PROPERTIES AND TESTS FOR SOME CLASSES OF LIFE DISTRIBUTIONS

by

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ABSTRACT

A life distribution and its survival function \( \bar{F} = 1 - F \) with finite mean \( \mu = \int_0^\infty \bar{F}(x)dx \) are said to be HNBUE (HNWUE) if \( \int_t^\infty \bar{F}(x)dx \leq \mu \exp(-t/\mu) \) for \( t \geq 0 \). The major part of this thesis deals with the class of HNBUE (HNWUE) life distributions. We give different characterizations of the HNBUE (HNWUE) property and present bounds on the moments and on the survival function \( \bar{F} \) when this is HNBUE (HNWUE). We examine whether the HNBUE (HNWUE) property is preserved under some reliability operations and study some test statistics for testing exponentiality against the HNBUE (HNWUE) property.

The HNBUE (HNWUE) property is studied in connection with shock models. Suppose that a device is subjected to shocks governed by a counting process \( N = \{N(t): t \geq 0\} \). The probability that the device survives beyond \( t \) is then
\[
\bar{H}(t) = \sum_{k=0}^{\infty} P(N(t)=k)\bar{P}_k,
\]
where \( \bar{P}_k \) is the probability of surviving \( k \) shocks. We prove that \( \bar{H} \) is HNBUE (HNWUE) under different conditions on \( N \) and \( \{\bar{P}_k\}_{k=0}^{\infty} \). For instance we study the situation when the interarrival times between shocks are independent and HNBUE (HNWUE).

We also study the Pure Birth Shock Model, introduced by A-Hameed and Proschan (1975), and prove that \( \bar{H} \) is IFRA and DMRL under conditions which differ from those used by A-Hameed and Proschan.

Further we discuss relationships between the total time on test transform \( H_F^{-1}(t) = \int_0^{F^{-1}(t)} \bar{F}(s)ds \), where \( F^{-1}(t) = \inf\{x: F(x) \geq t\} \), and different classes of life distributions based on notions of aging. Guided by properties of \( H_F^{-1} \) we suggest test statistics for testing exponentiality against IFR, IFRA, NBUE, DMRL and heavy-tailedness. Different properties of these statistics are studied.
Finally, we discuss some bivariate extensions of the univariate properties NBU, NBUE, DMRL and HNBUE and study some of these in connection with bivariate shock models.

Key words and phrases: Life distribution, survival function, exponential distribution, IFR, IFRA, NBUE, DMRL, HNBUE, shock model, total time on test transform, testing of exponentiality.
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This thesis consists of the present summary and the following six papers, which will be referred to in the text by the given letters.


[D] Some tests against aging based on the total time on test transform. Statistical Research Report No. 1979-4, Department of Mathematical Statistics, University of Umeå. (Revised version.)


1. Introduction

The notation of aging plays an important role e.g. in reliability and maintenance theory. Therefore several classes of life distributions have been introduced in order to model different aspects of aging. Among these are the classes of life distributions with the properties in the following definition.

**DEFINITION 1.1** Let \( F \) be a life distribution (i.e. a distribution function with \( F(0^+) = 0 \)) with survival function \( \bar{F} = 1 - F \) and finite mean \( \mu = \int_0^\infty \bar{F}(x)dx \) and let \( S = \{ t : \bar{F}(t) > 0 \} \). Then \( F \) and \( \bar{F} \) are said to be (or to have)

(i) **increasing failure rate (IFR)** if the conditional survival function

\[
\frac{t \sim \bar{F}(x+t)}{\bar{F}(t)}
\]

is a decreasing function of \( t \) on \( S \) for \( x \geq 0 \);

(ii) **increasing failure rate in average (IFRA)** if

\[
t \sim \frac{-\ln \bar{F}(t)}{t}
\]

is increasing on \( S \);

(iii) **new better than used (NBU)** if

\[
\bar{F}(x)\bar{F}(y) \geq \bar{F}(x+y)
\]

for \( x \geq 0 \) and \( y \geq 0 \);

(iv) **new better than used in expectation (NBUE)** if

\[
\bar{F}(x) \int_0^\infty \bar{F}(y)dy \geq \int_0^\infty \bar{F}(x+y)dy
\]

for \( x \geq 0 \);
(v) decreasing mean residual life (DMRL) if

\[ t \sim \frac{1}{\int_0^\infty \frac{F(x)}{F(t)} \, dx} \]

is decreasing on \( S \).

By reversing inequalities and changing decreasing (increasing) to increasing (decreasing) we get the dual classes DFR, DFRA, NWU, NWUE and IMRL. Here \( D = \) decreasing, \( I = \) increasing and \( W = \) worse.

The IFR, IFRA, NBU, NBUE and DMRL classes (and duals) and their properties were discussed e.g. by Bryson and Siddiqui (1969), Haines (1973) and Barlow and Proschan (1975).

Rolski (1975) introduced another class of life distributions named HNBUE, with dual HNWUE, by the following definition.

**DEFINITION 1.2** A life distribution \( F \) and its survival function \( \overline{F} = 1 - F \) with finite mean \( \mu = \int_0^\infty \frac{F(x)}{x} \, dx \) are said to be harmonic new better than used in expectation (HNBUE) if

\[ \int_0^\infty \frac{F(x)}{x} \, dx \leq \mu \exp(-t/\mu) \quad \text{for} \quad t \geq 0. \]

If the reversed inequality is true \( F \) and \( \overline{F} \) are said to be harmonic new worse than used in expectation (HNWUE).

Between these classes of life distributions the implications in Figure 1.1 on p. 3 (but no other) hold.

Most of this thesis deals with these classes of life distributions and particularly with the HNBUE and HNWUE classes. We present different properties of these classes, e.g. closure theorems under some shock models. We also present some statistics for testing the hypothesis

\[ H_0: \ F \text{ is the exponential distribution} \]
against

\[ H_1: \ F \text{ is } \mathbb{V} \text{ but not exponential} \]

where \( \mathbb{V} \) denotes one of IFR, IFRA, NBUE, HNBUE or DMRL. Further we discuss some alternative extensions of the aging properties to the bivariate situation.

In subsequent sections we summarize the main results of papers [A]-[F].

2. The HNBUE (HNWUE) class of life distributions - paper [A]

We present the HNBUE (HNWUE) class of life distributions in more detail. We also introduce another aging property for discrete distributions, named discrete HNBUE (discrete HNWUE), which is useful e.g. when we discuss different shock models in paper [B].

The reason for the name HNBUE is the following. Suppose for simplicity that \( \bar{F}(t) > 0 \) for \( t > 0 \) and let

\[
e_F(t) = \int_t^\infty \frac{\bar{F}(x)}{\bar{F}(t)} dx
\]

denote the mean residual life of a unit of age \( t \). Then the inequality (1.1) can be written
This inequality says that the integral harmonic mean value of \( e_F(x) \), 
\( 0 \leq x \leq t \), is less than or equal to \( e_F(0) \).

One reason why we find the HNBUE (HNWUE) class interesting is the following.

Let \( \xi_j \), with life distribution \( F_j \) and survival function \( F_j \), for 
j = 1, 2, ..., \( n \), denote the life lengths of \( n \) independent units. If 
\( \mu_s \) and \( \mu_p \) denote the mean lives of their series and parallel system, respectively, we have

\[
\mu_s = \frac{1}{t} \int_0^t \frac{1}{e^F(x) - 1} dx \leq \mu \quad \text{for} \quad t > 0.
\]

The integrals in (2.1) and (2.2) are often difficult to calculate. However, 
if \( F_j \) is HNBUE with mean \( \mu_j \) and \( \bar{G}_j(x) = \exp(-x/\mu_j), x \geq 0, \) for \( j = 1, 2, ..., n \), we have the bounds

\[
\mu_s \geq \frac{1}{t} \int_0^t \prod_{j=1}^n \bar{G}_j(x) dx
\]

and

\[
\mu_p \leq \frac{1}{t} \int_0^t \prod_{j=1}^n \bar{G}_j(x) dx
\]

which are simple to calculate.

In fact it follows from Theorem 3.2 on p. 33 in Barlow and Proschan (1975) that (2.3) and (2.4) are true under the weaker condition that \( \xi_1, \xi_2, ..., \xi_n \) are associated and each \( \xi_j \) is HNBUE with mean \( \mu_j \). That \( \xi_1, \xi_2, ..., \xi_n \) are associated means a form of dependence (see Barlow and Proschan (1975), Chapter 2).
The HNBUE property can be characterized in several ways. One of these is the following.

Let $T_F$ denote the *equilibrium distribution of $F$* defined by

$$T_F(x) = \frac{1}{\mu} \int_0^x F(s) ds \quad \text{for } x \geq 0.$$  

This life distribution is of interest e.g. in renewal theory (see e.g. Feller (1971), Chapter 11). By using $T_F$, the HNBUE property (1.1) can be written as

$$T_F(t) < G(t) \quad \text{for } t \geq 0,$$

where $T_F = 1 - T_F$ and $G(t) = \exp(-t/\mu)$, $t \geq 0$.

Some of the aging properties in Definition 1.1 can also be characterized by using $T_F$. It can be proved that $F$ is DMRL if and only if $T_F$ is IFR and that $F$ is IFR if and only if $T_F$ is unimodal (see e.g. Bhattacharjee (1980)). Furthermore, it is obvious that the NBUE property is equivalent to the inequality $T_F(x) \leq F(x)$ for $x \geq 0$.

We also characterize the HNBUE (HNWUE) property by using the Laplace transform and the total time on test (TTT-) transform.

We examine whether the HNBUE and HNWUE classes are closed under (i) formation of a coherent structure, (ii) convolution, and (iii) mixture. For instance we prove that a mixture of HNWUE life distributions is HNWUE. This is interesting since it is (as far as this author knows) still an open question whether the NWUE class is closed under mixture or not.

Some bounds on the moments and on the survival function of a HNBUE (HNWUE) life distribution are presented. For instance we have the following theorem.
THEOREM 2.1 Let \( F \) be a life distribution and let \( \mu_r = \int_0^\infty x^r dF(x) \) and \( \lambda_r = \mu_r / \Gamma(r+1) \) for \( r > 0 \). If \( F \) is HNBUE (HNWUE) then

\[
\begin{cases} 
\lambda_r \leq (\geq) \lambda_1^r & \text{for } r \geq 1 \\
\lambda_r \geq (\leq) \lambda_1^r & \text{for } 0 < r < 1.
\end{cases}
\]

The bounds are sharp.

A consequence of this theorem is that, for an HNBUE life distribution, the "Carleman condition" that \( \sum_{j=1}^\infty \mu_{2j}^{-1/2j} = \infty \) holds. Accordingly, the moments \( \mu_j, j = 1, 2, 3, \ldots \), uniquely determine \( F \) (see Feller (1971), p. 227). This was pointed out by Bhattacharjee (1980) in the NBU case.

Bounds on the survival function are useful e.g. in reliability since in a typical situation the only fact known a priori may be that the time to failure is HNBUE (HNWUE) with mean \( \mu \). If \( F \) is HNWUE we have

\[ 0 \leq \overline{F}(t) \leq \frac{\mu}{t}(1 - \exp(-t/\mu)) \quad \text{for } t \geq 0. \]

In the HNBUE case we have

\[
\overline{F}(t) \leq \begin{cases} 
1 & \text{for } t \leq \mu \\
\exp\left(\frac{\mu-t}{\mu}\right) & \text{for } t > \mu
\end{cases}
\]

and

\[
\overline{F}(t) \geq \begin{cases} 
\exp(-\alpha/\mu) & \text{for } 0 \leq t < \mu \\
0 & \text{for } t \geq \mu
\end{cases},
\]

where \( \alpha = \alpha(t) \) is the largest non-negative number for which

\[ (\alpha-t+\mu)\exp(-\alpha/\mu) - \mu + t = 0. \]
3. Aging properties and shock models - papers [B] and [C]

Suppose that a device is subjected to shocks governed by a counting process \( N = \{N(t): t \geq 0\} \). The probability that the device survives beyond \( t \) is then given by

\[
H(t) = \sum_{k=0}^{\infty} P(N(t) = k) \bar{P}_k,
\]

where \( \bar{P}_k \) denotes the probability of surviving \( k \) shocks. In paper [B] we prove for some different processes \( N \) that \( H(t) \) in (3.1) is HNBUE (HNWUE) if \( (\bar{P}_k)_{k=0}^{\infty} \) is discrete HNBUE (discrete HNWUE) as defined in paper [A]. For instance we study the case when the interarrival times between shocks are independent and HNBUE (HNWUE). This situation includes the cases when \( N \) is a Poisson process or a pure birth process.

Similar results in the IFR, IFRA, NBU, NBUE and DMRL cases (with duals) have been derived by Esary, Marshall and Proschan (1973), A-Hameed and Proschan (1973), A-Hameed and Proschan (1975) and Block and Savits (1978).

In paper [C] we study the following Pure Birth Shock Model considered by A-Hameed and Proschan (1975):

Shocks occur according to a Markov process; given that \( k \) shocks have occurred in \((0, t]\), the probability of a shock in \((t, t+\Delta t] \) is \( \lambda_k \lambda(t) \Delta t + o(\Delta t) \), while the probability of more than one shock in \((t, t+\Delta t] \) is \( o(\Delta t) \).

A-Hameed and Proschan (1975) proved that the survival function \( \bar{H}(t) \) in (3.1) is IFR, IFRA, NBU, NBUE and DMRL under suitable conditions on \( (\lambda_k)_{k=0}^{\infty} \), \( \lambda(t) \) and \( (\bar{P}_k)_{k=0}^{\infty} \). We prove that \( \bar{H}(t) \) is IFRA under weaker conditions on \( (\lambda_k)_{k=0}^{\infty} \) and \( (\bar{P}_k)_{k=0}^{\infty} \). We also prove that \( \bar{H}(t) \) is DMRL if \( \lambda(t) \) and \( (\lambda_k)_{k=0}^{\infty} \) are increasing and \( \left( \sum_{j=k}^{\infty} \bar{P}_j \right) / \bar{P}_k \) is decreasing. Dual theorems in the DFRA and IMRL cases are also presented.
4. Statistics for testing against aging properties

4.1 Testing against IFR, IFRA, NBUE, DMRL and heavy-tailedness - paper [D]

Some of the aging properties in Definition 1.1 can be translated to properties of the total time on test (TTT-) transform defined by

$$H_F^{-1}(t) = \int_0^t F(s)ds$$

for $0 \leq t \leq 1$,

where

$$F^{-1}(t) = \inf\{x: F(x) \geq t\}.$$

We summarize and discuss such correspondences given by Barlow and Campo (1975), Barlow (1979) and Bergman (1979). We also present some new results. Further we consider the connection between the TTT-transform and the concept of heavy-tailedness (as defined by Vännman (1975)).

Properties of the TTT-transform are used to get ideas for different test statistics for testing the hypothesis

$$H_0: F \text{ is the exponential distribution}$$

against

$$H_1: F \text{ is } \mathcal{V} \text{ but not exponential},$$

where $\mathcal{V}$ denotes IFR, IFRA, NBUE, DMRL or heavy-tailedness. Statistics for testing $H_0$ against $H_1$ have been proposed by several authors; for references see paper [D].

Let $0 = t(0) \leq t(1) \leq t(2) \leq \ldots \leq t(n)$ denote an ordered sample from a life distribution $F$. Further let

$$D_j = (n-j+1)(t(j)-t(j-1)) \text{ for } j = 1,2,\ldots,n$$

and

$$S_n = \sum_{j=1}^{n} t(j).$$
We consider the following test statistics.

Against IFR:

\[ A_1 = \frac{(D_n - D_1)}{S_n}; \]
\[ A_2 = \sum_{j=1}^{n} \alpha_j \frac{D_j}{S_n}, \]
with \( \alpha_j = \frac{1}{6} \{ (n+1)^3 j - 3(n+1)^2 j^2 + 2(n+1)j^3 \}. \)

Against IFRA:

\[ B = \sum_{j=1}^{n} \beta_j \frac{D_j}{S_n}, \]
with \( \beta_j = \frac{1}{6} (n^3 + 3n^2 j - 3nj - 3j^2 + j + 2n^2 + 2j^3). \)

Against DMRL:

\[ M = \sum_{j=1}^{n} \gamma_j \frac{D_j}{S_n}, \]
with \( \gamma_1 = 1 - n, \gamma_j = 4j - 3 - 2n, 2 \leq j \leq n. \)

Against heavy-tailedness:

\[ E = \frac{nD_n}{S_n}. \]

The asymptotic distributions of the test statistics are discussed. We also study the consistency. For instance, the tests based on \( A_2 \) and \( B \) are consistent against the classes of continuous IFR and IFRA life distributions, respectively. We give the Pitman efficiency values for testing exponentiality against some common families of life distributions and present a small sample study. We study by simulation the sample sizes \( n = 10 \) and \( n = 20 \) for some Weibull, gamma and Pareto alternatives and the life distribution

\[
F(t) = (1-e^{-3t})(1-e^{-7t}) \quad \text{for} \ t \geq 0,
\]
which is IFRA but not IFR. The power estimates which are obtained in this study confirm the asymptotic results shown by the efficiency values. For instance the IFRA statistic $B$ has good power values in the studied cases. Against $F$ in (4.1) $B$ has the largest power values of all the included statistics. The statistic $E$ for testing against heavy-tailedness has unpleasant small power values in many cases.

### 4.2 Testing against HNBUE - paper \[E\]

We propose some test statistics for testing exponentiality against the HNBUE (HNWUE) alternative. Two of these are based on the following idea.

If $F$ is HNBUE with mean $\mu$ then

\[
(4.2) \quad \int_0^\infty F^\nu(x)dx > \frac{\mu}{\nu} \quad \text{for } \nu = 2, 3, 4, ... 
\]

and

\[
(4.3) \quad \int_0^\infty (1 - F^\nu(x))dx \leq \mu \sum_{j=1}^{\nu} \frac{1}{j} \quad \text{for } \nu = 2, 3, 4, ... 
\]

If $F$ is HNBUE (with $F(0) \neq 0$) but not exponential then equality can not hold in any of (4.2) and (4.3). Therefore, if we can find a statistic $\Delta_3$ such that

\[
\mathcal{L} \left( \frac{n(\Delta_3 - \mu_3)}{\sigma_3} \right) \to N(0,1) \quad \text{when } n \to \infty,
\]

where

\[
\mu_3 = \int_0^\infty F^\nu(x)dx - \frac{\mu}{\nu}
\]

and $\sigma_3$ is independent of $n$, we get a test which is consistent against the HNBUE alternative.

We prove that

\[
\Delta_3 = \sum_{j=1}^{n} \frac{J_3(j/n)t(j)/S_n}{n},
\]
where
\[ J_3(u) = -1 + v^2(1-u)^{v-1}, \]
has this property for every \( v = 2, 3, 4, \ldots \). Also for
\[ \Delta_4 = \sum_{j=1}^{n} \frac{J_4(j/n)t(j)/s_n}{}, \]
where
\[ J_4(u) = \sum_{j=1}^{v} \frac{1}{J} - vu^{v-1}, \]
we have
\[ \mathcal{L} \left( \frac{\sqrt{n}(\Delta_4 - \mu_4)}{\sigma_4} \right) + N(0,1) \text{ when } n \rightarrow \infty, \]
where
\[ \mu_4 = \sum_{j=1}^{v} \frac{1}{J} - \int_{0}^{\infty} \left( 1 - F(x) \right) dx \]
and \( \sigma_4 \) is independent of \( n \). This means that a test based on \( \Delta_4 \) is consistent against the HNBUE alternative. We try to choose \( v \) so that \( \Delta_3 \) and \( \Delta_4 \) get as large efficiency values as possible when testing against some common families of life distributions. It is shown that the optimal value of \( v \) depends on the family of life distributions in the alternative hypothesis. We have chosen \( v = 3 \) as a compromise both for \( \Delta_3 \) and \( \Delta_4 \).

A small sample study shows that \( \Delta_3 \), with \( v = 3 \), is a good test statistic whose power values in many cases are better than e.g. those of the cumulative TTT-statistic (see e.g. Barlow et al. (1972)).
5. Bivariate aging properties - paper [F]

During the last years efforts have been made to define bivariate and multivariate extensions of the univariate aging properties IFR, IFRA, NBU NBUE and DMRL (with duals); for references see paper [F]. We present two new definitions of bivariate NBUE (NWUE) and several definitions of bivariate HNBUE (HNWUE). Four of the bivariate HNBUE (HNWUE) definitions are analogies to the multivariate aging properties given by Buchanan and Singpurwalla (1977). They gave in each one of the IFR, IFRA, NBU, NBUE and DMRL cases four different proposals of multivariate extensions with suffices VS (= Very Strong), S (= Strong), W (= Weak) and VW (= Very Weak). These are in the following definition named "jointly NBUE-*", where * denotes VS, S, W or VW.

**DEFINITION 5.1** Let $F(t_1, t_2)$ be a bivariate life distribution and

\[ \bar{F}(t_1, t_2) \]

the corresponding survival function. Suppose that the moments 

\[ \int_0^\infty \int_0^\infty t_1^i t_2^j dF(t_1, t_2) < \infty \]

for $i, j = 0, 1$. Then $F(t_1, t_2)$ and $\bar{F}(t_1, t_2)$ are said to be

**NBUE-VS** if 

\[ \int_0^\infty \int_0^\infty F(x+t_1, x+t_2) dt_1 dt_2 < \int_0^\infty F(x_1, x_2) dF(t_1, t_2) \]

for every $(x_1, x_2) \geq 0$;

**NBU-S** if 

\[ \int_0^\infty \int_0^\infty F(x+t_1, x+t_2) dt_1 dt_2 \leq \int_0^\infty F(x_1, x_2) dF(t_1, t_2) \]

for every $(x_1, x_2) \geq 0$;

**NBU-W** if 

\[ \int_0^\infty \int_0^\infty F(x+t_1, x+t_2) dt_1 dt_2 \leq \int_0^\infty F(x, x) dF(t_1, t_2) \]

for every $x \geq 0$;

**NBU-VW** if 

\[ \int_0^\infty \int_0^\infty F(x+t, x+t) dt_1 dt_2 \leq \int_0^\infty F(x, x) dF(t, t) \]

for every $t \geq 0$.

We shall say that $F(t_1, t_2)$ and $\bar{F}(t_1, t_2)$ are jointly NBUE-* if $F(t_1, t_2)$ is NBUE-* and the marginal distributions are (univariate) NBUE. Here * denotes VS, S, W or VW.
The reason for Buchanan and Singpurwalla (1977) to add the suffices VS, S, W and VW was that they thought that the chain

\[(5.1) \text{ jointly } \nabla-VS \rightarrow \text{ jointly } \nabla-S \rightarrow \text{ jointly } \nabla-W \rightarrow \text{ jointly } \nabla-VW\]

of implications is true when \( \nabla \) is IFR, IFRA, NB, NBUE or DMRL (with duals). We prove that (5.1) is not true in the NBUE and DMRL cases.

We also study two bivariate shock models. Suppose that two devices, I and II say, are subjected to shocks. Let \( T_1 \) and \( T_2 \) denote the life lengths of the devices and let \( N_j, j=1, 2 \), be the number of shocks on device \( j \) until failure. If further

\[\tilde{P}(k_1, k_2) = P(N_1 > k_1, N_2 > k_2) \quad \text{for } k_1, k_2 = 0, 1, 2, \ldots,\]

then the bivariate variable \((T_1, T_2)\) has the survival function

\[\tilde{H}(t_1, t_2) = P(T_1 > t_1, T_2 > t_2)\]

given by

\[\tilde{H}(t_1, t_2) = \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} P(S_I(t_1) = k_1, S_{II}(t_2) = k_2) \tilde{P}(k_1, k_2),\]

where \( S_I(t_1) \) and \( S_{II}(t_2) \) are the number of shocks for which the two devices have been exposed up to \( t_1 \) and \( t_2 \), respectively.

One possible model is obtained when the shocks occur according to a homogeneous Poisson process and upon occurrence of a shock both devices suffer damages. This model was named the Marshall-Olkin Shock Model by Block (1977). Recently Marshall and Shaked (1979) gave conditions on \( \tilde{P}(k_1, k_2), k_1, k_2 = 0, 1, 2, \ldots, \) under which \( \tilde{H}(t_1, t_2) \) is multivariate IFRA of different forms.

Another interesting shock model is obtained when three independent homogeneous Poisson processes \( Z_1(t), Z_2(t) \) and \( Z_{12}(t) \) deliver shocks in such a manner that \( S_I(t_1) = Z_1(t_1) + Z_{12}(t_1) \) and \( S_{II}(t_2) = Z_2(t_2) + Z_{12}(t_2) \). This shock model was studied by Buchanan and Sing-
purwalla (1975) when $N_1$ and $N_2$ are independent. Because of this, Block (1977) named the model the Buchanan-Singpurwalla Shock Model.

We study the Marshall-Olkin and Buchanan-Singpurwalla Shock Models and give sufficient conditions, containing $P(k_1, k_2)$, $k_1, k_2 = 0, 1, 2, \ldots$, under which $H(t_1, t_2)$ is bivariate NBU (NWU), bivariate NBUE (NWUE) and bivariate HNBUE (HNWUE) of different forms.
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