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**Weakly Approaching Sequences of Random Distribution
Laws with Applications to Resampling**

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Weakly approaching sequences of random distribution laws with applications to resampling

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Abstract

The standard definition of weak convergence of distribution laws (d.l.'s) is generalized to a definition of weak approach of d.l.'s to each other. For the case of random d.l.'s a definition is given for a sequence of random d.l.'s to weakly approach a sequence of non-random d.l.'s. The main result (Theorem 2) concerns this case providing necessary and sufficient conditions for weak approach in terms of characteristic functions of random and non-random d.l.'s. Applications of the theory are given to scheme of series of independent random variables and to linear regression.¹

Key words: random distribution laws, weak convergence, characteristic functions, resampling, Central Limit Resampling Theorem, linear regression

1991 AMS subject classification: Primary 62F12; secondary 62G09

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1 Introduction

Asymptotic methods are often used in the analysis of collected statistical data. A typical problem is to obtain an asymptotically exact approximation of an "unobserved" sequence of distribution laws (d.l.'s) by the use of some other "observed" sequence of d.l.'s when the volume of collecting data is increasing. For example one may wish to determine the d.l.'s of deviations of point estimators from some estimated parameters. Intensive computer methods, using bootstrap and resampling for example, can help to solve such problems. Usually it is assumed or implied under some assumptions, that the "unobserved" sequence of d.l.'s weakly converges to some limit d.l. which we call the *attractor d.l.* The existence of attractor d.l.'s can be a too strong restriction on statistical models and can limit the use of known statistical methods in the analysis of real statistical data. Moreover, the reduction of a problem to one with an attractor d.l. inevitably involves the use of stabilizing transformations, in the form of normalizing constants or matrices, for example, and then their unknown parameters need to be estimated. This can be difficult if the collected data have different structures and hence different probability distributions, Belyaev, and Gadjiev [3], White [11]. In this paper we suggest a way to avoid the use of attractor d.l.'s by defining what it means for sequences of d.l.'s to weakly approach each other. Sequences of random d.l.'s can also be considered, providing a justification of the use of resampling methods. The new type of convergence of sequences of d.l.'s is a natural generalization of the usual weak convergence of d.l.'s to some limit d.l., i.e. to some attractor d.l. It is shown that this generalized weak convergence has similar properties to standard weak convergence. The well-known continuity theorem (see e.g. Shiryaev [10], page 318) states relations between the weak convergence of d.l.'s and the pointwise convergence of characteristic functions (c.f.'s). One variant of the continuity theorem within the framework of ordinary weak convergence was given in Knight [6]. Theorems 1 and 2 in Section 2 are generalisations of the continuity theorem in the case of sequences of d.l.'s that weakly approach each other. These theorems can be generalised to the case of sequences of d.l.'s of vector-valued random variables (r.v.'s) that weakly approach each other, see forthcoming report Belyaev, and Sjöstedt [4]. Theorem 2 can be used to justify the resampling methods in many typical problems involving the point estimation of unknown parameters, and in some other problems. This paper is an extended version of Section 2 in the research report Belyaev [2]. The basic definitions and results in Belyaev [2] are restated. Here, the proofs are given only for new results.

Capital letters are used for r.v.'s, small letters for values of r.v.'s and for constants, Greek letters for parameters of d.l.'s, bold letters for vectors, contour fonts (such as $\mathbb{B}$) for collections of sets, and matrices, X^T for the transposed matrix X , calligraphic (bold) letters for sets and d.l.'s, $\mathcal{L}(X)$ denotes the d.l. (probability measure) of a r.v. X , $N_1(\mu, \sigma^2)$ denotes the one dimensional normal d.l. with

mean value μ and variance σ^2 , $\mathcal{C}(\mu, \theta)$ denotes the Cauchy d.l. centered at μ and with the scale parameter θ , $:=$ means definition by expression, $f(\cdot)$ denotes the "whole" function f and $f(x)$ denotes its value at x , $I(A)$ is an indicator function of an event A , $\mathbb{Z}^+ = \{1, 2, \dots\}$, A^c is the complement of a set A . We use $\xrightarrow{P}$ to denote the convergence in probability, and $\xrightarrow{d}$ for convergence in distribution.

We now outline a class of problems to which Theorem 2 is applicable. This is a scheme of series of observations of independent r.v.'s, for which we are interested in approximating the d.l.'s of sums of the r.v.'s.

Let $U_n := (U_{1n}, \dots, U_{nn})$ be an array of collections of real valued r.v.'s U_{in} , $EU_{in} = 0$, $i = 1, \dots, n$, $n = 1, 2, \dots$. For each r.v. U_{in} we have only one observation u_{in} . We will consider $u_n = (u_{1n}, \dots, u_{nn})$ as the observed statistical data, and denote by $\mathcal{L}(U_n)$ the d.l. of $U_n := U_{1n} + \dots + U_{nn}$. We seek to understand how, and under what assumptions, it is possible to approximate $\mathcal{L}(U_n)$, asymptotically accurately as $n \rightarrow \infty$, given only u_n . In Section 2 we cite the so-called Central Limit Resampling Theorem (CLRT) obtained in Belyaev [1]. The case with non i.i.d. sequence of r.v.'s has been considered in Liu [7], where the uniform convergence of d.l.'s was used.

2 Weakly approaching sequences of distribution laws

Distribution laws (probability measures) are points in some functional space (the space of all probability measures or all distribution functions on $\mathbf{R}^1$). Estimators of d.l.'s should be considered as random d.l.'s (random probability measures). It is necessary to define what is to be meant by the statement that two sequences of d.l.'s approach each other. Our definition of convergence is a modification of the standard notion of weak convergence. We restate the definitions and Lemmas 1 - 3, and 5 from Belyaev [2]. Let $\mathcal{C}_b(\mathbf{R}^1)$ be the set of all bounded, continuous, real valued functions on $\mathbf{R}^1$.

Definition 1 Two sequences of d.l.'s $(\mathcal{L}(X_n))_{n \geq 1}$, $(\mathcal{L}(Y_n))_{n \geq 1}$ of real valued random variables $(X_n)_{n \geq 1}$, and $(Y_n)_{n \geq 1}$ are said to weakly approach each other if for any function $c(\cdot) \in \mathcal{C}_b(\mathbf{R}^1)$

$$Ec(X_n) - Ec(Y_n) \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (1)$$

Correspondingly $(X_n)_{n \geq 1}$, and $(Y_n)_{n \geq 1}$ are said to approach each other in distribution.

We denote this convergence by $\mathcal{L}(X_n) \xrightarrow{w} \mathcal{L}(Y_n)$ or $X_n \xrightarrow{d} Y_n$, $n \rightarrow \infty$.

Remark. If $\mathcal{L}(Y_n) = \mathcal{L}(Y_0)$ for some r.v. Y_0 , and $n = 1, 2, \dots$, and (1) holds, then the sequence $(\mathcal{L}(X_n))_{n \geq 1}$ is said to weakly converge to $\mathcal{L}(Y_0)$, $\mathcal{L}(X_n) \xrightarrow{w} \mathcal{L}(Y_0)$

or $X_n \xrightarrow{d} Y_0$, as $n \rightarrow \infty$. This is the case usually considered in the literature (see e.g. Dudley [5], and Shiryaev [10]). $\square$

There is also a need to consider random d.l.'s. We restrict ourselves to the following situation. We assume that the d.l.'s of $(X_n)_{n \geq 1}$ are random d.l.'s in the sense that there are r.v.'s Z_n , taking values in some space $\mathcal{Z}_n$, such that the d.l. of X_n is the ordinary regular conditional d.l. given Z_n . We denote it by $\mathcal{L}(X_n | Z_n)$. In this case, we can consider conditional expectations $E(c(X_n) | Z_n)$, $c(\cdot) \in \mathcal{C}_b(\mathbf{R}^1)$, $n = 1, 2, \dots$, which are real valued r.v.'s.

Definition 2 Let $(X_n, Z_n)_{n \geq 1}$, and $(Y_n)_{n \geq 1}$ be two sequences of r.v.'s, where $X_n, Y_n \in \mathbf{R}^1$, $Z_n \in \mathcal{Z}_n$, the r.v.'s X_n, Z_n being defined on the same (basic) probability space $(\Omega, \mathfrak{B}(\Omega), P)$ for each n . The sequence of conditional d.l.'s $(\mathcal{L}(X_n | Z_n))_{n \geq 1}$ given Z_n is said to weakly approach $(\mathcal{L}(Y_n))_{n \geq 1}$ in probability (almost surely) along $(Z_n)_{n \geq 1}$, if for any $c(\cdot) \in \mathcal{C}_b(\mathbf{R}^1)$

$$E(c(X_n) | Z_n) - E(c(Y_n)) \rightarrow 0 \quad \text{as } n \rightarrow \infty, \quad (2)$$

in probability (almost surely). We denote this convergence by

$$\mathcal{L}(X_n | Z_n) \xrightarrow{wa(P)} \mathcal{L}(Y_n) \quad (\mathcal{L}(X_n | Z_n) \xrightarrow{wa(as)} \mathcal{L}(Y_n)) \quad \text{as } n \rightarrow \infty.$$

Remark. Here we combine weak convergence and convergence in probability. If $\mathcal{L}(Y_n) \equiv \mathcal{L}(Y_0)$, the sequence $(\mathcal{L}(X_n | Z_n))_{n \geq 1}$ is said to weakly converge to $\mathcal{L}(Y_0)$ in probability (almost surely) along $(Z_n)_{n \geq 1}$ as $n \rightarrow \infty$, if (2) holds almost surely. Note that P can also depend on n . $\square$

We recall that a sequence of real valued r.v.'s $(X_n)_{n \geq 1}$ is said to be uniformly tight if for any small $\varepsilon > 0$ there is a $k_\varepsilon < \infty$ such that

$$P(|X_n| > k_\varepsilon) < \varepsilon, \quad n = 1, 2, \dots$$

Definition 3 Let $(X_n | Z_n)_{n \geq 1}$ be a sequence of r.v.'s X_n given Z_n , $n = 1, 2, \dots$. The sequence $(X_n | Z_n)_{n \geq 1}$ is called uniformly tight along $(Z_n)_{n \geq 1}$ in probability if for any small $\varepsilon > 0$, and $\delta > 0$ there is a constant $k_{\varepsilon\delta} < \infty$, such that

$$P(P(|X_n| > k_{\varepsilon\delta} | Z_n) > \varepsilon) < \delta, \quad n = 1, 2, \dots \quad (3)$$

Lemma 1 If $(Y_n)_{n \geq 1}$ is uniformly tight and $\mathcal{L}(X_n) \xrightarrow{wa} \mathcal{L}(Y_n)$ as $n \rightarrow \infty$, then $(X_n)_{n \geq 1}$ is also a uniformly tight sequence of r.v.'s.

Proof. See Belyaev [2].

Lemma 2 Let $(X_n, V_n)_{n \geq 1}$ be a sequence of pairs of real valued r.v.'s defined on the same probability space, and $V_n \xrightarrow{P} 0$, $n \rightarrow \infty$. Assume also that $(Y_n)_{n \geq 1}$ is uniformly tight real valued r.v.'s and that $\mathcal{L}(X_n) \xrightarrow{wa} \mathcal{L}(Y_n)$ as $n \rightarrow \infty$. Then $\mathcal{L}(X_n + V_n) \xrightarrow{wa} \mathcal{L}(Y_n)$ as $n \rightarrow \infty$.

Proof. See Belyaev [2].

Definition 4 A sequence of real valued random variables $(X_n)_{n \geq 1}$ is said to be properly random if there is no sequence of real numbers $(a_n)_{n \geq 1}$ such that $X_n - a_n \xrightarrow{P} 0$ as $n \rightarrow \infty$.

If $\mathcal{L}(X_n) = N_1(0, \sigma_n^2)$, $0 < \sigma_-^2 \leq \sigma_n^2 \leq \sigma_+^2 < \infty$, then $(X_n)_{n \geq 1}$ is properly random.

Lemma 3 Let $(X_n)_{n \geq 1}$ be uniformly tight. Then $(X_n)_{n \geq 1}$ is properly random iff for any sequence of real numbers $(a_n)_{n \geq 1}$, there exists $c(\cdot) \in \mathcal{C}_b(\mathbb{R}^1)$ such that $\mathbb{E}c(X_n) - c(a_n) \not\rightarrow 0$ as $n \rightarrow \infty$.

Proof. See Belyaev [2].

Note. If $(Y_n)_{n \geq 1}$ is uniformly tight, properly random and $\mathcal{L}(X_n) \xrightarrow{w} \mathcal{L}(Y_n)$, as $n \rightarrow \infty$, then $(X_n)_{n \geq 1}$ is also uniformly tight and properly random. $\square$

Let $G_n(s) = \mathbb{E}I(Y_n \leq s)$, $F_n(s) = \mathbb{E}I(X_n \leq s)$, and $F_n(s | Z_n) = \mathbb{E}(I(X_n \leq s) | Z_n)$ be distribution functions (d.f.'s) of the r.v.'s Y_n, X_n , and X_n given Z_n . The sequence of d.f.'s $(G_n(\cdot))_{n \geq 1}$ is uniformly continuous iff

$$\sup_{n \geq 1} \sup_s |G_n(s+h) - G_n(s)| \rightarrow 0 \quad \text{as } h \rightarrow 0.$$

The following lemma and its corollary show that under natural assumptions, weakly approaching sequences of r.v.'s have uniformly approaching d.f.'s.

Lemma 4 Let $(X_n)_{n \geq 1}, (Y_n)_{n \geq 1}$ be two sequences of r.v.'s with d.f.'s $(F_n(\cdot))_{n \geq 1}, (G_n(\cdot))_{n \geq 1}$. If the sequence (Y_n) is uniformly tight, $(G_n(\cdot))_{n \geq 1}$ is uniformly continuous, and $\mathcal{L}(X_n) \xrightarrow{w} \mathcal{L}(Y_n)$, $n \rightarrow \infty$, then

$$\sup_s |F_n(s) - G_n(s)| \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (4)$$

Proof. The assumption of uniform continuity $(G_n(\cdot))_{n \geq 1}$ implies that for any s , and for any arbitrarily small $\varepsilon > 0$ we can find a $h_\varepsilon = h_\varepsilon(s) > 0$ such that

$$\sup_s (G_n(s+h_\varepsilon) - G_n(s-h_\varepsilon)) < \frac{\varepsilon}{10}, \quad n = 1, 2, \dots \quad (5)$$

For any s we define two bounded continuous functions of u

$$\begin{aligned} c_{0,\varepsilon}(u, s) &:= I(u \leq s+h_\varepsilon) - ((u-s)/h_\varepsilon)I(s < u \leq s+h_\varepsilon), \\ c_{-,\varepsilon}(u, s) &:= I(u \leq s) - ((u-(s-h_\varepsilon))/h_\varepsilon)I(s-h_\varepsilon < u \leq s). \end{aligned}$$

The assumption $\mathcal{L}(X_n) \xrightarrow{w} \mathcal{L}(Y_n)$, $n \rightarrow \infty$, and (5) imply that for all sufficiently large $n \geq n_\varepsilon(s)$ $|\mathbb{E}c_{i,\varepsilon}(X_n, s) - \mathbb{E}c_{i,\varepsilon}(Y_n, s)| \leq \varepsilon/10$, $i = 0, -$. Hence,

$$\begin{aligned} F_n(s) - G_n(s) &\leq \mathbb{E}c_{0,\varepsilon}(X_n, s) - \mathbb{E}c_{-,\varepsilon}(Y_n, s) \\ &= \mathbb{E}c_{0,\varepsilon}(X_n, s) - \mathbb{E}c_{0,\varepsilon}(Y_n, s) + \mathbb{E}c_{0,\varepsilon}(Y_n, s) - \mathbb{E}c_{-,\varepsilon}(Y_n, s) \\ &\leq \frac{\varepsilon}{10} + G_n(s+h_\varepsilon) - G_n(s-h_\varepsilon) \leq \frac{\varepsilon}{5}, \end{aligned} \quad (6)$$

and

$$\begin{aligned}
F_n(s) - G_n(s) &\geq Ec_{-, \varepsilon}(X_n, s) - Ec_{0, \varepsilon}(Y_n, s) \\
&= Ec_{-, \varepsilon}(X_n, s) - Ec_{-, \varepsilon}(Y_n, s) + Ec_{-, \varepsilon}(Y_n, s) - Ec_{0, \varepsilon}(Y_n, s) \\
&\geq -\frac{\varepsilon}{10} + G_n(s - h_\varepsilon) - G_n(s + h_\varepsilon) \geq -\frac{\varepsilon}{5},
\end{aligned} \tag{7}$$

It follows, that

$$|F_n(s) - G_n(s)| \leq \frac{\varepsilon}{5}, \quad \text{if } n \geq n_\varepsilon(s). \tag{8}$$

By Lemma 1 both sequences $(X_n)_{n \geq 1}$, and $(Y_n)_{n \geq 1}$ are uniformly tight. Therefore we can find sufficiently large $k_\varepsilon > 0$ such that

$$P(|X_n| > k_\varepsilon) < \varepsilon, \quad P(|Y_n| > k_\varepsilon) < \varepsilon, \quad n = 1, 2, \dots \tag{9}$$

We consider a finite set

$$S_{m, \varepsilon} = \left(s_{lm} : s_{lm} = \frac{lk_\varepsilon}{2^m}, \quad l = -2^m, -2^m + 1, \dots, 2^m - 1, 2^m \right),$$

with points s_{lm} , $|s_{l+1, m} - s_{lm}| \leq h_\varepsilon$. From (8) it follows that

$$\max_{s_{lm} \in S_{m, \varepsilon}} |F_n(s_{lm}) - G_n(s_{lm})| \leq \frac{\varepsilon}{5}, \quad \text{if } n \geq n_\varepsilon(m, k_\varepsilon), \tag{10}$$

$n_\varepsilon(m, k_\varepsilon) := \max(n_\varepsilon(s) : s \in S_{m, \varepsilon})$. We also have for s , $s_{lm} \leq s \leq s_{l+1, m}$, s_{lm} , and $s_{l+1, m} \in S_{m, \varepsilon}$, by using (5) and (10), that

$$\begin{aligned}
|F_n(s) - G_n(s)| &\leq F_n(s) - F_n(s_{lm}) + G_n(s) - G_n(s_{lm}) \\
&\quad + |F_n(s_{lm}) - G_n(s_{lm})| \\
&\leq F_n(s_{l+1, m}) - F_n(s_{lm}) + G_n(s_{l+1, m}) - G_n(s_{lm}) \\
&\quad + |F_n(s_{lm}) - G_n(s_{lm})| \\
&\leq |F_n(s_{l+1, m}) - G_n(s_{l+1, m})| + 2(G_n(s_{l+1, m}) - G_n(s_{lm})) \\
&\quad + 2|F_n(s_{lm}) - G_n(s_{lm})| < \varepsilon.
\end{aligned}$$

It follows that

$$\max_{s_{lm}, s_{l+1, m} \in S_{m, \varepsilon}} \sup_{s_{lm} \leq s \leq s_{l+1, m}} |F_n(s) - G_n(s)| < \varepsilon, \quad \text{if } n \geq n_\varepsilon(m, k_\varepsilon).$$

This inequality together with (9) imply

$$\sup_s |F_n(s) - G_n(s)| < \varepsilon, \quad \text{if } n \geq n_\varepsilon(m, k_\varepsilon),$$

and the desired result follows. $\square$

Corollary 1 If $(Y_n)_{n \geq 1}$ is the same as in Lemma 4, and $\mathcal{L}(X_n | Z_n) \xrightarrow{wa(P)} \mathcal{L}(Y_n)$, $n \rightarrow \infty$, then for any small $\varepsilon > 0$

$$P\left(\sup_s |F_n(s | Z_n) - G_n(s)| > \varepsilon\right) \rightarrow 0, \quad n \rightarrow \infty. \quad (11)$$

Proof. Let $\varepsilon > 0$ and $\delta > 0$ be any small numbers. We can find a sufficiently large $k_{\varepsilon\delta} > 0$ such that (3) holds with $\delta/2$ instead of δ , and such that

$$G_n(-k_{\varepsilon\delta}) + 1 - G_n(k_{\varepsilon\delta}) < \varepsilon/2. \quad (12)$$

We can write (3) in the equivalent form

$$P(F_n(-k_{\varepsilon\delta} - | Z_n) + 1 - F_n(k_{\varepsilon\delta} | Z_n) < \varepsilon) > 1 - \delta/2. \quad (13)$$

Let $k_\varepsilon = k_{\varepsilon\delta}$, m , and $S_{m\varepsilon}$ be as in Lemma 4. From the assumption $\mathcal{L}(X_n | Z_n) \xrightarrow{wa(P)} \mathcal{L}(Y_n)$, $n \rightarrow \infty$, and inequalities similar to (6) and (7) with $s \in S_{m\varepsilon}$, and with $F_n(s | Z_n)$ and $E(c_{i\varepsilon}(X_n, s) | Z_n)$ instead of $F_n(s)$ and $E c_{i\varepsilon}(X_n, s)$, we have

$$P\left(|F_n(s_{lm} | Z_n) - G_n(s_{lm})| \leq \frac{\varepsilon}{5}\right) \geq 1 - \frac{\delta}{2(2^{m+1} + 1)},$$

if n is sufficiently large, say $n \geq n'_\varepsilon(m, k_{\varepsilon\delta})$.

Further, by similar arguments to those in the proof of Lemma 4, we obtain

$$P\left(\max_{s_{lm}, s_{l+1}, m \in S_{m\varepsilon}} \sup_{s_{l+1}, m \leq s \leq s_{lm}} |F_n(s | Z_n) - G_n(s)| < \varepsilon\right) > 1 - \delta/2, \quad (14)$$

if $n \geq n'_\varepsilon(m, k_\varepsilon)$. Inequalities (12), (13) and (14) imply

$$P\left(\sup_s |F_n(s | Z_n) - G_n(s)| < \varepsilon\right) > 1 - \delta, \quad (15)$$

for all sufficiently large n . If $\varepsilon > 0$ is given we can choose $\delta > 0$ arbitrarily small. Hence, inequality (15) implies the assertion of Corollary 1. $\square$

Remark. Under natural assumptions, weakly approaching sequences of r.v.'s approach each other in the Levy metric. Let $\mathcal{C}_n = \{s : G_n(s-) = G_n(s+)\}$, i.e. $\mathcal{C}_n$ contains all points of continuity of $G_n(\cdot)$. If $\mathcal{C}_n = \mathcal{C}_0$, i.e. the sets $\mathcal{C}_n$ are the same, and at any $s \in \mathcal{C}_0$ for any $\varepsilon > 0$ exists $h_\varepsilon > 0$ such that

$$\sup_n |G_n(s + h_\varepsilon) - G_n(s - h_\varepsilon)| < \varepsilon,$$

then the Levy distance

$$d_L(\mathcal{L}(X_n | Z_n), \mathcal{L}(Y_n)) \xrightarrow{P} 0, \quad n \rightarrow \infty. \quad (16)$$

If $G_n(\cdot) = G_0(\cdot)$, $n = 1, 2, \dots$, then (16) is valid. $\square$

Together with the sequences of r.v.'s $(X_n)_{n \geq 1}$ and $(Y_n)_{n \geq 1}$, we consider their characteristic functions (ch.f.) $f_n(t) := E e^{itX_n}$, and $g_n(t) := E e^{itY_n}$, $n \in \mathbb{Z}^+$. Here $i = \sqrt{-1}$. We restate the well-known continuity theorem (see Shiryaev [10], page 320).

Proposition 1 (Continuity theorem)

(i) If $\mathcal{L}(Y_n) \xrightarrow{w} \mathcal{L}(Y_0)$ as $n \rightarrow \infty$, then $g_n(t) \rightarrow g_0(t)$ as $n \rightarrow \infty$, for each $t \in \mathbf{R}^1$ and $g_0(t) = Ee^{itY_0}$.

(ii) If $g(t) = \lim_{n \rightarrow \infty} g_n(t)$ for each t , and $g(\cdot)$ is continuous at $t = 0$, then $g(\cdot)$ is the ch.f. of some r.v. Y_0 , i.e. $g(\cdot) = g_0(\cdot)$, and $\mathcal{L}_n(Y_n) \xrightarrow{w} \mathcal{L}(Y_0)$ as $n \rightarrow \infty$.

The following theorem is an extension of the above cited continuity theorem.

Theorem 1 (Continuity theorem for sequences of d.l.'s that weakly approach each other) Let $(X_n)_{n \geq 1}$, and $(Y_n)_{n \geq 1}$ be two sequences of r.v.'s, and suppose $(Y_n)_{n \geq 1}$ is uniformly tight. Then

$$\mathcal{L}(X_n) \xrightarrow{wg} \mathcal{L}(Y_n) \quad \text{as } n \rightarrow \infty, \quad (17)$$

iff for each $t \in \mathbf{R}^1$

$$f_n(t) - g_n(t) \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (18)$$

Proof. (Necessity of (18)). We note that $\sin x$ and $\cos x$ are continuous and bounded functions. Hence, for any $t \in \mathbf{R}^1$ from (17) it follows that

$$|f_n(t) - g_n(t)| \leq |E \cos(tX_n) - E \cos(tY_n)| + |E \sin(tX_n) - E \sin(tY_n)| \rightarrow 0$$

as $n \rightarrow \infty$.

(Sufficiency of (18)). See Belyaev [2].

Lemma 5 If $(Y_n)_{n \geq 1}$ is a uniformly tight sequence of real valued r.v.'s, then the sequence of corresponding ch.f.'s $(g_n(\cdot))_{n \geq 1}$, $g_n(t) := Ee^{itY_n}$, $t \in \mathbf{R}^1$, is uniformly equicontinuous, i.e. for any $\varepsilon > 0$ there is a $\delta > 0$ such that

$$\sup_{n \in \mathbf{Z}^+} \sup_{h: |h| \leq \delta} \sup_{t \in \mathbf{R}^1} |g_n(t+h) - g_n(t)| \leq \varepsilon. \quad (19)$$

Proof. See Belyaev [2].

Let $(\mathcal{L}(X_n | Z_n))_{n \geq 1}$ and $(\mathcal{L}(Y_n))_{n \geq 1}$ be two sequences of d.l.'s as introduced in Definition 2. We denote by $f_n(t | Z_n) := E(e^{itX_n} | Z_n)$ the ch.f. of X_n given Z_n . As before let $g_n(t) := Ee^{itY_n}$.

Lemma 6 If $(Y_n)_{n \geq 1}$ is uniformly tight and if for any $t \in \mathbf{R}^1$

$$|f_n(t | Z_n) - g_n(t)| \xrightarrow{P} 0 \quad \text{as } n \rightarrow \infty, \quad (20)$$

then the sequence of r.v.'s $(X_n | Z_n)_{n \geq 1}$ is uniformly tight along $(Z_n)_{n \geq 1}$ in probability.

Proof. We recall that for a r.v. V with ch.f. $h(\cdot)$, and any $u > 0$, the following inequality is valid

$$P\left(|V| > \frac{1}{u}\right) \leq \frac{7}{u} \int_0^u (1 - \operatorname{Re} h(t)) dt,$$

see e.g. Shiryaev [10], page 324.

This inequality implies that a.s. for any given Z_n

$$\begin{aligned} P\left(|X_n| > \frac{1}{u} \mid Z_n\right) &\leq \frac{7}{u} \int_0^u (1 - \operatorname{Re} f_n(t \mid Z_n)) dt \leq \frac{7}{u} \int_0^u |1 - g_n(t)| dt \\ &\quad + \frac{7}{u} \int_0^u |g_n(t) - f_n(t \mid Z_n)| dt. \end{aligned} \quad (21)$$

Let ε be any small value ($0 < \varepsilon < 1$) and let $a_\varepsilon > 0$ be large enough such that

$$P(|Y_n| > a_\varepsilon) < \frac{\varepsilon}{42}. \quad (22)$$

We take $u_\varepsilon > 0$ such that $3u_\varepsilon a_\varepsilon \leq \varepsilon/42$. Then it follows

$$\begin{aligned} I_n(\varepsilon) &:= \sup_{0 \leq t \leq u_\varepsilon} E |1 - e^{itY_n}| I(|Y_n| \leq a_\varepsilon) \\ &\leq \sup_{0 \leq t \leq u_\varepsilon} E \left(2 \sin^2 \left(\frac{tY_n}{2} \right) + |\sin(tY_n)| \right) I(|Y_n| \leq a_\varepsilon) \\ &\leq 2 \sin^2 \left(\frac{u_\varepsilon a_\varepsilon}{2} \right) + |\sin(u_\varepsilon a_\varepsilon)| \leq 3u_\varepsilon a_\varepsilon \leq \frac{\varepsilon}{42}. \end{aligned}$$

By using (22) we have

$$\begin{aligned} \sup_{0 \leq t \leq u_\varepsilon} |1 - g_n(t)| &\leq I_n(\varepsilon) + \sup_{0 \leq t \leq u_\varepsilon} E(|1 - e^{itY_n}| I(|Y_n| > a_\varepsilon)) \\ &\leq \frac{\varepsilon}{42} + 2P(|Y_n| > a_\varepsilon) \leq \frac{\varepsilon}{14}. \end{aligned} \quad (23)$$

The inequality $|g_n(t) - f_n(t \mid Z_n)| \leq 2$, assumption (20) and Lebesgue's Dominated Convergence Theorem give

$$E \int_0^{u_\varepsilon} |g_n(t) - f_n(t \mid Z_n)| dt = \int_0^{u_\varepsilon} E |g_n(t) - f_n(t \mid Z_n)| dt \rightarrow 0$$

as $n \rightarrow \infty$.

It follows that

$$J_n(\varepsilon) := \frac{7}{u_\varepsilon} \int_0^{u_\varepsilon} |g_n(t) - f_n(t \mid Z_n)| dt = o_p(1), \quad n \rightarrow \infty. \quad (24)$$

From relations (21), and (23) we have

$$P\left(|X_n| > \frac{1}{u_\varepsilon} \mid Z_n\right) \leq 7 \sup_{0 \leq t \leq u_\varepsilon} |1 - g_n(t)| + J_n(\varepsilon) \leq \frac{\varepsilon}{2} + J_n(\varepsilon).$$

Therefore, by (24) it follows that

$$P\left(P\left(|X_n| > \frac{1}{u_\varepsilon} \mid Z_n\right) > \varepsilon\right) \leq P\left(\frac{\varepsilon}{2} + J_n(\varepsilon) > \varepsilon\right) = P\left(J_n(\varepsilon) > \frac{\varepsilon}{2}\right) \rightarrow 0$$

as $n \rightarrow \infty$. This relation holds for arbitrarily small $\varepsilon > 0$ and, with regard to Definition 3, for $k_\varepsilon = 1/u_\varepsilon$. $\square$

Lemma 7 *If $((X_n \mid Z_n))_{n \geq 1}$ is uniformly tight along $(Z_n)_{n \geq 1}$ in probability and*

$$w_n(\alpha \mid Z_n) := \sup_{|h| \leq \alpha} \sup_{t \in \mathbb{R}^1} |f_n(t+h \mid Z_n) - f_n(t \mid Z_n)|,$$

then for arbitrarily small $\varepsilon' > 0$

$$\sup_n P(w_n(\alpha \mid Z_n) > \varepsilon') \rightarrow 0 \quad \text{as } \alpha \rightarrow 0. \quad (25)$$

Proof. Let ε, δ be arbitrarily small positive numbers, and let $c_{\varepsilon\delta}$ be such that

$$\sup_n P(P(|X_n| > c_{\varepsilon\delta} \mid Z_n) > \varepsilon) < \delta. \quad (26)$$

We consider the events

$$A_{\varepsilon\delta n} = (\omega : P(|X_n| > c_{\varepsilon\delta} \mid Z_n) > \varepsilon),$$

$$A_{\varepsilon\delta n}^c = (\omega : P(|X_n| > c_{\varepsilon\delta} \mid Z_n) \leq \varepsilon).$$

(26) implies that $\sup_n P(A_{\varepsilon\delta n}) \leq \delta$, and we have

$$\begin{aligned} P(|X_n| > c_{\varepsilon\delta}) &= E(E(I(|X_n| > c_{\varepsilon\delta}) \mid Z_n)) \\ &= E(P(|X_n| > c_{\varepsilon\delta} \mid Z_n)I(A_{\varepsilon\delta n})) \\ &\quad + E(P(|X_n| > c_{\varepsilon\delta} \mid Z_n)I(A_{\varepsilon\delta n}^c)) \\ &\leq P(A_{\varepsilon\delta n}) + \varepsilon P(A_{\varepsilon\delta n}^c) \leq \delta + \varepsilon \end{aligned} \quad (27)$$

for any $n = 1, 2, \dots$.

If $|X_n| \leq c_{\varepsilon\delta}$ and $h \leq 1/c_{\varepsilon\delta}$, then

$$\begin{aligned} u_n(h \mid Z_n) &:= \sup_{t \in \mathbb{R}^1} |f_n(t+h \mid Z_n) - f_n(t \mid Z_n)| \\ &\leq E(|e^{ihX_n} - 1| \mid (I(|X_n| \leq c_{\varepsilon\delta}) + I(|X_n| > c_{\varepsilon\delta})) \mid Z_n) \\ &\leq 2 \sin^2(h c_{\varepsilon\delta}/2) + |\sin(h c_{\varepsilon\delta})| + 2P(|X_n| > c_{\varepsilon\delta} \mid Z_n) \\ &\leq 2(h c_{\varepsilon\delta} + P(|X_n| > c_{\varepsilon\delta} \mid Z_n)). \end{aligned} \quad (28)$$

If $|h| \leq \alpha \leq c_{\varepsilon\delta}^{-1}$, then from (27), (28) and the Markov inequality, with any $\varepsilon' > 0$, it follows that

$$\begin{aligned} P(w_n(\alpha \mid Z_n) > \varepsilon') &= P\left(\sup_{|h| \leq \alpha} u_n(h \mid Z_n) > \varepsilon'\right) \leq \frac{1}{\varepsilon'} E w_n(\alpha \mid Z_n) \\ &\leq \frac{1}{\varepsilon'} 2(c_{\varepsilon\delta} \alpha + P(|X_n| > c_{\varepsilon\delta})) \leq \frac{2}{\varepsilon'} (c_{\varepsilon\delta} \alpha + \delta + \varepsilon). \end{aligned} \quad (29)$$

For any given $\varepsilon' > 0$ we can choose so small $\varepsilon > 0$, and $\delta > 0$, that $\delta + \varepsilon \leq \varepsilon'/4$. Then we have $c_{\varepsilon\delta} < \infty$ and we can take $\alpha \leq \varepsilon'^2/(4c_{\varepsilon\delta})$. Hence (29) gives $P(w_n(\alpha | Z_n) > \varepsilon') \leq \varepsilon'$ for all $n = 1, 2, \dots$. We thus have

$$\sup_n P(w_n(\alpha | Z_n) > \varepsilon') \leq \varepsilon'. \quad (30)$$

Since ε' can be chosen arbitrarily small (30) is equivalent to (25), and so the lemma is proved. $\square$

Corollary 2 *There is a sequence $(\alpha_n)_{n \geq 1}$, $\alpha_n \downarrow 0$, $n \rightarrow \infty$, such that almost surely*

$$w_n(\alpha_n | Z_n) \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (31)$$

Proof. Let $\varepsilon_r \rightarrow 0$, $r \rightarrow \infty$ and let $\alpha_0 = 1$. By using Lemma 7 for any fixed $r = 1, 2, \dots$ we can find a small $\alpha_r > 0$ such that $\alpha_r < \alpha_{r-1}$

$$\sup_n P(w_n(\alpha_r | Z_n) > \varepsilon_r) < \frac{1}{2^r}.$$

Then, of course,

$$Q_r = P(w_r(\alpha_r | Z_r) > \varepsilon_r) < \frac{1}{2^r},$$

and $\sum_{r=1}^{\infty} Q_r < \infty$. By the Borel - Cantelli lemma almost surely there exists an $r(\omega)$ such that for all $r \geq r(\omega)$

$$0 \leq w_r(\alpha_r | Z_r) \leq \varepsilon_r,$$

and (31) follows. $\square$

The following theorem can be applied to justify certain resampling schemes used to obtain consistent estimation of the d.l.'s of sums of r.v.'s, and in many other applications.

Theorem 2 (Continuity theorem for sequences of random d.l.'s that weakly approach deterministic sequences of d.l.'s in probability)
Let $(X_n, Z_n)_{n \geq 1}$, and $(Y_n)_{n \geq 1}$ be two sequences of r.v.'s as in Definition 2, and let $(Y_n)_{n \geq 1}$ be uniformly tight. Then

$$\mathcal{L}(X_n | Z_n) \xrightarrow{wa(P)} \mathcal{L}(Y_n) \quad \text{as } n \rightarrow \infty, \quad (32)$$

iff for each $t \in \mathbb{R}^1$

$$f_n(t | Z_n) - g_n(t) \xrightarrow{P} 0 \quad \text{as } n \rightarrow \infty. \quad (33)$$

Proof. Assume that (32) holds. For any $t \in \mathbf{R}^1$ both functions $a_t(x) = \cos(tx)$ and $b_t(x) = \sin(tx)$, $x \in \mathbf{R}^1$, are in $C_b(\mathbf{R}^1)$. Therefore, for each $t \in \mathbf{R}^1$ we have as $n \rightarrow \infty$

$$E(a_t(X_n) | Z_n) - Ea_t(Y_n) \xrightarrow{P} 0, \quad E(b_t(X_n) | Z_n) - Eb_t(Y_n) \xrightarrow{P} 0. \quad (34)$$

We have $f_n(t | Z_n) = E(a_t(X_n) | Z_n) + iE(b_t(X_n) | Z_n)$, and $g_n(t) = Ea_t(Y_n) + iEb_t(Y_n)$. Hence, (34) implies (33).

Now assume that (32) does not hold. Then we can find $c_0(\cdot) \in C_b(\mathbf{R}^1)$ such that

$$D_n = E(c_0(X_n) | Z_n) - Ec_0(Y_n) \not\xrightarrow{P} 0 \quad \text{as } n \rightarrow \infty. \quad (35)$$

Hence there is a subsequence $(n_k)_{k \geq 1}$ such that for some small $\varepsilon > 0$

$$\gamma = \inf_{n_k} P(|D_{n_k}| \geq \varepsilon) > 0. \quad (36)$$

Let $a_n(Z_n) = E(c_0(X_n) | Z_n)$, $b_n = Ec_0(Y_n)$, $n \in \mathbb{Z}^+$. We have $a_{n_k}(Z_n), b_{n_k} \in [-\inf_{t \in \mathbf{R}^1} c_0(t), \sup_{t \in \mathbf{R}^1} c_0(t)]$, $k \in \mathbb{Z}^+$. Let b_0 be an accumulation point of $(b_{n_k})_{k \geq 1}$, e.g. we can choose one such point with the largest absolute value. Then there is a subsequence $(n_{k(l)})_{l \geq 1}$ of the sequence $(n_k)_{k \geq 1}$, along which $b_{n_{k(l)}} \rightarrow b_0$, as $l \rightarrow \infty$. The sequence of r.v.'s $(Y_{n_{k(l)}})_{l \geq 1}$ is still uniformly tight. Hence, by Prohorov's theorem (see Shiryaev [10], page 318) we can find a r.v. Y_0 and a new subsequence $(\tilde{n}_m)_{m \geq 1}$, $\tilde{n}_m = n_{k(l_m)}$, with the property $\mathcal{L}(Y_{\tilde{n}_m}) \xrightarrow{w} \mathcal{L}(Y_0)$ as $m \rightarrow \infty$, where Y_0 is a r.v. By the continuity theorem (see Proposition 1 (i)) for each $t \in \mathbf{R}^1$

$$g_{\tilde{n}_m}(t) \rightarrow g_0(t) \quad \text{as } m \rightarrow \infty. \quad (37)$$

(33), and (37) imply for each $t \in \mathbf{R}^1$

$$f_{\tilde{n}_m}(t | Z_n) - g_0(t) \xrightarrow{P} 0 \quad \text{as } m \rightarrow \infty. \quad (38)$$

For any fixed t it is possible to find a subsequence of $(f_{\tilde{n}_m}(t | Z_{\tilde{n}_m}))_{m \geq 1}$ that converges almost surely to $g_0(t)$ (see Shiryaev [10], p. 258). For any given $t_0 > 0$ and any $r \in \mathbb{Z}^+$, we consider $\mathcal{T}_r(t_0) := \{t_{kr} : t_{kr} = \frac{k}{2^r} t_0, k \in \{0, \pm 1, \pm 2, \pm 3, \dots, \pm 2^r\}\}$, and $\mathcal{T}_r(t_0) \subset \mathcal{T}_{r+1}(t_0)$. We can take a subsequence $(n_r(m))_{m \geq 1}$ of the sequence $(\tilde{n}_m)_{m \geq 1}$ such that

$$\max_{t \in \mathcal{T}_r(t_0)} |f_{n_r(m)}(t | Z_{n_r(m)}) - g_0(t)| \rightarrow 0 \quad \text{a.s. as } m \rightarrow \infty. \quad (39)$$

We can select these subsequences $(n_r(m))_{m \geq 1}$ in the following way. We start with $r = 1$ and find $(n_1(m))_{m \geq 1}$. For $r = 2$ we find $(n_2(m))_{m \geq 1}$ as a subsequence of $(n_1(m))_{m \geq 1}$ and so on. By Cantor's method, taking one term $n_r(r)$ from each subsequence $(n_r(m))_{m \geq 1}$, $r \in \mathbb{Z}^+$, we construct the universal subsequence $(n_r(r))_{r \geq 1}$ for which we have

$$\max_{t \in \mathcal{T}_0(t_0)} |f_{n_r(r)}(t | Z_{n_r(r)}) - g_0(t)| \xrightarrow{a.s.} 0 \quad \text{as } r \rightarrow \infty,$$

where r_0 is any fixed number, $r_0 \in \mathbb{Z}^+$. For each $u \in [-t_0, t_0]$ we can find a $t_r(u)$ such that $|t_{r_0}(u) - u| = \min_{t \in T_{r_0}} |t - u|$. For any r_0 , and r we will use the notation:

$$p_{r_0, r}(t_0) = \max_{t \in T_{r_0}(t_0)} |f_{n_r(r)}(t | Z_{n_r(r)}) - g_0(t)|,$$

$$q_{r_0}(t_0) = \max_{-t_0 \leq u \leq t_0} |g_0(t_{r_0}(u)) - g_0(u)|.$$

We have

$$\begin{aligned} & \max_{-t_0 \leq u \leq t_0} |f_{n_r(r)}(u | Z_{n_r(r)}) - g_0(u)| \\ & \leq \max_{-t_0 \leq u \leq t_0} |f_{n_r(r)}(u | Z_{n_r(r)}) - f_{n_r(r)}(t_{r_0}(u) | Z_{n_r(r)})| \\ & + p_{r_0, r}(t_0) + q_{r_0}(t_0) \leq w_{n_r(r)}(t_0/2^{r_0} | Z_{n_r(r)}) \\ & + p_{r_0, r}(t_0) + q_{r_0}(t_0). \end{aligned} \quad (40)$$

By Corollary 2, for any small given number $\beta > 0$, and for almost every $\omega \in \Omega$, we can find an $r'_0 = r'_0(\omega)$ such that for all $r \geq r'_0(\omega)$

$$w_{n_r(r)}(t_0/2^{r'_0} | Z_{n_r(r)}) < \beta/3.$$

The uniform continuity of the ch.f. $g_0(\cdot)$ implies the existence of an r''_0 , such that $q_r(t_0) < \beta/3$ for all $r \geq r''_0$. From the construction of the sequence $(n_r(r))_{r \geq 1}$ there almost surely exists $r''_0 = r''_0(\omega)$ such that for all $r \geq r''_0(\omega)$, we have $p_{r'_0, r}(t_0) < \beta/3$. We denote $r^*_0(\omega) = \max(r''_0, r'_0(\omega))$. Then for almost all $\omega \in \Omega$ for any $r \geq \max(r^*_0(\omega), r''_0(\omega))$ the inequality

$$\max_{-t_0 \leq u \leq t_0} |f_{n_r(r)}(u | Z_{n_r(r)}) - g_0(u)| \leq \beta$$

holds for arbitrary small $\beta > 0$. Hence for any $t \in [-t_0, t_0]$

$$f_{n_r(r)}(t | Z_{n_r(r)}) - g_0(t) \xrightarrow{a.s.} 0 \quad \text{as } r \rightarrow \infty. \quad (41)$$

By the same argument we can find a subsequence $(n_r(r, 2))_{r \geq 1}$ of the sequence $(n_r(r))_{r \geq 1}$ satisfying (41) on $[-2t_0, 2t_0]$. We can then find a subsequence $(n_r(r, 3))_{r \geq 1}$ of $(n_r(r, 2))_{r \geq 1}$ satisfying (41) on $[-4t_0, 4t_0]$, and so on. We take the subsequence $(n_r(r, r))_{r \geq 1}$ to be "universal" in the sense that, for any $t \in \mathbb{R}^1$, along almost every sequence $(Z_{n_r(r, r)})_{r \geq 1}$

$$f_{n_r(r, r)}(t | Z_{n_r(r, r)}) - g_0(t) \rightarrow 0 \quad \text{as } r \rightarrow \infty. \quad (42)$$

We stress that (42) holds for any $t \in \mathbb{R}^1$ when we have $(Z_{n_r(r, r)}(\omega))_{r \geq 1}$, i.e. we almost surely don't need to change ω when we consider all $t \in \mathbb{R}^1$. Therefore, we can use Theorem 1, and along almost every sequence $(Z_{n_r(r, r)})_{r \geq 1}$ we have

$$\mathcal{L}(X_{n_r(r, r)} | Z_{n_r(r, r)}) \xrightarrow{w} \mathcal{L}(Y_0) \quad \text{as } r \rightarrow \infty.$$

This implies that

$$a_{n_r(r,r)}(Z_{n_r(r,r)}) - b_{n_r(r,r)} \xrightarrow{a.s.} 0 \quad \text{as } n \rightarrow \infty.$$

Since this contradicts (36), it follows that (35) cannot hold. Thus the theorem is proved. $\square$

The following auxiliary r.v.'s will be used. Let $J_{1n}^*, \dots, J_{nn}^*$ be i.i.d. r.v.'s, $P^*(J_{in}^* = h) = 1/n$, $h = 1, \dots, n$, $M_{hn}^* := \sum_{j=1}^n I(J_{jn}^* = h)$, $M_n^* = M_{1n}^* + \dots + M_{nn}^* = n$. We use $P^*(X^* \in A)$ and E^*X^* for probability expectation when a random variable X^* is obtained via the r.v.'s $J_{1n}^*, \dots, J_{nn}^*$. We have

$$E^*M_{hn}^* = \sum_{j=1}^n E^*I(J_{jn}^* = h) = 1, \quad (43)$$

$$\begin{aligned} & E^*(M_{hn}^* - 1)^2 \\ &= \sum_{j=1}^n E^*I(J_{jn}^* = h) + \sum \sum_{j_1 \neq j_2} E^*I(J_{j_1n}^* = h)E^*I(J_{j_2n}^* = h) - 1 \\ &= 1 + n(n-1)/n^2 - 1 = 1 - 1/n, \end{aligned} \quad (44)$$

$$\begin{aligned} E^*(M_{hn}^* - 1)(M_{gn}^* - 1) &= \sum \sum_{j_1 \neq j_2} E^*I(J_{j_1n}^* = h)E^*I(J_{j_2n}^* = g) - 1 \\ &= n(n-1)/n^2 - 1 = -1/n, \quad h \neq g, \quad h, g \in \{1, 2, \dots, n\}. \end{aligned} \quad (45)$$

We also let $M_{hrn}^* := \sum_{j=1}^r I(J_{jn}^* = h)$, and have by similar calculations

$$E^*M_{hrn}^* = \frac{r}{n}, \quad E^*\left(M_{hrn}^* - \frac{r}{n}\right)^2 = \frac{r}{n} \left(1 - \frac{1}{n}\right), \quad (46)$$

$$E^*\left(M_{h_1rn}^* - \frac{r}{n}\right)\left(M_{h_2rn}^* - \frac{r}{n}\right) = -\frac{r}{n^2}, \quad h_1 \neq h_2. \quad (47)$$

One can regard $J_{1n}^*, \dots, J_{nn}^*$ as the results of n independent multinomial experiments with n possible outcomes $\{1, 2, \dots, n\}$. In fact, M_{hn}^* is the number of repetitions of outcome h after n independent multinomial experiments, and M_{hrn}^* after the first r independent multinomial experiments. We can use the r.v.'s $M_{hn}^*, M_{hrn}^*, h = 1, \dots, n$ to define others r.v.'s.

Now we recall one result of Belyaev [1] related to the problem of the asymptotically exact approximation of $\mathcal{L}(U_n)$ mentioned in the Introduction. The following assumptions will be used:

$$\begin{aligned} AI: & \quad U_{in}, i = 1, \dots, n, \text{ are independent r.v.'s for each } n = 1, 2, \dots, \\ AL(2 + \delta): & \quad \text{for } \tilde{U}_{in} := \sqrt{n}U_{in}, \quad E|\tilde{U}_{in}|^{2+\delta} \leq c(2 + \delta) < \infty, \\ & \quad i = 1, \dots, n, n = 1, 2, \dots \end{aligned}$$

The assumptions $AVar$ in Belyaev [1] can be dropped.

Let us define the r.v. $U_n^* := \sum_{h=1}^n M_{hn}^* u_{hn}$. It is possible to interpret U_n^* as a sum of values, n times randomly sampled with replacement from the components in the statistical data $u_n = (u_{1n}, \dots, u_{nn})$. After centering U_n^* by $u_n = \sum_{i=1}^n u_{in}$ we obtain

$$U_n^{*0} := \sum_{h=1}^n M_{hn}^* u_{hn} - u_n = \sum_{h=1}^n (M_{hn}^* - 1) u_{hn}. \quad (48)$$

In fact, the d.l. of the r.v. U_n^{*0} defined in (48) is the conditional d.l. given $U_n = u_n$, and we denote it by $\mathcal{L}(U_n^{*0} | u_n)$. We use the same notation for U_n^{*0} if we consider in (48) the r.v.'s U_{hn} instead of their fixed (observed) values u_{hn} , $h = 1, \dots, n$. In this case we write $\mathcal{L}(U_n^{*0} | U_n)$ for the d.l. of U_n^{*0} . Now we can formulate the following theorem, see Belyaev [1].

Theorem 3 (The Central Limit Resampling Theorem (CLRT)) *If assumptions $AI, AL(2 + \delta)$ hold then*

$$\mathcal{L}(U_n^{*0} | U_n) \xrightarrow{wa(P)} \mathcal{L}(U_n), \quad n \rightarrow \infty,$$

along the sequence of r.v.'s $(U_n)_{n \geq 1}$.

The d.l.'s of the r.v.'s $U_{rn}^{*0} = \sqrt{\frac{n}{r}} \sum_{h=1}^n (M_{hrn}^* - \frac{r}{n}) u_{hn}$ can also be used to approximate the d.l.'s $\mathcal{L}(U)$ if $r = r(n) \rightarrow \infty$, and $r = o(n)$, as $n \rightarrow \infty$. Theorem 3 can be applied to obtain approximations of the d.l.'s of deviations of ordinary least squares estimators (OLS-estimators) from true parameters of heteroscedastic linear regression.

Let $Y_i = \sum_{j=1}^k x_{ij} \beta_{j0} + W_i$ be an observed random response in the i th experiment, with a vector $x_i = (x_{i1}, \dots, x_{ik})^T$ of factors, a vector $\beta_0 = (\beta_{10}, \dots, \beta_{k0})^T$ of unknown true parameters, and with an unobserved value of random error W_i , $i = 1, \dots, n$. We assume that $W_1, \dots, W_n$ are independent r.v.'s, $EW_i = 0$, $EW_i^2 = \sigma_i^2$, where σ_i^2 are unknown and can be different for different i . Let $Y_n = (Y_1, \dots, Y_n)^T$, $X_n = (x_{ij})$, $i = 1, \dots, n$, $j = 1, \dots, k$, and further let $(X_n^T X_n)^+$ be the pseudoinverse matrix for $(X_n^T X_n)$ and $\text{tr}(X_n^T X_n)^+$ be the trace of $(X_n^T X_n)^+$, Rao [8]. We write $A \preceq B$ if $B - A$ is a non-negatively defined matrix. Let β_0 be estimable, i.e. $\beta_0 = (X_n^T X_n)^+ (X_n^T X_n) \beta_0$. Then the unbiased OLS-estimator $\hat{\beta}_n = (X_n^T X_n)^+ X_n^T Y_n$ exists, Searly [9]. Let $c = (c_1, \dots, c_k)^T$ be a given vector, $\|c\| = (c_1^2 + \dots + c_k^2)^{1/2} = 1$, $q_n(c) := (c^T (X_n^T X_n)^+ c)^{1/2}$, $a_{in}(c) := c^T (X_n^T X_n)^+ x_i / q_n(c)$. The typical problem, and one of great interest in heteroscedastic linear regression theory, is to determine the d.l. of the weighted deviations of the OLS-estimators from the true parameters, this is given by the r.v.

$$U_n(c) = c^T (\hat{\beta}_n - \beta_0) / q_n(c).$$

Let $Y_i - \hat{Y}_{in}$ be the residuals, $\hat{Y}_{in} = \mathbf{x}_i^T (\mathbf{X}_n^T \mathbf{X}_n)^+ \mathbf{X}_n^T \mathbf{Y}_n$, $i = 1, \dots, n$. We define the random variable

$$U_n^*(\mathbf{c}) = \sum_{i=1}^n a_{in}(\mathbf{c})(Y_i - \hat{Y}_{in})(M_{in}^* - 1),$$

where the r.v.'s M_{in}^* , $i = 1, \dots, n$ are as before.

Theorem 4 (Belyaev [1]) *Assume that*

$$(i) \quad \sup_n \text{tr}(\mathbf{X}_n^T \mathbf{X}_n)^+ / q_n(\mathbf{c}) < \infty, \quad \text{and} \quad \text{tr}(\mathbf{X}_n^T \mathbf{X}_n)^+ \rightarrow 0 \quad \text{as } n \rightarrow \infty;$$

$$(ii) \quad \sqrt{n} \sup_i \mathbf{c}^T (\mathbf{X}_n^T \mathbf{X}_n)^+ \mathbf{x}_i / q_n(\mathbf{c}) < \infty;$$

$$(iii) \quad \text{for some } 0 < \sigma_- \leq \sigma_+ < \infty \text{ and any } n = 1, 2, \dots$$

$$\sigma_-^2(\mathbf{X}_n^T \mathbf{X}_n) \preceq \sum_{i=1}^n \sigma_i^2 \mathbf{X}_i \mathbf{X}_i^T \preceq \sigma_+^2(\mathbf{X}^T \mathbf{X}).$$

Then as $n \rightarrow \infty$ along $(\mathbf{Y}_n)_{n \geq 1}$

$$\mathcal{L}(U_n^*(\mathbf{c}) \mid \mathbf{Y}_n) \xrightarrow{wa(P)} \mathcal{L}(U_n(\mathbf{c})).$$

Note. The d.l. of $U_n^*(\mathbf{c})$ given $\mathbf{Y}_n$ can be obtained by a simulation of the r.v.'s M_{in}^* , $i = 1, \dots, n$. Actually we obtain this d.l. by the random resampling from the centered weighted residuals $a_{in}(\mathbf{c})(Y_i - \hat{Y}_{in})$, $i = 1, \dots, n$, (compare with recommendations in Wu [12]). The left hand-side inequality in Theorem 4 (iii) can be dropped.

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