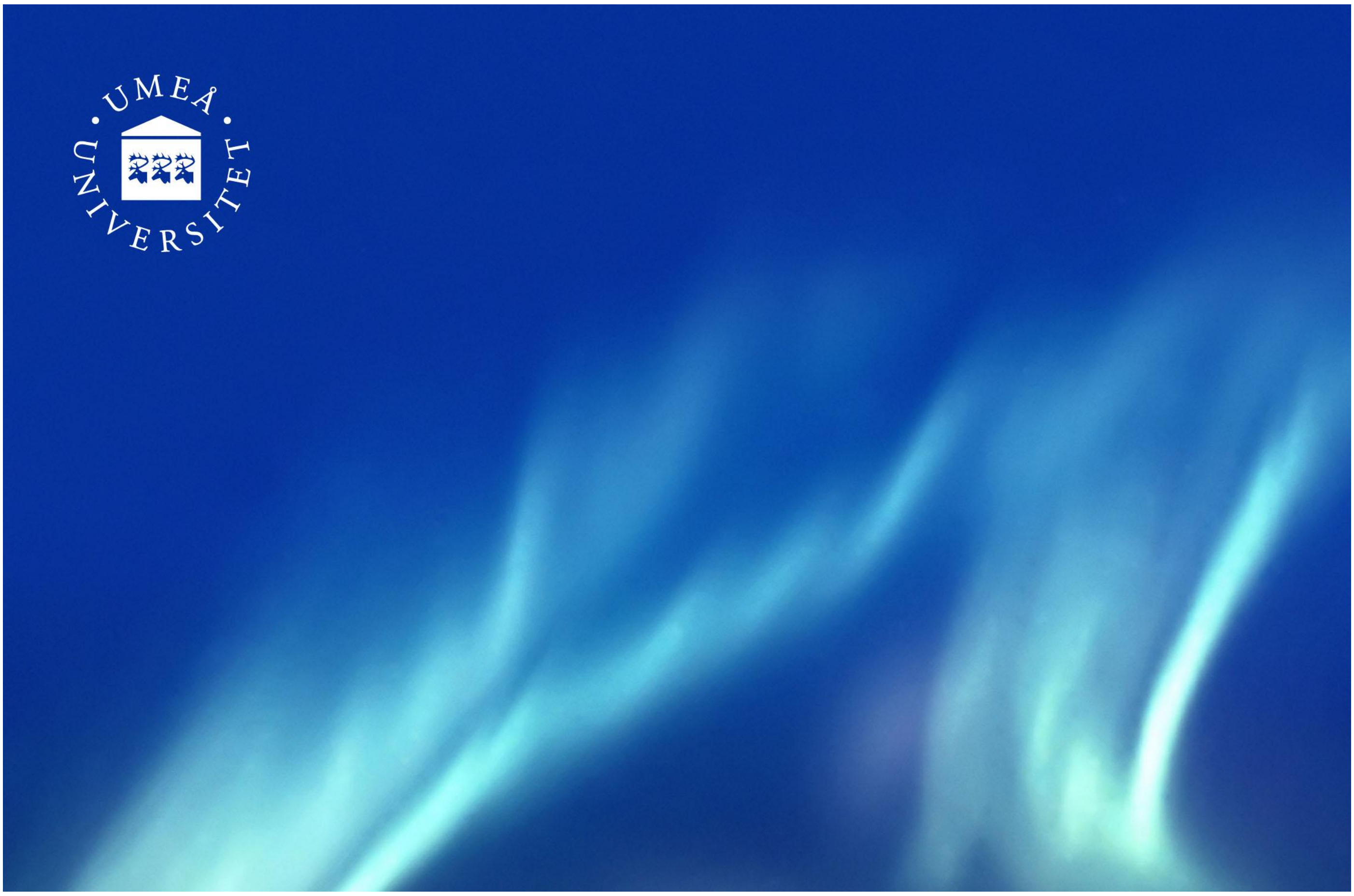

On the geometry of calibrated manifolds

with applications to electrodynamics

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Abstract

In this master thesis we study calibrated geometries, a family of Riemannian or Hermitian manifolds with an associated differential form, φ. We show that it is useful to introduce the concept of proper calibrated manifolds, which are in a sense calibrated manifolds where the geometry is derived from the calibration. In particular, the φ-Grassmannian is considered in the case of proper calibrated manifolds. The impact of proper calibrated manifolds as a model is studied, as well as the usefulness of pluripotential theory as tools for the model. The special Lagrangian calibration is an example of an important calibration introduced by Harvey and Lawson, which leads to the definition of the special Lagrangian differential equation. This partial differential equation can be formulated in three and four dimensions as $\det(\mathrm{H}(u)) = \Delta u$, where $\mathrm{H}(u)$ is the Hessian matrix of some potential u. We prove the existence of solutions and some other properties of this nonlinear differential equation and present the resulting 6- and 8-dimensional manifolds defined by the graph $\{x + i\nabla u(x)\}$. We also consider the physical applications of calibrated geometry, which have so far largely been restricted to string theory. However, we consider the manifold $(M, g, \mathbf{F})$, which is calibrated by the scaled Maxwell 2-form. Some geometrical properties of relativistic and classical electrodynamics are translated into calibrated geometry.

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Contents

Papers included in this thesis:

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II. Leijon, R. On the properties of the special Lagrangian differential equation, manuscript (2013).
III. Leijon, R. Electrodynamics on calibrated manifolds, manuscript (2013).

1. Introduction

Differential geometry is a subject which has had great impact both in mathematics and in other sciences. This is because of its powerful and natural use as a setting for models. In mathematics, it is such a powerful tool for studying the properties of surfaces that it needs no further motivation. In other sciences, it has revolutionized the construction of models — in physics, the differential geometric setting of general relativity is one of the most renowned achievements in physics [MTW].

The foundations for the modern viewpoint on geometry originated in the *Theorema Egregium* by Gauss [GC]. It could be argued that this marked the most important feature of modern geometry: the transition to viewing geometrical properties of surfaces and spaces as entirely intrinsic properties — that is, properties that the surfaces and spaces possess independent of their embeddability in Euclidean space. As a tool for building physical and mathematical models, seeing things like curvature as intrinsic is necessary.

The famous speech by Riemann [RB] laid the basis for modern manifold theory. In his vision of the field, every aspect of geometry is something studied in itself and not in relation to the Euclidean space. Thus, we are able to study shapes and curves themselves. From his work, the well-known Riemannian geometry derives its name. It is a natural geometry to endow any *real* manifold with, since the tangent bundle of every real differentiable manifold admits a Riemannian metric tensor [LJ].

However, complex manifolds have very different properties from real manifolds, as one might imagine [DJP], due to the requirement of preservation of complex structure. An analogue to the Riemannian geometry in real differential geometry is the Hermitian geometry, since every *complex* manifold admits a Hermitian metric [KN]. Important classes of modern manifolds are for example the well-known Kähler manifolds, which are a special case of the Hermitian manifolds, and the symplectic manifolds. The symplectic manifolds are differentiable manifolds (real or complex) endowed with a symplectic form, and the Kähler manifolds are therefore a special case.

The symplectic and Hermitian manifolds have had enormous uses in many different types of mathematics and physics. One of the more recent developments in differential geometry is the introduction of the theory of calibrations due to Harvey and Lawson [HL1]. This theory has proven rich in several ways — it has in particular provided us with natural ways to construct volume-minimizing submanifolds. In a sense this is an extension of the geodesics often studied in Riemannian geometry, which are curves that locally minimize length. Calibrated submanifolds locally minimize volume, and as such they are interesting due to their applicability.

The main application is that of minimization problems, which have been studied extensively, and classically the calculus of variations has been one of the most important tools to solve them, like the brachistochrone problem solved by Johann Bernoulli in the late 17th century. Other important minimization problems in mathematics are minimal surfaces problems, such as the double bubble conjecture [MF]. Many physical laws demand the minimization of some quantities — examples of this include Hamilton's principle, which can be stated as follows [TM]:

Of all the possible paths along which a dynamical system may move from one point to another within a specified time interval (consistent with any constraints), the actual path followed is that which minimizes the time integral of the difference between the kinetic and potential energies.

Because of this, the applications of optimization in physics are virtually endless.

Thus, the use of calibrated geometry as a model to solve these problems have, since the development of the theory of calibrations, been used to a significant extent. A calibration φ on an orientable Riemannian n-dimensional manifold (M, g) is a closed k-form, $1 \leq k \leq n$ such that for all unit simple k-vectors ξ, $\varphi(\xi) \leq 1$. These calibrations have for example been used to prove the angle conjecture [LG]. However, it has also proved to be a useful extension of the tool of pluripotential theory, and the construction of calibrated pluripotential theory adds to the value of the calibrated geometry as a model, as well as the use of pluripotential theory [HL2].

The calibrated manifolds have a strong (although there is little or no research on the subject at the point of writing) relationship with the multisymplectic manifolds, a generalization of the symplectic manifolds, which are simply manifolds with an associated closed nondegenerate k-form [CIL]. The symplectic manifolds has many useful applications in physics, where they may for example represent the phase space of a closed system in classical mechanics [CDS].

A problem that has not been formulated previously but is somewhat natural is this: why do we look to associate a calibration to a Riemannian manifold — could we just derive the metric tensor from the calibration as we do in the Kähler case? What properties would these "proper calibrated" manifolds have? This is the main problem we will look into in Paper I.

What we hope to achieve with these proper calibrated manifolds is natural. Consider the relationship between a complex symplectic manifold (M, J, g, ω) with symplectic form ω and metric g, and a Kähler manifold (M, J, ω). These are both useful formulations, but the association of the symplectic form to the geometry via the relationship $g(X, JY) = \omega(X, Y)$ makes the Kähler manifold the most important class of symplectic (and Hermitian) manifolds [KN]. Entirely analogously, we can consider the relationship between a calibrated manifold and a proper calibrated manifold. In the special case in which the calibration is a Kähler form, the usefulness of the proper calibrated manifold is obvious because it is then exactly the Kähler manifold, and is therefore a type of generalization of a Kähler manifold.

However, in this master thesis we will show that some additional properties hold for the considered class of calibrated manifolds. Thus, this class of manifolds may prove useful in geometrical applications due to the intrinsic volume-minimizing property of its submanifolds.

Consider a calibrated complex manifold (M, J, g, φ), with a given vector basis $\{\partial_1, \bar{\partial}_1, \ldots, \partial_n, \bar{\partial}_n\}$ and the linear operator $r_i \colon TM \to TM$ defined by

$$r_i \partial_j = \partial_l, \ i + j = l \ (\mathrm{mod}\, n), \ 1 \leq l \leq n.$$

Now consider the functions given by

$$g_{ij} = \varphi(\partial_i, \bar{\partial}_j, r_1\partial_i, r_1\bar{\partial}_j \ldots, r_{k-1}\partial_i, r_{k-1}\bar{\partial}_j). \tag{1}$$

We call these the components of the proper metric. Then we may present the main result from Paper I as follows:

Main Theorem A: *Let (M, φ) be an orientable n-dimensional complex manifold M and multinondegenerate (k, k)-calibration φ. Then the mapping $g\colon TM \times TM \to \mathbb{R}$ with components defined by equation* (1) *is a bilinear and nondegenerate metric.*

This proves that we can construct a metric tensor from a calibration in a meaningful way. We call this the *proper metric*. The following theorem is an indication of the usefulness of the proper calibrated manifolds, since it is in a sense a stronger version of Theorem 1.4 in [HL2]. Thus, we introduce a secondary use for φ-plurisubharmonic functions aside from the normal applications in geometry and convexity. On proper calibrated manifolds, there is a strong link between the calibration and pluripotential theory.

Main Theorem B: *Let (M, J, φ) be a proper calibrated manifold, with a function f that is φ-plurisubharmonic on M. Then f is plurisubharmonic on M.*

Many problems in calibrated geometry are unsolved, among them the characterization of the special Lagrangian submanifolds which we shall deal with in Paper II. This paper is a survey on the properties of the special Lagrangian differential equation and the properties of the special Lagrangian submanifolds, where we do not present any new results in particular, but rather reintroduce some results and expand on the usefulness of this equation.

One of the main result that we consider is not a new result, but proves the usefulness of the special Lagrangian submanifolds — this is the regularity of solutions to the special Lagrangian differential equation, which is important, since it proves that all C^2 submanifolds calibrated by the special Lagrangian submanifolds are not only volume-minimizing, but also real-analytic.

The second result we consider, however, is subtler — we introduce in Paper II a more intuitive way of considering volume-minimization. The next theorem is equivalent to the statement that a straight line is a geodesic in $\mathbb{R}^n$, but it is more useful than one might think due to the new perspective on volume-minimization. This is not a new result, but it is easily extended and generalized to higher dimensions.

Main Theorem C: *For each $C \in \mathbb{R}$, the graph $\{(x, C)\colon x \in \Omega \subset \mathbb{R}\}$ is volume-minimizing.*

Finally, the question of further physical applications for the theory of calibrations is a problem that we we will look at in Paper III, where we present some possible applications to electrodynamics. Physical applications of calibrated geometry have so far been largely restricted to string theory. However, due to the strong link between electrodynamics and the exterior algebra, we show that it is natural to consider Maxwells equations from a calibrated perspective.

We look at the two main results of Paper III. The first is a combination of the applications of calibrated pluripotential theory and the use of constant calibrations, often studied by for example Joyce [JD1]. We consider a constant (electrical) current J, which is common in physical applications. From this we prove that we may derive an associated calibration $\mathbf{J}$, and arrive at the following result

Main Theorem D: *Let $(M, g, \mathbf{J})$ be a 4-dimensional, calibrated, oriented and contractible manifold, on which $\mathbf{J}$ is a constant calibration. Then the following statements hold on $(M, g, \mathbf{J})$:*

i. The 1-form $\star\mathbf{J}$ is a calibration.
ii. Let f be a convex function. Then it it is $\star\mathbf{J}$-plurisubharmonic.
iii. The $\star\mathbf{J}$-submanifolds are the real lines parallel to $\star\mathbf{J}$.

Consider the applications to the electromagnetic Lagrangian V. We find the following result in Paper III, which indicates that the electrodynamically calibrated submanifolds provide certain energy-conserving properties.

Main Theorem E: *Let N be a flat, four-dimensional, connected, contractible and orientable manifold with metric g. Further, let $M = U_1 \times U_2$ be a smooth submanifold of N and let U_1 and U_2 be $\mathbf{F}$-submanifolds. Then the Lagrangian V satisfies*

$$\int_M V = \frac{\mathrm{vol}\,(U_1)\mathrm{vol}\,(U_2)}{2}\sqrt{|\det(g)|}.$$

We now move on to discuss some basic theory of calibrations. To understand this report, the reader is assumed to have an understanding of differential geometry, complex manifolds, the calculus of several complex variables, partial differential equations as well as some linear algebra and geometric measure theory. Recommended reading for these subjects are [LJ], [DJP], [KS], [EL], for the first four subjects, respectively. The book [FH] can be useful for the study of geometric measure theory. No knowledge of physics is required as a prerequisite, although it is strongly encouraged for the enjoyment of the reader.

In the next section, basic theory of calibrated geometry is presented — some rudimentary proofs (often merely in the case of $\mathbb{R}^n$ or $\mathbb{C}^n$) will be given to certain theorems. First, a short summary of the fundamental principles of calibrations is given. Then the two important calibrations of the Kähler calibration and the special Lagrangian calibration are presented along with some theory developed by Harvey, Lawson and Joyce. Finally, we will present a section on pluripotential theory in calibrated geometry.

The last section will contain some potential future research in calibrated geometry and related fields. Five different projects are presented, and some preliminary thoughts are given.

2. Preliminaries

In this section, we seek to introduce the reader to calibrated geometry. At least the most rudimentary theory is presented here, although the reader is strongly encouraged to consult the papers by Harvey and Lawson [HL1] and [HL2] as well as [H1], [JD1], [JD2] and [MF2].

2.1. **Introduction to manifolds.** We start by defining three concepts: embeddings, submanifolds and orientability. These concepts are crucial to the theory presented here, and at least the definition of submanifolds is often left out. In general we require all manifolds and submanifolds to be orientable.

Definition 2.1. Let M and N be smooth manifolds. A map $\iota\colon M \to N$ is an *embedding* if ι is a homeomorphism onto $\iota(M)$ and for every point $p \in M$, $T_p\iota\colon T_pM \to T_{\iota(p)}N$ is an injection.

Now we show what we mean by a submanifold, what is sometimes called a regular submanifold (or embedded submanifold) [LJ].

Definition 2.2. Let U be a subset of a smooth manifold M. If the inclusion map $\iota\colon U \to M$ is an embedding, then U is a *submanifold* of M.

The analogue of a real submanifold in complex geometry is of course a complex submanifold. We define these next, for clarity.

Definition 2.3. Let (M, J) be a complex manifold M with complex structure J. Let U be a submanifold of M. Then U is a *complex submanifold* of M if for each $p \in U$, $J(T_pU) = T_pU$.

The set of complex submanifolds is therefore a subset of the set of submanifolds. This is because they are required to preserve the complex structure of (M, J).

Definition 2.4. Let V be a k-dimensional vector space, E, M smooth manifolds and $\pi\colon E \to M$ a smooth map such that (E, M, π, V) is a smooth vector bundle of rank k. Then (E, M, π, V) is *orientable* if there exists a smooth, nowhere vanishing global section of the bundle $\bigwedge^k E^* \to M$. A manifold is orientable if its tangent bundle is orientable.

An orientation is an equivalence class of the smooth sections discussed in Definition 2.4. To find an oriented submanifold, take an orientable submanifold and choose an orientation. We therefore never specify *how* any manifold or submanifold should be oriented, only that it *can* be in a suitable way.

We explain some further terms of differential geometry that will be used. First, we call a manifold flat if the *curvature tensor*

$$R(u, v)w = \nabla_u\nabla_v w - \nabla_v\nabla_u w - \nabla_{[u,v]}w$$

is identically zero for all vector fields u, v, w. The simplest example of this is when the metric tensor has only constant components. Second, we call a manifold M associated with a symmetric bilinear form g semi-Riemannian (Riemannian) if g is nondegenerate (positive definite). Associating a complex structure J with a Riemannian manifold (M, g) such that J is compatible with g in the sense that $g(u, v) = g(Ju, Jv)$ defines a Hermitian manifold (M, J, g) (without the metric, it is merely a complex manifold).

Since we use differential forms extensively, we briefly define the four most important operations on them — the wedge product $\bigwedge$, the exterior derivative, the contraction operator and the Hodge star operator. We consider $L^k_{\mathrm{alt}}(U)$ to be the space of antisymmetric multilinear k-maps $\alpha\colon U \to \mathbb{R}$.

Definition 2.5. Let $\alpha \in L^k_{\text{alt}}(U)$ and $\beta \in L^l_{\text{alt}}(U)$ be smooth mappings. Then the *wedge product* $\alpha \wedge \beta$ is defined as

$$\alpha \wedge \beta(u_1, \dots, u_k, u_{k+1}, \dots, u_{k+l}) = \frac{1}{k!l!} \sum_{\sigma} \operatorname{sgn}(\sigma) \alpha(u_{\sigma(1)}, \dots, u_{\sigma(k)}) \beta(u_{\sigma(k+1)}, \dots, u_{\sigma(k+l)})$$

where σ is a permutation of the numbers $1, \dots, k+l$ and $\operatorname{sgn}(\sigma)$ is the signature of the permutation.

We consider the exterior derivative next. We consider $\Omega^k(U)$ to be the space of smooth differential k-forms on U.

Definition 2.6. Let $\big(U, \xi = (x^1, \dots, x^n)\big)$ be some chart on a manifold M. Then the *exterior derivative* is a map $d \colon \Omega^k(U) \to \Omega^{k+1}(U)$ such that $d \circ d = 0$ and satisfying the following two conditions. First, for any smooth function $f \colon U \to \mathbb{R}$,

$$df = \sum_i \frac{\partial f}{\partial x^i} dx^i.$$

Since we may write any $\alpha \in \Omega^k(U)$ in local coordinates as $\alpha = \sum_I f_I dx^I$ for some smooth functions f_I and increasing multiindices I, d need also satisfy

$$d\alpha = \sum_I d(f_I) \wedge dx^I.$$

Definition 2.7. Let $\big(U, \xi = (x^1, \dots, x^n)\big)$ be some chart on a manifold M, and let $\alpha \in \Omega^k(U)$. Then we define the *contraction* $i \colon \Omega^k \to \Omega^{k-1}$ of α by a vector field u as

$$i_u \alpha(v_1, \dots, v_{k-1}) = \alpha(u, v_1, \dots, v_{k-1}),$$

for any smooth vector fields $v_1, \dots, v_{k-1}$.

Finally, we define the Hodge operator or Hodge star operator, which we define in terms of the Levi-Civita tensor ϵ.

Definition 2.8. Let (M, g) be an n-dimensional semi-Riemannian manifold, and let $\alpha \in \Omega^k(M)$. Then if we write α in local coordinates as $\alpha = \frac{1}{k!} \sum_I f_I dx^I$, we define the *Hodge star operator* $\star \colon \Omega^k(M) \to \Omega^{n-k}(M)$:

$$\star\alpha = \sqrt{|\det(g)|} \sum_{1 < i_{k+1} < \cdots < i_n < n} f^{i_1, \dots, i_k} \epsilon_{i_1, \dots, i_k, i_{k+1}, \dots, i_n} dx^{k+1} \wedge \cdots \wedge dx^n.$$

2.2. **Calibrated geometry.**

Definition 2.9. Let (M, g) be an orientable Riemannian manifold. Then the Grassmannian $G(k, T_pM)$ is defined to be the set

$$G(k, T_pM) = \{\xi \in \bigwedge^k T_pM \colon \xi \text{ is a unit simple } k\text{-vector}\}.$$

Let φ be a smooth closed k-form. Then the φ-Grassmannian is defined as

$$G(\varphi) = \bigcup_{p \in M} \{\xi_p \in G(k, T_pM) \colon \varphi_p(\xi_p) = 1\}.$$

Definition 2.10. Let M be an orientable manifold. Then a smooth k-form ω is a *calibration* on M if it is

i. Closed: $d\omega = 0$.
ii. Area-restricted: for each $p \in M$, ω satisfies

$$\omega(\xi) \leq 1,$$

for all $\xi \in G(k, TM)$.

Now we define what we mean by volume minimizing (area-minimizing) in terms of a submanifold. This is also an important concept when we discuss calibrations, because of the relationship between differential forms and integration on orientable manifolds. First, we know that for any orientable n-dimensional manifold M we have a volume form, that is a nowhere vanishing n-form. We always define the *standard volume form* on the manifold M with metric g in local coordinates as the n-form $\lambda_M = \sqrt{|\det(g)|}dx^1 \wedge \cdots \wedge dx^n$.

Definition 2.11. Let U be a submanifold of a manifold M. Then U is *volume-minimizing* if for every submanifold $V \subset M$ such that $\partial V = \partial U$,

$$\text{vol}(U) \leq \text{vol}(V).$$

Here, the volume of U is defined to be $\text{vol}(U) = \int_U \lambda_U$. Now we arrive at the fundamental theorem of calibrations. It can be thought of as a way to generalize the statement that a straight line in $\mathbb{R}^n$ minimizes length. Intuitively it can be though of as a way to decide if a submanifold is volume-minimizing.

Theorem 2.12 (The Fundamental Theorem of Calibrations)**.** *Let M be an orientable manifold, and let U be some closed, oriented k-dimensional submanifold in M. Let ω be a k-calibration on M. Let e_U be the volume vector form on U. If*

$$\omega(e_U) = 1,$$

for all points $p \in U$, then U is volume-minimizing in M.

Proof. We prove this for the case $M = \mathbb{R}^n$. See [HL1] for a proof of the general case. Let U be some relatively compact open subset with smooth boundary $\partial U = \partial V$, where V is some submanifold of $\mathbb{R}^n$. Since $\mathbb{R}^n$ is contractible, we know that any closed form is exact, and thus $\omega = d\alpha$, for some $\alpha \in \bigwedge^{k-1} T^*M$. Then we also know that

$$\int_U \omega - \int_V \omega = \int_{U-V} \omega = \int_{U-V} d\alpha = \int_{\partial U - \partial V} \alpha = 0,$$

by Stoke's Theorem. Thus, $\text{vol}\,(U) = \int_U \omega = \int_V \omega = \int_V \omega(V)\lambda_V \leq \int_V \lambda_V = \text{vol}\,(V)$. This proves the theorem. □

We now consider the most important object for understanding calibrated geometries — the calibrated manifolds. Calibrated manifolds, or special cases thereof, will in general be the setting used throughout this master thesis.

Definition 2.13. A Riemannian n-dimensional manifold (M, g) associated with a k-calibration ω is called a *calibrated manifold.*

A popular example of a calibrated manifold is the *Cayley* calibration. We present a short calculation of the properties of this calibration next.

Example 2.14. Let $(\mathbb{O}, \langle\cdot,\cdot\rangle)$ denote the octonions with inner product $\langle\cdot,\cdot\rangle$. Then

$$\Phi(X, Y, Z, W) = \langle X, Y \times Z \times W\rangle$$

is the Cayley calibration.

We prove that this is an alternating form by proving that $X \times Z \times W = X(\bar{Z}W)$, for X, Y, Z, W orthogonal. This is because $X \times Z = \frac{1}{2}(X\bar{Z} - \bar{Z}X) = X\bar{Z}$, and thus $X\bar{Z} \times W = X(\bar{Z}W)$.

Hence, Φ is an alternating form on $\mathbb{O}$, since

$$\langle X, X \times Z \times W\rangle = |X|^2\langle Z, W\rangle = 0.$$

□

We list now some common general calibrations which have important properties and have been introduced in the works of Harvey and Lawson [HL1], and have been used for example in the works of Lotay [LJD] and the physical works of Gutowski and Papadopoulos [GP].

Example 2.15. The *exceptional geometries* are the calibrations introduced in [HL1] on the space of the octonions (and the imaginary octonions). The following are calibrations:

i. The *associative* calibration, $\psi(u, v, w) = \langle u, vw\rangle$ on $\operatorname{Im}\mathbb{O}$.
ii. The *coassociative* calibration, $\chi = \star\psi$ on $\operatorname{Im}\mathbb{O}$.
iii. The Cayley calibration on $\mathbb{O}$, introduced in Example 2.14.

Here, $\star\psi$ is the Hodge dual of ψ.

Now we look at the thoroughly studied parallel calibrations with constant coefficients on $\mathbb{R}^n$ for $n \leq 8$. All these calibrations have been characterized, and a comprehensive survey is presented by Joyce [JD1].

Example 2.16. Let $(\mathbb{R}^n, \varphi)$ be a calibrated manifold with a k-calibration φ for $n \leq 5$ with constant coefficients. Then there are orthonormal coordinates on $\mathbb{R}^n$ such that the following holds:

i. If $k = 1$, then $\varphi = dx^1$. Thus, the φ-submanifolds are the real lines parallel to $(1, 0, \dots, 0)$.
ii. If $k = 2$, then $\varphi = \psi_m = \sum_{j=1}^m dx^{2j-1} \wedge dx^{2j}$, for some $1 \leq m \leq n/2$. The φ-submanifolds take the form $\Sigma \times (x^{2m+1}, \dots, x^n)$ for a complex surface Σ with complex coordinates $(z_1, \dots, z_m) = (x_1+ix_2, \dots, x_{2m-1}+ix_{2m})$.
iii. If $k = n-2$, then $\varphi = \psi_m = \sum_{j=1}^m dx^{2j-1} \wedge dx^{2j}$, for some $1 \leq m \leq n/2$. The φ-submanifolds take the form $\Sigma \times (x^{2m+1}, \dots, x^n)$ for a complex surface Σ in $\mathbb{R}^{2m}$.
iv. If $k = n - 1$, then $\varphi = dx^2 \wedge \dots \wedge dx^n$. The φ-submanifolds are the real hyperplanes $(c, x^2, \dots, x^n)$, $c \in \mathbb{R}$.
v. If $k = n$, then $\varphi = dx^1 \wedge dx^2 \wedge \dots \wedge dx^n$.

□

Finally, in the next sections, we consider the special Lagrangian and Kähler calibrations, which are two powerful tools used throughout calibrated geometry.

2.3. **The special Lagrangian calibration.** Consider $\mathbb{C}^n$ with the standard Hermitian inner product, with a standard complex volume form $dz = dz^1 \wedge \dots \wedge dz^n$. We consider this from the real perspective since $\mathbb{C}^n \cong \mathbb{R}^{2n}$ and the volume form $dz = (dx^1 + idy^1) \wedge \dots \wedge (dx^1 + idy^n)$. Then we find the following result, found in [HL1]:

Theorem 2.17. *Let dz be the volume form on $\mathbb{R}^{2n}$ as stated above. Then the real n-form $\phi = \mathrm{Re}(dz)$ is a calibration on $\mathbb{R}^n$ called the special Lagrangian calibration.*

Proof. See the proof to Theorem 7.26 in [H1]. □

Corollary 2.18. *Each special Lagrangian submanifold U of $\mathbb{R}^n$ is volume-minimizing.*

Proof. This follows from Theorem 2.12 and Theorem 2.17. □

Finally, we look at the fundamental question of the special Lagrangian calibration that we seek to study thoroughly in this master thesis: suppose M is the graph of a smooth function f and a special Lagrangian submanifold — is this statement equivalent to some differential equation in f?

Proposition 2.19. *Let f be the gradient of a function $\varphi \in C^2(\Omega)$, $\Omega \subset \mathbb{R}^n$. Then the graph of f on Ω, $M = \{x + if(x) \colon x \in \Omega\}$ is a* special Lagrangian submanifold *if and only if φ satisfies*

$$\sum_{k=0}^{\lfloor (n-1)/2 \rfloor} (-1)^k \alpha_{2k+1} \left(\mathrm{H}(\varphi)\right) = 0, \tag{2}$$

or

$$\mathrm{Im}\left(\det_{\mathbb{C}}(\mathrm{I} + \mathrm{i}\mathrm{H}(\varphi))\right),$$

where $\alpha_j(\mathrm{H}(\varphi)) = \sum_{1 \le i_1 < \cdots < i_j \le n} \lambda_{i_1} \cdots \lambda_{i_j}$, where λ_i is the ith eigenvalue of the Hessian matrix of φ, $\mathrm{H}(\varphi)$.

Proof. See the proof to Theorem 7.53 in [H1]. □

Equation (2) is called the *special Lagrangian differential equation* and it is a nonlinear differential equation for $n \geq 3$, and for $n < 3$ it takes the form of $\Delta\varphi = 0$, which is the definition of harmonic functions.

Now, although everything here has been done in $\mathbb{R}^n$, it can be generalized as follows. Consider a Calabi-Yau manifold M with suitably normalized holomorphic volume form Ω. Then $\mathrm{Re}\,(\Omega)$ is a calibration on M, with special Lagrangian calibrated submanifolds [JD2]. The theory of special Lagrangian calibrations on $\mathbb{C}^n$ can be extended to this calibration on the Calabi-Yau manifold M.

2.4. **The Kähler calibration.** Commonly, the standard Kähler form is used as a good example of a calibration. Indeed, on the standard Kähler manifold $(\mathbb{C}^n, J, \omega)$, the calibrated submanifolds are precisely the complex submanifolds [H1]. Consequently, the complex submanifolds are locally volume minimizing. We study this calibration in this section, and its extensions.

In this master thesis, a great deal of work will be put on higher-order analogues of the Kähler forms. Therefore it is important to understand classical Kähler geometry and this is a very important subject in modern differential geometry — see for example the works of Yau [YS]. A good reference for understanding complex Kähler and Hermitian manifolds is the book by Demailly, [DJP]. A good work on symplectic manifolds which may be equally useful for the reader are the lecture notes by Cannas da Silva, [CDS].

Definition 2.20. Let (M, J, h) be a Hermitian manifold. Then the *standard Kähler form* $\omega\colon T\mathbb{C}^n \to \mathbb{C}$ on M is defined by

$$\omega = \frac{i}{2}\sum_{j=1}^{n} dz^j \wedge d\bar{z}^j.$$

We concern ourselves with this ω in the rest of the section, and in fact this is a popular calibration which has been studied extensively [H1], [HL1].

Proposition 2.21. *Let ω be the standard Kähler form. Then the form $\varphi = \frac{\omega^k}{k!}$ is a $2k$-calibration on the manifold (M, J, g).*

Proof. Recall that this proposition follows from Wirtinger's inequality. We prove this for $\mathbb{C}^n$. For the first part of this proof, we prove that φ is closed. Since $d\varphi = \frac{1}{k!}d(\omega^k)$, and $d\omega = 0$, this is clearly the case.
For the second part of this proof, we prove that φ fulfills the requirement that

$$\varphi(\xi) \leq 1 \text{ for all } \xi \in \mathrm{G}(2k, \mathbb{C}^n).$$

Now, inspired by the proof of Wirtinger's inequality by Federer (see section 1.8.2 in [FH]), we check this for the case $k = 1$ first. We let u, v denote an orthonormal basis for ξ, and we notice that $\varphi(\xi)$ reduces to

$$\varphi(\xi) = \omega(\xi) = \langle Ju, v\rangle \leq |Ju||v| = 1, \tag{3}$$

by the Cauchy-Schwarz inequality. Now we aim to reduce the general case to the $k = 1$ case. This is based on the restriction of ω to the $2k$ subspaces $P(\xi)$ where $\xi \in \mathrm{G}(2k, \mathbb{C}^n)$. Then $\omega|_P$ is a 2-form that can be written

$$\omega|_P = \sum_{i=1}^{r} \lambda_i \alpha^{2i-1} \wedge \alpha^{2i},$$

where $\lambda_i \geq \lambda_{i+1} > 0$ and the set $\{\alpha^i\}$ forms an orthonormal basis for the dual space to P, P^*. Let us denote the dual basis for P by $e_1, e_2, \ldots, e_k$. Now, by (3)

$$\lambda_i = \omega(e_{2i-1}, e_{2i}) \leq 1.$$

Now, since

$$\varphi|_P = \lambda_1 \cdots \lambda_k \alpha^1 \wedge \cdots \wedge \alpha^k,$$

we have that relative ξ,

$$\varphi(\xi) = \lambda_1 \cdots \lambda_k \leq 1.$$

This concludes our proof. □

2.5. **Pluripotential theory on calibrated manifolds.** Pluripotential theory has proven its usefulness as a tool in many fields. These fields of application include geometric- and analytic convexity [KS], algebraic geometry [DJP2] as well as the important Monge-Ampère equation [YS2]. Of course this makes the field of classical pluripotential theory interesting in itself, but the works of Harvey and Lawson have given insight in useful analogues of pluripotential theory in the setting of calibrated geometry [HL2], [HL3].

These tools are largely unused except in the field of convexity, which is their primary use. In this master thesis we explore the field of pluripotential theory in calibrated geometry and we will therefore need some of the theory developed in the paper [HL2].

Definition 2.22. Let Ω be some open subset of $\mathbb{C}^n$. Let $f\colon \Omega \to [-\infty, \infty)$ be upper semicontinuous which is not identically $-\infty$ on any connected component of Ω. If $z \mapsto f(a + bz)$ is subharmonic or $-\infty$ on the set

$$\{z \in \mathbb{C}\colon a + bz \in \Omega\},\ a \in \Omega,\ b \in \mathbb{C}^n,$$

then f is a *plurisubharmonic* function.

There are many equivalent statements to this definition. However, we are satisfied in stating that plurisubharmonicity is a local property of a function. Further, the family of plurisubharmonic functions on Ω is a convex cone.

Example 2.23. Let $f \not\equiv 0$ be a holomorphic function defined on $\mathbb{C}^n$. Then the function $g = \log |f|$ is plurisubharmonic on $\mathbb{C}^n$. □

Consider now the following definition, which is a natural extension of plurisubharmonicity onto a smooth manifold.

Definition 2.24. Let M be a complex manifold, and let $f\colon M \to \mathbb{R} \cup \{-\infty\}$ be an upper semicontinuous function, $f \not\equiv -\infty$. Then f is plurisubharmonic on M if for every chart $(\mathcal{U}, x)$, the mapping $f \circ x^{-1}\colon x(\mathcal{U}) \to \mathbb{R} \cup \{-\infty\}$ is plurisubharmonic in the sense of Definition 2.22.

We now move on to the definitions relevant to calibrated geometry — namely the introduction of φ-plurisubharmonic functions.

Definition 2.25. Let (M, g, φ) be a calibrated manifold, and consider a function $f \in C^1(M)$. Then $d^\varphi f$ is defined as the contraction $i_{\nabla f}\varphi$.

The d^φ operator is related to the complex exterior derivative d^c, and is in a sense a generalization, since they are equivalent for certain calibrations (for example the standard Kähler calibration). However, this operator defines calibrated plurisubharmonic functions by the following definition.

Definition 2.26. Let (M, g, φ) be a calibrated n-dimensional manifold. Then a function $f \in C^2(M)$ is φ-plurisubharmonic if

$$dd^\varphi f(\zeta) - \nabla_{\nabla f}\varphi(\zeta) \geq 0,$$

for all $\zeta \in G(\varphi)$.

This is the most intuitive way to classify φ-plurisubharmonic functions, although alternative definitions extend to functions which are not in C^2 — see for example [HL2]. It should be added that most papers on calibrated plurisubharmonic functions instead define them in the class C^∞. For our purposes, C^2 functions are good enough.

The special case when φ is parallel, that is, $\nabla_X\varphi = 0$ for all $X \in TM$, gives us the simpler expression for φ-plurisubharmonicity: $dd^\varphi f(\zeta) \geq 0$. Further, consider the following theorem, which is one of the defining results of φ-plurisubharmonic functions.

Theorem 2.27. *Let (M, φ) be a calibrated manifold and let f be a φ-plurisubharmonic function on M. Let U be a φ-submanifold on M. Then f is subharmonic on U.*

Next, we present a common example of φ-plurisubharmonic functions. This example is found in [HL2].

Example 2.28. Let φ be a parallel k-calibration on $\mathbb{R}^n$. Further, let $E\colon \mathbb{R}^n - \{0\} \to \mathbb{R}$ be a smooth function defined by

$$E(x) = \begin{cases} \frac{-1}{k-2}\frac{1}{|x|^{k-2}} & \text{if } k \geq 3, \\ \ln|x| & \text{if } k = 2. \end{cases}$$

Then E is φ-plurisubharmonic on $\mathbb{R}^n - \{0\}$. □

This example of a φ-plurisubharmonic function is fundamental, in a sense. The relationship between this function and the fundamental solution to the Laplace equation should be noted by the reader. Further, we consider the characterization of φ-plurisubharmonic functions on constant coefficent calibrations on $\mathbb{R}^n$, $n \leq 5$. This is analogous to the results presented in Example 2.16. First, we introduce k-plurisubharmonic functions in the sense of Harvey and Lawson, which are in a certain sense nearly convex. These functions have for example been studied in [HL4].

Definition 2.29. A function $f \in C^2(\Omega)$ is k-plurisubharmonic in Ω if at each point $p \in \Omega$, the eigenvalues λ of the Hessian $\mathrm{H}(f)$ satisfies

$$\sum_{i=1}^{k} \lambda_i \geq 0,$$

for $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$.

Now, we are ready to move on to characterize the φ-plurisubharmonic functions on the manifolds from Example 2.16.

Example 2.30. Let $(\mathbb{R}^n, \varphi)$ be a calibrated manifold with a k-calibration φ for $n \leq 5$ with constant coefficients. This is the same calibrated manifolds studied in Example 2.16. Then there are orthonormal coordinates on $\mathbb{R}^n$ such that we may characterize the φ-plurisubharmonic functions as follows:

i. If $k = 1$, then the φ-plurisubharmonic functions are the convex functions.
ii. If $2 \leq k \leq n-1$, then the φ-plurisubharmonic functions are the k-plurisubharmonic functions.
iii. If $k = n$, then the φ-plurisubharmonic functions are the subharmonic functions.

Note that any k-plurisubharmonic function is also $k+1$-plurisubharmonic. All k-plurisubharmonic functions are subharmonic, however the most interesting case is when $k = n-1$. Then an equivalent statement to saying that a function f is k-plurisubharmonic is saying that

$$(\Delta f)I - \mathrm{H}(f) \geq 0.$$

□

3. Future research

In this section we consider some questions that are as of yet unanswered, but they are problems which are relevant to the fields of calibrated geometries and pluripotential theory.

Quasiplurisubharmonic functions and ω-plurisubharmonic functions. The quasi-plurisubharmonic functions studied by for example Demailly give us a new way of finding interesting properties of plurisubharmonic functions on compact Kähler manifolds. In [GZ], these ω-plurisubharmonic functions are defined in terms of the Kähler form ω, and are seen to have some very useful properties. Let a $(1, 1)$-calibration ω be exactly a Kähler form on a complex Hermitian (Kähler) manifold. Then what is the significance of these ω-plurisubharmonic functions? The connection between the quasi-plurisubharmonic functions and the calibrated ω-plurisubharmonic functions have not been thoroughly understood, which brings us to the following question.

Problem: Can the relationship between ω-plurisubharmonic functions in a calibrated sense and ω-plurisubharmonic functions in the sense of Demailly be characterized in a way that is useful for the field of pluripotential theory and (ultimately) for calibrated geometry?

Calibrations as multisymplectic forms. Multisymplectic manifolds have been studied for example by Cantrijn et al. [CIL]. These manifolds consist of a smooth manifold and an associated nondegenerate closed form. The study of these manifolds and their properties have so far been completely separate from the notion of calibrations. This is despite the fact that hyper-Kähler manifolds are considered as an example of multisymplectic forms. These ideas hint at the possibility that there may exist a very strong and natural link between the two theories, which brings us to the main question.

Problem: Is it possible to formulate a connection between the multisymplectic geometry and the calibrated geometry — or more importantly, can we improve either theory by such a link?

Regularity properties of plurisubharmonic functions. An open problem in pluripotential theory is this: when is a subharmonic function also a plurisubharmonic function? Through the tools presented to us by the potential theory in calibrated geometry by Harvey and Lawson in [HL2] we find that on a Kähler calibrated manifold (M, J, g, φ), all φ-plurisubharmonic functions are plurisubharmonic functions, and the φ-submanifolds are exactly the complex manifolds. This hints that we might find a criterion for a "regularity" theorem for subharmonic and plurisubharmonic functions. So the problem is this:

Problem: Can we define in a meaningful way a criterion on a calibrated manifold (M, g, φ) so that, for example, certain subharmonic functions are plurisubharmonic on the φ-submanifolds?

Special Lagrangian differential equation. Due to the usefulness of the special Lagrangian submanifolds, a thorough paper on the properties of the special Lagrangian differential equation would be of use. The calibrated submanifolds of, for example, a Calabi-Yau manifold calibrated by this differential form have proven to be extremely useful. This is chiefly due to the geometrical property of the special Lagrangian submanifolds being volume-minimizing.

This leads us to consider a potential for future research into the field of nonlinear partial differential equations, which will probably also have applications in industrial economy and transport theory due to the relationship between the special Lagrangian differential equations and the partial differential equations of Monge-Ampère type [EL2]. Hence, we formulate the following problem:

Problem: How can the solutions to the equation be characterized? What properties will the solutions have if they are not in C^2? Can the application of the special Lagrangian differential equation prove useful in transport theory?

Geometrical currents in electrodynamics and calibrated geometry. The study of geometrical currents, which are functionals on the space of differential forms, have been very influential through for example Federers work [FH] and the work of Demailly [DJP]. It is instrumental in the theory of calibrations as a tool for investigating the geometry of calibrated manifolds [HL1]. Geometrical currents have had a very limited use in physical applications, despite the important role that other functionals have played in the development of for example quantum physics.

This brings up important questions surrounding the potential for the use of φ-currents and its possible applications as functionals of, for example, the Maxwell 2-form in the covariant formulation of electrodynamics. The natural properties of φ-currents hint at their usefulness in applications.

Problem: Can the use of (geometrical) currents and φ-currents in the covariant formulation of electrodynamics provide new insight into the geometrical aspects of electrodynamics?

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