that
\[
K(\Delta) < 4 + 4/\pi \quad \text{since } \Delta < r.
\]
Q. E. D.
Proof of Theorem 3.3. The usual real interpolation formula [4, p. 67]
gives
\[(4.8) \quad (f - P_{2p-1,k})(x) = \frac{f(2p)(\xi)}{(2p)!} (x - x_{k-1})^p(x - x_k)^p, \quad x \text{ and } \xi \in \Delta_k.\]

The function \( f - P_{2p-1,k} \) has a zero of order \( p \) at both \( x_{k-1} \) and \( x_k \), which means that \( (f - P_{2p-1,k})^2 \) has a zero of order \( p-1 \) at those points, and according to Rolle's theorem it has a zero at \( z = c \) inside \( [x_{k-1}, x_k] \). This means that
\[(4.9) \quad f'(x) - P_{2p-1,k}(x) = \frac{f(2p)(\xi)}{(2p-1)!} (x-x_{k-1})^{p-1}(x-x_k)^{p-1}(x-c),\]
where \( x, \xi_1, \) and \( c \in \Delta_k \).

The first factor on the right side is the derivative of order \( 2p-1 \) of \( f' \) divided by \( (2p-1)! \). By using the definition of \( S_{2p-1,0}(\tau)(z) \) and by iterating \( j \) times we get
\[(4.10) \quad (f^{(j)} - S_{2p-1,0}^{(j)}(\tau))(x) = \frac{f(2p)(\xi)}{(2p-j)!} (x-x_{k-1})^{p-j}(x-x_k)^{p-j}(x-c_1)(x-c_2)\ldots\]
\[\ldots (x-c_j),\]
where \( x, \xi, c_1, c_2, \ldots, c_j \in [x_{k-1}, x_k] \) and \( k \in \{1,2,\ldots, N\} \).

Let
\[(4.11) \quad g(x) = \left| (x-x_{k-1})^{p-j}(x-x_k)^{p-j}(x-c_1)(x-c_2)\ldots (x-c_j) \right|,\]
where \( x, c_1, c_2, \ldots, c_j \in \Delta_k \).

We see that \( g(x) \) has its largest maximum when the points \( c_i \) are
either equal to \( x_{k-1} \) or to \( x_k \).

If we let \( \ell c_1:s \) equal \( x_{k-1} \) and \( j-\ell c_1:s \) equal \( x_k \), and if we
then determine the maximum of \( g \), we get
\[(4.12) \quad g_{\max} = (p-j+\ell)^{p-j+\ell}(p-\ell)^{p-\ell} \frac{x_{k-1} - x_k}{2p-j} \cdot\]

Now, if we let \( \ell > j-\ell \), we see that \( \ell = j \) gives the largest maximum
of \( g \)
\[(4.13) \quad g_{\max} = p^{p-j} \frac{x_{k-1} - x_k}{2p-j}.\]
From the Cauchy integral we get

\[(4.14)\quad f^{(2p)}(z) = \frac{(2p)!}{2\pi i} \int_{\gamma} \frac{f(t)}{(t-z)^{2p+1}} \, dt,\]

where \(\gamma = \{t| |t-z| = r\} \).

From this we see

\[(4.15)\quad \max_{z \in \Omega} \left| \frac{f^{(2p)}(z)}{(2p)!} \right| \leq \max_{z \in \Omega} |f(z)| \cdot \frac{1}{r^{2p}}.

Now (4.15), (4.13) and (4.10) give (3.5).

Q. E. D.

5. **Piecewise Padé approximation in the complex plane.**

Let the function \(f\) be analytic in the domain

\(\Omega = \{z = x+iy| |x| \leq r, |y| \leq r\}\), where \(r\) is a real number which is greater than zero. This implies that for an arbitrarily chosen point \(z_0 \in \Omega\) we can write

\[(5.1)\quad f(z) = \sum_{k=0}^{\infty} a_k (z-z_0)^k.

If we put \(P_n(z) = \sum_{k=0}^{n} a_k (z-z_0)^k\) and \(Q_m(z) = \sum_{k=0}^{m} b_k (z-z_0)^k\), we see from (1.1), (1.2) and (1.3) how to define the \([n,m]\) Padé approximant to \(f\) at this point.

We now divide \(\Omega\) into disjoint subsquares and determine the \([n,m]\) Padé approximant to \(f\) at the center of every subsquare. We shall make two different partitions of \(\Omega\) and compare the results.

The first partition.

Let \(p\) and \(j\) be positive integers such that \(p \leq 4(2^j+1) - 3\).

Let \(z_{j,p} = x_{j,p} + iy_{j,p}\), where

\[
\begin{align*}
\{ x_{j,p} = \frac{(2p-1)r}{2^{j+1}} \\
y_{j,p} = r(1-3/2^{j+1})
\end{align*}
\]

when \(p \leq 2^j - 1\) and

\[
\begin{align*}
\{ x_{j,p} = r(1-3/2^{j+1}) \\
y_{j,p} = r(2-(5+2p)/2^{j+1})
\end{align*}
\]

when \(2^j \leq p \leq 2^j+1 - 3\).
Let \( z_{j,p} = x_{j,p} + iy_{j,p} \), where
\[
\begin{align*}
x_{j,p} &= r(1-3/2^j+1) \\
y_{j,p} &= (-r/2^j+1)(2p-2^j+2 + 5)
\end{align*}
\]
for \( 2^j+1 - 3 < p < 3 \cdot 2^j - 4 \),
and
\[
\begin{align*}
x_{j,p} &= (r/2^j+1)(2^j+3 - 11 - 2p) \\
y_{j,p} &= -r(1-3/2^j+1)
\end{align*}
\]
for \( 3 \cdot 2^j - 4 < p < 2(2^j+1 - 3) \).
For \( 2(2^j+1 - 3) < p \leq 4(2^j+1 - 3) \) we define \( z_{j,p} \) in a similar way.

Now we define
\[
\Omega(j,p) = \{ z = x+iy \mid |x-x_{j,p}| < r/2^j+1, \ |y-y_{j,p}| < r/2^j+1 \},
\]
and
\[
\begin{align*}
y_{j,p} &= \{ z = x+iy \mid |x-x_{j,p}| = \frac{3r}{2^j+1}, \ y_{j,p} - \frac{3r}{2^j+1} \leq y \leq y_{j,p} + \frac{3r}{2^j+1} \} \\
U \{ z = x+iy \mid x_{j,p} - \frac{3r}{2^j+1} \leq x \leq x_{j,p} + \frac{3r}{2^j+1}, \ |y-y_{j,p}| = \frac{3r}{2^j+1} \}.
\end{align*}
\]
(See fig. 1!)

We also define
\[
S_{n,m}(\lambda)(z) = \{(P_{n,j,p}/Q_{m,j,p})(z) \mid P_{n,j,p}/Q_{m,j,p} \text{ is the } [n,m] \text{ Padé approximant to } f \text{ at the center of } \Omega(j,p), \text{ and } j \leq \lambda \}.
\]

Theorem 5.1. Let \( f \) be analytic in \( \Omega \) and let \( \eta > 0 \) be arbitrarily chosen.

Then there exists a real number \( k \) so that, when \( n > k \cdot m \) and \( z \in \Omega(j,p) \), we have
\[
|f - S_{n,m}(\lambda)(z)| \leq \frac{6}{\pi} \max_{\Omega} |f| \cdot \frac{1}{2^j} \nu \left( \frac{r}{3} \right)^{n+1-km}
\]
except on a set \( F \) of capacity \( \leq \eta \).

Remark. Here \( k \) is the smallest real number that satisfies
\[
\frac{15(2r\sqrt{2})^N (\sqrt{2})^{k+1}}{4 \eta} \leq 1, \text{ where } N = N(\lambda) = 4 \sum_{s=1}^{\lambda} (2^s+1 - 3)
\]
is the number of squares \( \Omega(j,p) \) in \( \Omega \).
The second partition.

Let $s$ be a real number such that $r/s = N$, where $N > 1$ is an integer.

Let $p$ and $j$ be positive integers that satisfy the relation $p \leq 4(2j-1)$ and $j < N$. Now we define (see fig. 2)

$$\mathcal{O}(j,p) = \begin{cases} 
\{z = x+iy \mid (p-1)s \leq x \leq ps, (j-1)s \leq y \leq js\}, & \text{if } p \leq j \\
\{z = x+iy \mid (j-1)s \leq x \leq js, (2j-p-1)s \leq y \leq (2j-p)s\}, & \text{if } j < p \leq 2j-1,
\end{cases}$$

$$\mathcal{O}(j,p) = \begin{cases} 
\{z = x+iy \mid (j-1)s \leq x \leq js, (p-2j-1)s \leq y \leq (p-2j)s\}, & \text{if } 2j < p \leq 3j-1 \\
\{z = x+iy \mid (4j-p-2)s \leq x \leq (4j-p-1)s, (j-1)s \leq y \leq js\}, & \text{if } 3j < p \leq 4j-2.
\end{cases}$$

When $2(2j-1) < p \leq 4(2j-1)$ we define $\mathcal{O}(j,p)$ in a similar way.

If $z_{j,p} = x_{j,p} + iy_{j,p}$ is the center of $\mathcal{O}(j,p)$ we define

$$\gamma_{j,p} = \{z = x+iy \mid |x-x_{j,p}| = \frac{3s}{2}, |y-y_{j,p}| \leq \frac{3s}{2}\} \cup \{z = x+iy \mid |x-x_{j,p}| \leq \frac{3s}{2}, |y-y_{j,p}| = \frac{3s}{2}\}.$$

We also define

$$\gamma_j = \{z = x+iy \mid |x| = (j+1)s, |y| \leq (j+1)s\} \cup \{z = x+iy \mid |x| \leq (j+1)s, |y| = (j+1)s\},$$

and $S_{n,m}(z) = \{(P_{n,j,p}/Q_{m,j,p})(z)\}$ is the $[n,m]$ Padé approximant to $f$ at the center of $\mathcal{O}(j,p)$.

See fig. 2!

With these definitions we get

**Theorem 5.2.** Let $f$ be analytic in $\mathcal{O}$ and let $\eta > 0$ be arbitrarily chosen. Then there exists a real number $k$ so that, when $n > k \cdot m$ and $z \in \mathcal{O}(j,p)$, we have

$$(5.4) \quad |(f - S_{n,m})(z)| \leq \frac{6}{\pi} \max_{\gamma_j} |f| \cdot \left(\frac{\sqrt{2}}{3}\right)^{n-km+1}$$

except on a set $F$ of capacity $\leq \eta$. 
Figure 1.

Denotation: \( j, \rho = \Omega(i, \rho) \)
Figure 2.

\textbf{Denotation: } \( j, p = \Omega(j, p) \)
Remark. Here $k$ is the smallest real number that satisfies

$$\begin{align*}
(5.5) \quad \frac{15s}{4r} \left(\frac{\sqrt{2}}{3}\right)^{k+1} \left(\frac{2r\sqrt{2}}{\eta}\right)^M &< 1, \\
\text{where } M &= 4\left(\frac{s}{s} - 1\right)^2
\end{align*}$$

denotes the number of squares $Ω(j,p)$ in $Ω$.

The proofs are omitted since the technique is the same as before.

We use Hermite's formula to get an error estimate and the "subadditivity" of the capacity to measure the exceptional set.

Theorems 5.1 and 5.2 show that the degree $n$ becomes very large, if the estimate is to be valid in the vicinity of the boundary, or if the exceptional set is to be small. The use of Hermite's formula in the estimation procedure is responsible for the fact that we cannot reach the boundary of $Ω$, when we estimate the error.

Remark. If $n + m = p$, where $p$ is a fixed integer, we see from (5.2) and (5.4), that if we put $m = 0$, we get at least as good error estimates as with piecewise Padé approximation, and we do not get any exceptional set.

When we use the first partition we see from (5.2) that the approximation procedure gives its best result near the boundary of $Ω$, and from (5.4) we see that if we use the second partition the approximation is best in the middle of $Ω$. 
References


