IFRA OR DMRL SURVIVAL UNDER THE
PURE BIRTH SHOCK PROCESS

by

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Key words and phrases: Shock model, birth process, survival function,
variation diminishing property, total positivity, IFRA, DFRA, DMRL, IMRL.
Suppose that a device is subjected to shocks and that \( \bar{p}_k, k = 0, 1, 2, \ldots, \infty \) denotes the probability of surviving \( k \) shocks. Then \( \bar{H}(t) = \sum_{k=0}^{\infty} P(N(t) = k) \bar{p}_k \) is the probability that the device will survive beyond \( t \), where \( N = \{N(t): t \geq 0\} \) is the counting process which governs the arrival of shocks. A-Hameed and Proschan (1975) considered the survival function \( \bar{H}(t) \) under what they called the Pure Birth Shock Model. In this paper we shall prove that \( \bar{H}(t) \) is IFRA and DMRL under conditions which differ from those used by A-Hameed and Proschan (1975).
1. Introduction

Suppose that a device is subjected to shocks and that the probability of surviving \( k \) shocks is \( \overline{P}_k \), \( k = 0, 1, 2, \ldots \), where \( 1 = \overline{P}_0 > \overline{P}_1 > \overline{P}_2 > \ldots \).

If \( N = \{N(t): t \geq 0\} \) is the counting process which governs the arrival of shocks, the probability \( \overline{H}(t) \) that the device will survive beyond \( t \) is given by

\[
\overline{H}(t) = \sum_{k=0}^{\infty} P(N(t) = k)\overline{P}_k.
\]

Shock models of this kind have been studied by several authors; e.g. Esary, Marshall and Proschan (1973), A-Hameed and Proschan (1973, 1975), Block and Savits (1978) and Klefsjö (1980).

In this paper we shall study the following Pure Birth Shock Model considered by A-Hameed and Proschan (1975):

Shocks occur according to a Markov process; given that \( k \) shocks have occurred in \( (0, t] \), the probability of a shock in \( (t, t + \Delta t] \) is \( \lambda_k \lambda(t) \Delta t + o(\Delta t) \), while the probability of more than one shock in \( (t, t + \Delta t] \) is \( o(\Delta t) \).

For this shock model the survival function \( \overline{H}(t) \) in (1.1) can be written as

\[
\overline{H}(t) = \overline{S}(\Lambda(t))
\]

with

\[
\overline{S}(t) = \sum_{k=0}^{\infty} z_k(t)\overline{P}_k,
\]

where

\[
z_k(t) = P(\text{exactly } k \text{ shocks have occurred in } (0, t] \text{ when } \lambda(t) = 1)
\]

and

\[
\Lambda(t) = \int_{0}^{t} \lambda(x)dx
\]
This means that \( \bar{S} \) is the survival function when the shocks occur according
to a stationary birth process for which the interarrival times between
the shocks number \( k \) and \( k + 1, k = 0, 1, 2, \ldots \), are independent and
exponentially distributed with mean \( \lambda_k^{-1} \).

We are only interested in sequences \( (\lambda_k)_{k=0}^{\infty} \) for which the probability
of infinitely many shocks in \((0,t]\) is zero, i.e. for which \( \sum_{k=0}^{\infty} z_k(t) = 1 \).
By the Feller-Lundberg Theorem (see Feller (1968), p. 452) this is equivalent
to the condition that \( \sum_{k=0}^{\infty} \lambda_k^{-1} = \infty \). This condition is not explicitly

A-Hameed and Proshcan (1975) proved that \( \bar{H}(t) \) in (1.2) is IFR, IFRA,
NBU, NBUE and DMRL (for definitions see e.g. Barlow and Proshcan (1975))
under suitable conditions on \( (\lambda_k)_{k=0}^{\infty}, (\bar{P}_k)_{k=0}^{\infty} \). For instance
they proved that

(a) \( \bar{H}(t) \) is IFRA if \( (\lambda_k)_{k=0}^{\infty} \) is increasing, \( \Lambda(t) \) is starshaped (i.e.
\( \Lambda(t)/t \) is increasing for \( t \geq 0 \) and \( \Lambda(0) = 0 \)) and \( (\bar{P}_k)_{k=0}^{\infty} \) has
the discrete IFRA property. (Theorem 2.6 in A-Hameed and Proshcan
(1975).)

(b) \( \bar{H}(t) \) is DMRL if \( \lambda(t) \) is increasing and \( \left( \sum_{j=k}^{\infty} \bar{P}_j \lambda_j^{-1} \right)/\bar{P}_k \) is decreasing
in \( k = 0, 1, 2, \ldots \). (Theorem 2.10 in A-Hameed and Proshcan (1975).)

In this paper we shall prove that \( \bar{H}(t) \) is IFRA under weaker condi-
tions on \( (\lambda_k)_{k=0}^{\infty} \) and \( (\bar{P}_k)_{k=0}^{\infty} \) than those used by A-Hameed and Proshcan
(1975). We shall also prove that \( \bar{H}(t) \) is DMRL when \( \lambda(t) \) and \( (\lambda_k)_{k=0}^{\infty} \)
are increasing and \( (\bar{P}_k)_{k=0}^{\infty} \) has the discrete DMRL property. Finally, we
present dual theorems in the DFRA and IMRL cases.

2. IFRA survival

The main theorem of this section is the following one.
**Theorem 2.1** The survival function $H(t)$ in (1.2) is IFRA if

(i) $\Lambda(t)$ is starshaped

and

(ii) for every $\theta$ with $0 < \theta < \lambda_0$ the sequence $\prod_{j=0}^{k-1} (1 - \theta \lambda_j^{-1})$, $k = 1, 2, 3, \ldots$, changes sign at most once, and if once, from $+$ to $-$

and

(iii) there is a $j_0$ such that $\lambda_j \geq \lambda_0$ for $j \geq j_0$.

To prove that theorem we use the variation diminishing property of the totally positive kernel $z_k(t)$ in $k = 1, 2, 3, \ldots$, and $t \in [0, \infty)$ (see Karlin (1968), Ch. 1). The fact that $z_k(t)$ is totally positive follows from Theorem 3 in Karlin and Proshcan (1960). We also need the following lemma.

**Lemma 2.2** Suppose that there is a $j_0$ such that $0 < \theta \leq \lambda_j$ for $j \geq j_0$.

Let $E_0 = 1$ and $E_k = \prod_{j=0}^{k-1} (1 - \theta \lambda_j^{-1})$, $k = 1, 2, 3, \ldots$, and let $\bar{L}(t) = \sum_{k=0}^{\infty} z_k(t) E_k$, where $z_k(t)$ is given by (1.4). Then $\bar{L}(t) = \exp(-\theta t)$ if $\sum_{k=0}^{\infty} \lambda_k^{-1} = \infty$.

**Proof** Let $\pi_k(s)$ denote the Laplace transform of $z_k(t)$, i.e.

$$\pi_k(s) = \int_0^{\infty} e^{-st} z_k(t) dt.$$ 

Then

$$\pi_0(s) = \frac{1}{\lambda_0 + s}$$

and

$$\pi_k(s) = \left( \prod_{j=0}^{k-1} \frac{\lambda_j}{\lambda_j + s} \right) \frac{1}{\lambda_k + s} \text{ for } k = 1, 2, 3, \ldots$$

(see Feller (1971), p. 489). From this follows that the Laplace transform $g(s)$ of $\bar{L}(t)$ is given by

$$g(s) = \int_0^{\infty} e^{-st} \bar{L}(t) dt = \sum_{k=0}^{\infty} E_k \pi_k(s) = \frac{1}{\lambda_0 + s} + \sum_{k=1}^{\infty} \left( \prod_{j=0}^{k-1} \frac{\lambda_j}{\lambda_j + s} \right) \frac{1}{\lambda_k + s}$$

is this series converges.

With

$$\theta_0 = \frac{1}{s + \theta}$$
and

\[ \beta_k = \frac{1}{s+\theta} \left( \prod_{j=0}^{k-1} \frac{\lambda_j - \theta}{\lambda_j + s} \right) \] for \( k = 1, 2, 3, \ldots \)

we can write the terms of the series in (2.1) as

\[
\begin{align*}
\frac{1}{\lambda_0 + s} &= \beta_0 - \beta_1 \\
\left( \prod_{j=0}^{k-1} \frac{\lambda_j - \theta}{\lambda_j + s} \right) \frac{1}{\lambda_k + s} &= \beta_k - \beta_{k+1} \quad \text{for} \quad k = 1, 2, 3, \ldots
\end{align*}
\]

Accordingly, the nth partial sum is equal to

\[
\sum_{k=0}^{n} E_k \pi_k(s) = \beta_0 - \beta_{n+1} = \frac{1}{s+\theta} \left( 1 - \prod_{j=0}^{n} \frac{\lambda_j - \theta}{\lambda_j + s} \right)
\]

Using the uniqueness of the Laplace transform (see Feller (1971), p. 430) it is now sufficient to prove that, for \( s \geq 0 \),

\[
(2.2) \quad \lim_{n \to \infty} \prod_{j=0}^{n} \frac{\lambda_j - \theta}{\lambda_j + s} = 0 \quad \text{if} \quad \sum_{j=0}^{\infty} \lambda_j^{-1} = \infty.
\]

But from the Mean Value Theorem we get that

\[
\ln(\lambda_j + s) - \ln(\lambda_j - \theta) \geq \frac{s + \theta}{\lambda_j + s} \quad \text{for} \quad j \geq j_0
\]

and hence that

\[
-\ln \prod_{j=j_0}^{n} \frac{\lambda_j - \theta}{\lambda_j + s} \geq (s + \theta) \sum_{j=j_0}^{n} \frac{1}{\lambda_j + s}.
\]

From this inequality follows (2.2) and the proof is complete.

**PROOF OF THEOREM 2.1** From Lemma 2.1 in A.-Hameed and Proschan (1975) it follows that the survival function \( H(t) \) in (1.2) is IFRA if \( S(t) \) in (1.3) is IFRA and \( \Lambda(t) \) is starshaped. Accordingly, it is sufficient to prove that \( S(t) \) is IFRA if \( (\lambda_k)_{k=0}^{\infty} \) and \( (\bar{F}_k)_{k=0}^{\infty} \) satisfy the conditions (ii) and (iii). This will be done by showing that, for every \( \theta > 0 \), \( S(t) - \exp(-\theta t) \) changes sign at most once, and if once, from + to −.
That this is sufficient for \( \bar{S}(t) \) to be IFRA follows from Theorem 2.18 in Barlow and Proschan (1975), p. 89.

Since
\[
\bar{S}(t) = z_0(t) + \sum_{k=1}^{\infty} z_k(t) \bar{P}_k \geq z_0(t) = \exp(-\lambda_0 t)
\]
we get that \( \bar{S}(t) \geq \exp(-\theta t) \) if \( \theta \geq \lambda_0 \). Now suppose \( 0 < \theta < \lambda_0 \). Then Lemma 2.2 gives that
\[
\bar{S}(t) - \exp(-\theta t) = \bar{S}(t) - \bar{L}(t) = \sum_{k=1}^{\infty} z_k(t)(\bar{P}_k - E_k),
\]
where
\[
\begin{cases}
E_0 = 1 \\
E_k = \prod_{j=0}^{k-1} (1 - \theta \lambda_j^{-1}) \text{ for } k = 1, 2, 3, \ldots.
\end{cases}
\]
Since \( z_k(t) \) is totally positive in \( k = 1, 2, 3, \ldots \) and \( t \geq 0 \) it follows from the variation diminishing property that \( \bar{S}(t) - \exp(-\theta t) \) has the same sign changes as \( \bar{P}_k - E_k, k = 1, 2, 3, \ldots \). This completes the proof.

We shall now prove that our conditions (ii) and (iii) in Theorem 2.1 hold when \( (\lambda_k)_{k=0}^{\infty} \) and \( (\bar{P}_k)_{k=0}^{\infty} \) satisfy the conditions used by A-Hameed and Proschan (1975) in their Theorem 2.6 (cf. (a) on p. 2).

**THEOREM 2.3** The conditions (ii) and (iii) in Theorem 2.1 hold if \( (\lambda_k)_{k=0}^{\infty} \) is increasing and \( (\bar{P}_k)_{k=0}^{\infty} \) has the discrete IFRA property.

**PROOF** That condition (iii) is satisfied is obvious. To prove that (ii) holds, take a \( \theta \) with \( 0 < \theta < \lambda_0 \). Since \( (\lambda_k)_{k=0}^{\infty} \) is increasing it can be proved by induction that the geometric mean
\[
M_k(\theta) = \left\{ \prod_{j=0}^{k-1} (1 - \theta \lambda_j^{-1}) \right\}^{1/k}
\]
is increasing in \( k \). Since further \( (\bar{P}_k)_{k=1}^{\infty} \) is decreasing if \( (\bar{P}_k)_{k=0}^{\infty} \)
is IFRA we get that \( \frac{1}{k} - M_k(\theta) \) is decreasing in \( k \). Accordingly, \( \frac{1}{k} - M_k(\theta) \), \( k = 1, 2, 3, \ldots \), changes sign at most once, and if one change occurs, it occurs from + to -. Since \( \prod_{j=0}^{k-1} (1 - \theta \lambda_j^{-1}) \) has the same sign as \( \frac{1}{k} - M_k(\theta) \) the proof is complete.

We shall now give an example where conditions (ii) and (iii) in Theorem 2.1 hold although the sequence \( (\bar{P}_k)_{k=0}^{\infty} \) of survival probabilities is not IFRA and \( (\lambda_k)_{k=0}^{\infty} \) is not increasing. Together with Theorem 2.3 this shows that the conditions on \( (\lambda_k)_{k=0}^{\infty} \) and \( (\bar{P}_k)_{k=0}^{\infty} \) used in Theorem 2.1 are weaker than those used by A-Haneed and Proschan (1975) in their Theorem 2.6.

Choose \( \bar{P}_0 = 1, \bar{P}_1 = 3/4, \bar{P}_2 = 5/8, \bar{P}_3 = 2/5, \bar{P}_k = 0 \) for \( k \geq 4 \), \( \lambda_0 = 1 \), \( \lambda_1 = 2 \), \( \lambda_2 = 1 \) and \( \lambda_k = k \) for \( k \geq 3 \). Then \( (\bar{P}_k)_{k=0}^{\infty} \) is not IFRA since \( \bar{P}_1 < \bar{P}_2^{1/2} \). Further \( (\lambda_k)_{k=0}^{\infty} \) is not increasing. But (iii) holds and

\[
\gamma_k = \frac{1}{k} - \prod_{j=0}^{k-1} (1 - \theta \lambda_j^{-1}) \quad \text{for} \quad k = 1, 2, 3, \ldots
\]

satisfies the condition (ii) in Theorem 2.1 since straight-forward calculations show that \( \gamma_1 > \gamma_2 > \gamma_3 \) and \( \gamma_k < 0 \) for \( k \geq 4 \).

3. DMRL survival

In this section we shall prove the following theorem.

**Theorem 3.1** The survival function \( \bar{H}(t) \) in (1.2) is DMRL if \( \lambda(t) \) and \( (\lambda_k)_{k=0}^{\infty} \) are increasing and \( (\bar{P}_k)_{k=0}^{\infty} \) has the discrete DMRL property.

**Proof** From Lemma 2.2 in A-Haneed and Proschan (1975) it follows that \( \bar{H}(t) \) is DMRL if \( \lambda(t) \) is increasing and \( \bar{S}(t) \) in (1.3) is DMRL. Furthermore, Theorem 2.10 is A-Haneed and Proschan (1975) gives that \( \bar{S}(t) \) is DMRL if \( \left( \sum_{j=k}^{\infty} \bar{P}_j \lambda_j^{-1} \right) / \bar{P}_k \) is decreasing in \( k = 0, 1, 2, \ldots \). Accordingly, it is sufficient to prove that
\[
\bar{P}_k^{-1} \sum_{j=k}^{\infty} \bar{P}_j \lambda_j^{-1} - \bar{P}_{k+1}^{-1} \sum_{j=k+1}^{\infty} \bar{P}_j \lambda_j^{-1} \geq 0 \quad \text{for} \quad k = 0, 1, 2, \ldots
\]
if \((\bar{P}_k)_{k=0}^{\infty}\) is DMRL and \((\lambda_k)_{k=0}^{\infty}\) is increasing.

Since \((\lambda_k)_{k=0}^{\infty}\) is increasing \(\lim_{k \to \infty} \lambda_k^{-1} = \lambda_k^{-1} \geq 0\) exists. From this follows that

\[
\bar{P}_k^{-1} \sum_{j=k}^{\infty} \bar{P}_j \lambda_j^{-1} - \bar{P}_{k+1}^{-1} \sum_{j=k+1}^{\infty} \bar{P}_j \lambda_j^{-1} = \\
= \left[ \lambda_k^{-1} \left( \bar{P}_k^{-1} \sum_{j=k}^{\infty} \bar{P}_j - \bar{P}_{k+1}^{-1} \sum_{j=k+1}^{\infty} \bar{P}_j \right) \right] + \\
+ \left[ \bar{P}_k^{-1} \sum_{j=k}^{\infty} \bar{P}_j (\lambda_j^{-1} - \lambda_j^{-1}) - \bar{P}_{k+1}^{-1} \sum_{j=k+1}^{\infty} \bar{P}_j (\lambda_j^{-1} - \lambda_j^{-1}) \right] = I + II.
\]

Here the expression I is non-negative since \((\bar{P}_k)_{k=0}^{\infty}\) is DMRL. If we write

\[
\lambda_j^{-1} - \lambda_j^{-1} = \sum_{\nu=0}^{\infty} (\lambda_{\nu}^{-1} - \lambda_{\nu+1}^{-1})
\]
and then change the summation order we can write the expression II as

\[
= \sum_{\nu=0}^{\infty} (\lambda_{\nu}^{-1} - \lambda_{\nu+1}^{-1}) \bar{P}_k^{-1} \sum_{j=k}^{\infty} \bar{P}_j - \sum_{\nu=0}^{\infty} (\lambda_{\nu}^{-1} - \lambda_{\nu+1}^{-1}) \bar{P}_{k+1}^{-1} \sum_{j=k+1}^{\infty} \bar{P}_j = \\
\left[ \bar{P}_k^{-1} \sum_{j=k}^{\infty} \bar{P}_j - \bar{P}_{k+1}^{-1} \sum_{j=k+1}^{\infty} \bar{P}_j \right] + \sum_{\nu=0}^{\infty} (\lambda_{\nu}^{-1} - \lambda_{\nu+1}^{-1}) \left[ \bar{P}_k^{-1} \sum_{j=k}^{\infty} \bar{P}_j - \bar{P}_{k+1}^{-1} \sum_{j=k+1}^{\infty} \bar{P}_j \right].
\]

If

\[
\phi_k(\nu) = \bar{P}_k^{-1} \sum_{j=k}^{\infty} \bar{P}_j - \bar{P}_{k+1}^{-1} \sum_{j=k+1}^{\infty} \bar{P}_j \geq 0
\]
for \(\nu = k + 1, k + 2, k + 3, \ldots\) the proof is complete. But this follows from the facts that

\[
\phi_k(\nu) - \phi_k(\nu + 1) = \bar{P}_{\nu+1}^{-1} (\bar{P}_{k+1}^{-1} - \bar{P}_k^{-1}) \geq 0
\]
since \((\bar{P}_k)_{k=0}^{\infty}\) is decreasing, and that

\[
\lim_{\nu \to \infty} \phi_k(\nu) = \bar{P}_k^{-1} \sum_{j=k}^{\infty} \bar{P}_j - \bar{P}_{k+1}^{-1} \sum_{j=k+1}^{\infty} \bar{P}_j \geq 0
\]
since \((\bar{P}_k)_{k=0}^{\infty}\) is DMRL. \(\square\)
Note that the conditions on \((\lambda_k)_{k=0}^\infty\) and \((\bar{P}_k)_{k=0}^\infty\) are equivalent and analogous, respectively, to those used by A-Hameed and Proschan (1975) in the IFR, IFRA and NBU cases and by Block and Savits (1978) in the NBUE case.

We shall now illustrate that \(\left(\sum_{j=k}^\infty \bar{P}_j \lambda_j^{-1}/\bar{P}_k\right)\), \(k = 1, 2, 3, \ldots\), may be decreasing although neither \((\lambda_k)_{k=0}^\infty\) is increasing nor \((\bar{P}_k)_{k=0}^\infty\) is DMRL. This means that the conditions on \((\lambda_k)_{k=0}^\infty\) and \((\bar{P}_k)_{k=0}^\infty\) used in Theorem 3.1 is stronger than those used by A-Hameed and Proschan (1975) in their Theorem 2.10 (cf. (b) on p. 2).

Choose \(\bar{P}_0 = 1, \bar{P}_1 = 3/4, \bar{P}_2 = 1/2, \bar{P}_k = 2^{-k}\) for \(k \geq 3\), \(\lambda_0 = \lambda_1 = 1, \lambda_2 = 1/2\) and \(\lambda_k = 1\) for \(k \geq 3\). Then \((\lambda_k)_{k=0}^\infty\) is not increasing and \((\bar{P}_k)_{k=0}^\infty\) is not DMRL since \(\left(\sum_{j=2}^\infty \bar{P}_j \lambda_j^{-1}/\bar{P}_2\right) = 3/2\) and \(\left(\sum_{j=3}^\infty \bar{P}_j \lambda_j^{-1}/\bar{P}_3\right) = 2\). But

\[
\sum_{j=k}^\infty \bar{P}_j \lambda_j^{-1}/\bar{P}_k = \begin{cases} 
3 & \text{for } k = 0 \\
8/3 & \text{for } k = 1 \\
5/2 & \text{for } k = 2 \\
2 & \text{for } k \geq 3
\end{cases}
\]

and therefore \(\left(\sum_{j=k}^\infty \bar{P}_j \lambda_j^{-1}/\bar{P}_k\right)\), \(k = 0, 1, 2, 3, \ldots\), is decreasing.

4. The dual cases

By reversing all inequalities and the directions of monotonicity we get dual results to Theorem 2.1 and Theorem 3.1.

**Theorem 4.1.** The survival function \(\bar{H}(t)\) in (1.2) is DFRA if

(i) \(\Lambda(t)\) is antistarshaped (i.e. \(\Lambda(t)/t\) is decreasing for \(t \geq 0\) and \(\Lambda(0) = 0\))

and

(ii) for every \(\theta\) with \(0 < \theta < \lambda_0\) the sequence \(\bar{P}_k - \prod_{j=0}^{k-1} (1 - \theta \lambda_j^{-1})\), \(k = 1, 2, 3, \ldots\), changes sign at most once, and if once, from \(-\) to \(+\)

and

(iii) there is a \(j_0\) such that \(\lambda_j \geq \lambda_0\) for \(j \geq j_0\).
THEOREM 4.2 The survival function \( \bar{H}(t) \) in (1.2) is IMRL if \( \lambda(t) \) and \( (\lambda_k)_{k=0}^{\infty} \) are decreasing and \( (\bar{F}_k)_{k=0}^{\infty} \) has the discrete IMRL property.

Acknowledgement

The author is very grateful to Dr Bo Bergman, Professor Gunnar Kulldorff and Dr Kerstin Vännman for valuable comments.

References


